Temporalizing digraphs via linear-size balanced bi-trees

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joint work with :
Alkida Balliu, Filippo Brunelli, Pierluigi Crescenzi, Dennis Olivetti
and :
Stéphane Bessy, Stéphan Thomassé

Île du Saussay

Directed graph temporalisation

Define the temporal reachability of a temporal graph as the number of node pairs that are (strictly) temporally connected.

Problem : given a strongly connected multi-digraph, assign one time label per edge so that temporal reachability is maximal.

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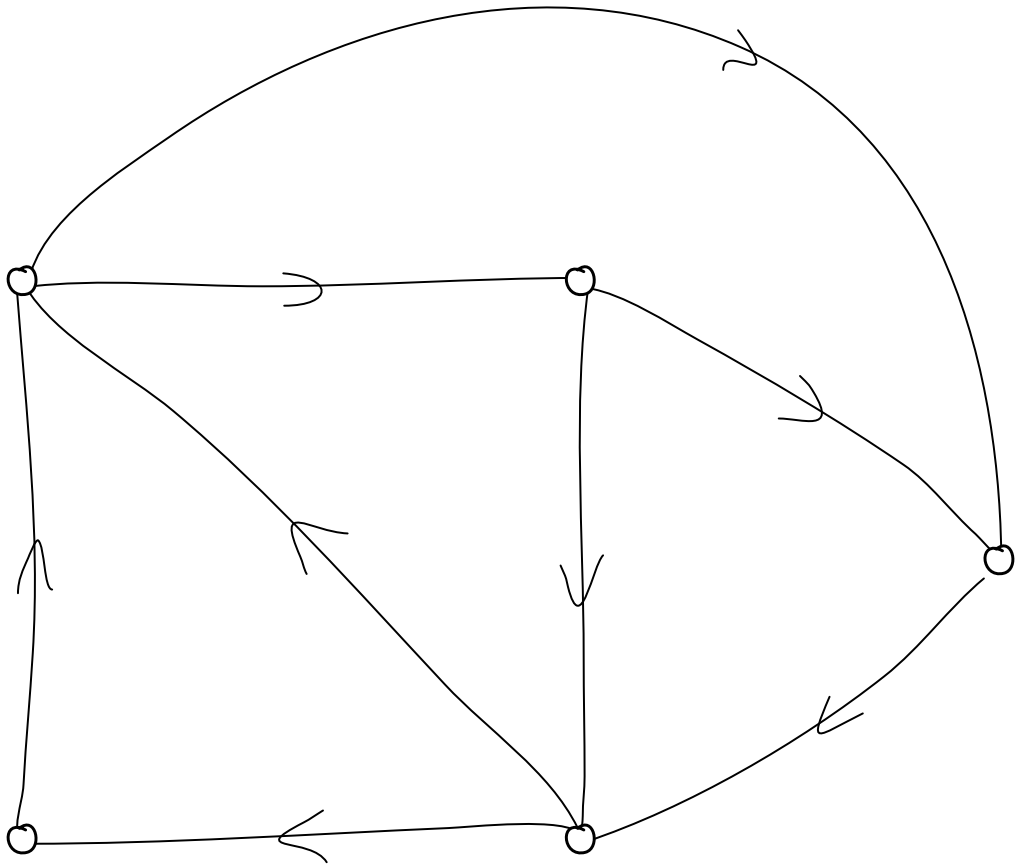

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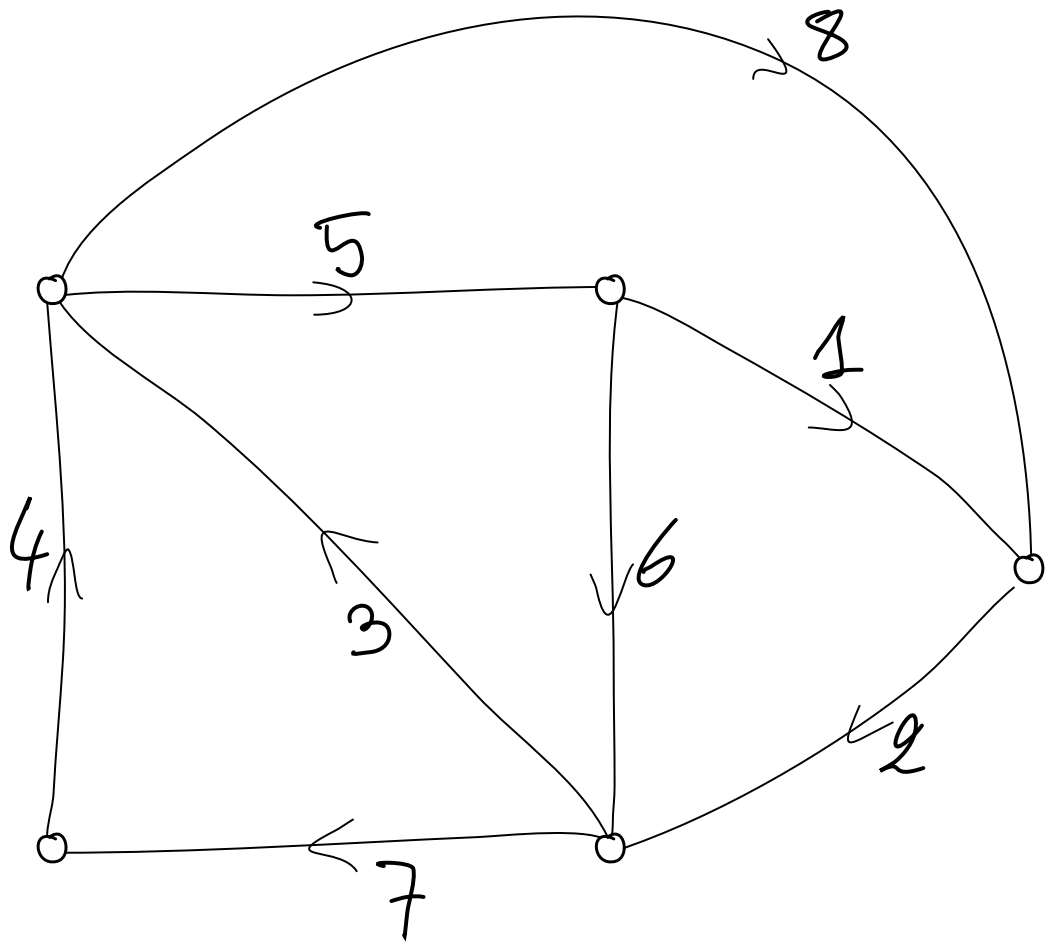

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Original motivation : optimize the schedule of a public transit network.

In a social network : imagine a conference where attendees to talks are known, and where the order of talks is free.

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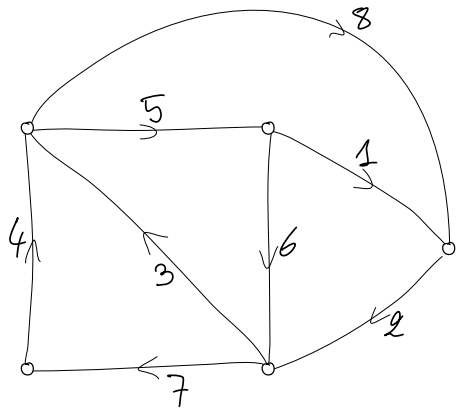

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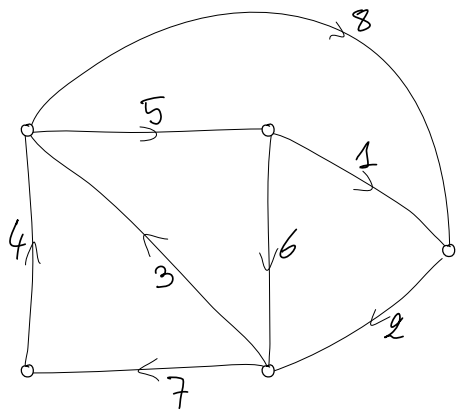

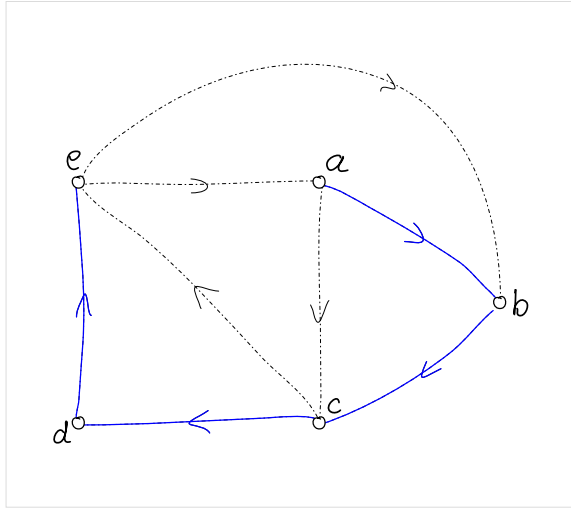

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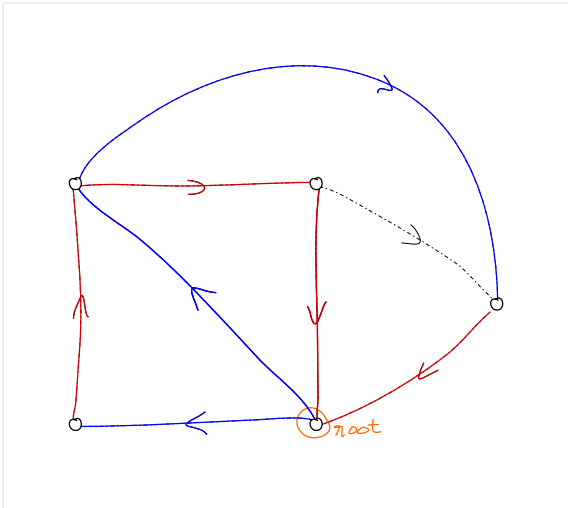

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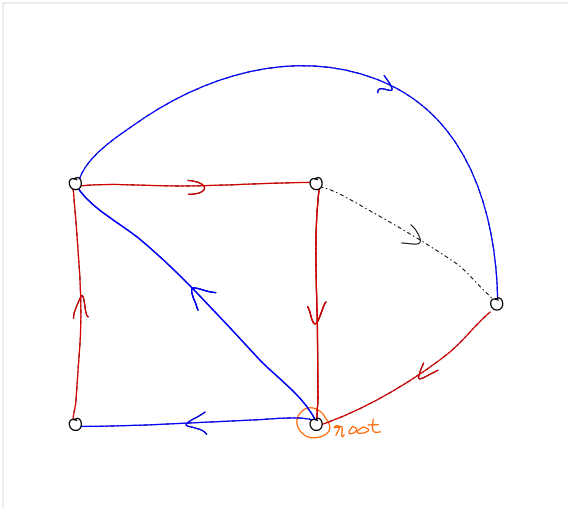

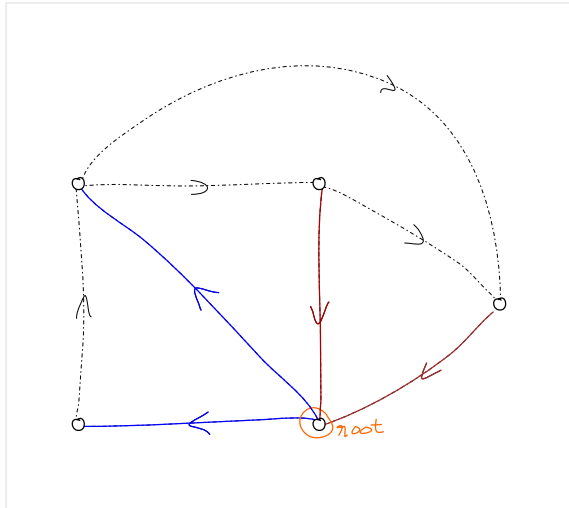

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Related work

Undirected graph : deciding "label connectivity" is NP-complete [Göbel, Cerdeira, Veldman 1991], approximation is easy.

Minimizing $|\lambda|$, i.e. the number of labels, for achieving temporal connectivity is NP-hard [Klobas, Mertzios, Molter, Spirakis 2022].

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Result 1 : hardness

Given a strong digraph D, deciding whether there exists an assignment of one time label per edge such that all pairs are temporally connected is NP-complete. [Balliu, Brunelli, Crescenzi, Olivetti, V. 2023]

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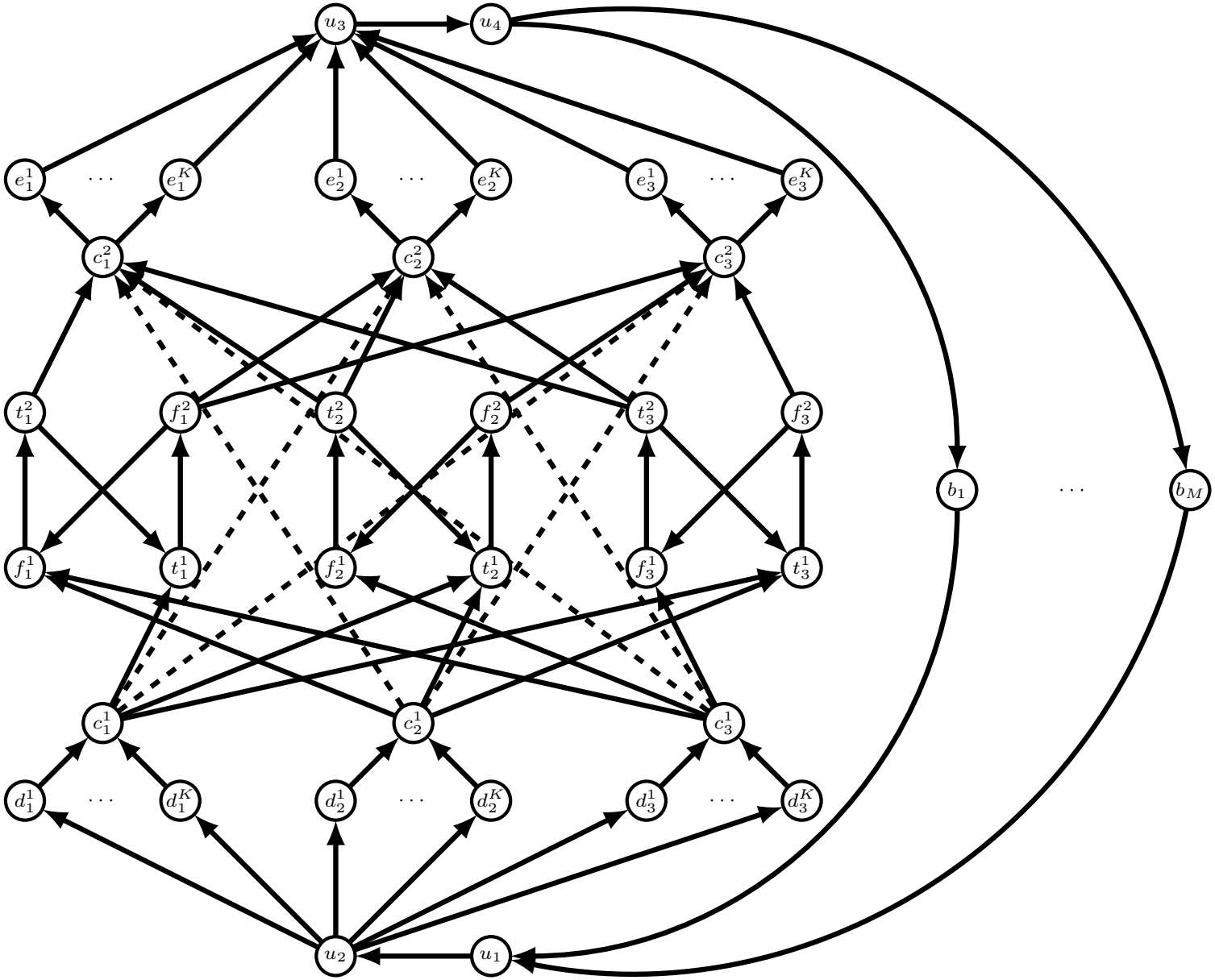
Proof of hardness : reduction from 3SAT



$$(\mathbf{x}_1 \vee \mathbf{x}_2 \vee \mathbf{x}_3) \wedge (\neg\mathbf{x}_1 \vee \mathbf{x}_2 \vee \mathbf{x}_3) \wedge (\neg\mathbf{x}_1 \vee \neg\mathbf{x}_2 \vee \neg\mathbf{x}_3)$$

Result 2 : approximation

Any strong digraph D has a pair of node-disjtoint in-tree and out-tree both spanning $n/6$ nodes that can be computed in $O(n^2)$ time [Bessy, Thomassé, V. 2023].

Lemma : any strong digraph (V, A) has a balanced cyclic separator C, that is V can be partionned in I, C, O such that :

- C is spanned by a directed cycle,
- there are no arcs from I to O (directed separator),
- both $I \cup C$ and $I \cup O$ has size at least $n/3$ (balanced).

Main tool : a "left-maximal" DFS tree.

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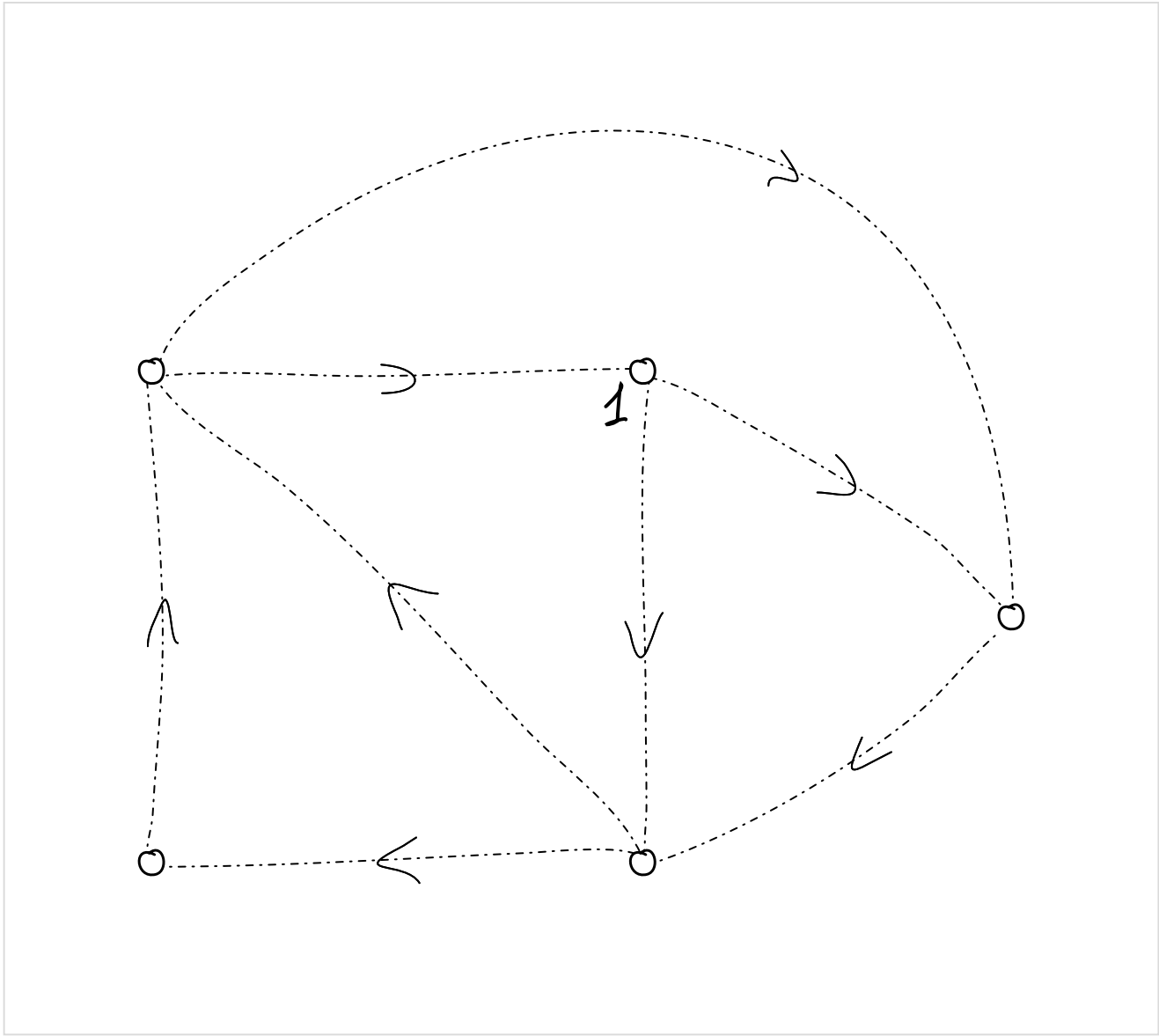

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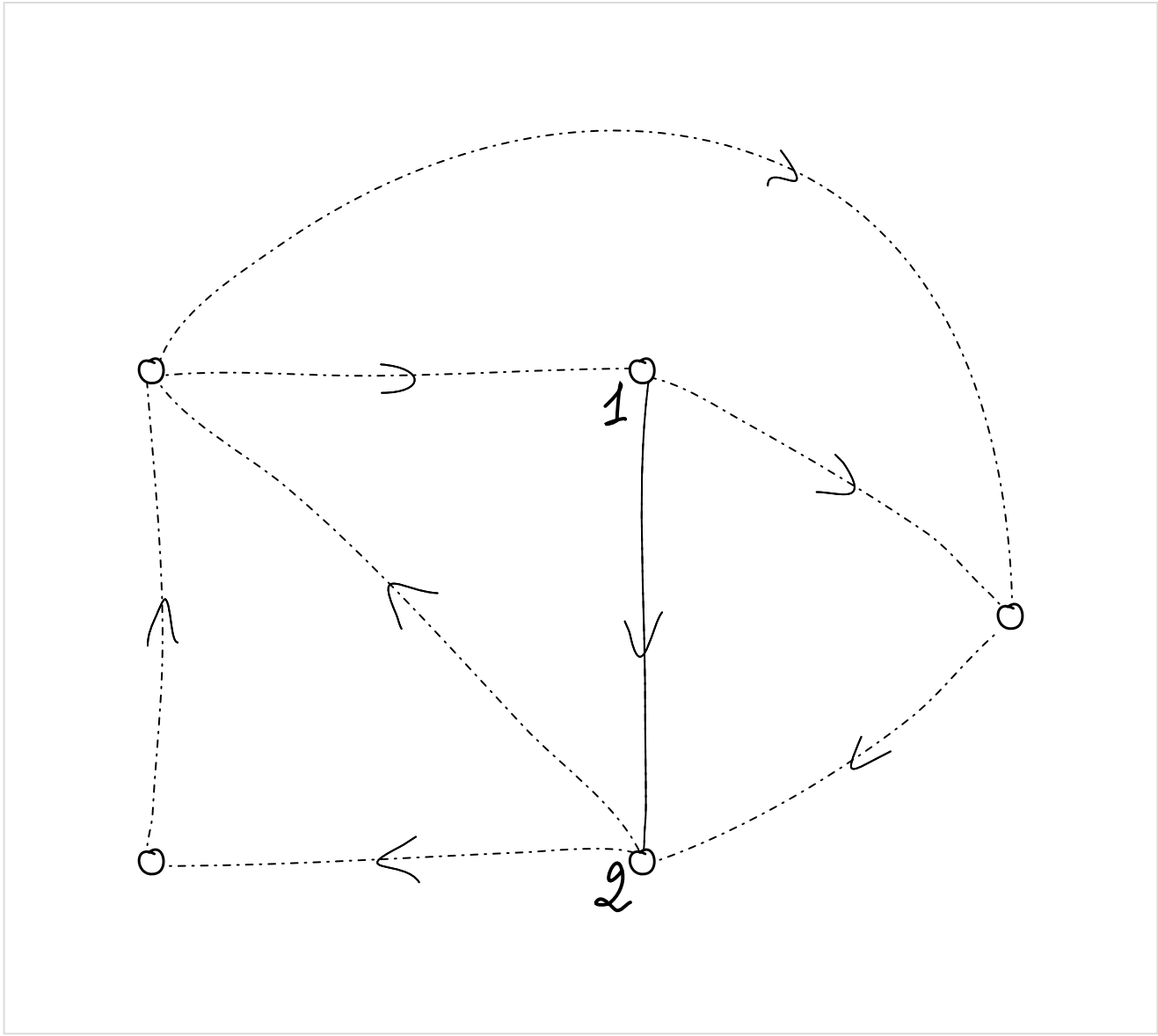

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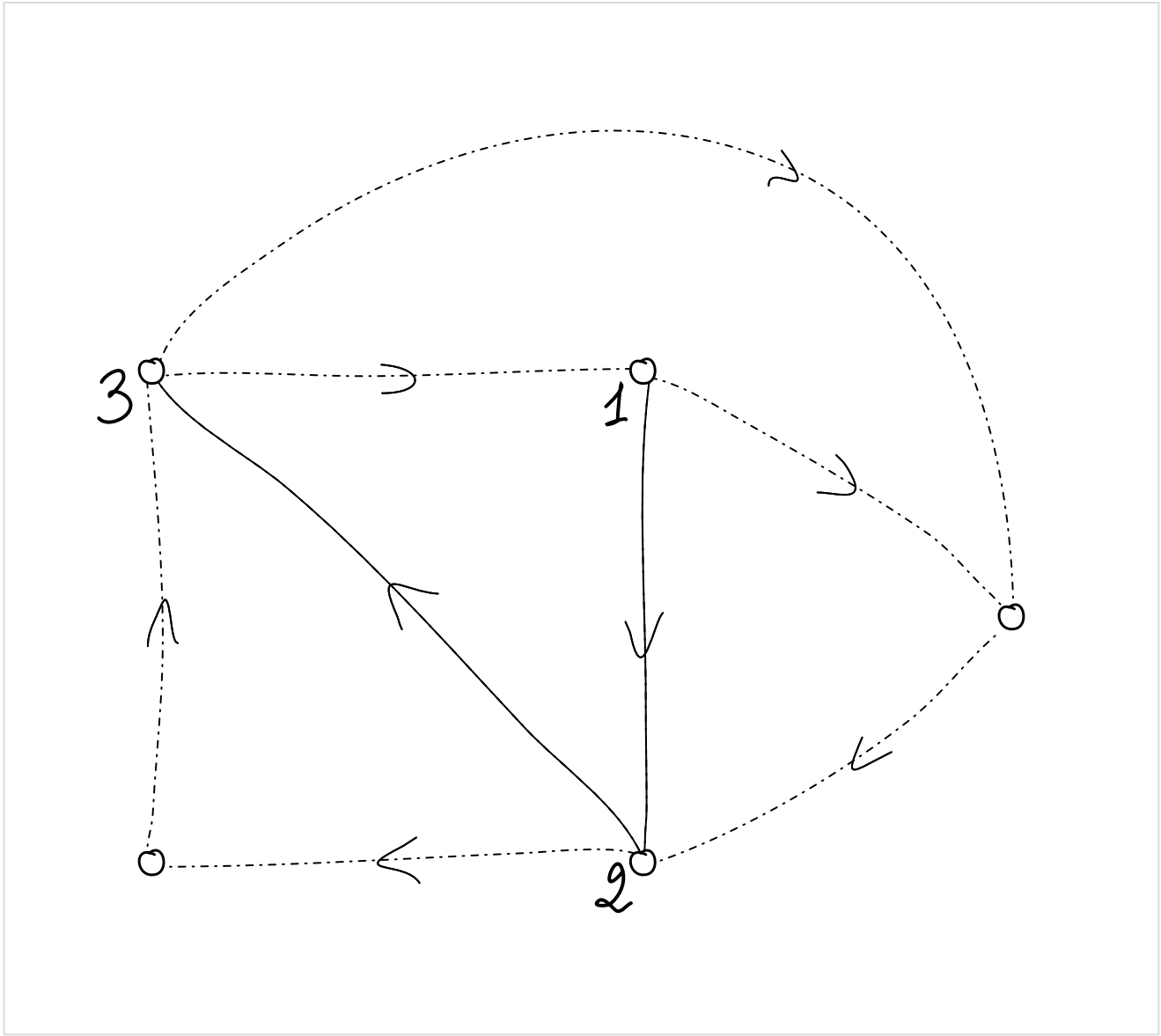

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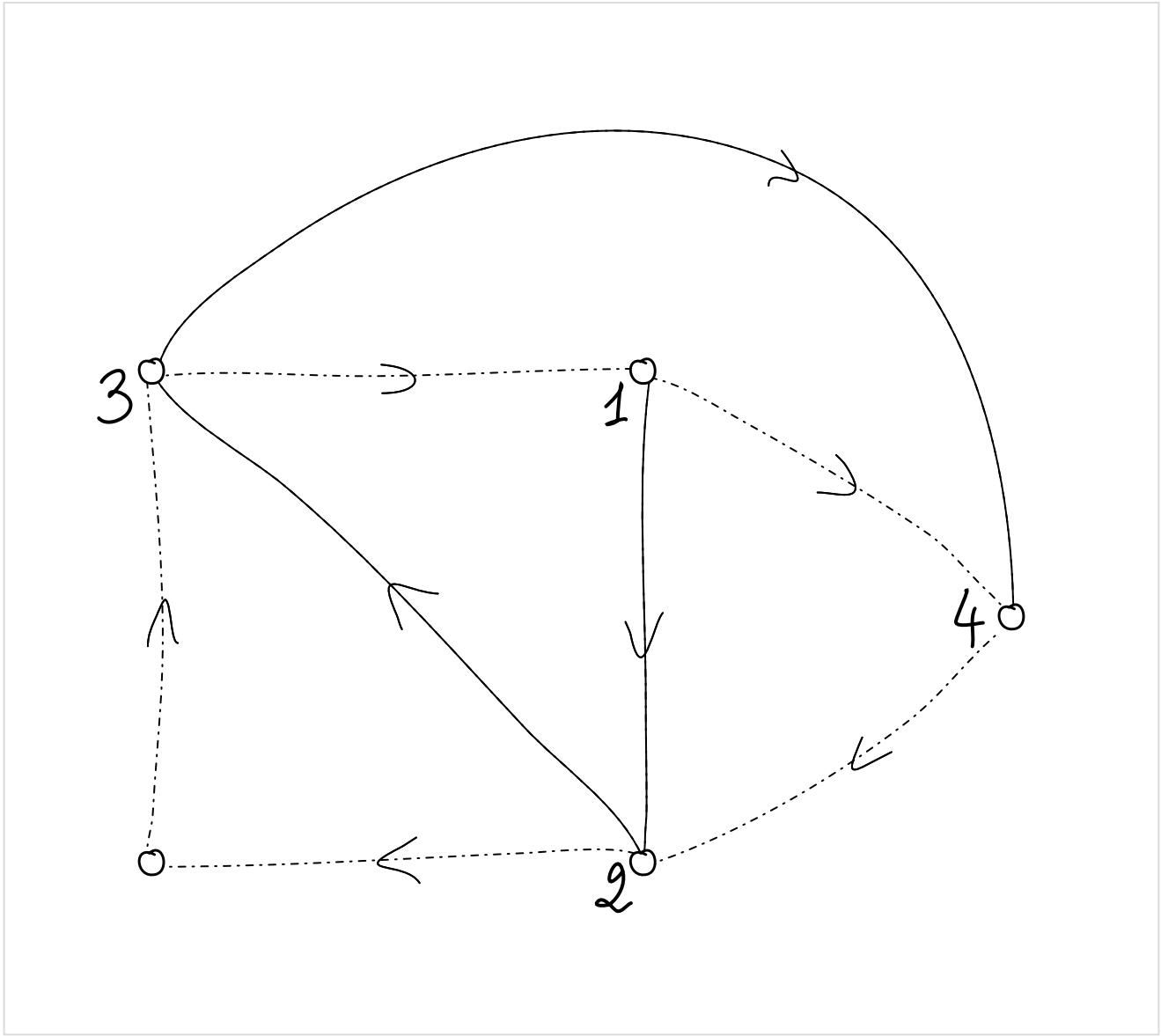

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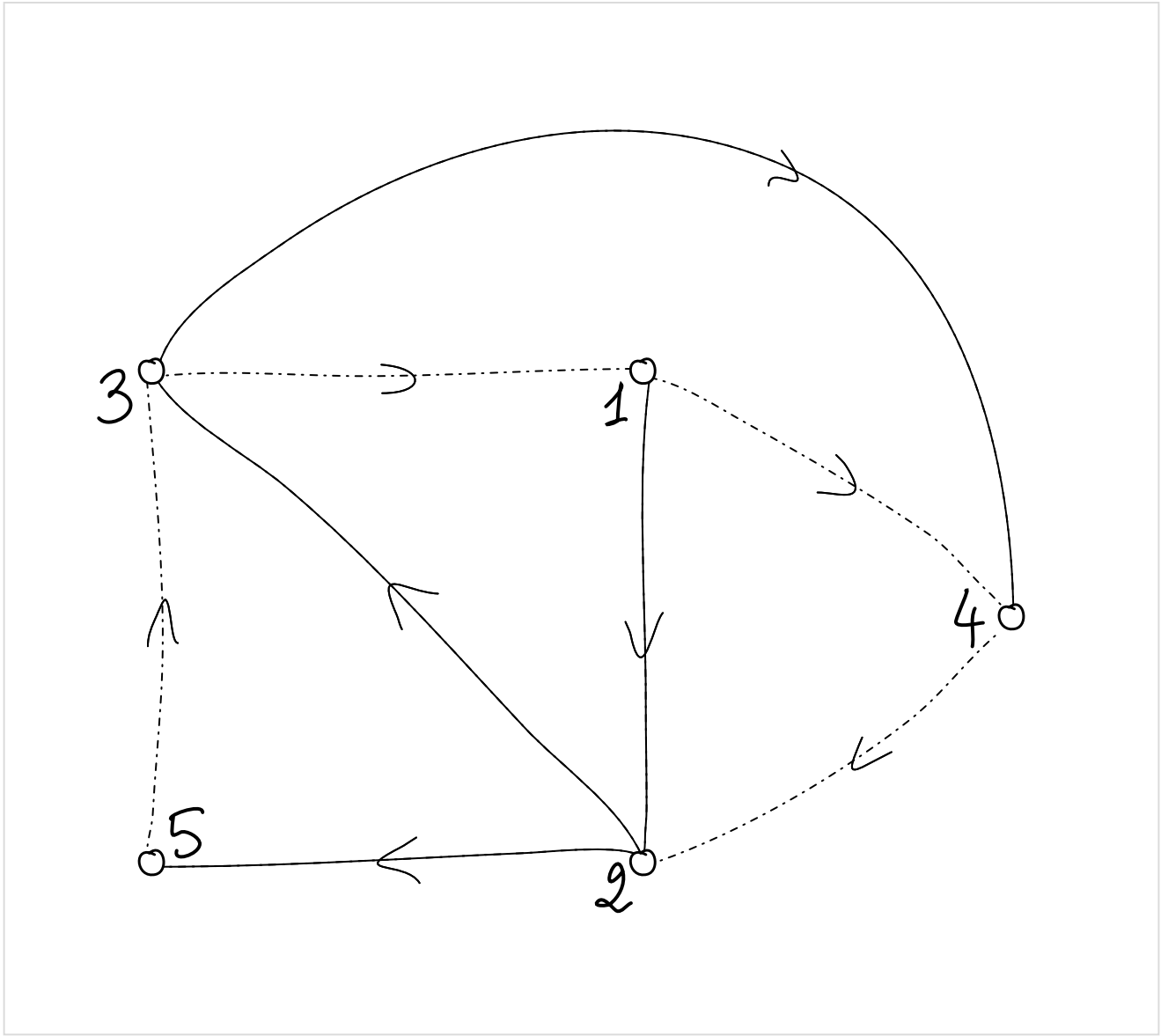

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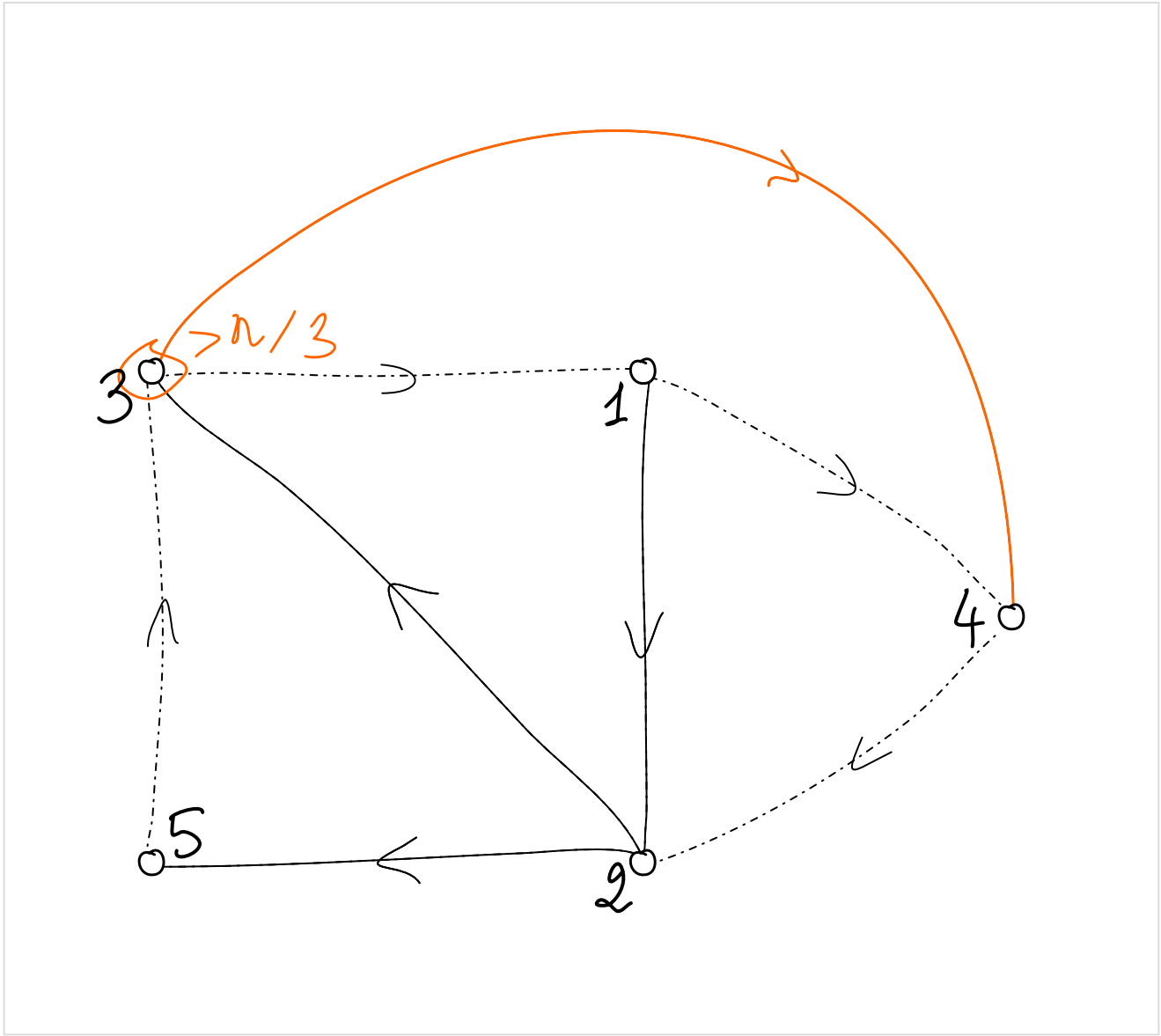

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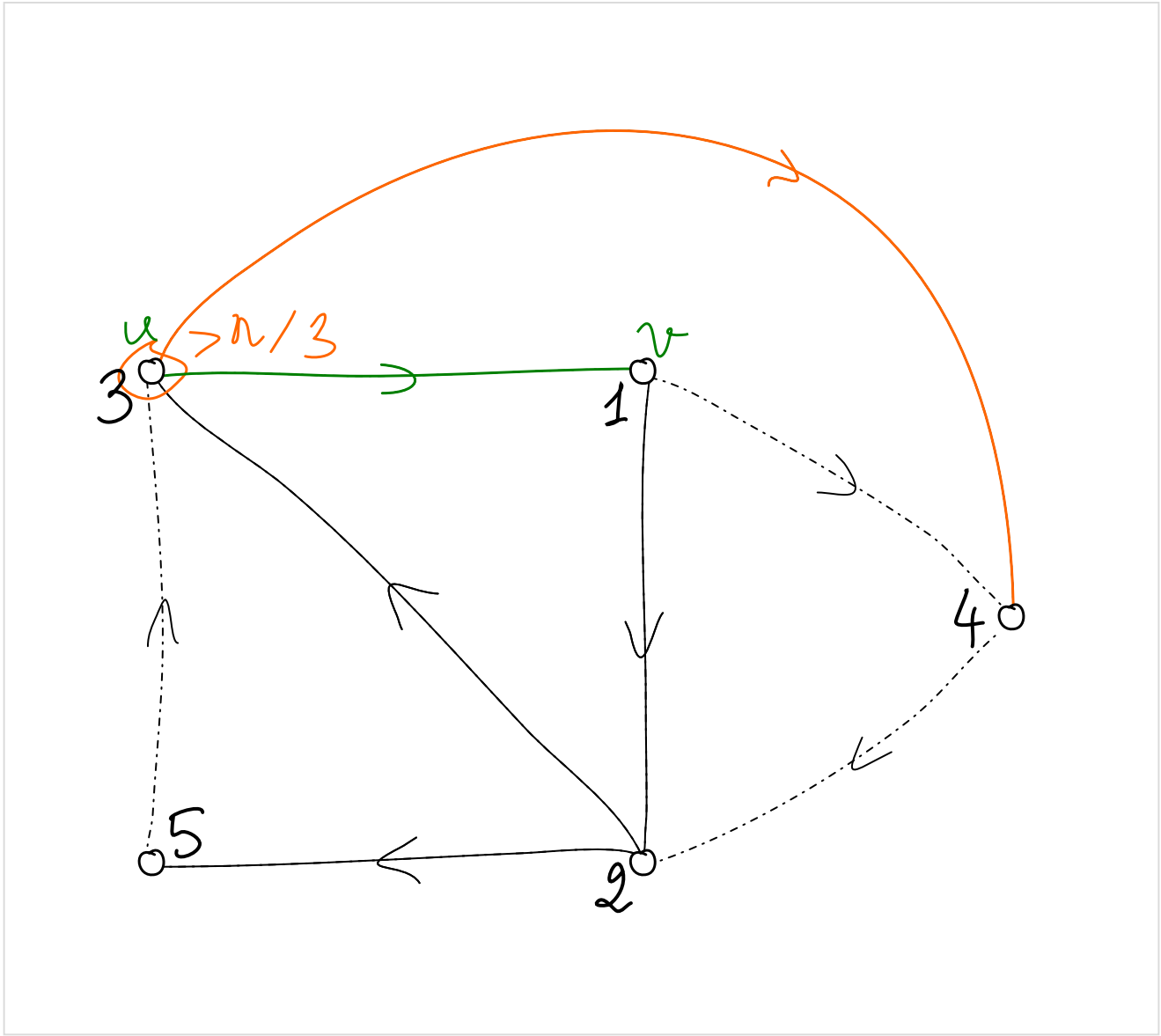

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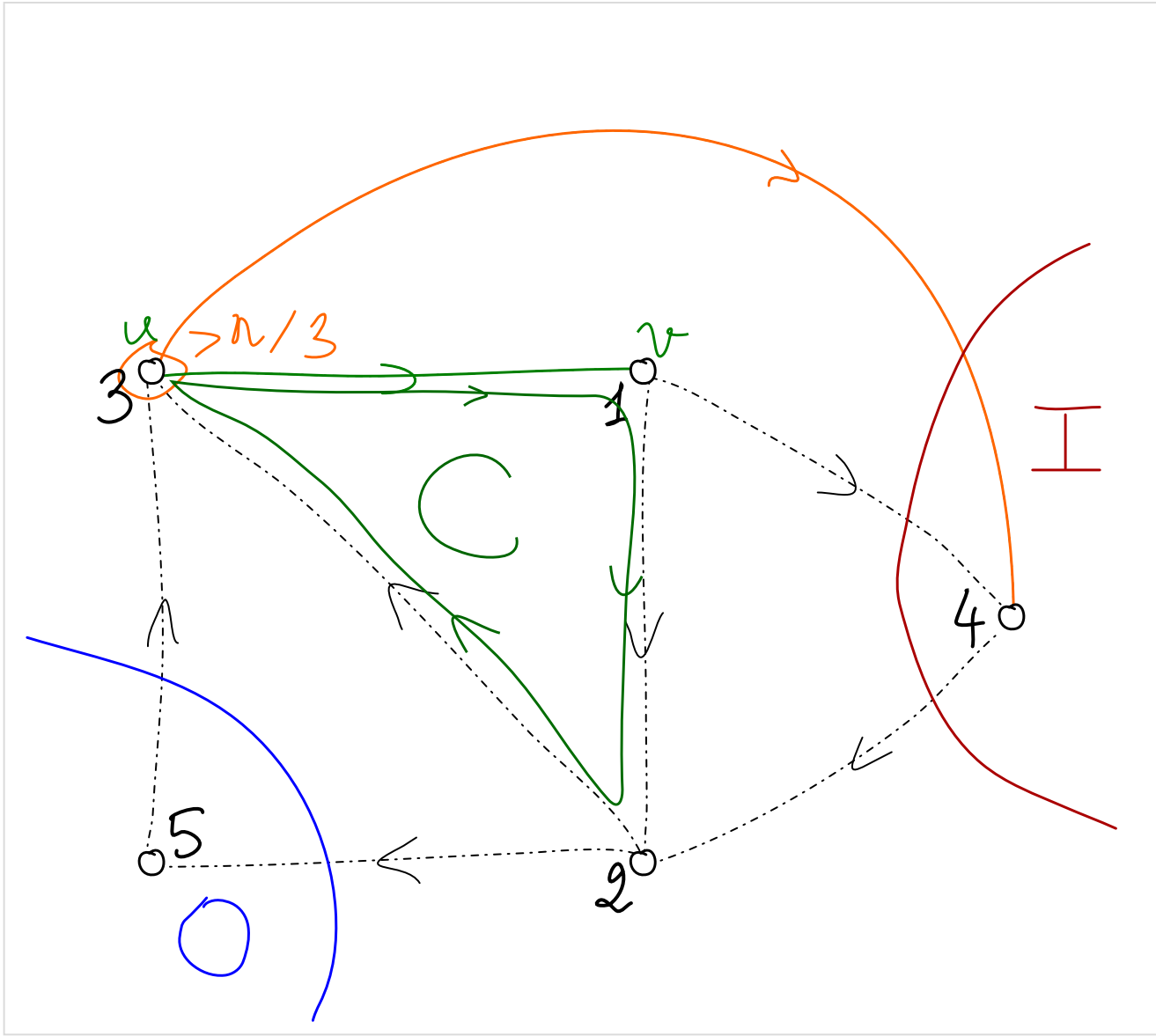

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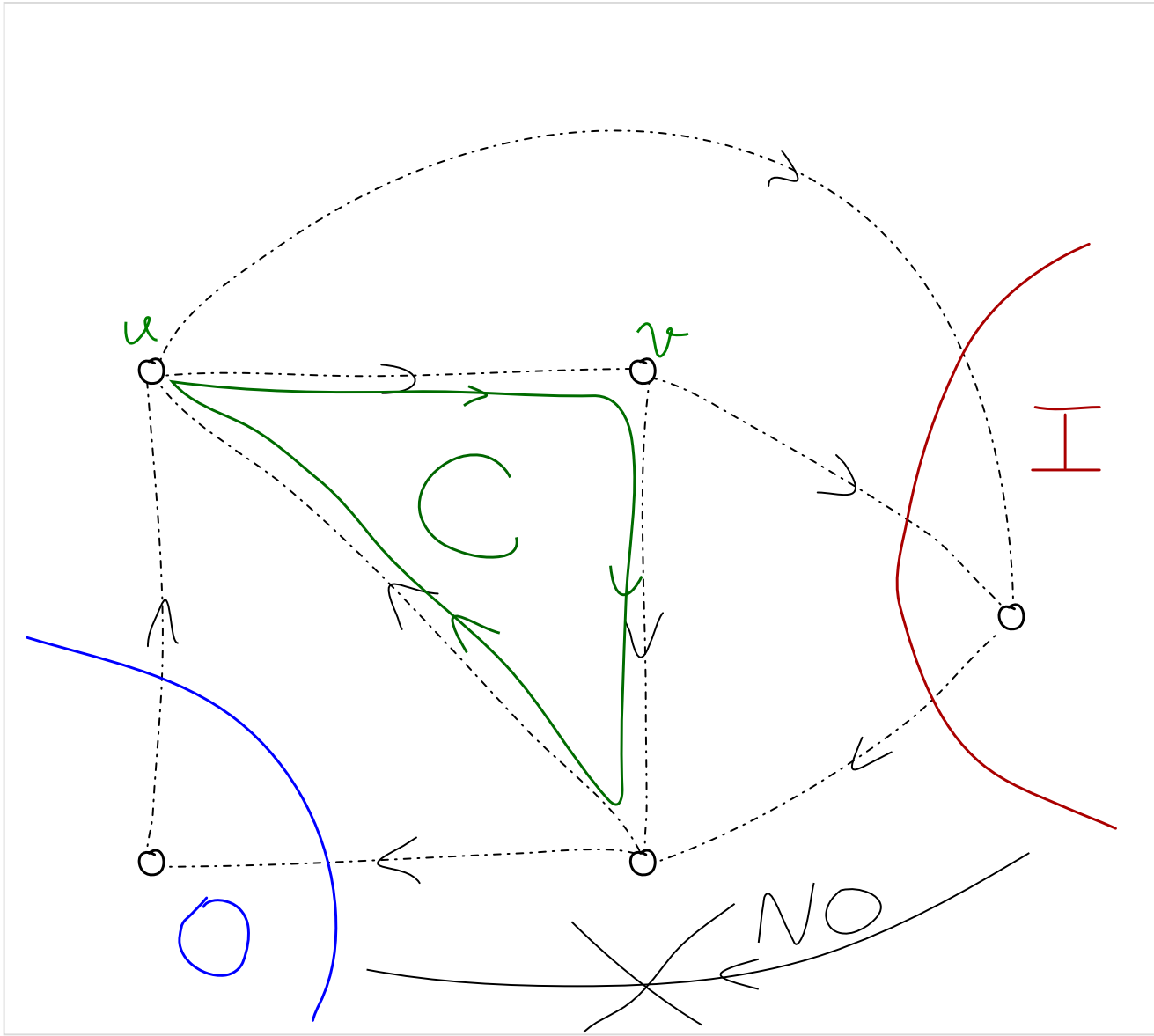

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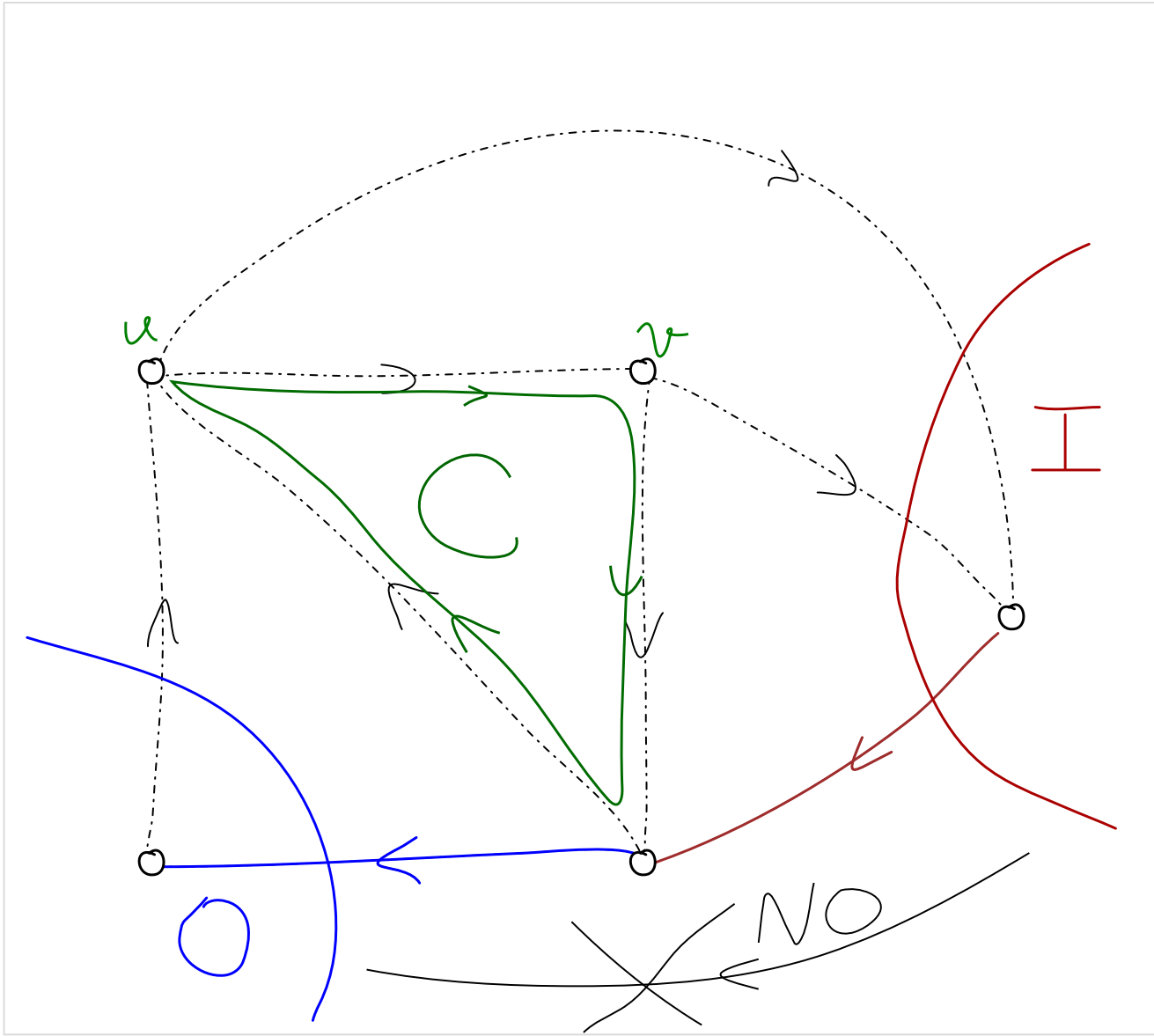

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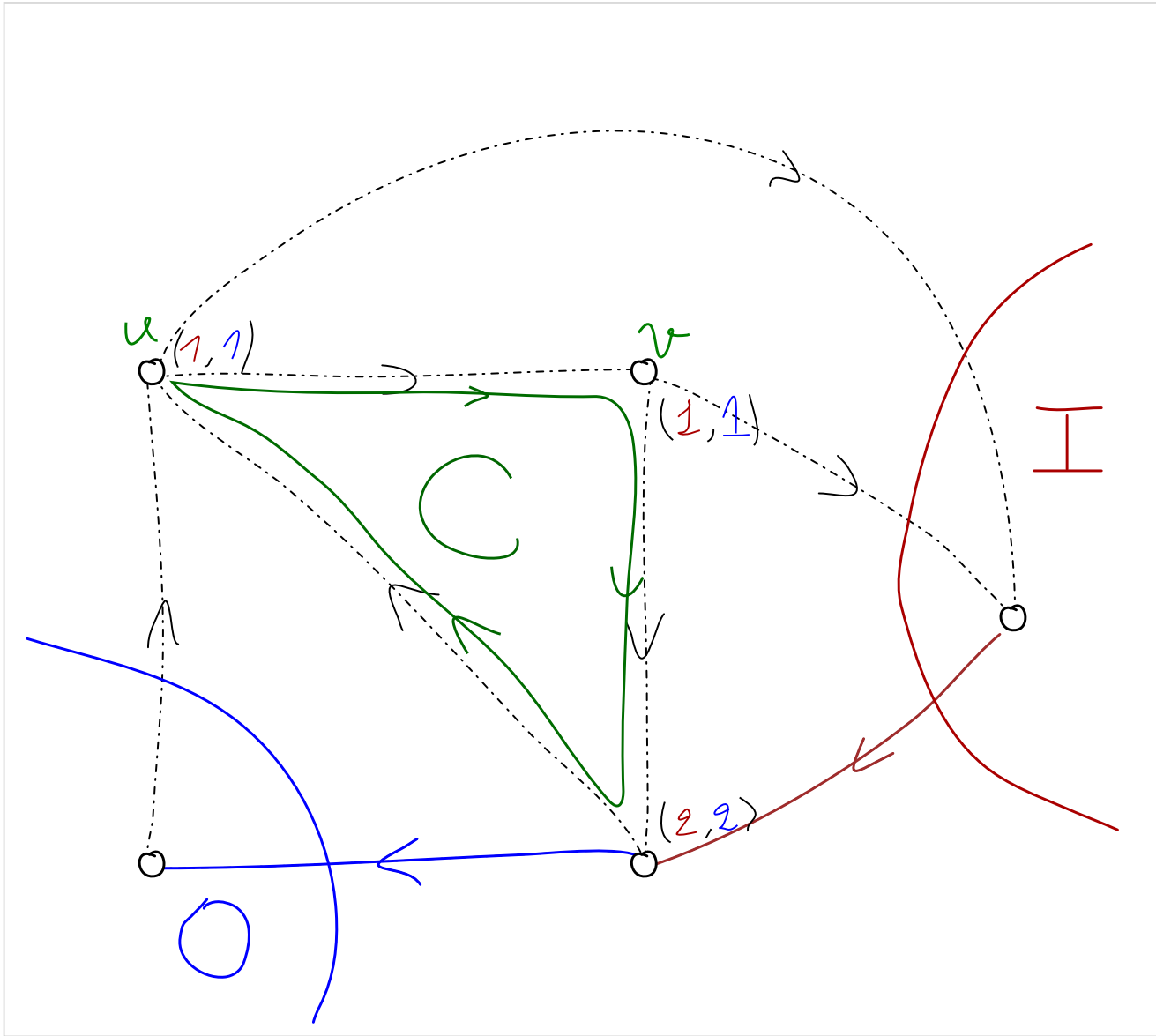

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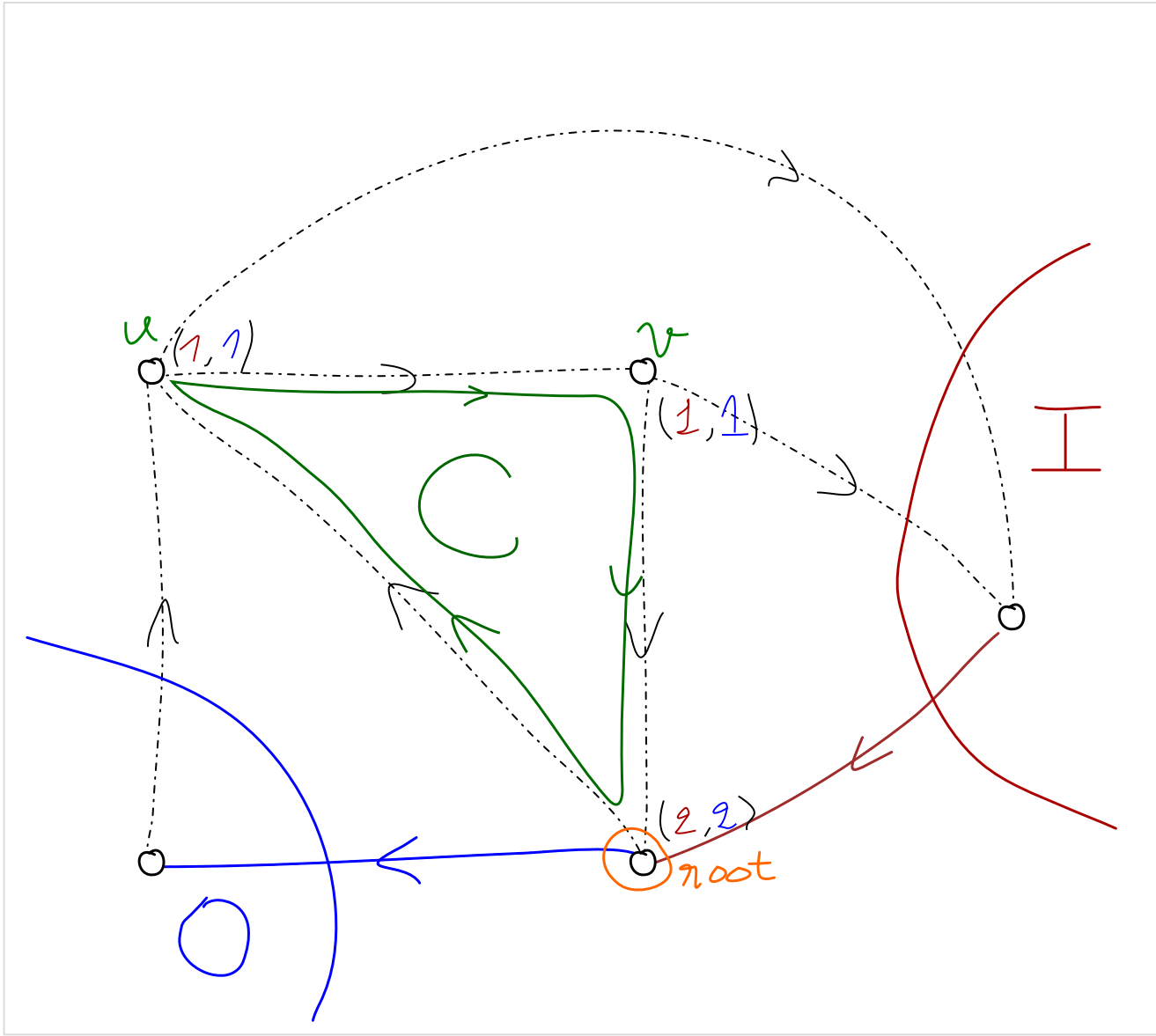

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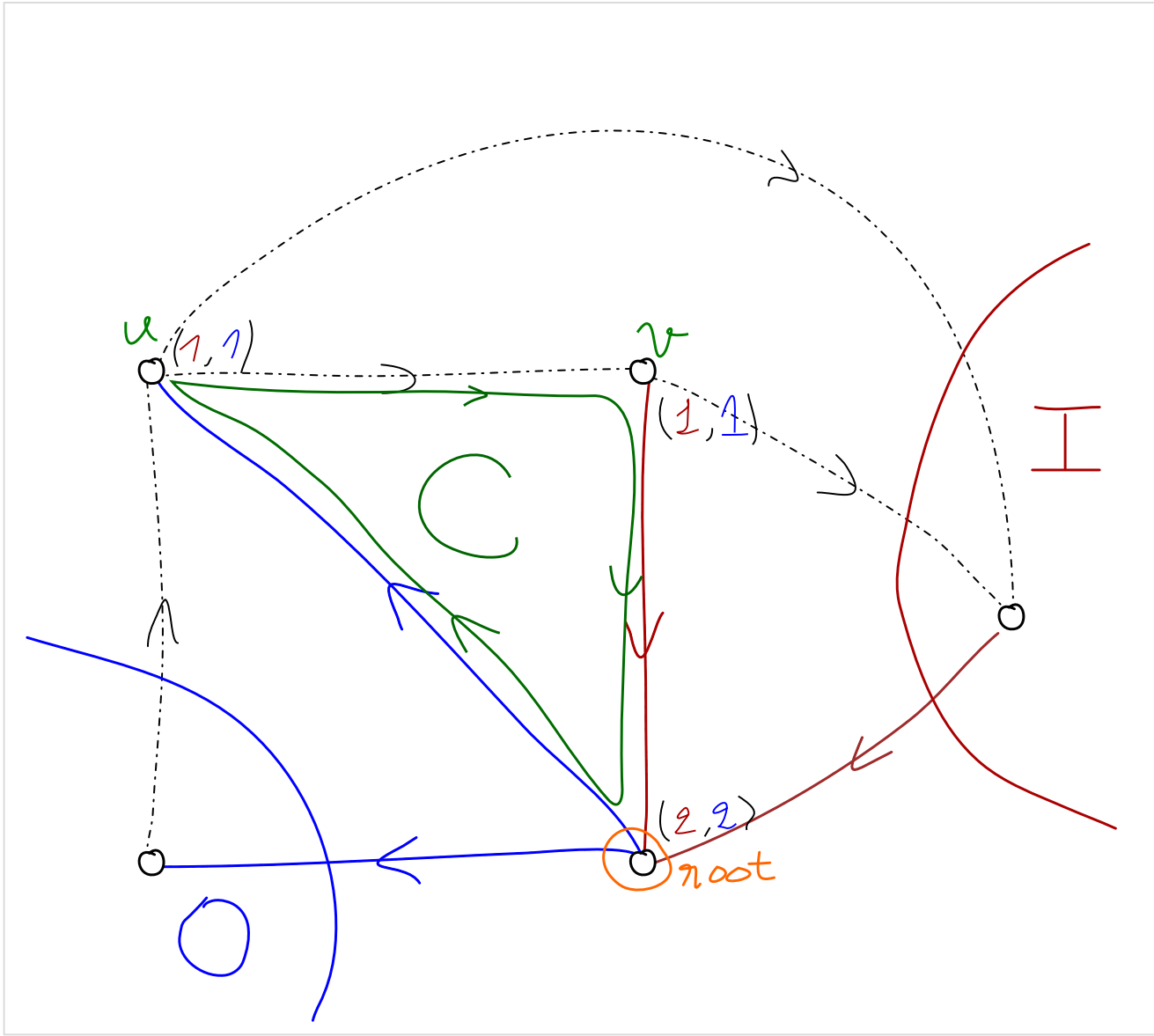

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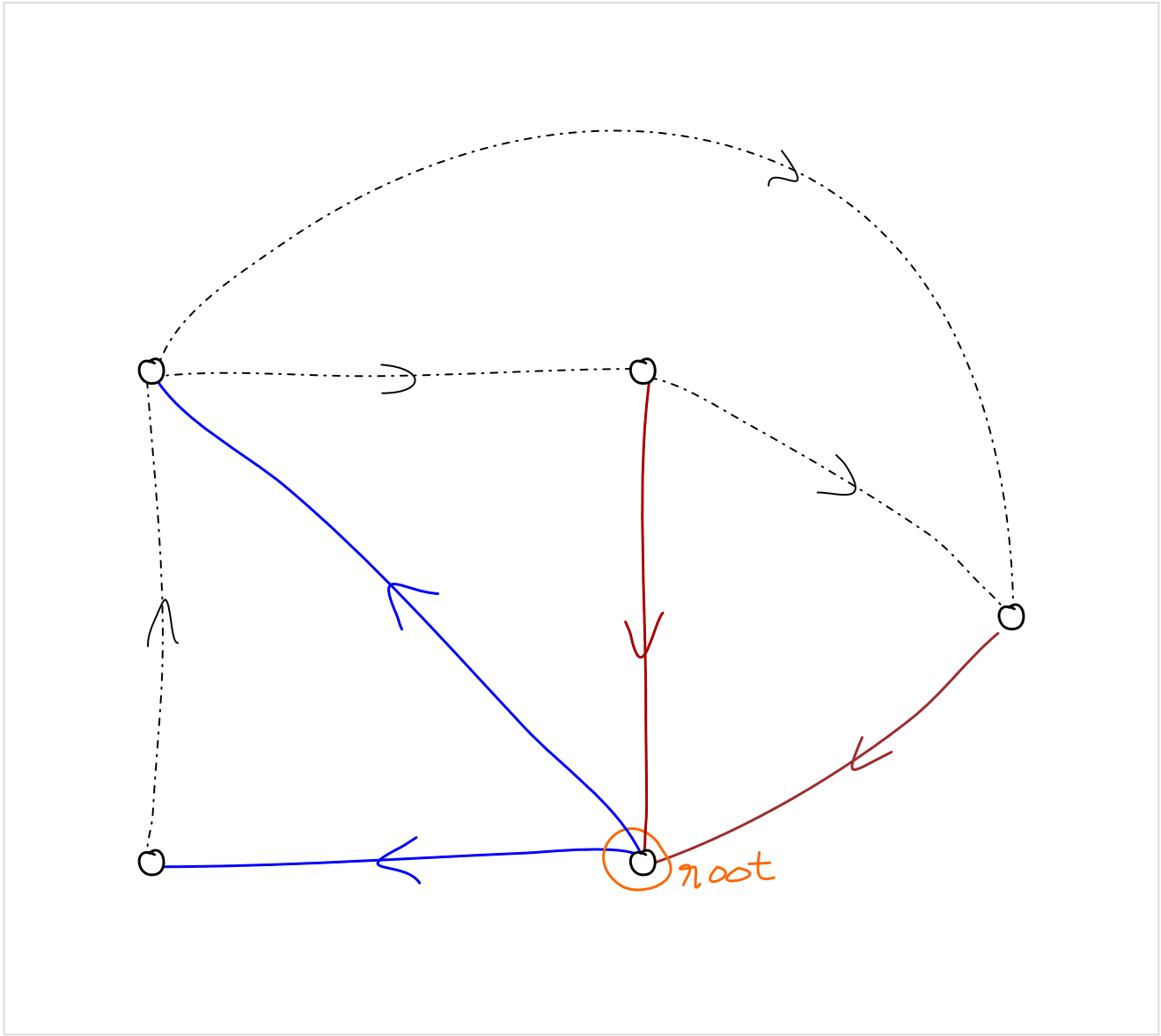

Extensions

The bitree construction generalizes to node weights.

And thus to a subset of nodes U (only pairs in $\binom{\binom{U}{2}}{2}$ are counted).

But not to an arbitrary set $R \subset \binom{\binom{V}{2}}{2}$ of requested pairs.

Conjecture : every strong digraph has a $O(\log n)$ forward cover, i.e. $k = O(\log n)$ node orderings such that any pair $\{x, y\}$ is connected by a path respecting one of the k orderings.

Question : what is the complexity of left-maximal DFS ?

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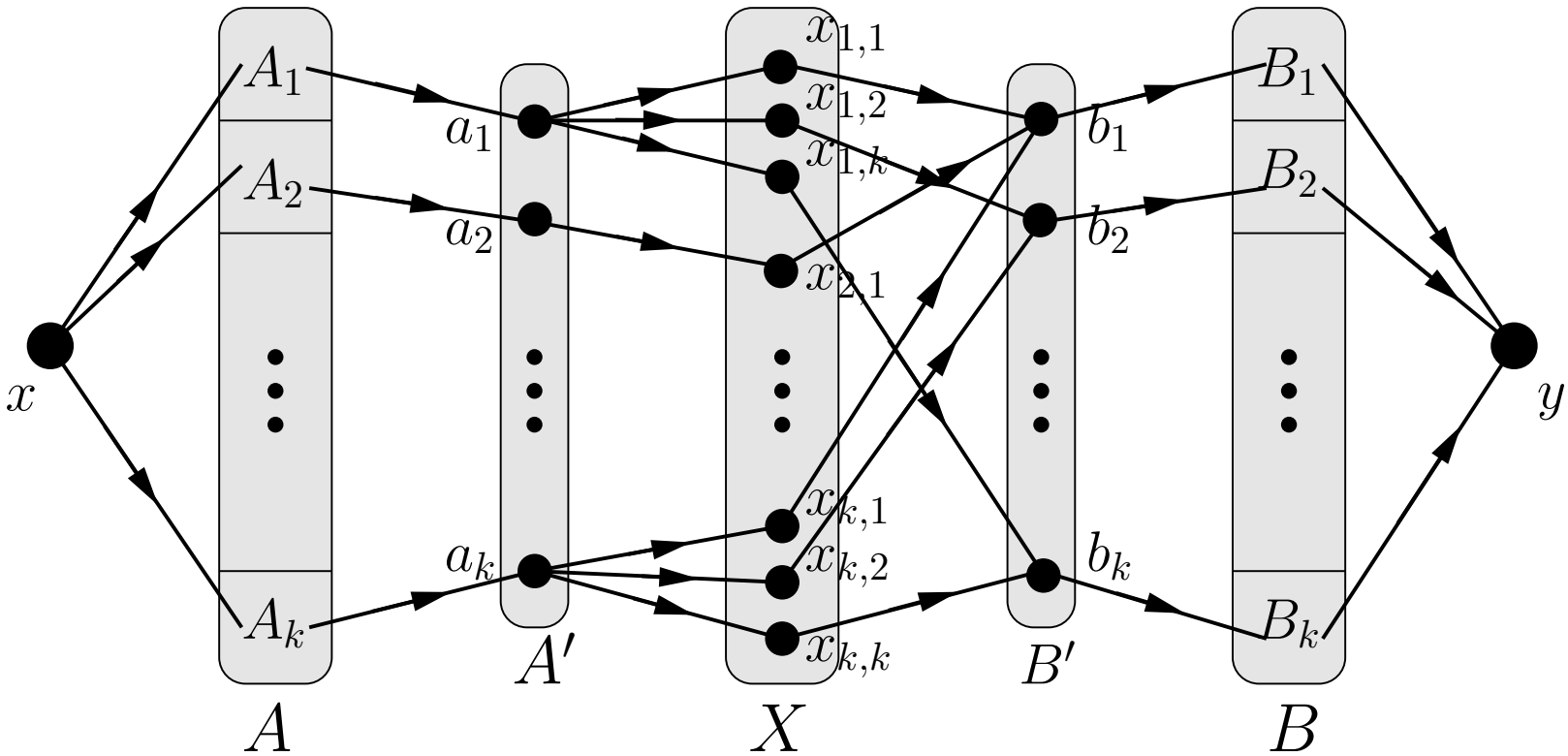
A good node ordering may give poor bitrees.



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Related problem : Find an in-tree and an out-tree with same root that are edge disjoint and have size $\Omega(n)$.

It is NP-hard to decide if a strong digraph has such a pair. [Bang-Jensen 1991]

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Given an undirected graph G, deciding whether there exists an assignment of one time label per edge such that all pairs are temporally connected is NP-complete. [Göbel, Cerdeira, Veldman 1991]

Approximation is obvious : take a spanning tree and assign time labels that connect $(n/2)^2$ pairs.

Related (Gossip/telephone problem [Bumby 1981]) : The minimum number of time labels allowing to temporally connect all pairs at least $2n-4$, and equals $2n-4$ if G has a C_4 (one or two time labels per edge).

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