

# The number of labelled intervals in the $m$ -Tamari lattices

Mireille Bousquet-Mélou, CNRS, LaBRI

Guillaume Chapuy, CNRS, LIAFA

Louis-François Prévaille-Ratelle, LACIM, UQAM, Montréal

*<http://www.labri.fr/~bousquet>*

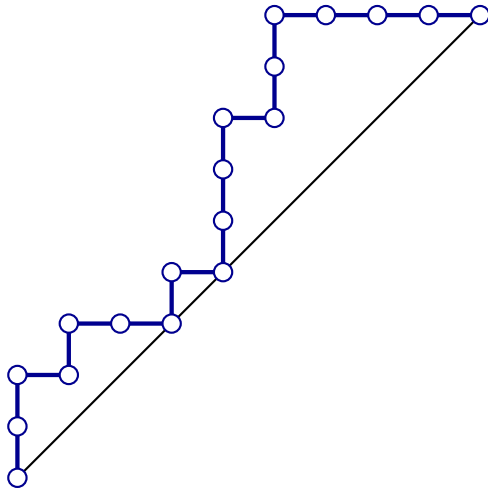
# Ballot paths

An  $m$ -ballot path of size  $n$ :

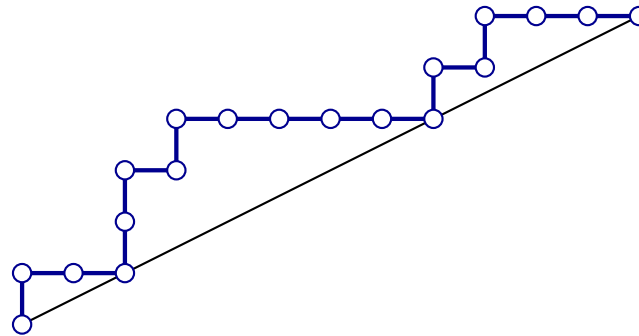
- starts at  $(0, 0)$ ,
- ends at  $(mn, n)$ ,
- never goes below the line  $\{x = my\}$ .

Examples:

$m = 1$



$m = 2$



$m = 1$ : The (usual) Tamari lattice  $\mathcal{T}_n$

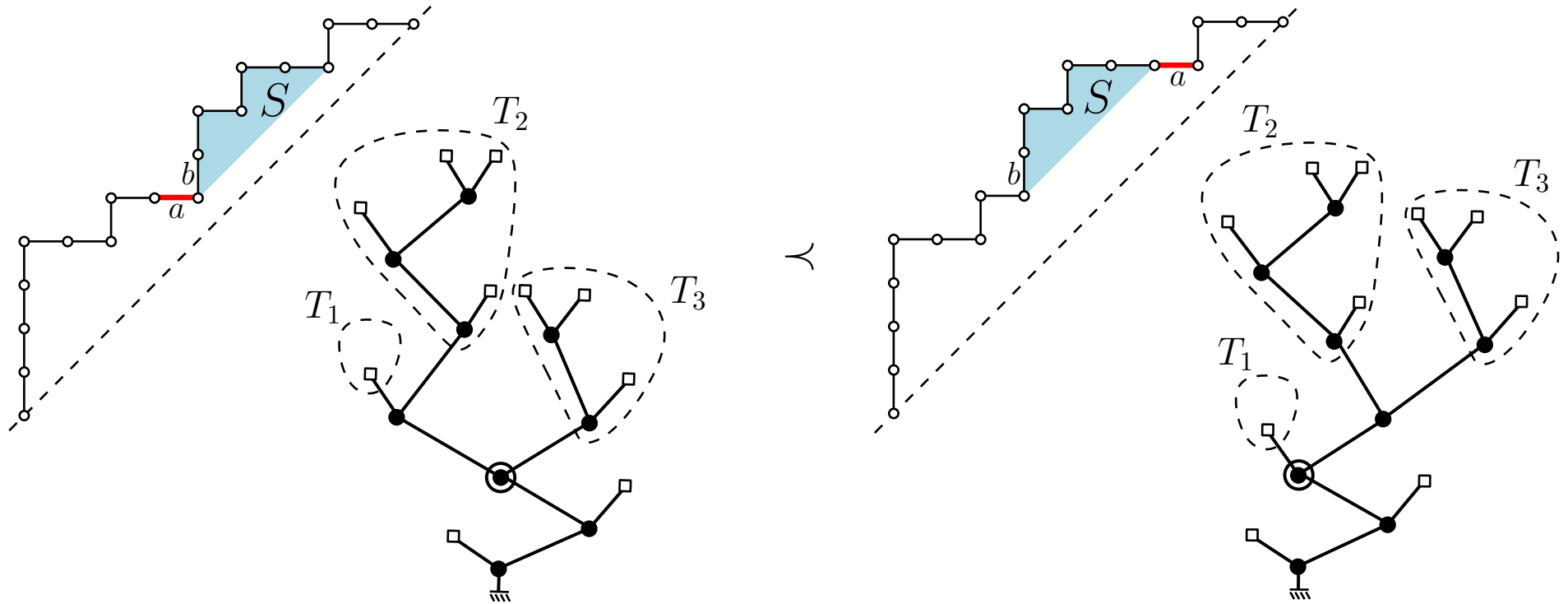
Covering relation:



[Huang-Tamari 72]

$m = 1$ : The (usual) Tamari lattice  $\mathcal{T}_n$

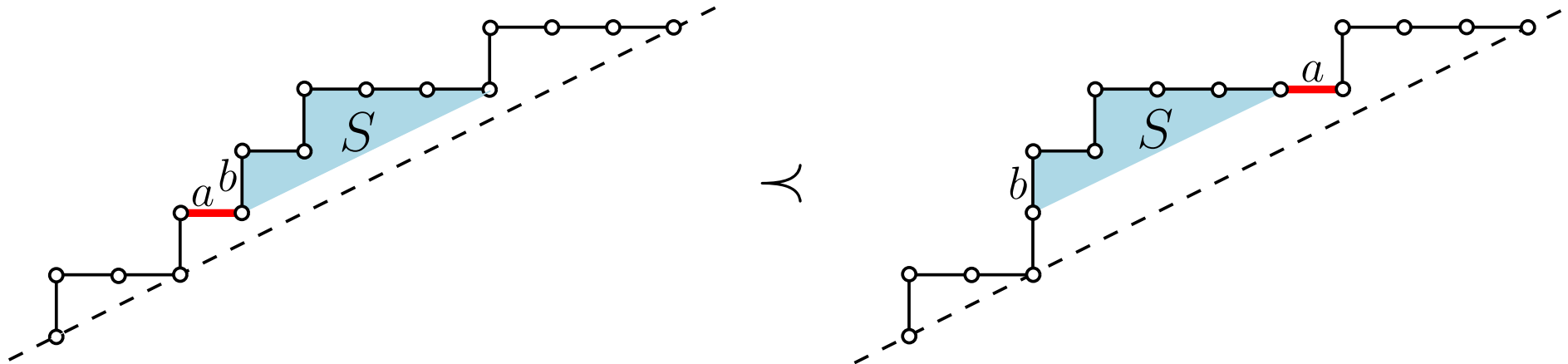
Covering relation:



[Huang-Tamari 72]

# The $m$ -Tamari lattice $\mathcal{T}_n^{(m)}$

Covering relation:



[Bergeron 10]

**Proposition:** Defines a lattice  
[mbm-Fusy-Préville-Ratelle 11]

## Last year's intervals

**Proposition:** Let  $m \geq 1$  and  $n \geq 1$ . The number of intervals in the Tamari lattice  $\mathcal{T}_n^{(m)}$  is

$$\frac{m+1}{n(mn+1)} \binom{n(m+1)^2+m}{n-1}$$

- Map-like numbers!
- When  $m = 1$ : proved by [Chapoton 06]

$$\frac{2}{n(n+1)} \binom{4n+1}{n-1}.$$

This is also the number of 3-connected planar triangulations on  $n+3$  vertices [Tutte 62]  $\Rightarrow$  Bijection found by [Bernardi & Bonichon 09]

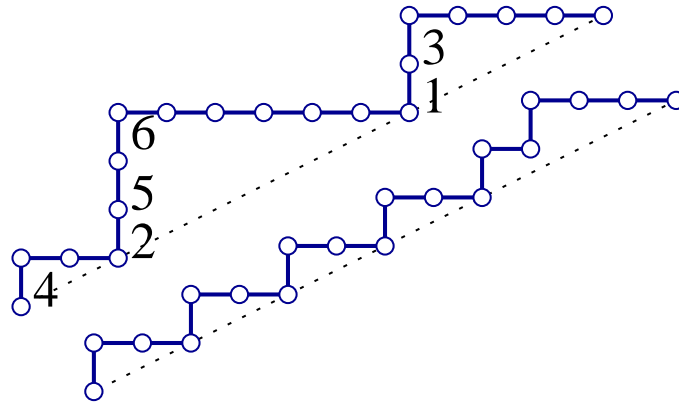
- Proved in [mbm–Fusy–Préville-Ratelle 11] for a generic  $m$ ... but no bijection

# In 2012, Tamari intervals get labels



## Labelled Tamari intervals

The up steps of the upper path are labelled from 1 to  $n$ , in such a way the labels increase along each ascent.



**Proposition:** Let  $m \geq 1$  and  $n \geq 1$ . The number of **labelled** intervals in the Tamari lattice  $\mathcal{T}_n^{(m)}$  is

$$f_n^{(m)} = (m + 1)^n (mn + 1)^{n-2}$$

- For  $m = 1$ ,

$$f_n^{(1)} = 2^n (n + 1)^{n-2}$$

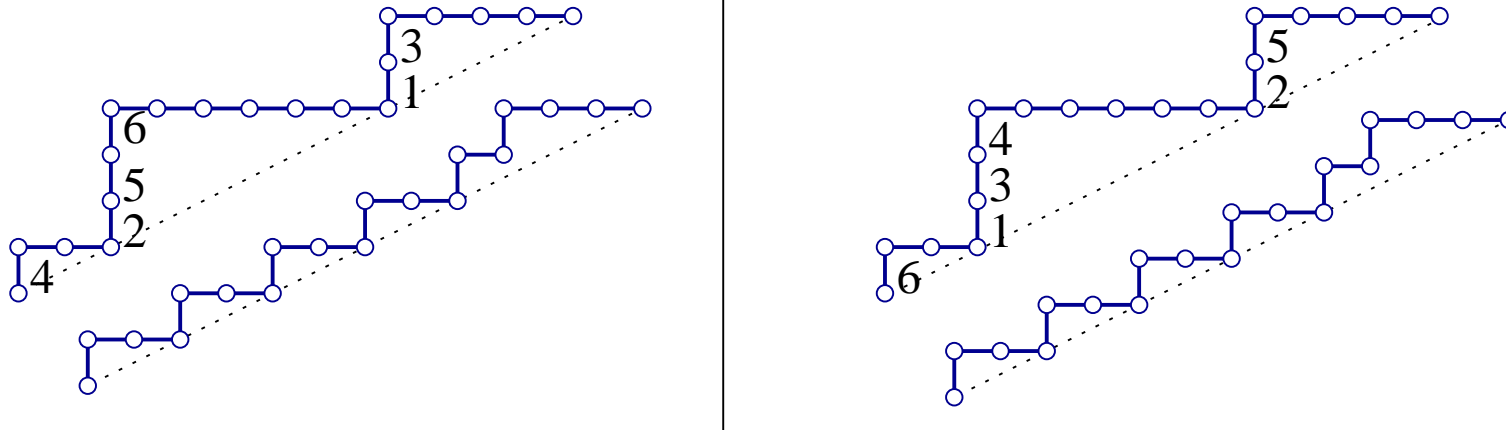
Since there are  $(n + 1)^{n-1}$  Cayley trees with  $n + 1$  nodes, these are again “map-like numbers”, that is, conjugacy classes of trees.



## A refined result: the action of $\mathfrak{S}_n$ on Tamari intervals

Permutations of  $\mathfrak{S}_n$  act on  $m$ -Tamari intervals of size  $n$  by permuting (and then reordering) labels.

**Example:** if  $\sigma = 2\ 3\ 5\ 6\ 1\ 4$ ,



**Proposition:** Let  $\sigma \in \mathfrak{S}_n$  have cycle type  $\lambda = (\lambda_1, \dots, \lambda_\ell)$ . The number of labelled  $m$ -Tamari intervals of size  $n$  left unchanged under the action of  $\sigma$  is

$$\chi_m(\sigma) = (mn + 1)^{\ell-2} \prod_{1 \leq i \leq \ell} \binom{(m+1)\lambda_i}{\lambda_i}.$$

• When  $\sigma = \text{id}$ , i.e.,  $\lambda = (1, \dots, 1)$ , one recovers the total number of  $m$ -Tamari intervals:

$$\chi_m(\text{id}) = (mn + 1)^{n-2} (m + 1)^n.$$

# I. Functional equations

## Generating functions

Let  $I = [P, Q]$  be an  $m$ -Tamari interval. A **contact** of  $I$  is a contact of the **lower** path  $P$  with the line  $\{x = my\}$ .

The **initial rise** of  $I$  is the length of the first sequence of up steps of the **upper** path  $Q$ .

We denote by  $U^{(m)}(t; x, y) \equiv U(x, y)$  the ordinary generating function of unlabelled  $m$ -Tamari intervals, where  $t$  counts the size,  $x$  the number of contacts and  $y$  the initial rise.

Similarly,  $L^{(m)}(t; x, y) \equiv L(x, y)$  is the exponential generating function of labelled  $m$ -Tamari intervals, counted according to the same parameters (exponential in the variable  $t$ ).

## Functional equations

**Proposition:** For  $m \geq 1$ , the generating functions of unlabelled and labelled  $m$ -Tamari intervals satisfy:

$$U(x, y) = x + xy t (U(x, 1) \cdot \Delta)^{(m)} U(x, y),$$

$$\frac{\partial L}{\partial y}(x, y) = xt (L(x, 1) \cdot \Delta)^{(m)} L(x, y),$$

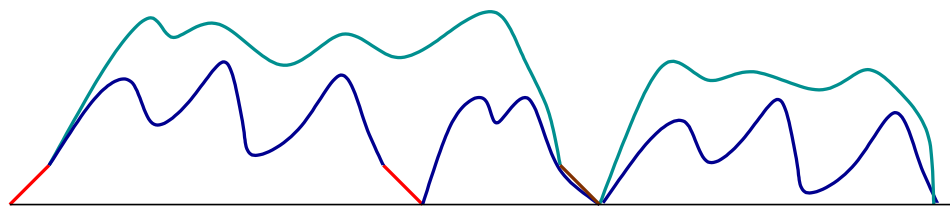
with the initial condition  $L(x, 0) = x$ .

Here,  $\Delta$  is the divided difference operator

$$\Delta S(x) = \frac{S(x) - S(1)}{x - 1},$$

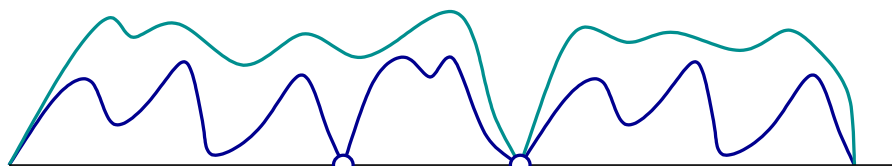
and the power  $m$  means that the operators are applied  $m$  times.

# Proof: a recursive description of intervals (here $m = 1$ )



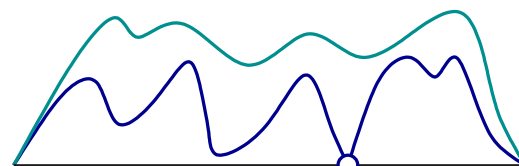
Tamari interval

↓ 1-reduction

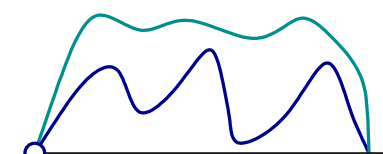


$p_1$

$q_1$



$p_1$



$$U(x, y) = x +$$

$$ty \sum_i U_i(y) (x + \dots + x^i) \times U(x, 1)$$

$$txy \sum_i U_i(y) \frac{x^i - 1}{x - 1}$$

$$txy \frac{U(x, y) - U(x, 1)}{x - 1}$$

## Generating function for the action of $\mathfrak{S}_n$

- Let  $p = (p_1, p_2, \dots)$  be a list of indeterminates and

$$F^{(m)}(t, p; x, y) = \sum_{I=[P,Q], \text{ labelled}} \frac{t^{|I|}}{|I|!} x^{c(P)} \sum_{\sigma \in \text{Stab}(I)} y^{r_\sigma(Q)} p_{\lambda(\sigma)},$$

where  $p_\lambda = \prod_{i \geq 1} p_{\lambda_i}$ ,  $c(P)$  is the number of contacts of  $P$ , and  $r_\sigma(Q)$  the number of cycles of  $\sigma$  contained in the first rise of  $Q$ .

- Then  $F(x, 0) = x$  and

$$\frac{\partial F}{\partial y}(x, y) = \sum_{k \geq 1} \frac{p_k}{k} \left( tx(F(x, 1)\Delta)^{(m)} \right)^{(k)} F(x, y).$$

## Generating function for the action of $\mathfrak{S}_n$

- Let  $p = (p_1, p_2, \dots)$  be a list of indeterminates and

$$F^{(m)}(t, p; x, y) = \sum_{I=[P,Q], \text{ labelled}} \frac{t^{|I|}}{|I|!} x^{c(P)} \sum_{\sigma \in \text{Stab}(I)} y^{r_\sigma(Q)} p_{\lambda(\sigma)},$$

where  $p_\lambda = \prod_{i \geq 1} p_{\lambda_i}$ ,  $c(P)$  is the number of contacts of  $P$ , and  $r_\sigma(Q)$  the number of cycles of  $\sigma$  contained in the first rise of  $Q$ .

- Then  $F(x, 0) = x$  and

$$\frac{\partial F}{\partial y}(x, y) = \sum_{k \geq 1} \frac{p_k}{k} \left( tx(F(x, 1)\Delta)^{(m)} \right)^{(k)} F(x, y).$$

- For  $(p_1, p_2, \dots) = (1, 0, 0, \dots)$ , only the identity contributes and one recovers

$$\frac{\partial F}{\partial y}(x, y) = \left( tx(F(x, 1)\Delta)^{(m)} \right) F(x, y).$$

## II. Solution



## Functional equations when $m = 1$

**Proposition:** The generating functions of unlabelled and labelled 1-Tamari intervals satisfy:

$$U(x, y) = x + xyU(x, 1) \cdot \frac{U(x, y) - U(1, y)}{x - 1},$$

$$\frac{\partial L}{\partial y}(x, y) = xyL(x, 1) \cdot \frac{L(x, y) - L(1, y)}{x - 1},$$

with the initial condition  $L(x, 0) = x$ .

## Solution in the unlabelled case ( $m = 1$ )

0. The equation:

$$U(x, y) = x + xyU(x, 1) \cdot \frac{U(x, y) - U(1, y)}{x - 1} \quad (1)$$

1. Determine the series  $U(x, 1) = U_1(x)$ .

2. Solve (1) with  $U(x, 1)$  replaced by  $U_1(x)$  (a linear equation in  $U(x, y)$ ).

## Solution in the unlabelled case ( $m = 1$ )

0. The equation:

$$U(x, y) = x + xyU(x, 1) \cdot \frac{U(x, y) - U(1, y)}{x - 1} \quad (1)$$

1. Determine the series  $U(x, 1) = U_1(x)$ . We have

$$U_1(x) = x + xU_1(x) \cdot \frac{U_1(x) - U_1(1)}{x - 1}$$

and this can be solved using the quadratic method [Brown 60s].

## Solution in the unlabelled case ( $m = 1$ )

0. The equation:

$$U(x, y) = x + xytU(x, 1) \cdot \frac{U(x, y) - U(1, y)}{x - 1} \quad (1)$$

1. Determine the series  $U(x, 1) = U_1(x)$ . We have

$$U_1(x) = x + xtU_1(x) \cdot \frac{U_1(x) - U_1(1)}{x - 1}$$

and this can be solved using the **quadratic method** [Brown 60s].

2. Solve (1) with  $U(x, 1)$  replaced by  $U_1(x)$ , that is,

$$\left(1 - \frac{xytU_1(x)}{x - 1}\right) U(x, y) = x - \frac{xytU_1(x)}{x - 1} \cdot U(1, y).$$

This can be solved using the **kernel method**.

One remains in the world of **algebraic series**.

## Solution in the unlabelled case ( $m$ general)

Proposition: Set

$$t = z(1 - z)^{m^2 + 2m}, \quad x = \frac{1 + u}{(1 + zu)^{m+1}}, \quad \text{and} \quad y = \frac{1 + v}{(1 + zv)^{m+1}}.$$

Then

$$\frac{yU^{(m)}(t; x, y)}{x - y} = \frac{(1 + u)(1 + zu)(1 + v)(1 + zv)}{(u - v)(1 - zuv)(1 - z)^{m+2}}.$$

In particular,  $yU^{(m)}(t; x, y)$  is a **symmetric** series in  $x$  and  $y$ ... a combinatorial mystery

[mbm–Fusy–Préville–Ratelle 11]

## Solution in the labelled case ( $m = 1$ )

0. The equation:  $L(x, 0) = x$  and

$$\frac{\partial L}{\partial y}(x, y) = xtL(x, 1) \cdot \frac{L(x, y) - L(1, y)}{x - 1}. \quad (2)$$

1. Guess the series  $L(x, 1) \equiv L_1(x)$  (hard).

2. Solve (2) with  $L(x, 1)$  replaced by  $L_1(x)$  (a linear equation in  $L(x, y)$ )

3. Check that the series  $L(x, y)$  thus obtained satisfies  $L(x, 1) = L_1(x)$ .

## Solution in the labelled case ( $m = 1$ )

0. The equation:  $L(x, 0) = x$  and

$$\frac{\partial L}{\partial y}(x, y) = xtL(x, 1) \cdot \frac{L(x, y) - L(1, y)}{x - 1}. \quad (2)$$

1. Guess the series  $L(x, 1) \equiv L_1(x)$  (hard). Write

$$t = ze^{-2z} \quad \text{and} \quad x = (1 + u)e^{-zu}, \quad (3)$$

and  $\bar{u} = 1/u$ . Then  $L(x, 1) = L_1(x)$  with

$$\frac{L_1(x)}{x - 1} = (1 + \bar{u})e^{2z + zu}$$

## Solution in the labelled case ( $m = 1$ )

0. The equation:  $L(x, 0) = x$  and

$$\frac{\partial L}{\partial y}(x, y) = xtL(x, 1) \cdot \frac{L(x, y) - L(1, y)}{x - 1}. \quad (2)$$

1. Guess the series  $L(x, 1) \equiv L_1(x)$  (hard). Write

$$t = ze^{-2z} \quad \text{and} \quad x = (1 + u)e^{-zu}, \quad (3)$$

and  $\bar{u} = 1/u$ . Then  $L(x, 1) = L_1(x)$  with

$$\frac{L_1(x)}{x - 1} = (1 + \bar{u})e^{2z + zu}$$

2. Solve (2) with  $L(x, 1)$  replaced by  $L_1(x)$ .



## Solution in the labelled case ( $m = 1$ )

0. The equation:  $L(x, 0) = x$  and

$$\frac{\partial L}{\partial y}(x, y) = xtL(x, 1) \cdot \frac{L(x, y) - L(1, y)}{x - 1}. \quad (2)$$

1. Guess the series  $L(x, 1) \equiv L_1(x)$  (hard). Write

$$t = ze^{-2z} \quad \text{and} \quad x = (1 + u)e^{-zu}, \quad (3)$$

and  $\bar{u} = 1/u$ . Then  $L(x, 1) = L_1(x)$  with

$$\frac{L_1(x)}{x - 1} = (1 + \bar{u})e^{2z + zu}$$

2. Solve (2) with  $L(x, 1)$  replaced by  $L_1(x)$ . Denoting  $\tilde{L}(z; u, y) \equiv L(t; x, y)$  after the change of variables (3), Eq.(2) reads

$$\frac{\partial \tilde{L}}{\partial y}(u, y) = z(1 + u)(1 + \bar{u}) \left( \tilde{L}(u, y) - \tilde{L}(0, y) \right),$$

with initial condition  $\tilde{L}(u, 0) = (1 + u)e^{-zu}$ .

## 2. Solve

$$\frac{\partial \tilde{L}}{\partial y}(u, y) = z(1+u)(1+\bar{u}) \left( \tilde{L}(u, y) - \tilde{L}(0, y) \right),$$

with initial condition  $\tilde{L}(u, 0) = (1+u)e^{-zu}$ .

- **Key observation:** the term  $(1+u)(1+\bar{u})$  is symmetric in  $u$  and  $\bar{u}$ . Hence

$$\frac{\partial \tilde{L}}{\partial y}(\bar{u}, y) = z(1+u)(1+\bar{u}) \left( \tilde{L}(\bar{u}, y) - \tilde{L}(0, y) \right).$$

- **Take the difference:** an homogeneous linear DE for  $\tilde{L}(u, y) - \tilde{L}(\bar{u}, y)$ !

$$\frac{\partial}{\partial y} \left( \tilde{L}(u, y) - \tilde{L}(\bar{u}, y) \right) = z(1+u)(1+\bar{u}) \left( \tilde{L}(u, y) - \tilde{L}(\bar{u}, y) \right),$$

$$\Rightarrow \tilde{L}(u, y) - \tilde{L}(\bar{u}, y) = (1+u)e^{yz(1+u)(1+\bar{u})} \left( e^{-zu} - \bar{u}e^{-z\bar{u}} \right).$$

- **Extraction** of the non-negative powers of  $u$  (plus condition  $\tilde{L}(-1, y) = 0$ ):

$$\tilde{L}(u, y) = (1+u)[u^{\geq}] \left( e^{yz(1+u)(1+\bar{u})} \left( e^{-zu} - \bar{u}e^{-z\bar{u}} \right) \right)$$

where  $[u^{\geq}]S(u)$  denotes the part of  $S(u)$  with non-negative powers of  $u$ .

3. Check that the series  $L(x, y)$  thus obtained satisfies  $L(x, 1) = L_1(x)$ : simple!  
(when  $m = 1 \dots$ )

## Solution in the labelled case ( $m$ general)

- The equation:

$$\frac{\partial L}{\partial y}(x, y) = xt (L(x, 1) \cdot \Delta)^{(m)} L(x, y)$$

(involves  $L(1, y), L'(1, y), \dots, L^{(m-1)}(1, y)$ )

- Set

$$t = ze^{-m(m+1)z} \quad \text{and} \quad x = (1 + u)e^{-mzu}.$$

- Then

$$\frac{L(x, 1)}{x - 1} = (1 + \bar{u})e^{(m+1)z+zu}.$$

+ a horrible expression for  $L(x, y)$ .

## The action of $\mathfrak{S}_n$ on Tamari intervals

- The equation:

$$\frac{\partial F}{\partial y}(x, y) = \sum_{k \geq 1} \frac{p_k}{k} \left( tx(F(x, 1)\Delta)^{(m)} \right)^{(k)} F(x, y).$$

- Let

$$L = \sum_{k \geq 1} \frac{p_k}{k} \binom{(m+1)k}{k} z^k, \quad K(u) = \sum_{k \geq 1} \frac{p_k}{k} z^k \sum_{i=1}^k \binom{(m+1)k}{k-i} u^i,$$

and set

$$t = ze^{-mL} \quad \text{and} \quad x = (1+u)e^{-mK(u)}.$$

- Then

$$\frac{F(x, 1)}{x-1} = (1+\bar{u})e^{K(u)+L}$$

with  $\bar{u} = 1/u$ .

+ a horrible expression for  $F(t, p; x, y)$ .

**III. Motivations: a (conjectured) link with  
diagonal coinvariant spaces**

## The diagonal action of $\mathfrak{S}_n$

- Let  $k, n \geq 1$ , and consider  $k$  alphabets of size  $n$ :

$$\begin{array}{cccc} a_1, & a_2, & \dots, & a_n \\ b_1, & b_2, & \dots, & b_n \\ c_1, & c_2, & \dots, & c_n \\ & & & \dots \end{array}$$

- Then  $\mathfrak{S}_n$  acts on polynomials in these variables:

$$\sigma P(a_1, \dots, a_n, b_1, \dots, b_n, \dots) = P(a_{\sigma(1)}, \dots, a_{\sigma(n)}, b_{\sigma(1)}, \dots, b_{\sigma(n)}, \dots).$$

- Let  $\mathcal{I}_n^{(k)}$  be the ideal generated by polynomials (with non constant term) that are invariant under this action (for instance,  $a_1 + a_2, a_1 b_1^2 + a_2 b_2^2, \dots$ )
- Finally, let  $\mathcal{R}_n^{(k)}$  be the quotient of  $\mathbb{Q}[a_1, \dots, a_n, b_1, \dots, b_n, \dots]$  by  $\mathcal{I}_n^{(k)}$ .

## Dimension of the quotient

- $k = 1$ , variables  $a_1, \dots, a_n$ :

$$\dim \mathcal{R}_n^{(1)} = n!,$$

the number of **permutations** of size  $n$ .

Basis:  $\{a_1^{i_1} \cdots a_n^{i_n} : 0 \leq i_j < j\}$



## Dimension of the quotient

- $k = 1$ , variables  $a_1, \dots, a_n$ :

$$\dim \mathcal{R}_n^{(1)} = n!,$$

the number of **permutations** of size  $n$ .

Basis:  $\{a_1^{i_1} \cdots a_n^{i_n} : 0 \leq i_j < j\}$

- $k = 2$ , variables  $a_1, \dots, a_n, b_1, \dots, b_n$ :

$$\dim \mathcal{R}_n^{(2)} = (n + 1)^{n-1},$$

the number of **parking functions** of size  $n$  [Haiman 02]

## Dimension of the quotient

- $k = 1$ , variables  $a_1, \dots, a_n$ :

$$\dim \mathcal{R}_n^{(1)} = n!,$$

the number of **permutations** of size  $n$ .

Basis:  $\{a_1^{i_1} \cdots a_n^{i_n} : 0 \leq i_j < j\}$

- $k = 2$ , variables  $a_1, \dots, a_n, b_1, \dots, b_n$ :

$$\dim \mathcal{R}_n^{(2)} = (n + 1)^{n-1},$$

the number of **parking functions** of size  $n$  [Haiman 02]

- $k = 3$ , variables  $a_1, \dots, a_n, b_1, \dots, b_n, c_1, \dots, c_n$ :

$$\dim \mathcal{R}_n^{(3)} = 2^n (n + 1)^{n-2},$$

the number of **1-Tamari intervals** of size  $n$

**Conjecture** [Haiman 94, Bergeron–Préville–Ratelle 10]

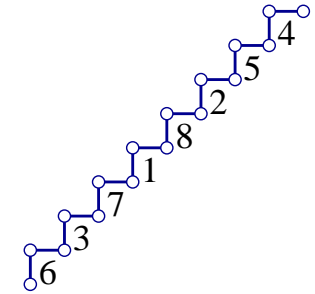
## Dimension of the quotient

- $k = 1$ , variables  $a_1, \dots, a_n$ :

$$\dim \mathcal{R}_n^{(1)} = n!,$$

the number of **permutations** of size  $n$ .

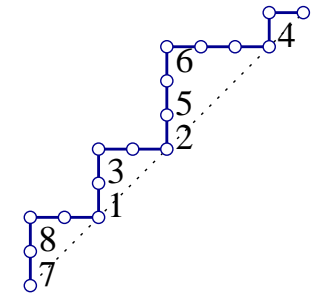
Basis:  $\{a_1^{i_1} \cdots a_n^{i_n} : 0 \leq i_j < j\}$



- $k = 2$ , variables  $a_1, \dots, a_n, b_1, \dots, b_n$ :

$$\dim \mathcal{R}_n^{(2)} = (n + 1)^{n-1},$$

the number of **parking functions** of size  $n$  [Haiman 02]

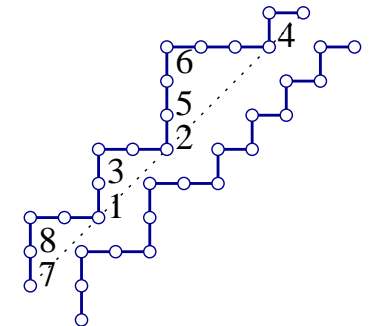


- $k = 3$ , variables  $a_1, \dots, a_n, b_1, \dots, b_n, c_1, \dots, c_n$ :

$$\dim \mathcal{R}_n^{(3)} = 2^n (n + 1)^{n-2},$$

the number of **1-Tamari intervals** of size  $n$

**Conjecture** [Haiman 94, Bergeron–Préville–Ratelle 10]



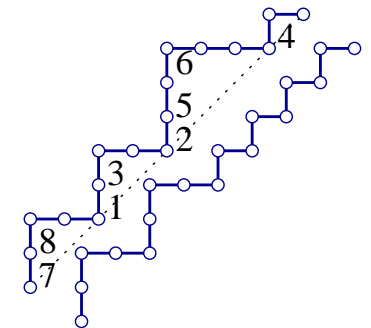
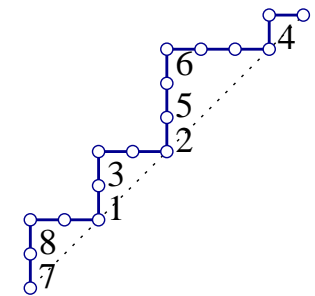
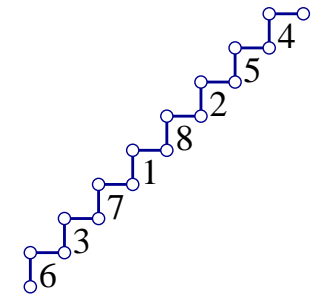
## Representation of $\mathfrak{S}_n$ on the quotient

- $k = 1$ , variables  $a_1, \dots, a_n$ : regular representation of  $\mathfrak{S}_n$

- $k = 2$ , variables  $a_1, \dots, a_n, b_1, \dots, b_n$ : action<sup>(\*)</sup> of  $\mathfrak{S}_n$  on parking functions of size  $n$  [Haiman 02]

- $k = 3$ , variables  $a_1, \dots, a_n, b_1, \dots, b_n, c_1, \dots, c_n$ : action<sup>(\*)</sup> of  $\mathfrak{S}_n$  on 1-Tamari intervals of size  $n$

Conjecture [Haiman 94, Bergeron–Préville–Ratelle 10]



(\*) tensored by the signature

## Some questions

- Prove the formulas without guessing
- Bijective proofs? Connections with certain maps?
- [unlabelled case] The joint distribution of the number of non-initial contacts of the lower path and the initial rise of the upper path is symmetric: why?