

An Introduction to Temporal Spanners

Arnaud Casteigts

LaBRI, Université de Bordeaux

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Based on joint works with:



Jason Schoeters
(Cambridge)



Joseph Peters
(Vancouver)



Michael Raskin
(Bordeaux)



Malte Renken
(Berlin)



Viktor Zamaraev
(Liverpool)



Daniele Carnevale
(L'Aquila)



Timothée Corsini
(Bordeaux)

Temporal graphs

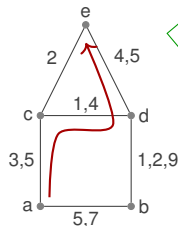
(a.k.a. time-varying, time-dependent, evolving, dynamic,...)

Basic definition:

$\mathcal{G} = (\underline{V}, E, \lambda)$, where $\lambda : E \rightarrow 2^{\mathbb{N}}$ assigns *presence times* to edges.

⏟
footprint of \mathcal{G}

Example:



Can also be viewed as a sequence of *snapshots* $\{G_i = \{e \in E : i \in \lambda(e)\}\}$

Temporally connected

Temporal paths

- ▶ e.g. $\langle (a, c, 3), (c, d, 4), (d, e, 5) \rangle$ (strict)
- ▶ e.g. $\langle (a, c, 3), (c, d, 4), (d, e, 4) \rangle$ (non-strict)

Temporal connectivity: \exists temporal paths between all vertices.

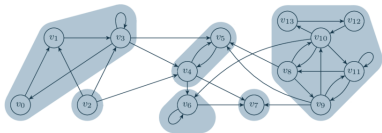
→ Warning: Reachability is non-symmetrical... and **non-transitive**!

Some adjectives: *simple* ($\lambda : E \rightarrow \mathbb{N}$); *proper* (λ locally-injective), *happy* (both).

Impact of non-transitivity

(Example: connected components)

In static graphs



- Components define a partition
- Easy to compute

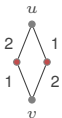
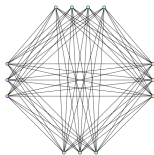
In temporal graphs



- Maximal components may overlap
- Can be exponentially many

MAX COMPONENT is NP-hard! (from CLIQUE)

Bui-Xuan, Ferreira, Jarry, 2003



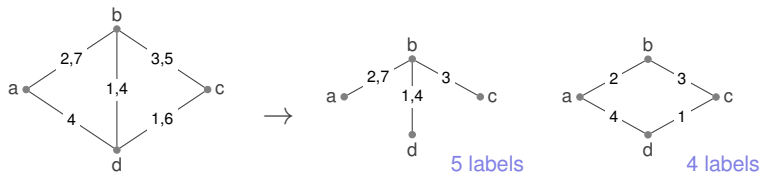
- Replace edges with semaphore gadgets
- Cliques become temporal components

Temporal spanners

Input: a graph \mathcal{G} that is temporally connected ($\mathcal{G} \in TC$)

Output: a subgraph $\mathcal{G}' \subseteq \mathcal{G}$ that preserves temporal connectivity ($\mathcal{G}' \in TC$)

Cost measure: size of the spanner (# labels or # edges)



Can we do better?

► $2n - 4$ labels always needed for proper graphs (or strict journeys) (Bumby'79)

Do spanners of size $2n - 4$ always exist?

► Some graphs with $\Theta(n \log n)$ labels are non-simplifiable (Kleinberg, Kempe, Kumar, 2000)

► Some graphs with $\Theta(n^2)$ labels are non-simplifiable !! (Axiotis, Fotakis, 2016)

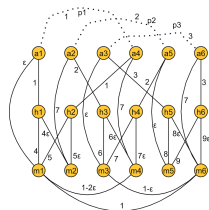
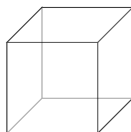
Complexity?

► Minimum-size spanner is APX-hard (Akrida, Gasieniec, Mertzios, Spirakis, 2017)

Bad news and good news

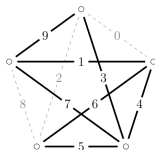
Recall the bad news:

- ▶ $\Omega(n \log n)$ - easy
- ▶ $\Omega(n^2)$ - rather unexpected



Good news 1: (C., Peters, Schoeters, ICALP 2019):

- ▶ Spanners of size $O(n \log n)$ always exist in **complete** temporal graphs

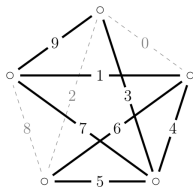


Good news 2: (C., Raskin, Renken, Zamaraev, FOCS 2021):

- ▶ Nearly optimal spanners (of size $2n + o(n)$) almost surely exist in **random** temporal graphs, as soon as the graph is temporally connected

Good news 1:

Temporal cliques admit sparse spanners



(with)



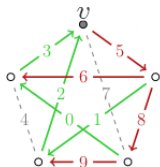
Some promising techniques ? (spoiler: yes and no)

Pivotability

Pivot node v and time t such that:

- ▶ all nodes can reach v before t
- ▶ v can reach all nodes after t

Then **in-tree** \cup **out-tree** = spanner of size $2n - 2$ (in fact $2n - 3$)

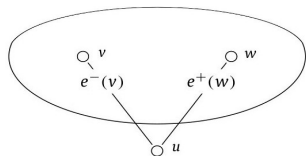


Dismountability

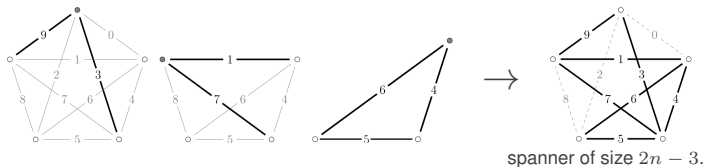
Three nodes u, v, w such that:

- ▶ $uv = \min\text{-edge}(v)$
- ▶ $uw = \max\text{-edge}(w)$

Then $\text{spanner}(\mathcal{G}) := \text{spanner}(\mathcal{G}[V \setminus u]) + uv + uw$



Recursively,



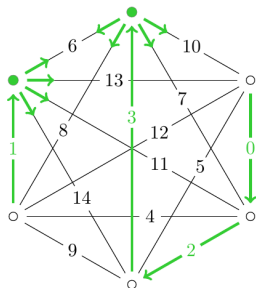
k -hop dismountability

If u has a *journey* towards v that ends at $e^+(v)$ and a journey from w that starts at $e^-(w)$, then u is k -hop dismountable.

Fireworks spanners

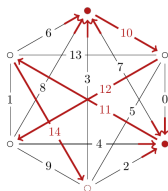
Principle

- ▶ Forest of all minimum edges, oriented temporally
- ▶ Only the leaves broadcast (leaves = **emitters**)
- ▶ Spanner = the forest + all edges of emitters



Backward fireworks also possible

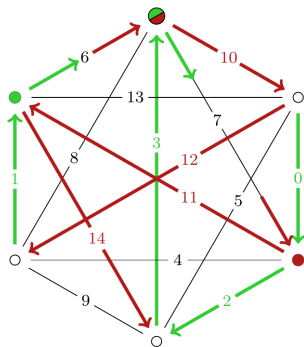
- ▶ Forest of all maximum edges, oriented temporally
- ▶ Only the leaves aggregate (leaves = **collectors**)
- ▶ Spanner = the forest + all edges of collectors



Combining both directions

- ▶ Each vertex reaches at least one emitter u through u 's min edge
- ▶ All emitters reach all collectors (directly)
- ▶ Each vertex can be reached by a collector v through v 's max edge

→ Spanner = min edges + max edges
+ edges between emitters and collectors



Recurse or sparsify?

If #emitters $\geq n/2$ or #collectors $\geq n/2$

Then one of them is 2-hop dismantlable (call it v).

→ recurse on $\mathcal{G} \setminus \{v\}$.

Now, both are $\leq n/2$...

Case 1: emitters \cup collectors $\neq V$

Lemma: At least one node is internal in both forests

→ this node is dismantlable

→ recurse on $\mathcal{G} \setminus \{v\}$

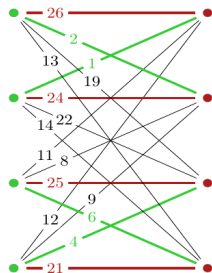
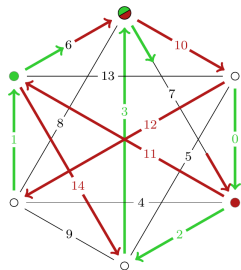
Case 2: emitters \cup collectors $= V$

→ All vertices are either emitters or collectors!

A lot of structure to work with:

- ▶ Complete bipartite graph \mathcal{H} between emitters and collectors
- ▶ Min edges and max edges form two perfect matchings
- ▶ Min edges (and max edges) are *reciprocal* in \mathcal{H}

→ Sparsify \mathcal{H} while preserving reachability from emitters to collectors



Sparsification of the bipartite graph

New objective:

→ Sparsify \mathcal{H} while preserving journeys from each **emitter** to all **collectors**

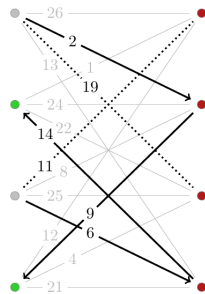
Technique: Partial delegations among emitters

- ▶ Find a 2-hop journey from one emitter to another, arriving through a “locally small” edge
- ▶ Pay extra edges (penalty) to reach missed collectors

Iterative procedure:

In each step i :

- ▶ Half of the **emitters** partially delegate to other half
- ▶ Penalty doubles in each step, but #emitters halves
- ▶ $O(n)$ edges over $O(\log n)$ iterations → $O(n \log n)$ edges.



Conclusion:

∃ spanner of size $O(n \log n)$ □

Open questions (deterministic)

Better spanners for temporal cliques?

- ▶ Is $O(n \log n)$ optimal for cliques? Is $O(n)$ possible?
- ▶ Even better, does $2n - 4 \leq OPT \leq 2n - 3$?
(so far, no counter-example found)

Relaxing the complete graph assumption

- ▶ Can more general classes of dense graphs be sparsified?
 - Recall that \exists unsparsifiable graphs of density $\Theta(n^2)$
 - Is there a family of graphs of density < 1 which admits sparse spanners?

What about random temporal graphs?

Good news 2:

Spanners of size $2n + o(n)$ almost surely exist
in random temporal graphs

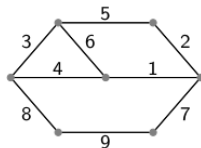
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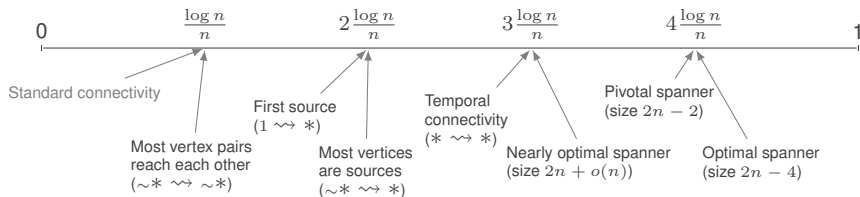
Sharp thresholds in random temporal graphs (C., Raskin, Renken, Zamaraev, 2021)

Random simple temporal graphs:

1. Pick an Erdős-Rényi $G \sim G_{n,p}$
2. Permute the edges randomly, interpret as (unique) presence time



Timeline for p (as $n \rightarrow \infty$):



All the thresholds are sharp.

(sharp: $\exists \epsilon(n) = o(1)$, not true at $(1 - \epsilon(n))p$, true at $(1 + \epsilon(n))p$)