

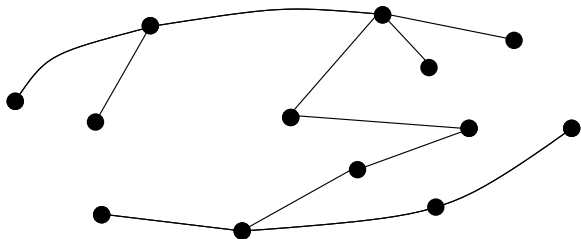
Compter des cartes planaires (colorées)

Mireille Bousquet-Mélou, CNRS, LaBRI, Université de Bordeaux

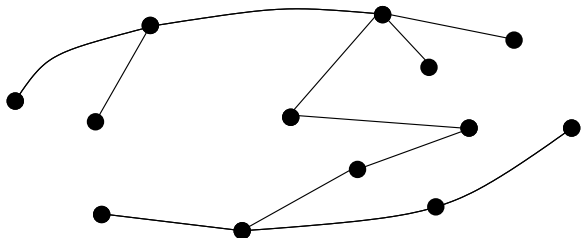


I. Trees and maps

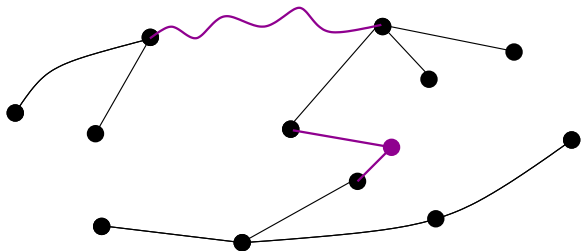
Plane trees



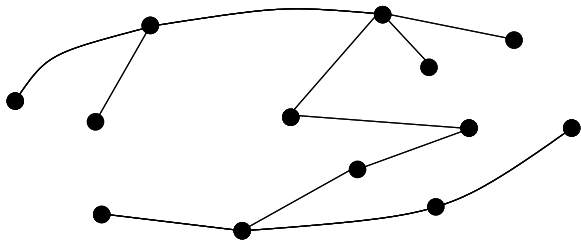
Plane trees



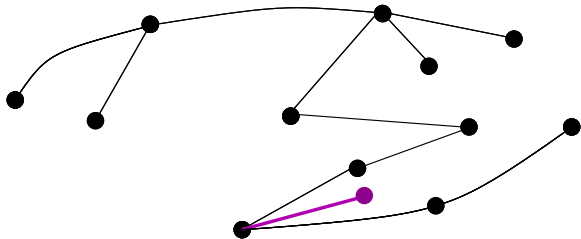
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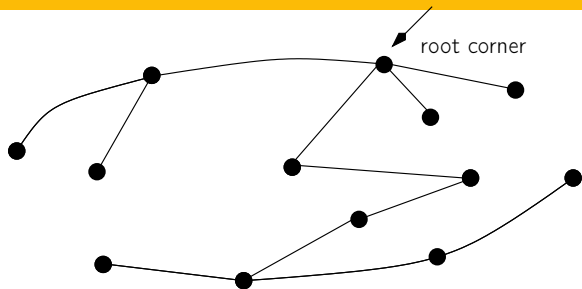
Plane trees



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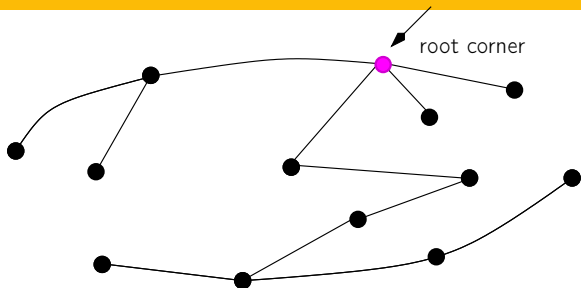


Plane trees: rooted version



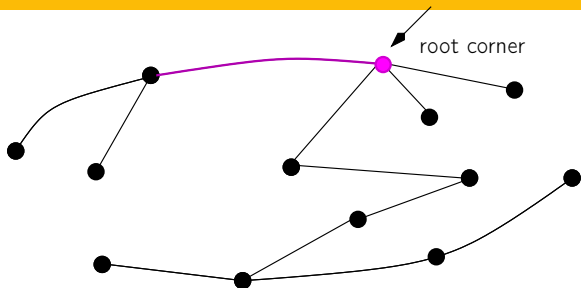
Mark a corner

Plane trees: rooted version



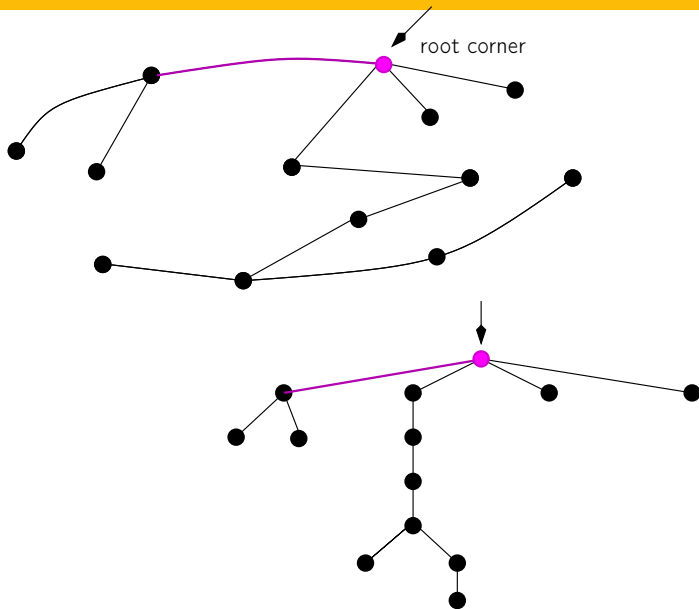
Mark a corner

Plane trees: rooted version

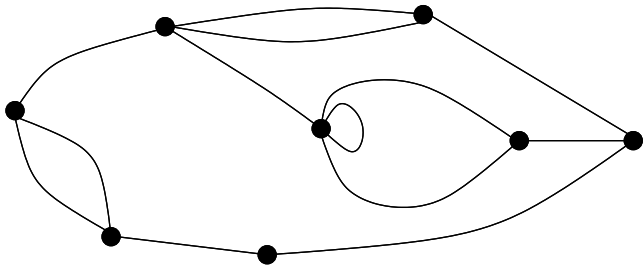


Mark a corner

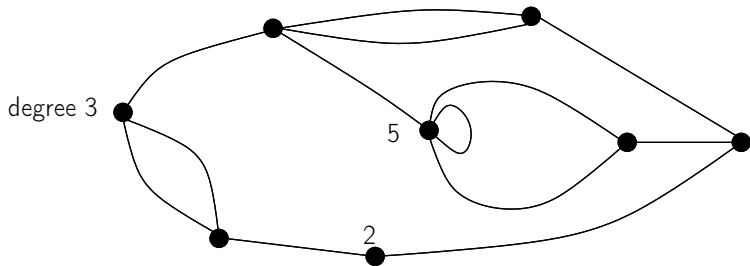
Plane trees: rooted version



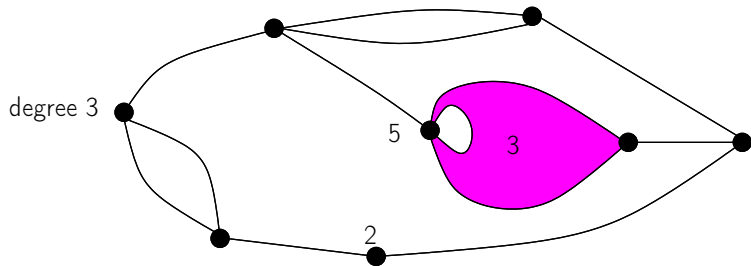
Plane maps



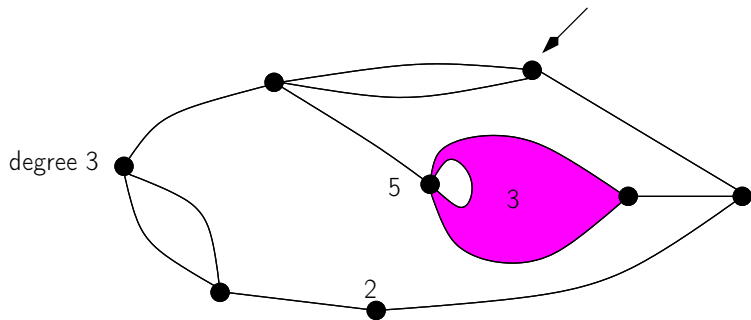
Plane maps



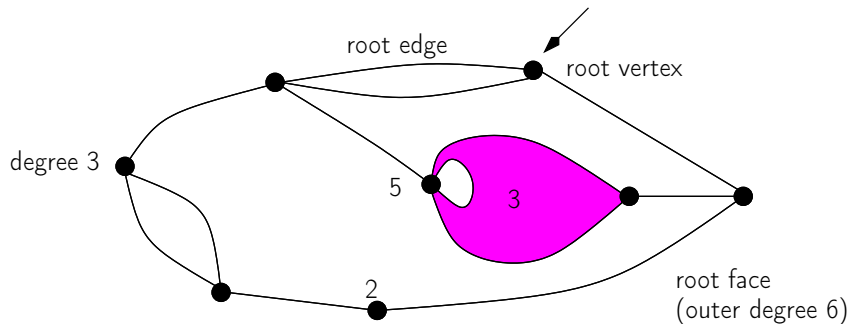
Plane maps



Plane maps: rooted version

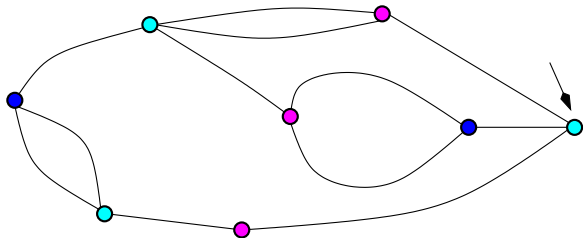


Plane maps: rooted version

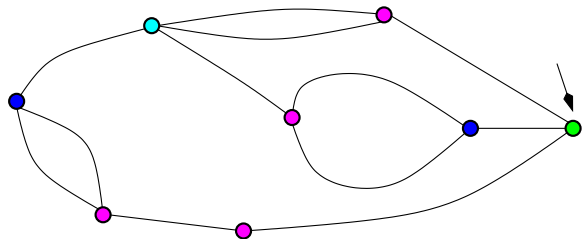


Coloured plane maps

Proper colouring:

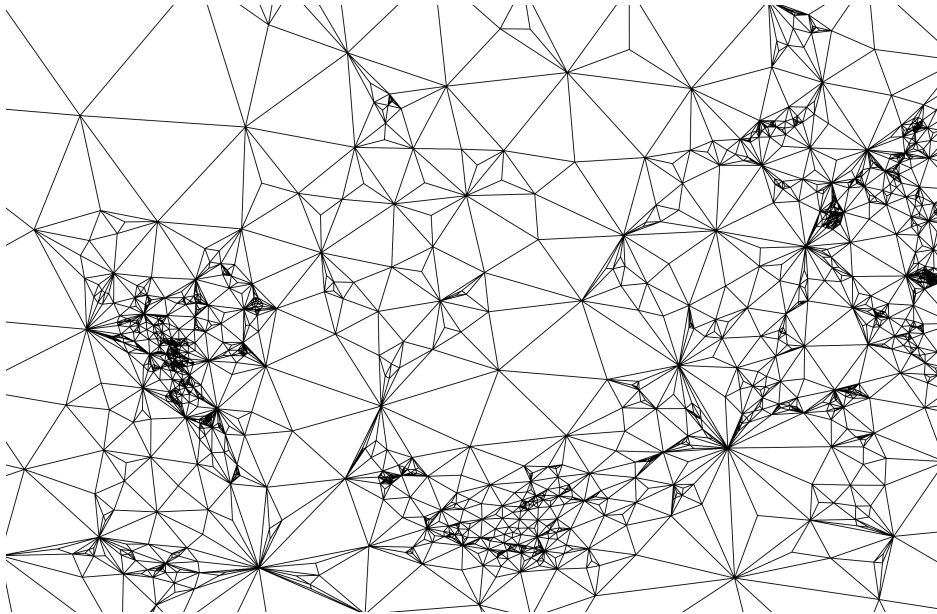


Non-proper colouring:



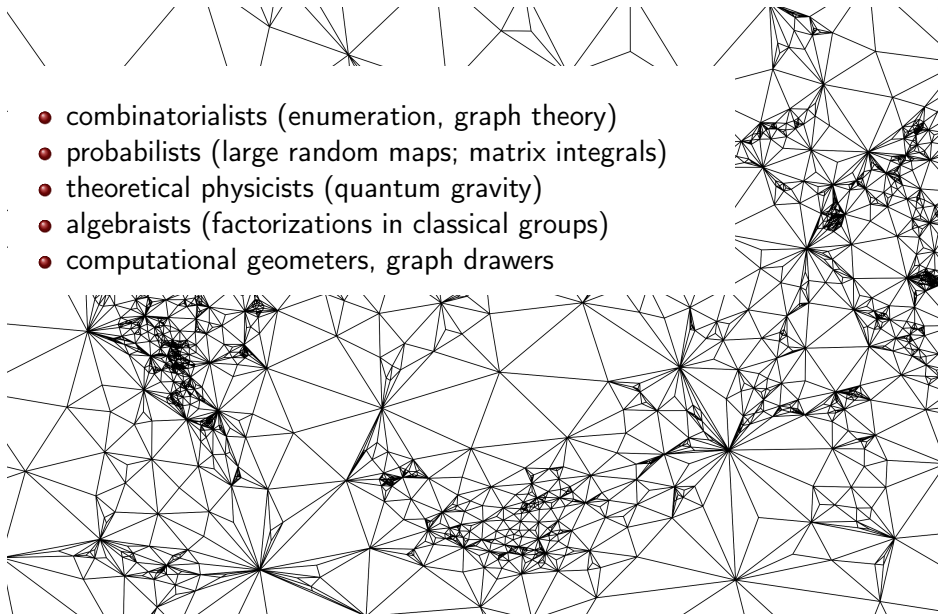
Monochromatic edge

Who studies plane maps?



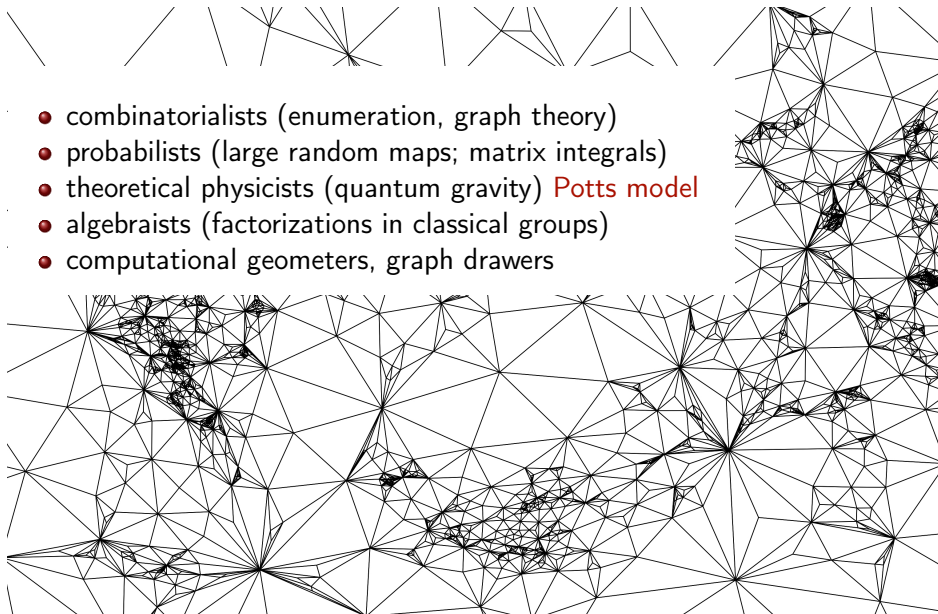
Who studies plane maps?

- combinatorialists (enumeration, graph theory)
- probabilists (large random maps; matrix integrals)
- theoretical physicists (quantum gravity)
- algebraists (factorizations in classical groups)
- computational geometers, graph drawers



Who studies plane maps?

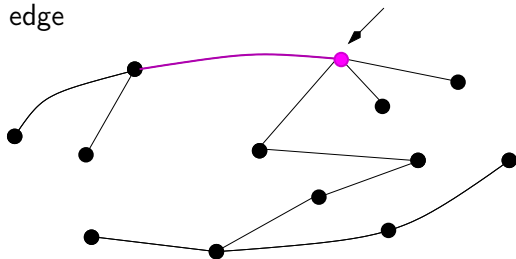
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II. Recursive descriptions and enumeration

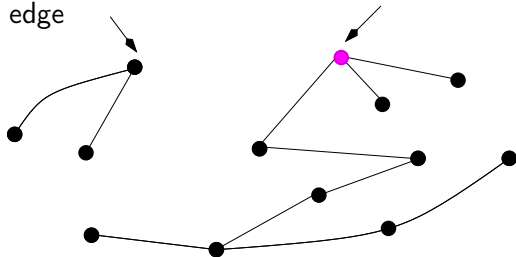
A recursive description of trees

Delete the root edge



A recursive description of trees

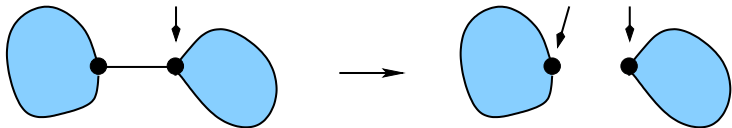
Delete the root edge



\Rightarrow An ordered pair of trees

A recursive description of trees

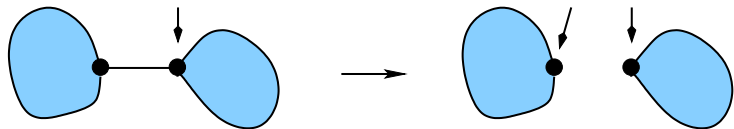
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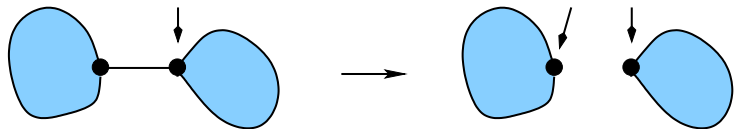
⇒ An ordered pair of trees

Enumeration: let $a(n)$ be the number of rooted plane trees with n edges. Then $a(0) = 1$ and

$$a(n) = \sum_{i+j=n-1} a(i)a(j)$$

A recursive description of trees

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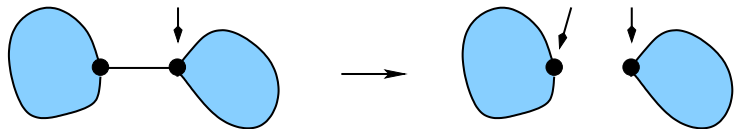
$$a(n) = \sum_{i+j=n-1} a(i)a(j)$$

Generating function: the associated formal power series

$$A := \sum_{n \geq 0} a(n)t^n = \sum_{T \text{ tree}} t^{e(T)}$$

A recursive description of trees

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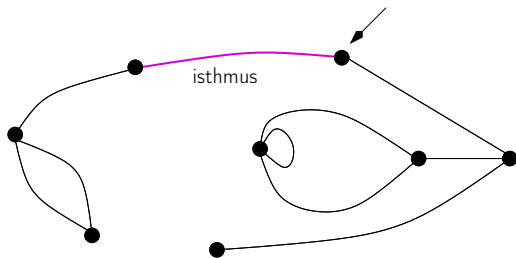
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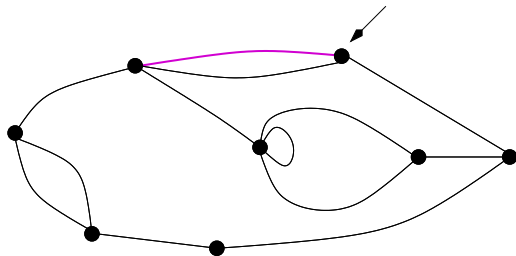
Functional equation:

$$A = 1 + tA^2$$

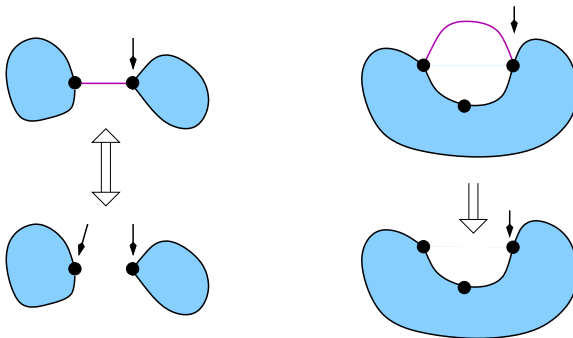
A recursive description of maps: delete the root edge



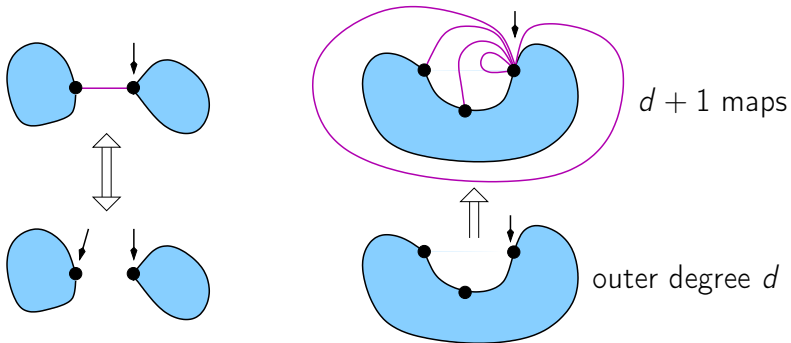
A recursive description of maps: delete the root edge



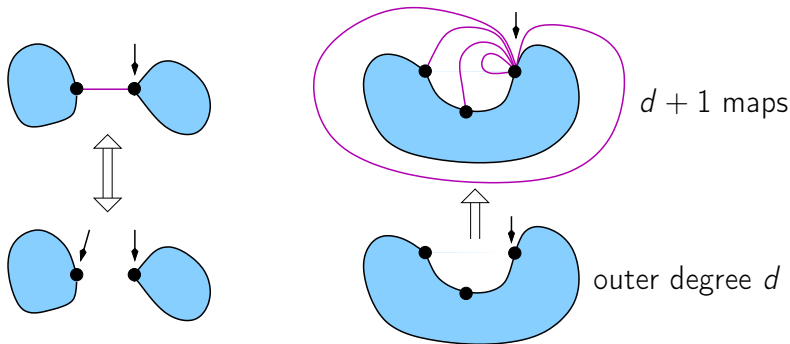
A recursive description of maps: delete the root edge



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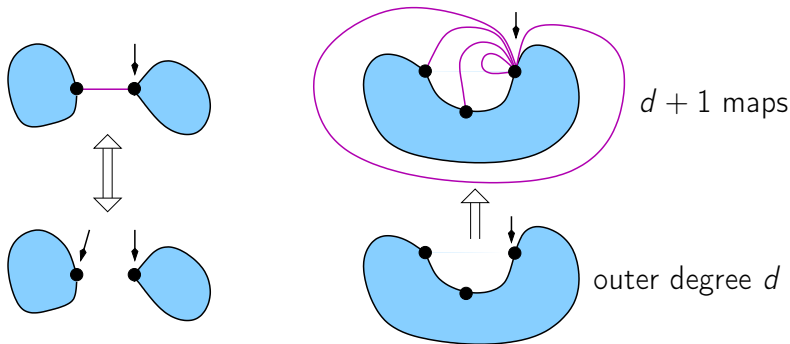


Enumeration: let $b(n)$ be the number of rooted plane maps with n edges. Then $b(0) = 1$ and

$$b(n) = \sum_{i+j=n-1} b(i)b(j) + \sum_{d \geq 0} b(n-1; d)(d+1)$$

where $b(n; e)$ is the number of maps with n edges and outer degree e .

A recursive description of maps: delete the root edge



Enumeration: let $b(n)$ be the number of rooted plane maps with n edges. Then $b(0) = 1$ and

$$b(n; e) = \sum_{\substack{i+j=n-1 \\ c+d=e-2}} b(i; c)b(j; d) + \sum_{d \geq e-1} b(n-1; d)$$

where $b(n; e)$ is the number of maps with n edges and outer degree e .

Generating function of plane maps

Enumeration: let $b(n; d)$ be the number of rooted plane maps with n edges and outer degree d . Then $b(0; e) = \delta_{e,0}$ and

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Generating function: the associated **bivariate** formal power series

$$B(x) := \sum_{n,d \geq 0} b(n; d)t^n x^d = \sum_{M \text{ map}} t^{e(M)} x^{\text{df}(M)}$$

Note: $B(1) = \sum_n b(n)t^n$ is the GF we want to compute

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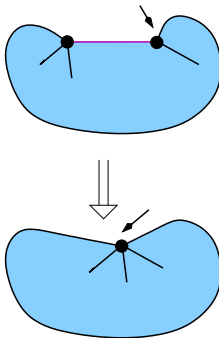
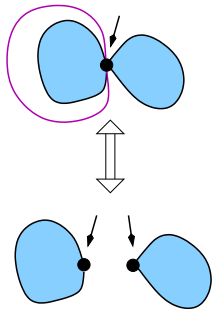
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Functional equation:

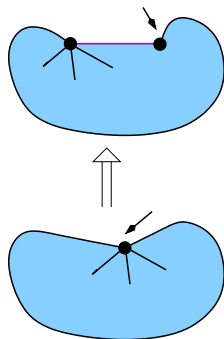
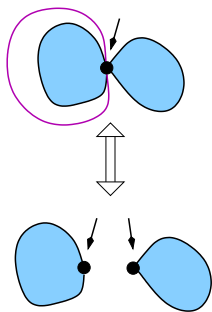
$$B(x) = 1 + tx^2 B(x)^2 + tx \frac{x B(x) - B(1)}{x - 1}$$

A discrete differential equation

Another description of maps: contract the root edge



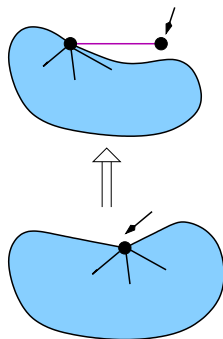
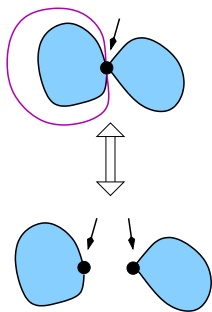
Another description of maps: contract the root edge



$d + 1$ maps

degree of root vertex
 $= d$

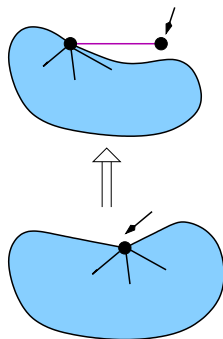
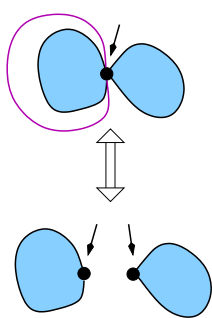
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Enumeration: let $\tilde{b}(n; e)$ be the number of rooted plane maps with n edges and root degree e .

Then $\tilde{b}(0; e) = \delta_{e,0}$ and

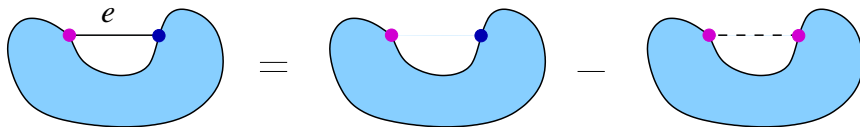
$$\tilde{b}(n; e) = \sum_{\substack{i+j=n-1 \\ c+d=e-2}} \tilde{b}(i; c)\tilde{b}(j; d) + \sum_{d \geq e-1} \tilde{b}(n-1; d)$$

Same recurrence relation

A recursive description of coloured maps: delete/contract the root edge

- Deletion/contraction relation for the chromatic polynomial: the number $\chi_M(q)$ of proper q -colourings of the map M satisfies:

$$\chi_M(q) = \chi_{M \setminus e}(q) - \chi_{M \leftrightarrow e}(q)$$



A recursive description of coloured maps: delete/contract the root edge

- **Deletion/contraction relation for the chromatic polynomial:** the number $\chi_M(q)$ of proper q -colourings of the map M satisfies:

$$\chi_M(q) = \chi_{M \setminus e}(q) - \chi_{M \leftrightarrow e}(q)$$

- More generally, if $P_M(q, \nu)$ counts all q -colourings of M with a weight ν per monochromatic edge:

$$P_M(q, \nu) = P_{M \setminus e}(q, \nu) + (\nu - 1) P_{M \leftrightarrow e}(q, \nu)$$

\Rightarrow Use this to write a recurrence relation (then a functional equation) to count coloured maps, with **two additional parameters**.

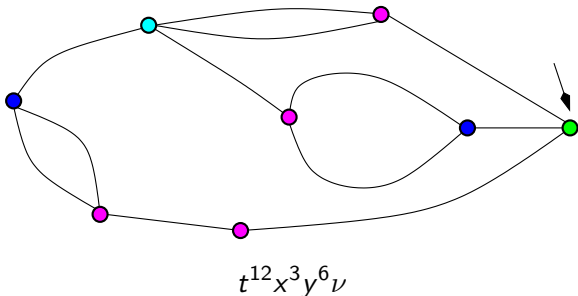
Remark. The polynomial $P_M(q, \nu)$ is equivalent to the **Tutte polynomial** of M

A functional equation for coloured maps

- Let

$$\begin{aligned}C(x, y) &= \frac{1}{q} \sum_M P_M(q, \nu) t^{e(M)} x^{dv(M)} y^{df(M)} \\ &= \frac{1}{q} \sum_{M \text{ } q\text{-coloured}} t^{e(M)} x^{dv(M)} y^{df(M)} \nu^{m(M)}\end{aligned}$$

where $dv(M)$ (resp. $df(M)$) is the degree of the root-vertex (resp. root-face) and $m(M)$ the number of monochromatic edges.



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where $dv(M)$ (resp. $df(M)$) is the degree of the root-vertex (resp. root-face) and $m(M)$ the number of monochromatic edges.

Proposition [Tutte 68]

$$\begin{aligned}C(x, y) &= 1 + xyt((\nu - 1)(y - 1) + qy) C(x, y)C(1, y) \\ &\quad + xyt(x\nu - 1)C(x, y)C(x, 1) \\ &\quad + xyt(\nu - 1) \frac{x C(x, y) - C(1, y)}{x - 1} + xyt \frac{y C(x, y) - C(x, 1)}{y - 1}\end{aligned}$$

A discrete partial differential equation (in two variables)

This equation has been sleeping for 40 years

Our bestiary

- Plane trees

$$A = 1 + tA^2$$

- Plane maps

$$B(x) = 1 + tx^2B(x)^2 + tx \frac{x B(x) - B(1)}{x - 1}$$

- Coloured plane maps

$$\begin{aligned} C(x, y) = & 1 + xyt((\nu - 1)(y - 1) + qy) C(x, y)C(1, y) \\ & + xyt(x\nu - 1)C(x, y)C(x, 1) \\ & + xyt(\nu - 1) \frac{x C(x, y) - C(1, y)}{x - 1} + xyt \frac{y C(x, y) - C(x, 1)}{y - 1} \end{aligned}$$

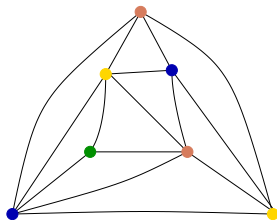
Recursive constructions are robust

- Plane triangulations (every finite face has degree 3):

$$B(x) = tx(1 + xB(x))^2 + t \frac{B(x) - B(0)}{x}$$

- Properly q -coloured triangulations [Tutte 73]:

$$T(x, y) = x(q - 1) + xyzT(x, y)T(1, y) \\ + xz \frac{T(x, y) - T(x, 0)}{y} - x^2yz \frac{T(x, y) - T(1, y)}{x - 1}$$



II. Guessing

Recurrences produce coefficients



Guessing with Maple and Gfun

Given the first terms of a series $(a(0), a(1), a(2), \dots)$, Gfun can guess:

- linear rec. relations with polynomial coefficients for $a(n)$

$$p_\ell(n)a(n + \ell) + \dots + p_0(n)a(n) = 0$$

- equivalently, linear diff. equations with polynomial coefficients for the GF $A(t)$

$$P_d(t)A^{(d)}(t) + \dots + P_0(t)A(t) = 0$$

- also polynomial equations with polynomial coefficients for $A(t)$

$$P_d(t)A(t)^d + \dots + P_0(t) = 0$$

Gfun: [Salvy & Zimmermann 94]

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Gfun: [Salvy & Zimmermann 94]

Guessing the number/generating function of plane maps

```
> b:=proc(n,e) option remember:
```

```
  if n=0 and e=0 then 1 elif n=0 or e=0 then 0
  else add(add(b(i,c)*b(n-1-i,e-2-c),i=0..n-1)
    ,c=0..e-2) + add(b(n-1,d),d=e-1..2*n):
  fi: end:
```

```
> liste:=[seq(b(n,1),n=1..10)];
```

```
      liste := [1, 2, 9, 54, 378, 2916, 24057, 208494, 1876440
```

```
> with(gfun):
```

```
> listtorec(liste,bb(n));
```

```
      [((-12 n - 6) bb(n) + (n + 3) bb(n + 1), bb(0) = 1
```

```
> listtodiffeq(liste,B(t));
```

```
      [ {6 B(t) + (30 t - 3) (d/dt B(t)) + (12 t^2 - t) (d^2/dt^2 B(t)) } B(0) = 1,
```

```
> listtoalgeq(liste,B(t));
```

```
      [16 t - 1 + (-18 t + 1) B(t) + 27 t^2 B(t)^2, ogf
```

Oh?

Is it true that

$$B(x) = 1 + tx^2B(x)^2 + tx \frac{x B(x) - B(1)}{x - 1}$$

implies

$$16t - 1 + (1 - 18t)B(1) + 27t^2B(1)^2 = 0$$

that is,

$$B(1) = \frac{(1 - 12t)^{3/2} - 1 + 18t}{54t^2} = \sum_{n \geq 0} 2 \cdot 3^n \frac{(2n)!}{n!(n+2)!} t^n \quad ?$$

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• Cf. for plane trees:

$$A = 1 + tA^2 \quad \text{and} \quad A = \frac{1 - (1 - 4t)^{1/2}}{2t} = \sum_{n \geq 0} \frac{(2n)!}{n!(n+1)!} t^n$$

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Guessing with Maple and Gfun

Given the first terms of a series $(a(0), a(1), a(2), \dots)$, Gfun can guess:

- linear rec. relations with polynomial coefficients for $a(n)$

$$p_\ell(n)a(n + \ell) + \dots + p_0(n)a(n) = 0$$

- equivalently, linear diff. equations with polynomial coefficients for the GF $A(t)$

$$P_d(t)A^{(d)}(t) + \dots + P_0(t)A(t) = 0$$

- also polynomial equations with polynomial coefficients for $A(t)$

$$P_d(t)A(t)^d + \dots + P_0(t) = 0$$

More generally

- linear relations with polynomial coefficients between given series $A_0(t), \dots, A_d(t)$

$$P_d(t)A_d(t) + \dots + P_0(t)A_0(t) = 0$$

(Important) example: non-linear differential equations

Guessing the number of properly q -coloured triangulations

- Triangulations with n vertices and outer-degree 2:

$$(c(n))_{n \geq 2} = q - 1, (q - 1)(q - 2), (4q - 9)(q - 1)(q - 2), \\ 3(q - 1)(q - 2)(8q^2 - 37q + 43), (176q^3 - 1245q^2 + 2951q - 2344)(q - 1)(q - 2) \dots$$

$$T(x, y) = x(q - 1) + xyz T(x, y) T(1, y) \\ + xz \frac{T(x, y) - T(x, 0)}{y} - x^2 yz \frac{T(x, y) - T(1, y)}{x - 1}$$

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- Differential equation:

$$2(1 - q)t + (t + 10C - 6tC')C'' + (4 - q)(20C - 18tC' + 9t^2C'') = 0$$

- Recurrence relation:

$$(n + 1)(n + 2)c(n + 2) = (q - 4)(3n - 1)(3n - 2)c(n + 1) \\ + 2 \sum_{i=1}^n i(i + 1)(3n - 3i + 1)c(i + 1)c(n + 2 - i),$$

with $c(2) = q - 1$.

[Tutte 84] — A combinatorial mystery

III. Solving discrete differential equations?

$$B(x) = 1 + tx^2 B(x)^2 + tx \frac{x B(x) - B(1)}{x - 1}$$

Our bestiary

- **Plane trees:** no derivative (polynomial equation)

$$A = 1 + tA^2$$

- **Plane maps:** discrete ordinary diff. eq.

$$B(x) = 1 + tx^2B(x)^2 + tx \frac{x B(x) - B(1)}{x - 1}$$

- **Coloured plane maps:** discrete partial diff. eq.

$$\begin{aligned} C(x, y) = & 1 + xyt((\nu - 1)(y - 1) + qy) C(x, y)C(1, y) \\ & + xyt(x\nu - 1)C(x, y)C(x, 1) \\ & + xyt(\nu - 1) \frac{x C(x, y) - C(1, y)}{x - 1} + xyt \frac{y C(x, y) - C(x, 1)}{y - 1} \end{aligned}$$

A hierarchy of formal power series



A hierarchy of formal power series

- Rational series

$$A(t) = \frac{P(t)}{Q(t)}$$

- Algebraic series

$$P(t, A(t)) = 0$$

- Differentially finite series (D-finite)

$$\sum_{i=0}^d P_i(t) A^{(i)}(t) = 0$$

- D-algebraic series

$$P(t, A(t), A'(t), \dots, A^{(d)}(t)) = 0$$



Our bestiary

- **Plane trees:** no derivative (polynomial equation)

$$A = 1 + xA^2$$

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$$B(x) = 1 + tx^2B(x)^2 + tx \frac{x^2B(x) - B(1)}{x - 1}$$

- **Coloured plane maps:** discrete partial diff. eq.

$$\begin{aligned} C(x, y) = & 1 + xyt((\nu - 1)(y - 1) + qy) C(x, y)C(1, y) \\ & + xyt(x\nu - 1)C(x, y)C(x, 1) \\ & + xyt(\nu - 1) \frac{x^2C(x, y) - C(1, y)}{x - 1} + xyt \frac{y^2C(x, y) - C(x, 1)}{y - 1} \end{aligned}$$

Discrete **ordinary** differential equations are simple

Theorem [mbm-Jehanne 2006] (after [Knuth 72] and [Brown 65])

Every series $A(t; x) \equiv A(x)$ solution of a (proper) discrete ordinary differential equation of any degree and order is algebraic:

$$P(t, x, A(t; x)) = 0$$

for some polynomial P .

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- Plane maps

$$B(x) = 1 + tx^2B(x)^2 + tx \frac{x B(x) - B(1)}{x - 1}$$
$$\Rightarrow 27t^2B(1)^2 + (1 - 18t)B(1) + 16t - 1 = 0$$

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Every series $A(t; x) \equiv A(x)$ solution of a (proper) discrete ordinary differential equation of any degree and order is algebraic:

$$P(t, x, A(t; x)) = 0$$

for some polynomial P .

- Plane 3-constellations (degree 3, order 2)

$$A(x) = 1 + txA(x)^3 + tx \frac{(A(x) - A(1))(2A(x) + A(1))}{x - 1} + \frac{tx(A(x) - A(1) - (x - 1)A'(1))}{(x - 1)^2}.$$

$$\Rightarrow A(1) = 1 - 47t + 3t^2 + 3t(22 - 9t)A(1) + 9t(9t - 2)A(1)^2 - 81t^2A(1)^3$$

Discrete **partial** differential equations are not that easy! Even in the linear case

- Algebraic

$$A(x, y) = 1 + txyA + t\Delta_x A + t\Delta_y A$$

- D-finite but not algebraic

$$A(x, y) = 1 + t(x + y)A + t\Delta_x A + t\Delta_y A$$

- Not D-finite

$$A(x, y) = 1 + tx(1 + y)A + t\Delta_x A + t\Delta_y A$$

$$\Delta_x A = \frac{A(x) - A(0)}{x}$$

Discrete **partial** differential equations are not that easy! Even in the linear case

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$$A(x, y) = 1 + txyA + t\Delta_x A + t\Delta_y A$$

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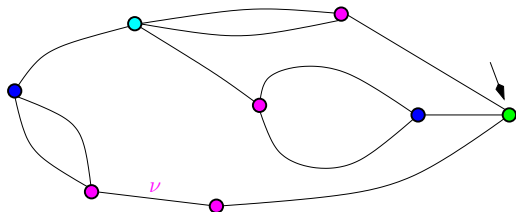
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Now a complete classification: [mbm-Mishna 10], [Bostan-Kauers 10], [Kurkova-Raschel 12], [Mishna-Rechnitzer 07], [Melczer-Mishna 13], [Bostan-Raschel-Salvy 14]

What about coloured maps?

A non-linear discrete differential equation in two variables:

$$\begin{aligned} C(x, y) = & 1 + xyt ((\nu - 1)(y - 1) + qy) C(x, y)C(1, y) \\ & + xyt(x\nu - 1)C(x, y)C(x, 1) \\ & + xyt(\nu - 1) \frac{x C(x, y) - C(1, y)}{x - 1} + xyt \frac{y C(x, y) - C(x, 1)}{y - 1}. \end{aligned}$$



... sleeping since 1968 [Tutte]

In the footsteps of W. Tutte

- For the GF $T(x, y)$ of properly q -coloured triangulations:

$$T(x, y) = x(q - 1) + xyzT(x, y)T(1, y) \\ + xz \frac{T(x, y) - T(x, 0)}{y} - x^2yz \frac{T(x, y) - T(1, y)}{x - 1}$$

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[Tutte 73] Chromatic sums for rooted planar triangulations: the cases $\lambda = 1$ and $\lambda = 2$

[Tutte 73] Chromatic sums for rooted planar triangulations, II : the case $\lambda = \tau + 1$

[Tutte 73] Chromatic sums for rooted planar triangulations, III : the case $\lambda = 3$

[Tutte 73] Chromatic sums for rooted planar triangulations, IV : the case $\lambda = \infty$

[Tutte 74] Chromatic sums for rooted planar triangulations, V : special equations

[Tutte 78] On a pair of functional equations of combinatorial interest

[Tutte 82] Chromatic solutions

[Tutte 82] Chromatic solutions II

[Tutte 84] Map-colourings and differential equations



[Tutte 95]: Chromatic sums revisited

In the footsteps of W. Tutte

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Theorem [Tutte]

- For $q = 2 + 2 \cos \frac{2\pi}{m}$, $q \neq 0, 4$, the series $T(1, y)$ satisfies a discrete differential equation in one variable y .

In the footsteps of W. Tutte

- For the GF $T(x, y)$ of properly q -coloured triangulations:

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Theorem [Tutte]

- For $q = 2 + 2 \cos \frac{2\pi}{m}$, $q \neq 0, 4$, the series $T(1, y)$ satisfies a discrete differential equation in one variable y .
- When q is generic, the generating function of properly q -coloured planar triangulations is differentially algebraic in t :

$$2(1 - q)t + (t + 10H - 6tH')H'' + (4 - q)(20H - 18tH' + 9t^2H'') = 0$$

with $H(t) = t^2 T(1, 0)$ and $t = \sqrt{z}$.

Our results

- Let $C(x, y)$ be the generating function of q -coloured planar maps:

$$C(x, y) = \frac{1}{q} \sum_{M \text{ } q\text{-coloured}} t^{e(M)} x^{\text{dv}(M)} y^{\text{df}(M)} \nu^{\text{m}(M)},$$

where $\text{dv}(M)$ (resp. $\text{df}(M)$) is the degree of the root-vertex (resp. root-face) and $\text{m}(M)$ the number of monochromatic edges.

Theorem [Bernardi-mbm]

- For $q = 2 + 2 \cos \frac{j\pi}{m}$, $q \neq 0, 4$, the series $C(1, y) \equiv C(y)$ satisfies a discrete ordinary differential equation in variable y ,

[Bernardi-mbm 09] Counting coloured planar maps: algebraicity results.

[Bernardi-mbm 14? 15?] Counting coloured planar maps: differential equations

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Theorem [Bernardi-mbm]

- For $q = 2 + 2 \cos \frac{j\pi}{m}$, $q \neq 0, 4$, the series $C(1, y) \equiv C(y)$ satisfies a discrete ordinary differential equation in variable y , and the generating function $C(x, y)$ is algebraic.

[Bernardi-mbm 09] Counting coloured planar maps: algebraicity results.

[Bernardi-mbm 14? 15?] Counting coloured planar maps: differential equations

Our results

- Let $C(x, y)$ be the generating function of q -coloured planar maps:

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- For $q = 2 + 2 \cos \frac{j\pi}{m}$, $q \neq 0, 4$, the series $C(1, y) \equiv C(y)$ satisfies a discrete ordinary differential equation in variable y , and the generating function $C(x, y)$ is algebraic.
- When q is generic, $C(1, 1)$ is differentially algebraic in t :
(an explicit system of differential equations)

[Bernardi-mbm 09] Counting coloured planar maps: algebraicity results.

[Bernardi-mbm 14? 15?] Counting coloured planar maps: differential equations

$q = 2$: The Ising model on planar maps

Let A be the series in t , with polynomial coefficients in ν , defined by

$$A = t \frac{(1 + 3\nu A - 3\nu A^2 - \nu^2 A^3)^2}{1 - 2A + 2\nu^2 A^3 - \nu^2 A^4}.$$

Then the generating function of bicoloured planar maps is

$$C(2, \nu, t; 1, 1) = \frac{1 + 3\nu A - 3\nu A^2 - \nu^2 A^3}{(1 - 2A + 2\nu^2 A^3 - \nu^2 A^4)^2} P(\nu, A)$$

where

$$P(\nu, A) = \nu^3 A^6 + 2\nu^2(1 - \nu)A^5 + \nu(1 - 6\nu)A^4 \\ - \nu(1 - 5\nu)A^3 + (1 + 2\nu)A^2 - (3 + \nu)A + 1.$$

↪ **Asymptotics**: Phase transition at $\nu_c = \frac{3+\sqrt{5}}{2}$, critical exponents...

$q = 2$: The Ising model on planar maps

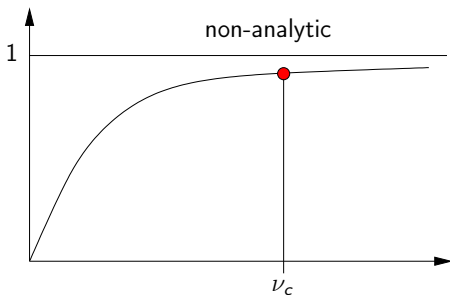
Fix n (large) and choose a bi-coloured planar map M with n edges at random, with probability proportional to

$$\nu^m(M).$$

When $\nu = 0$, the colouring is proper.

When ν is small, we favour maps with few monochromatic edges.

When ν is large, we favour maps with many monochromatic edges.



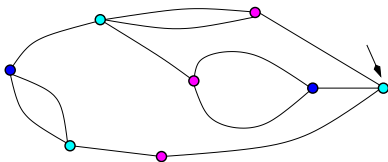
$q = 3, \nu = 0$: Properly 3-coloured planar maps

Let A be the quartic series in t defined by

$$A = t \frac{(1 + 2A)^3}{1 - 2A^3}.$$

Then the generating function of properly 3-coloured planar maps is

$$C(3, 0, t; 1, 1) = \frac{(1 + 2A)(1 - 2A^2 - 4A^3 - 4A^4)}{(1 - 2A^3)^2}$$



IV. Bijections, at last!

Simple formulas cry for bijective proofs

Plane trees

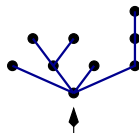
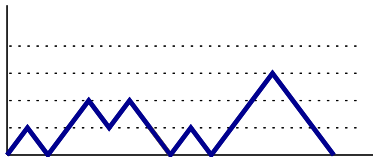
$$a(n) = \frac{1}{n+1} \binom{2n}{n} = \frac{(2n)!}{n!(n+1)!}$$

Plane maps

$$b(n) = \frac{2 \cdot 3^n}{(n+2)(n+1)} \binom{2n}{n} = 2 \cdot 3^n \frac{(2n)!}{n!(n+2)!}$$

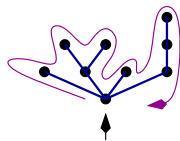
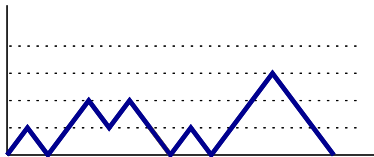
Bijjective enumeration of rooted plane trees

A tree with n edges is a Dyck path of length $2n$



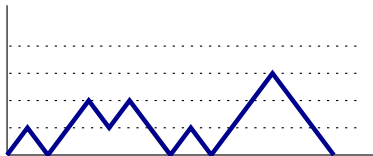
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Bijjective enumeration of rooted plane trees

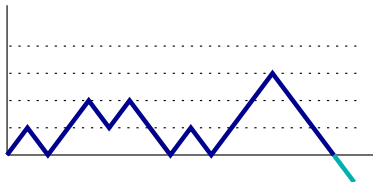
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$$a(n) =$$

Bijjective enumeration of rooted plane trees

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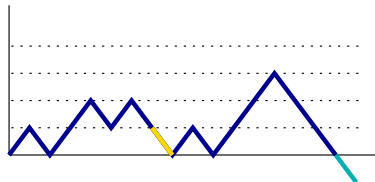


add a tail...

$$a(n) =$$

Bijjective enumeration of rooted plane trees

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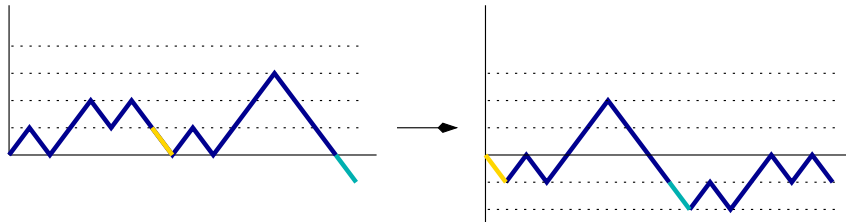
add a tail...

mark a step $((2n + 1)$ choices)...

$$(2n + 1)a(n) =$$

Bijection enumeration of rooted plane trees

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add a tail...

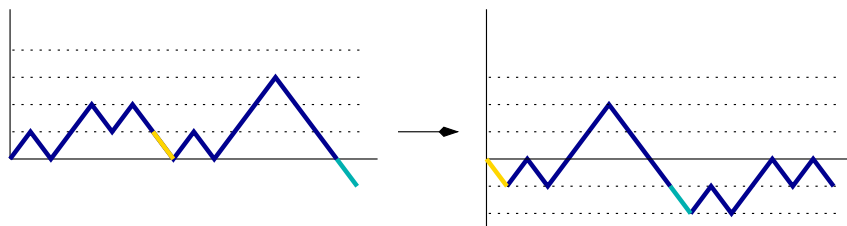
mark a step ($(2n + 1)$ choices)...

and read the path cyclically starting from that step

$$(2n + 1)a(n) = \binom{2n + 1}{n}$$

Bijection enumeration of rooted plane trees

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add a tail...

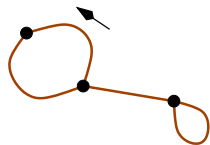
mark a step $((2n + 1)$ choices)...

and read the path cyclically starting from that step

$$(2n + 1)a(n) = \binom{2n + 1}{n} \Rightarrow a(n) = \frac{(2n)!}{n!(n + 1)!}$$

Bijjective enumeration of rooted plane maps

A plane map with n edges is a 4-valent map with n vertices

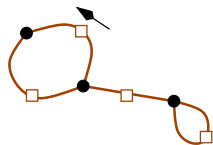


$b(n)$

[Schaeffer 97]

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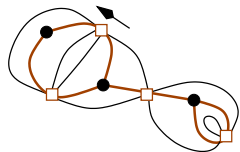


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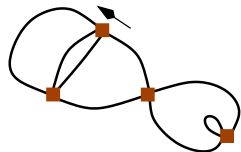


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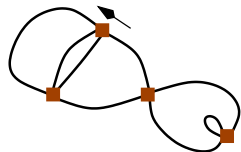


$b(n)$

[Schaeffer 97]

Bijjective enumeration of rooted plane maps

A plane map with n edges is a 4-valent map with n vertices or a **balanced blossoming binary tree** with n vertices

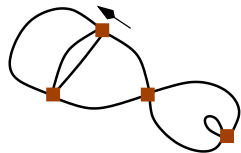


$b(n)$

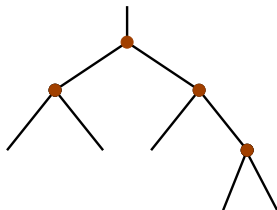
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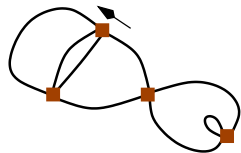


$$\frac{1}{(n+1)} \binom{2n}{n}$$

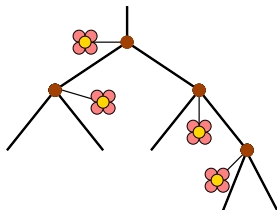
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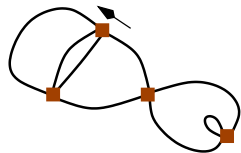


$$\frac{3^n}{(n+1)} \binom{2n}{n}$$

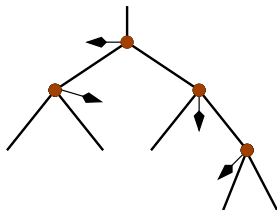
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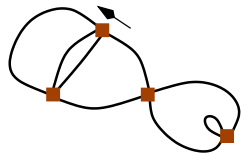


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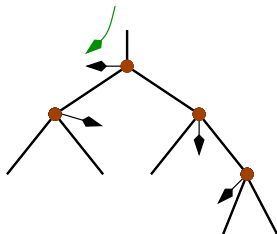
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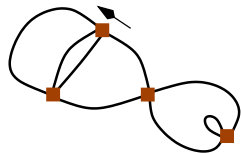


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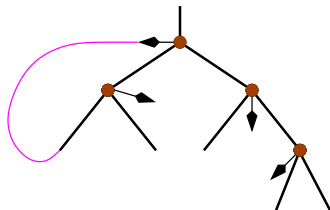
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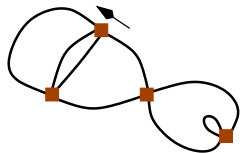


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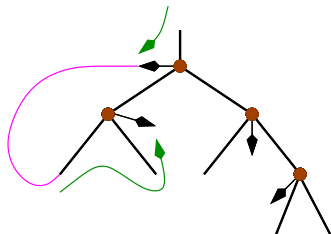
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Bijection enumeration of rooted plane maps

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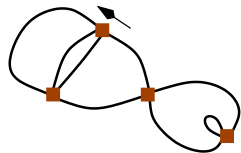


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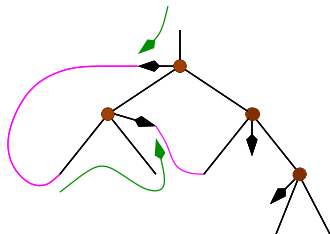
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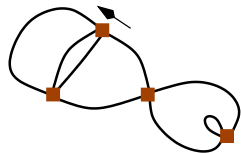


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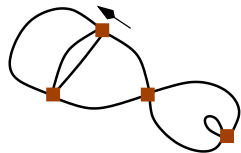


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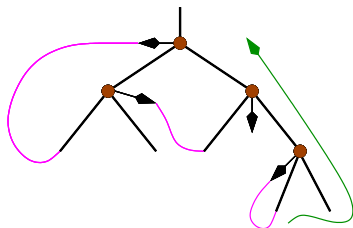
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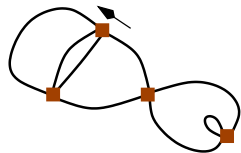


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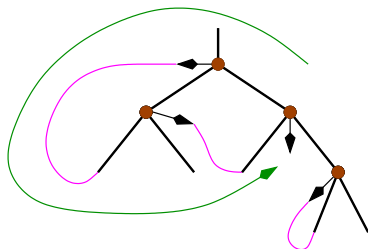
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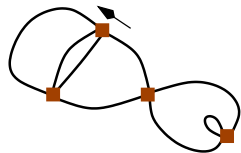


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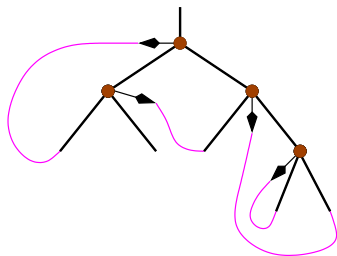
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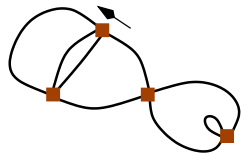


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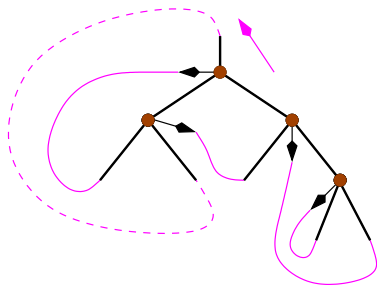
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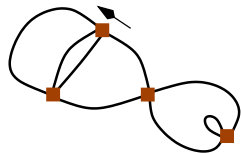


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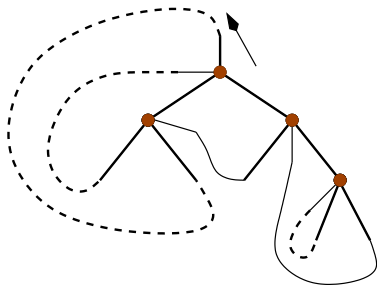
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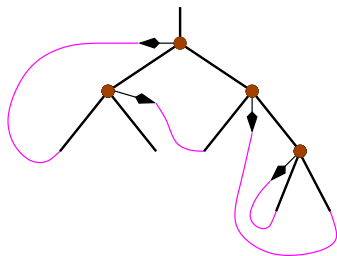
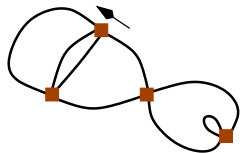


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=

$$\frac{3^n}{(n+1)} \binom{2n}{n} \times \frac{2}{n+2}$$

[Schaeffer 97]

The strength of bijections

- Improve our understanding of the nature of planar maps
 - ▶ nice numbers
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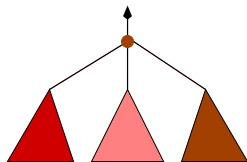
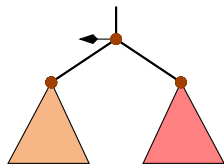
Plane maps (n edges) \iff **balanced** blossoming trees (n nodes)

- Blossoming trees:

$$BI = 1 + 3tBI^2$$

- An **unbalanced blossoming tree** is a 3-tuple of blossoming subtrees [Bouttier et al. 02]:

$$\Rightarrow B = BI - tBI^3.$$



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- Random generation
- Give a handle on other parameters: the average diameter of a random map of size n scales like $n^{1/4}$ [Chassaing-Schaeffer 02]
- The starting point of many recent results in probability theory on the asymptotic properties of large random maps [Le Gall, Miermont, Marckert, Paulin...]



More bijections

- Many, many families of uncoloured maps are now well-understood [Arquès, Bernardi, mbm, Bouttier, Chapuy, Cori, Di Francesco, Fusy, Guitter, Poulalhon, Schaeffer, Vauquelin]
- What about coloured maps? (**D-algebraic**)

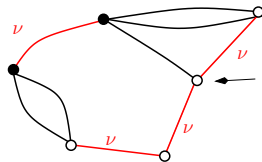


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The Ising model: 2-coloured planar maps
(**algebraic**)
[mbm-Schaeffer 02], [Bouttier et al. 04]



More bijections

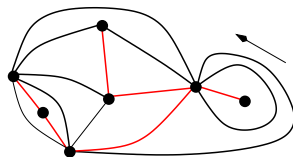
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Maps equipped with a spanning tree

($q = 0, \nu = 1$; **D-finite**)

[Mullin 67], [Bernardi 07]



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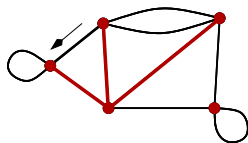
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Maps equipped with a spanning forest

($q = 0$; **D-algebraic**)

[Bouttier et al. 07], [mbm-Courtial 13a]



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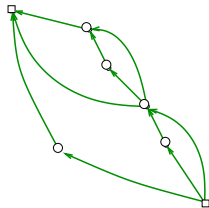
Maps equipped with a bipolar orientation

$((-1)^{v(M)} \chi'_M(1); \text{D-finite})$

[Felsner-Fusy-Noy-Orden 08],

[Fusy-Poulalhon-Schaeffer 08],

[Bonichon-mbm-Fusy 08]



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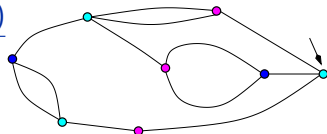


Properly 3-coloured planar maps
(**algebraic**; **OPEN**)

$$C(3, 0, t; 1, 1) = \frac{(1 + 2A)(1 - 2A^2 - 4A^3 - 4A^4)}{(1 - 2A^3)^2}$$

with

$$A = t \frac{(1 + 2A)^3}{1 - 2A^3}.$$



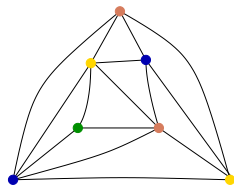
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Properly q -coloured triangulations
(**D-algebraic**; **OPEN**)

$$(n+1)(n+2)c(n+2) = \\ (q-4)(3n-1)(3n-2)c(n+1) \\ + 2 \sum_{i=1}^n i(i+1)(3n-3i+1)c(i+1)c(n+2-i)$$

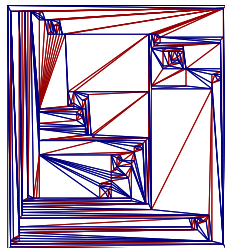


Pour résumer...



Objets aléatoires

Transitions de phase



Combinatoire
énumérative

Calcul formel
Divination

Dessin de graphes

Bijections

Séries formelles

$$B(x) = 1 + tx^2B(x)^2 + tx \frac{x B(x) - B(1)}{x-1}$$

