

Functional equations in enumerative combinatorics

Mireille Bousquet-Mélou, CNRS, Université de Bordeaux, France



European
Mathematical
Society

Enumerative combinatorics and generating functions

- Let \mathcal{A} be a set of combinatorial objects equipped with an integer size $|\cdot|$, and assume that for each n , the set

$$\{w \in \mathcal{A} : |w| = n\},$$

is finite. Let $a(n)$ be its cardinality.

- The **generating function** of the objects of \mathcal{A} , counted by the size, is

$$A(t) \equiv A := \sum_{n \geq 0} a(n)t^n = \sum_{w \in \mathcal{A}} t^{|w|}.$$

Enumerative combinatorics and generating functions

- Let \mathcal{A} be a set of combinatorial objects equipped with an integer size $|\cdot|$, and assume that for each n , the set

$$\{w \in \mathcal{A} : |w| = n\},$$

is finite. Let $a(n)$ be its cardinality.

- The **generating function** of the objects of \mathcal{A} , counted by the size, is

$$A(t) \equiv A := \sum_{n \geq 0} a(n)t^n = \sum_{w \in \mathcal{A}} t^{|w|}.$$

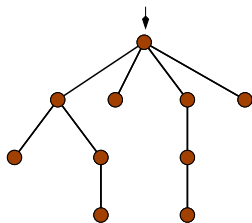
- Refined enumeration:

$$A(y; t) \equiv A(y) := \sum_{n \geq 0} a(k; n)y^k t^n = \sum_{w \in \mathcal{A}} y^{p(w)} t^{|w|}$$

for some parameter p .

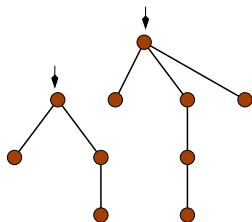
I. A collection of examples

Ex. 1: Plane trees



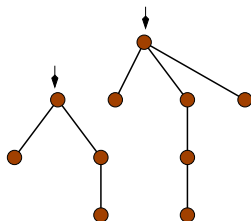
Ex. 1: Plane trees

Delete the root edge
 \Rightarrow An ordered pair of trees



Ex. 1: Plane trees

Delete the root edge
 \Rightarrow An ordered pair of trees

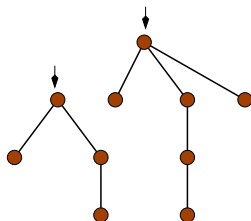


Counting: let $a(n)$ be the number of plane trees with n edges. Then $a(0) = 1$ and

$$a(n) = \sum_{i+j=n-1} a(i)a(j)$$

Ex. 1: Plane trees

Delete the root edge
⇒ An ordered pair of trees



Counting: let $a(n)$ be the number of plane trees with n edges. Then $a(0) = 1$ and

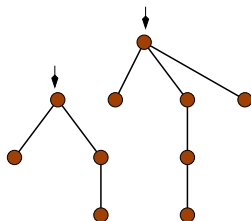
$$a(n) = \sum_{i+j=n-1} a(i)a(j)$$

Generating function: the associated formal power series

$$A := \sum_{n \geq 0} a(n)t^n = \sum_{T \text{ tree}} t^{e(T)}$$

Ex. 1: Plane trees

Delete the root edge
⇒ An ordered pair of trees



Counting: let $a(n)$ be the number of plane trees with n edges. Then $a(0) = 1$ and

$$a(n) = \sum_{i+j=n-1} a(i)a(j)$$

Generating function: the associated formal power series

$$A := \sum_{n \geq 0} a(n)t^n = \sum_{T \text{ tree}} t^{e(T)}$$

Functional equation:

$$A = 1 + tA^2$$

Ex. 1: Plane trees

Functional equation:

$$A = 1 + tA^2 \Rightarrow A = \frac{1 - \sqrt{1 - 4t}}{2t}$$

Ex. 1: Plane trees

Functional equation:

$$A = 1 + tA^2 \Rightarrow A = \frac{1 - \sqrt{1 - 4t}}{2t}$$

Are we happy?

Ex. 1: Plane trees

Functional equation:

$$A = 1 + tA^2 \Rightarrow A = \frac{1 - \sqrt{1 - 4t}}{2t}$$

Are we happy? YES!

- Expand:

$$a(n) = \frac{1}{n+1} \binom{2n}{n} \sim \frac{1}{\sqrt{\pi}} 4^n n^{-3/2}$$

Ex. 1: Plane trees

Functional equation:

$$A = 1 + tA^2 \Rightarrow A = \frac{1 - \sqrt{1 - 4t}}{2t}$$

Are we happy? YES!

- Expand:

$$a(n) = \frac{1}{n+1} \binom{2n}{n} \sim \frac{1}{\sqrt{\pi}} 4^n n^{-3/2}$$

- Linear recurrence relation:

$$(n+2)a(n+1) = 2(2n+1)a(n)$$

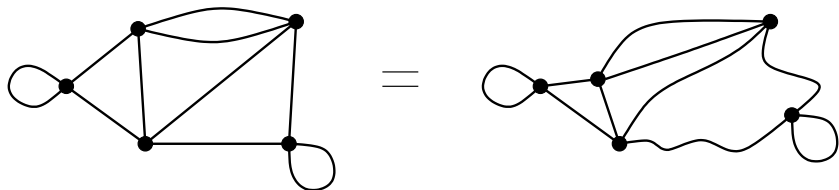
\Rightarrow Fast computation of coefficients.

\Rightarrow Bruno Salvy, tomorrow

Ex. 2: Planar maps

Definition

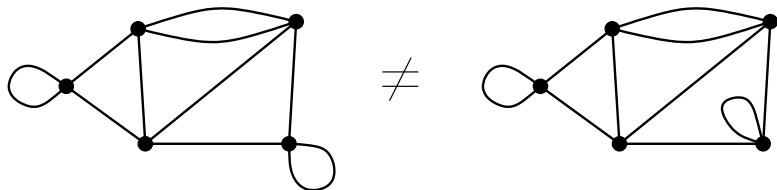
Planar map = connected planar graph
+ embedding of this graph in the plane,
taken up to continuous deformation



Ex. 2: Planar maps

Definition

Planar map = connected planar graph
+ embedding of this graph in the plane,
taken up to continuous deformation

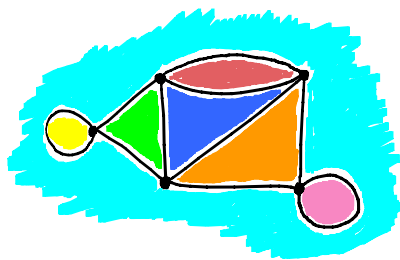


Ex. 2: Planar maps

Definition

Planar map = connected planar graph
+ embedding of this graph in the plane,
taken up to continuous deformation

faces

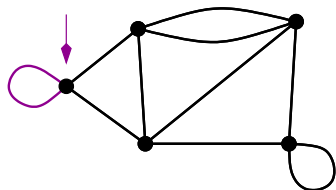


degree of the outer face: 7

Ex. 2: Planar maps

Definition

Planar map = connected planar graph
+ embedding of this graph in the plane,
taken up to continuous deformation

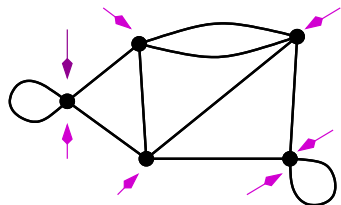


Maps are **rooted** at an external corner.
The next edge (in cc order) is the **root edge**.

Ex. 2: Planar maps

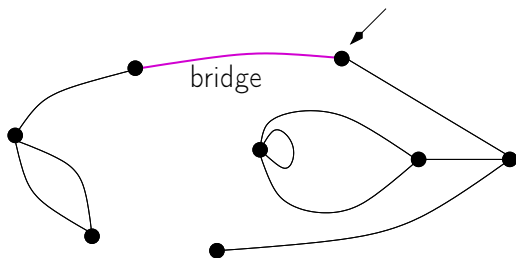
Definition

Planar map = connected planar graph
+ embedding of this graph in the plane,
taken up to continuous deformation

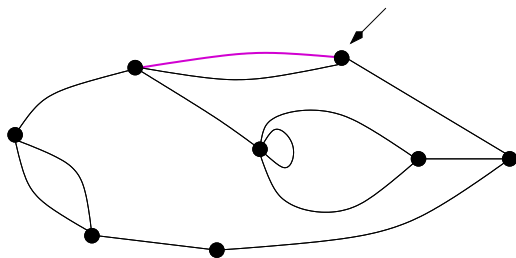


Maps are **rooted** at an external corner.
The next edge (in cc order) is the **root edge**.

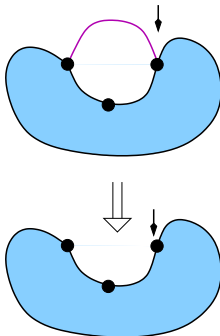
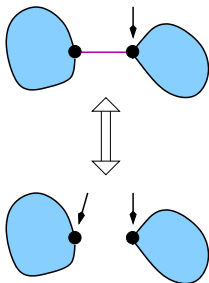
A recursive description of maps: delete the root edge



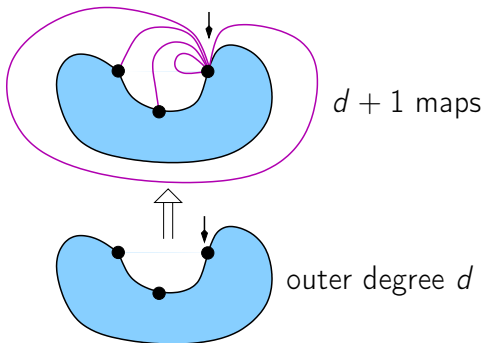
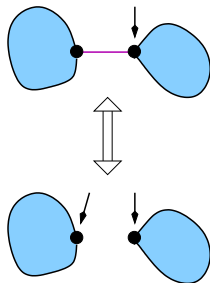
A recursive description of maps: delete the root edge



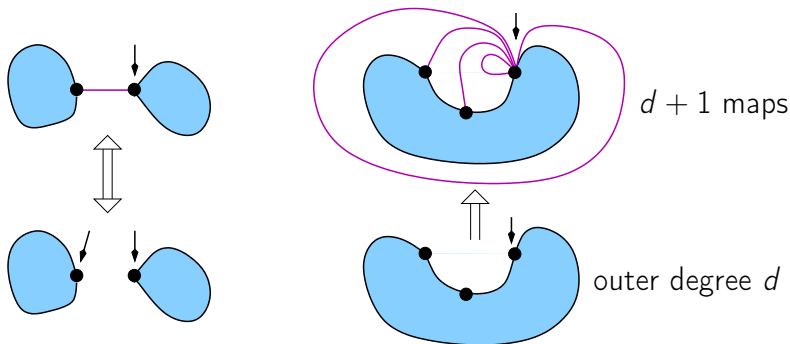
A recursive description of maps: delete the root edge



A recursive description of maps: delete the root edge



A recursive description of maps: delete the root edge



Functional equation: the series $M(y) := \sum_M \text{map} t^{e(M)} y^{\text{od}(M)}$ satisfies:

$$M(y) = 1 + ty^2 M(y)^2 + ty \frac{yM(y) - M(1)}{y - 1}$$

Note: $M(1) = \sum_M t^{e(M)}$ is the GF we want to compute

An equation with one **catalytic variable** [Zeilberger 00]

A recursive description of maps: delete the root edge

Functional equation:

$$M(y) = 1 + ty^2M(y)^2 + ty \frac{yM(y) - M(1)}{y - 1}$$

Are we happy?

A recursive description of maps: delete the root edge

Functional equation:

$$M(y) = 1 + ty^2M(y)^2 + ty \frac{yM(y) - M(1)}{y - 1}$$

Are we happy? NO!

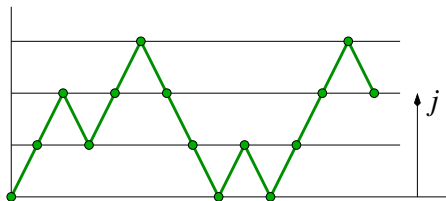
The solution is an algebraic function with nice coefficients:

$$\begin{aligned} M(1) &= \sum_M t^{e(M)} = \frac{(1 - 12t)^{3/2} - 1 + 18t}{54t^2} \\ &= \sum_{n \geq 0} t^n \frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n} \end{aligned}$$

Ex. 3: Walks on a half-line

Count walks on the non-negative half-line by the length and height j of the endpoint:

$$H(y) = \sum_{w \text{ walk}} t^{|w|} y^{j(w)}.$$



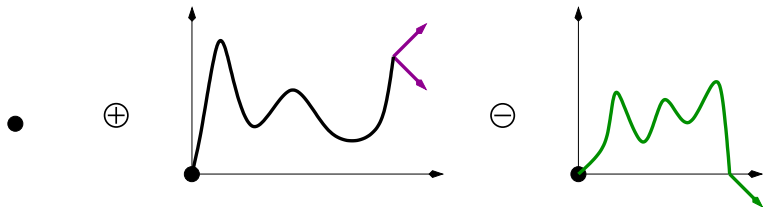
Ex. 3: Walks on a half-line

Count walks on the non-negative half-line by the length and height j of the endpoint:

$$H(y) = \sum_{w \text{ walk}} t^{|w|} y^{j(w)}.$$

Then

$$H(y) = 1 + t(y + 1/y)H(y) - t/y H(0).$$



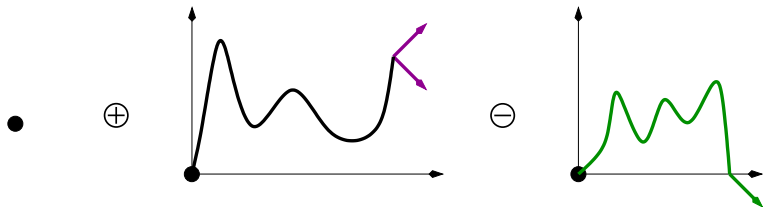
Ex. 3: Walks on a half-line

Count walks on the non-negative half-line by the length and height j of the endpoint:

$$H(y) = \sum_{w \text{ walk}} t^{|w|} y^{j(w)}.$$

Then

$$H(y) = 1 + t(y + 1/y)H(y) - t/y H(0).$$



Are we happy? NO!

The solution is algebraic with nice coefficients

$$H(0) = \frac{1 - \sqrt{1 - 4t^2}}{2t^2} = \sum_n \frac{1}{n+1} \binom{2n}{n} t^{2n}$$

Ex. 4: Walks in the first quadrant

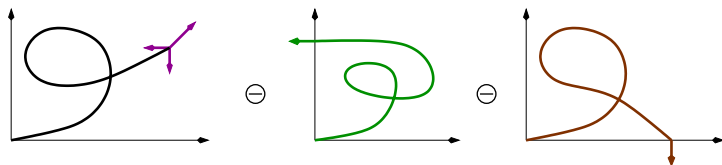


Count walks with steps NE, W, S starting from $(0,0)$ and confined in the first quadrant, by the length and coordinates (i,j) of the endpoint:

$$Q(x, y) = \sum_{w \text{ walk}} t^{|w|} x^{i(w)} y^{j(w)}.$$

Then, writing $\bar{x} = 1/x$ and $\bar{y} = 1/y$,

$$Q(x, y) = 1 + t(xy + \bar{x} + \bar{y})Q(x, y) - t\bar{x}Q(0, y) - t\bar{y}Q(x, 0)$$



An equation with two **catalytic variables**.

Ex. 4: Walks in the first quadrant

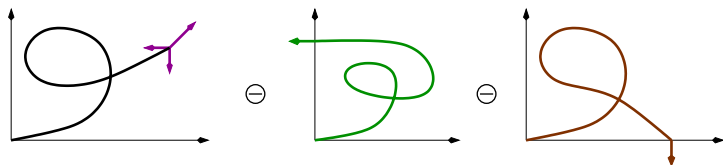


Count walks with steps NE, W, S starting from $(0,0)$ and confined in the first quadrant, by the length and coordinates (i,j) of the endpoint:

$$Q(x, y) = \sum_{w \text{ walk}} t^{|w|} x^{i(w)} y^{j(w)}.$$

Then, writing $\bar{x} = 1/x$ and $\bar{y} = 1/y$,

$$Q(x, y) = 1 + t(xy + \bar{x} + \bar{y})Q(x, y) - t\bar{x}Q(0, y) - t\bar{y}Q(x, 0)$$



An equation with two **catalytic variables**.

BUT: The series $Q(x, y; t)$ is again **algebraic**
[Gessel 86, mbm 02].

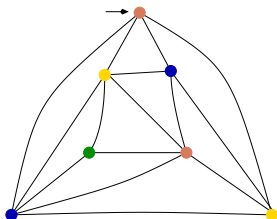
$$(\text{Pol}(x, y, t, Q) = 0)$$

Ex. 5: q -Coloured triangulations

- Let $T(x, y; t) \equiv T(x, y)$ be the unique formal power series in t , with polynomial coefficients in x and y , satisfying

$$T(x, y) = xq(q-1) + \frac{xyt}{q} T(x, y) T(1, y) \\ + xt \frac{T(x, y) - T(x, 0)}{y} - x^2 yt \frac{T(x, y) - T(1, y)}{x-1}$$

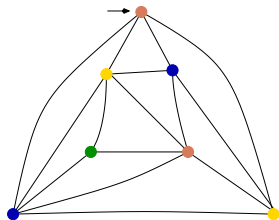
- Then $T(1, 0)$ counts properly q -coloured triangulations by the number of faces. [Tutte 73]



- The number $c(n)$ of q -coloured triangulations with $2n$ faces satisfies:

$$q(n+1)(n+2)c(n) = q(q-4)(3n-1)(3n-2)c(n-1) + 2 \sum_{i=1}^n i(i+1)(3n-3i+1)c(i-1)c(n-i),$$

with $c(0) = q(q-1)$.



- The number $c(n)$ of q -coloured triangulations with $2n$ faces satisfies:

$$q(n+1)(n+2)c(n) = q(q-4)(3n-1)(3n-2)c(n-1) + 2 \sum_{i=1}^n i(i+1)(3n-3i+1)c(i-1)c(n-i),$$

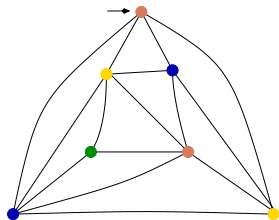
with $c(0) = q(q-1)$.

- The associated generating function

$$C(t) = \sum_n c(n)t^{n+2}$$

is **differentially algebraic**, and satisfies

$$2q^2(1-q)t + (qt + 10C - 6tC')C'' + q(4-q)(20C - 18tC' + 9t^2C'') = 0$$



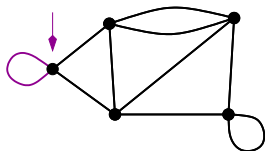
Summary

The combinatorial structure of discrete objects

often yields functional equations

that are not of the “right” type.

◁ ◁ ◊ ▷ ▷



$$M(y) = 1 + ty^2 M(y)^2 + ty \frac{yM(y) - M(1)}{y - 1}$$

vs.

$$M(1) = \sum_M t^{e(M)} = \frac{(1 - 12t)^{3/2} - 1 + 18t}{54t^2}$$

What are the “right” types?

- Rational series

$$A(t) = \frac{P(t)}{Q(t)}$$

- Algebraic series

$$P(t, A(t)) = 0$$

- Differentially finite series (D-finite)

$$\sum_{i=0}^d P_i(t) A^{(i)}(t) = 0$$

- D-algebraic series

$$P(t, A(t), A'(t), \dots, A^{(d)}(t)) = 0$$



What are the “right” types?

- Rational series

$$A(t) = \frac{P(t)}{Q(t)}$$

- Algebraic series

$$P(t, A(t)) = 0$$

- Differentially finite series (D-finite)

$$\sum_{i=0}^d P_i(t) A^{(i)}(t) = 0$$

- D-algebraic series

$$P(t, A(t), A'(t), \dots, A^{(d)}(t)) = 0$$

Multi-variate series: one DE per variable



A hierarchy of formal power series



A hierarchy of formal power series

plane trees
planar maps
walks on a half-line



A hierarchy of formal power series

plane trees
planar maps
walks on a half-line



q -coloured triangs.

A hierarchy of formal power series

plane trees

planar maps

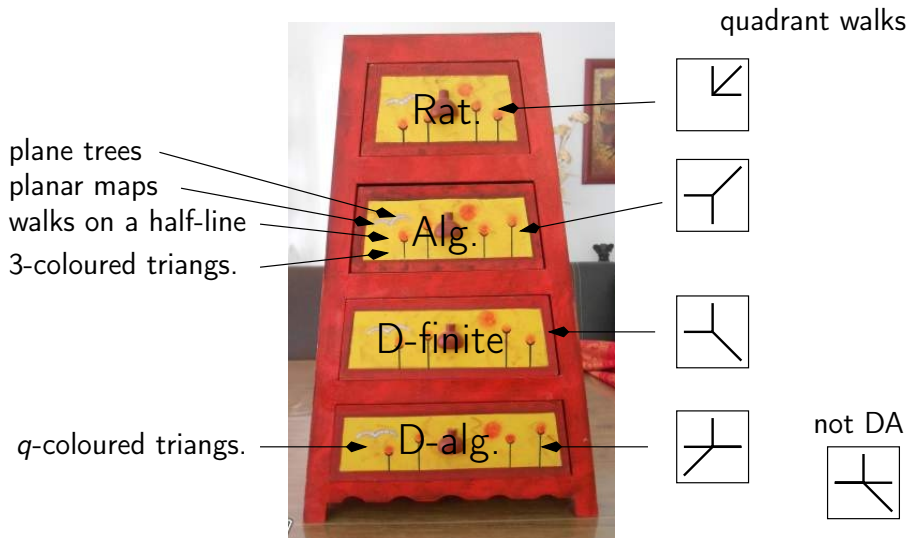
walks on a half-line

3-coloured triangs.

q -coloured triangs.



A hierarchy of formal power series



II. Equations with one catalytic variable: algebraicity

$$\text{Pol}(A(y), A_1, \dots, A_k, t, y) = 0$$

- Walks on a half-line:

$$H(y) = 1 + tyH(y) + t \frac{H(y) - H(0)}{y}$$

- Planar maps:

$$M(y) = 1 + ty^2M(y)^2 + ty \frac{yM(y) - M(1)}{y - 1}$$

- Form a square:

$$(2ty^2(y-1)M(y) + ty^2 - y + 1)^2 = \\ (y-1 - y^2t)^2 - 4ty^2(y-1)^2 + 4t^2y^3(y-1)M_1 := \Delta(y)$$

$\Delta(y)$ is a **polynomial** in y

- Form a square:

$$\begin{aligned} (2ty^2(y-1)M(y) + ty^2 - y + 1)^2 = \\ (y-1 - y^2t)^2 - 4ty^2(y-1)^2 + 4t^2y^3(y-1)M_1 := \Delta(y) \end{aligned}$$

$\Delta(y)$ is a **polynomial** in y

- There exists a (unique) series $Y \equiv Y(t)$ that cancels the LHS:

$$Y = 1 + tY^2 + 2tY^2(Y-1)M(Y).$$

\Rightarrow characterizes inductively the coefficient of t^n

- Form a square:

$$(2ty^2(y-1)M(y) + ty^2 - y + 1)^2 = (y-1 - y^2t)^2 - 4ty^2(y-1)^2 + 4t^2y^3(y-1)M_1 := \Delta(y)$$

$\Delta(y)$ is a **polynomial** in y

- There exists a (unique) series $Y \equiv Y(t)$ that cancels the LHS:

$$Y = 1 + tY^2 + 2tY^2(Y-1)M(Y).$$

\Rightarrow characterizes inductively the coefficient of t^n

- This series Y must be a root of $\Delta(y)$, and in fact a **double** root.

- Form a square:

$$(2ty^2(y-1)M(y) + ty^2 - y + 1)^2 = (y-1 - y^2t)^2 - 4ty^2(y-1)^2 + 4t^2y^3(y-1)M_1 := \Delta(y)$$

$\Delta(y)$ is a **polynomial** in y

- There exists a (unique) series $Y \equiv Y(t)$ that cancels the LHS:

$$Y = 1 + tY^2 + 2tY^2(Y-1)M(Y).$$

\Rightarrow characterizes inductively the coefficient of t^n

- This series Y must be a root of $\Delta(y)$, and in fact a **double** root.
- **Algebraic consequence:** The discriminant of $\Delta(y)$ w.r.t. y is zero:

$$27t^2M_1^2 + (1-18t)M_1 + 16t - 1 = 0$$

or

$$M_1 = \frac{(1-12t)^{3/2} - 1 + 18t}{54t^2}$$

Equations with one catalytic variable

Theorem [mbm-Jehanne 06]

Let $P(A(y), A_1, \dots, A_k, t, y)$ be a polynomial equation in one catalytic variable y . Under reasonable hypotheses (*),

- The equation

$$P'_1(A(Y), A_1, \dots, A_k, t, Y) = 0$$

admits k solutions Y_1, \dots, Y_k (Puiseux series in t)

- Each of them is a double root of $\Delta(A_1, \dots, A_k, t, y)$, the discriminant of P w.r.t. its first variable
- The series $A(y)$ and the A_i 's are algebraic.

(*) The equation defines uniquely $A(y), A_1, \dots, A_k$ as formal power series in t (with polynomial coefficients in y for $A(y)$).

Equations with one catalytic variable

Theorem [mbm-Jehanne 06]

Let $P(A(y), A_1, \dots, A_k, t, y)$ be a polynomial equation in one catalytic variable y . Under reasonable hypotheses (*),

- The equation

$$P'_1(A(Y), A_1, \dots, A_k, t, Y) = 0$$

admits k solutions Y_1, \dots, Y_k (Puiseux series in t)

- Each of them is a double root of $\Delta(A_1, \dots, A_k, t, y)$, the discriminant of P w.r.t. its first variable
- The series $A(y)$ and the A_i 's are algebraic.

\Rightarrow A special case of an Artin approximation theorem with “nested” conditions [Popescu 85], [Swan 98]

Polynomials with k common roots

The discriminant $\Delta(A_1, \dots, A_k, t, y)$ of P has k double roots in y .

Or: Δ and Δ'_y have k roots in common.

Polynomials with k common roots

The discriminant $\Delta(A_1, \dots, A_k, t, y)$ of P has k double roots in y .
Or: Δ and Δ'_y have k roots in common.

Proposition

Two polynomials $P(y) = \sum p_i y^i$ and $Q(y) = \sum q_j y^j$ of degrees m and n have k roots in common iff the rank of their Sylvester matrix is less than $m + n - 2k$.

$$\text{Sylv}(P, Q) = \begin{pmatrix} p_2 & p_1 & p_0 & 0 & 0 \\ 0 & p_2 & p_1 & p_0 & 0 \\ 0 & 0 & p_2 & p_1 & p_0 \\ q_3 & q_2 & q_1 & q_0 & 0 \\ 0 & q_3 & q_2 & q_1 & q_0 \end{pmatrix}$$

Polynomials with k common roots

The discriminant $\Delta(A_1, \dots, A_k, t, y)$ of P has k double roots in y .
Or: Δ and Δ'_y have k roots in common.

Proposition

Two polynomials $P(y) = \sum p_i y^i$ and $Q(y) = \sum q_j y^j$ of degrees m and n have k roots in common iff the rank of their Sylvester matrix is less than $m + n - 2k$. Equivalently, iff k (well-chosen) minors vanish.

$$\text{Sylv}(P, Q) = \begin{pmatrix} p_2 & p_1 & p_0 & 0 & 0 \\ 0 & p_2 & p_1 & p_0 & 0 \\ 0 & 0 & p_2 & p_1 & p_0 \\ q_3 & q_2 & q_1 & q_0 & 0 \\ 0 & q_3 & q_2 & q_1 & q_0 \end{pmatrix}$$

Example: planar maps

- Planar maps

$$P(M(y), M_1, t, y) = -M(y) + 1 + ty^2 M(y)^2 + ty \frac{yM(y) - M_1}{y-1} = 0$$

Example: planar maps

- Planar maps

$$P(M(y), M_1, t, y) = -M(y) + 1 + ty^2 M(y)^2 + ty \frac{yM(y) - M_1}{y-1} = 0$$

- The equation $P'_1(M(Y), M_1, t, Y) = 0$ reads

$$-1 + 2tY^2 M(Y) + tY^2 \frac{M(Y)}{Y-1} = 0 \Rightarrow Y = 1 + tY^2 + 2tY^2(Y-1)M(Y)$$

and has indeed a unique solution Y .

Example: planar maps

- Planar maps

$$P(M(y), M_1, t, y) = -M(y) + 1 + ty^2 M(y)^2 + ty \frac{yM(y) - M_1}{y-1} = 0$$

- The equation $P'_1(M(Y), M_1, t, Y) = 0$ reads

$$-1 + 2tY^2 M(Y) + tY^2 \frac{M(Y)}{Y-1} = 0 \Rightarrow Y = 1 + tY^2 + 2tY^2(Y-1)M(Y)$$

and has indeed a unique solution Y .

- This series Y is a double root of $\Delta(M_1, t, y)$.

Example: planar maps

- Planar maps

$$P(M(y), M_1, t, y) = -M(y) + 1 + ty^2 M(y)^2 + ty \frac{yM(y) - M_1}{y-1} = 0$$

- The equation $P'_1(M(Y), M_1, t, Y) = 0$ reads

$$-1 + 2tY^2 M(Y) + tY^2 \frac{M(Y)}{Y-1} = 0 \Rightarrow Y = 1 + tY^2 + 2tY^2(Y-1)M(Y)$$

and has indeed a unique solution Y .

- This series Y is a double root of $\Delta(M_1, t, y)$.
- **Algebraic consequence:** The discriminant of Δ w.r.t. y is zero:

$$0 = 27 t^2 M_1^2 + (1 - 18 t) M_1 + 16 t - 1$$

“The discriminant (in y) of the discriminant of P (in its first variable) vanishes”

Example: planar maps

- Planar maps

$$P(M(y), M_1, t, y) = -M(y) + 1 + ty^2 M(y)^2 + ty \frac{yM(y) - M_1}{y-1} = 0$$

- The equation $P'_1(M(Y), M_1, t, Y) = 0$ reads

$$-1 + 2tY^2 M(Y) + tY^2 \frac{M(Y)}{Y-1} = 0 \Rightarrow Y = 1 + tY^2 + 2tY^2(Y-1)M(Y)$$

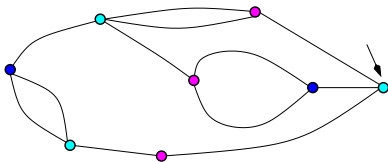
and has indeed a unique solution Y .

- This series Y is a double root of $\Delta(M_1, t, y)$.
- **Algebraic consequence:** The discriminant of Δ w.r.t. y is zero:

$$0 = (27 t^2 M_1^2 + (1 - 18 t) M_1 + 16 t - 1) (1 - t M_1)^2$$

“The discriminant (in y) of the discriminant of P (in its first variable) vanishes”

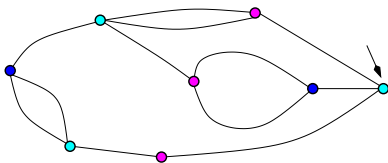
Example: Properly 3-coloured planar maps [Bernardi-mbm]



$$P(M(y), M_0, M_1, M_2, t, y) = 0$$

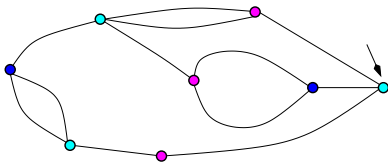
.

Example: Properly 3-coloured planar maps [Bernardi-mbm]



$$\begin{aligned}
 &P(M(y), M_0, M_1, M_2, t, y) = 0 \\
 &= 36 y^6 t^3 (2y+1)(y-1)^3 M(y)^4 + 2 t^2 y^4 (y-1)^2 (42 t y^3 + 12 y^2 t - 26 y^3 - 39 y^2 + 39 y + 26) M(y)^3 \\
 &\quad + (-36 y^6 t^3 (y-1)^2 M_0 + (y-1) y^2 t (32 t^2 y^5 + 4 y^4 t^2 + 2 t y^5 - 120 t y^4 + 8 y^5 + 78 t y^3 + 38 y^4 \\
 &\quad + 40 y^2 t - 25 y^3 - 71 y^2 + 25 y + 25)) M(y)^2 + (-36 y^5 t^3 (y-1)^2 M_0^2 - 6 t^2 (y-1) y^4 (6 y^2 t - 2 y t \\
 &\quad - 9 y^2 + 5 y + 4) M_0 - 12 M_1 t^3 y^7 + 24 M_1 t^3 y^6 + 4 y^7 t^3 - 12 y^5 t^3 M_1 + 10 t^2 y^7 - 42 t^2 y^6 - 26 t y^7 \\
 &\quad + 28 t^2 y^5 + 52 t y^6 + 4 y^4 t^2 + 32 t y^5 - 4 y^6 - 94 t y^4 - 2 y^5 + 14 t y^3 + 16 y^4 + 22 y^2 t - 16 y^2 + 2 y + 4) M \\
 &\quad - 36 y^4 t^3 (y-1)^2 M_0^3 - 2 t^2 (y-1) y^3 (22 y^2 t - 16 y t - 33 y^2 + 27 y + 6) M_0^2 - 2 y^2 t (18 M_1 t^2 y^4 \\
 &\quad - 36 M_1 t^2 y^3 + 6 y^4 t^2 + 18 M_1 t^2 y^2 - 6 y^3 t^2 - 4 t y^4 + 2 y^2 t^2 - 7 t y^3 + 16 y^4 + 13 y^2 t - 23 y^3 - 2 y t \\
 &\quad + 5 y + 2) M_0 - (y-1) (12 y^5 t^3 M_1 + 2 M_2 t^3 y^5 - 8 y^4 t^3 M_1 - 22 M_1 t^2 y^5 - 2 y^4 t^3 M_2 \\
 &\quad + 18 M_1 t^2 y^4 + 4 M_1 t^2 y^3 - 11 t y^5 + 21 t y^4 - 4 y^5 - 9 t y^3 - 6 y^4 - y^2 t + 10 y^3 + 10 y^2 - 6 y - 4).
 \end{aligned}$$

Example: Properly 3-coloured planar maps [Bernardi-mbm]



$$P(M(y), M_0, M_1, M_2, t, y) = 0$$

Theorem

The generating function of properly 3-coloured planar maps is

$$M_0 = \frac{(1 + 2A)(1 - 2A^2 - 4A^3 - 4A^4)}{(1 - 2A^3)^2}$$

where A is the unique series in t with constant term 0 such that

$$A = t \frac{(1 + 2A)^3}{1 - 2A^3}.$$

What's wrong?

- Wrong approach to this counting problem!

What's wrong?

- Wrong approach to this counting problem!
- Improve the elimination procedure

What's wrong?

- Wrong approach to this counting problem!
- Improve the elimination procedure
- Unavoidable factorizations? Iterated discriminants and resultants are known to factor, and to have repeated factors
[Henrici 1868, Lazard & McCallum 09, Busé & Mourrain 09]

$$\begin{aligned} \text{disc}_y(\text{disc}_x(x^4 + px^3 + qx^2 + yx + s)) = \\ - 256 s (p^4 - 8 p^2 q + 16 q^2 - 64 s)^2 \times \\ (27 s p^4 - p^2 q^3 - 108 p^2 q s + 3 q^4 + 72 q^2 s + 432 s^2)^3 \end{aligned}$$

What's wrong?

- Wrong approach to this counting problem!
- Improve the elimination procedure
- Unavoidable factorizations? Iterated discriminants and resultants are known to factor, and to have repeated factors
[Henrici 1868, Lazard & McCallum 09, Busé & Mourrain 09]

$$\begin{aligned} \text{disc}_y(\text{disc}_x(x^4 + px^3 + qx^2 + yx + s)) = \\ - 256 s (p^4 - 8 p^2 q + 16 q^2 - 64 s)^2 \times \\ (27 s p^4 - p^2 q^3 - 108 p^2 q s + 3 q^4 + 72 q^2 s + 432 s^2)^3 \end{aligned}$$

More equations to solve!

III. Tutte's invariant method:

From two to one catalytic variable(s)
(and algebraicity again)

- Quadrant walks:

$$Q(x, y) = 1 + txyQ(x, y) + t \frac{Q(x, y) - Q(0, y)}{x} + t \frac{Q(x, y) - Q(x, 0)}{y}$$

- q -Coloured triangulations:

$$T(x, y) = xq(q-1) + \frac{xyt}{q} T(x, y) T(1, y) \\ + xt \frac{T(x, y) - T(x, 0)}{y} - x^2yt \frac{T(x, y) - T(1, y)}{x-1}$$

[Tutte 1973 → 1984]

Equations with two catalytic variables are much harder than those with one...



Algebraic

$$(1 - t(\bar{x} + \bar{y} + xy))_{xy}Q(x, y) = xy - tyQ(0, y) - txQ(x, 0)$$



D-finite transcendental

$$(1 - t(y + \bar{x} + x\bar{y}))_{xy}Q(x, y) = xy - tyQ(0, y) - tx^2Q(x, 0)$$



Not D-finite

$$(1 - t(x + \bar{x} + \bar{y} + xy))_{xy}Q(x, y) = xy - tyQ(0, y) - txQ(x, 0)$$

In particular, their solutions are not systematically algebraic.
Still, some of them DO have an algebraic solution...



- Quadrant walks with NE, W and S steps:

$$\begin{aligned} (1 - t(\bar{x} + \bar{y} + xy))xyQ(x, y) &= xy - txQ(x, 0) - tyQ(0, y) \\ &= xy - R(x) - S(y) \end{aligned}$$

- The equation $1 - t(\bar{x} + \bar{y} + xy) = 0$ can be written in a decoupled form:

$$l_1(x) := \frac{t}{x^2} - x - \frac{t}{x} = \frac{t}{y^2} - y - \frac{t}{y} =: l_2(y)$$

The functions $(l_1(x), l_2(y))$ form a pair of invariants.



- Quadrant walks with NE, W and S steps:

$$\begin{aligned}(1 - t(\bar{x} + \bar{y} + xy))xyQ(x, y) &= xy - txQ(x, 0) - tyQ(0, y) \\ &= xy - R(x) - S(y)\end{aligned}$$

- The equation $1 - t(\bar{x} + \bar{y} + xy) = 0$ can be written in a decoupled form:

$$l_1(x) := \frac{t}{x^2} - x - \frac{t}{x} = \frac{t}{y^2} - y - \frac{t}{y} =: l_2(y)$$

The functions $(l_1(x), l_2(y))$ form a pair of invariants.

- Combined with $xy - R(x) - S(y) = 0$, this gives also:

$$J_1(x) := -R(x) - \frac{1}{x} + \frac{1}{t} = S(y) + \frac{1}{y} =: J_2(y).$$

The functions $(J_1(x), J_2(y))$ form another pair of invariants.

The invariant lemma



Two pairs of invariants:

$$I_1(x) := \frac{t}{x^2} - x - \frac{t}{x} = \frac{t}{y^2} - y - \frac{t}{y} =: I_2(y)$$

$$J_1(x) := -R(x) - \frac{1}{x} + \frac{1}{t} = S(y) + \frac{1}{y} =: J_2(y).$$

The invariant lemma



Two pairs of invariants:

$$I_1(x) := \frac{t}{x^2} - x - \frac{t}{x} = \frac{t}{y^2} - y - \frac{t}{y} =: I_2(y)$$

$$J_1(x) := -R(x) - \frac{1}{x} + \frac{1}{t} = S(y) + \frac{1}{y} =: J_2(y).$$

The invariant lemma

There are few invariants: $I_2(y)$ must be a polynomial in $J_2(y)$ whose coefficients are series in t .

The invariant lemma



Two pairs of invariants:

$$I_1(x) := \frac{t}{x^2} - x - \frac{t}{x} = \frac{t}{y^2} - y - \frac{t}{y} =: I_2(y)$$

$$J_1(x) := -R(x) - \frac{1}{x} + \frac{1}{t} = S(y) + \frac{1}{y} =: J_2(y).$$

The invariant lemma

There are few invariants: $I_2(y)$ must be a polynomial in $J_2(y)$ whose coefficients are series in t .

$$I_2(y) = \frac{t}{y^2} - \frac{1}{y} - ty = t \left(tyQ(0, y) + \frac{1}{y} \right)^2 - \left(tyQ(0, y) + \frac{1}{y} \right) + c$$

Expanding at $y = 0$ gives the value of c .

The invariant lemma



Two pairs of invariants:

$$I_1(x) := \frac{t}{x^2} - x - \frac{t}{x} = \frac{t}{y^2} - y - \frac{t}{y} =: I_2(y)$$

$$J_1(x) := -R(x) - \frac{1}{x} + \frac{1}{t} = S(y) + \frac{1}{y} =: J_2(y).$$

The invariant lemma

There are few invariants: $I_2(y)$ must be a polynomial in $J_2(y)$ whose coefficients are series in t .

$$I_2(y) = \frac{t}{y^2} - \frac{1}{y} - ty = t \left(tyQ(0, y) + \frac{1}{y} \right)^2 - \left(tyQ(0, y) + \frac{1}{y} \right) - 2t^2Q(0, 0).$$

Expanding at $y = 0$ gives the value of c .

The invariant lemma



Two pairs of invariants:

$$I_1(x) := \frac{t}{x^2} - x - \frac{t}{x} = \frac{t}{y^2} - y - \frac{t}{y} =: I_2(y)$$

$$J_1(x) := -R(x) - \frac{1}{x} + \frac{1}{t} = S(y) + \frac{1}{y} =: J_2(y).$$

The invariant lemma

There are few invariants: $I_2(y)$ must be a polynomial in $J_2(y)$ whose coefficients are series in t .

$$I_2(y) = \frac{t}{y^2} - \frac{1}{y} - ty = t \left(tyQ(0, y) + \frac{1}{y} \right)^2 - \left(tyQ(0, y) + \frac{1}{y} \right) - 2t^2Q(0, 0).$$

Expanding at $y = 0$ gives the value of c .

Polynomial equation with one catalytic variable $\Rightarrow Q(0, y; t)$ is algebraic

What invariants are good for

- start with an equation with two catalytic variables x and y (degree 1 in the main series $Q(x, y)$)
- construct a pair of invariants in y from the coefficients of $Q(x, y)^1$ and $Q(x, y)^0$
- relate them algebraically (the invariant lemma)
- obtain an equation with one catalytic variable only \Rightarrow algebraicity

Other equations: are there invariants?

Thm. [Bernardi, mbm, Raschel 15(a)]

Exactly 4 quadrant models with small steps have two invariants \Rightarrow uniform algebraic solution via the solution of an equation with one catalytic variable



Other equations: are there invariants?

Thm. [Bernardi, mbm, Raschel 15(a)]

Exactly 4 quadrant models with small steps have two invariants \Rightarrow uniform algebraic solution via the solution of an equation with one catalytic variable



Gessel's model

- conjecture for $q(0, 0; n)$ [Gessel $\simeq 00$]
- proof of this conjecture [Kauers, Koutschan & Zeilberger 08]
- $Q(x, y; t)$ are algebraic! [Bostan & Kauers 09a]
- new proof via complex analysis [Bostan, Kurkova & Raschel 13(a)]
- an elementary and constructive proof [mbm 15(a)]

Other equations: are there invariants?

Thm. [Bernardi, mbm, Raschel 15(a)]

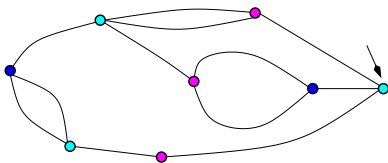
Exactly 4 quadrant models with small steps have two invariants \Rightarrow uniform algebraic solution via the solution of an equation with one catalytic variable



Thm. [Tutte 74], [Bernardi-mbm 11]

For q -coloured triangulations (or planar maps, and even for the q -state Potts model), there is one invariant for any q , and a second one if $q = 4 \cos^2 \frac{k\pi}{m}$, with $q \neq 0, 4$. \Rightarrow Algebraicity

Example: Properly 3-coloured planar maps



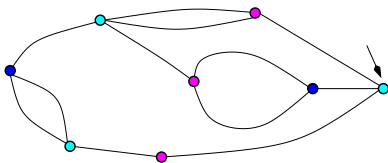
- Two catalytic variables [Tutte 68]

$$M(x, y) = 1 + xyt (1 + 2y) M(x, y)M(1, y) - xytM(x, y)M(x, 1) \\ - xyt \frac{xM(x, y) - M(1, y)}{x - 1} + xyt \frac{yM(x, y) - M(x, 1)}{y - 1}$$

- One catalytic variable [Bernardi-mbm 11]

$$P(M(1, y), M_0, M_1, M_2, t, y) = 0$$

Example: Properly 3-coloured planar maps



$$\begin{aligned}
 & P(M(1, y), M_0, M_1, M_2, t, y) = 0 \\
 & = 36 y^6 t^3 (2y+1)(y-1)^3 M(y)^4 + 2 t^2 y^4 (y-1)^2 (42 t y^3 + 12 y^2 t - 26 y^3 - 39 y^2 + 39 y + 26) M(y)^3 \\
 & \quad + (-36 y^6 t^3 (y-1)^2 M_0 + (y-1) y^2 t (32 t^2 y^5 + 4 y^4 t^2 + 2 t y^5 - 120 t y^4 + 8 y^5 + 78 t y^3 + 38 y^4 \\
 & \quad + 40 y^2 t - 25 y^3 - 71 y^2 + 25 y + 25)) M(y)^2 + (-36 y^5 t^3 (y-1)^2 M_0^2 - 6 t^2 (y-1) y^4 (6 y^2 t - 2 y t \\
 & \quad - 9 y^2 + 5 y + 4) M_0 - 12 M_1 t^3 y^7 + 24 M_1 t^3 y^6 + 4 y^7 t^3 - 12 y^5 t^3 M_1 + 10 t^2 y^7 - 42 t^2 y^6 - 26 t y^7 \\
 & \quad + 28 t^2 y^5 + 52 t y^6 + 4 y^4 t^2 + 32 t y^5 - 4 y^6 - 94 t y^4 - 2 y^5 + 14 t y^3 + 16 y^4 + 22 y^2 t - 16 y^2 + 2 y + 4) M \\
 & \quad - 36 y^4 t^3 (y-1)^2 M_0^3 - 2 t^2 (y-1) y^3 (22 y^2 t - 16 y t - 33 y^2 + 27 y + 6) M_0^2 - 2 y^2 t (18 M_1 t^2 y^4 \\
 & \quad - 36 M_1 t^2 y^3 + 6 y^4 t^2 + 18 M_1 t^2 y^2 - 6 y^3 t^2 - 4 t y^4 + 2 y^2 t^2 - 7 t y^3 + 16 y^4 + 13 y^2 t - 23 y^3 - 2 y t \\
 & \quad + 5 y + 2) M_0 - (y-1) (12 y^5 t^3 M_1 + 2 M_2 t^3 y^5 - 8 y^4 t^3 M_1 - 22 M_1 t^2 y^5 - 2 y^4 t^3 M_2 \\
 & \quad + 18 M_1 t^2 y^4 + 4 M_1 t^2 y^3 - 11 t y^5 + 21 t y^4 - 4 y^5 - 9 t y^3 - 6 y^4 - y^2 t + 10 y^3 + 10 y^2 - 6 y - 4).
 \end{aligned}$$

Some questions

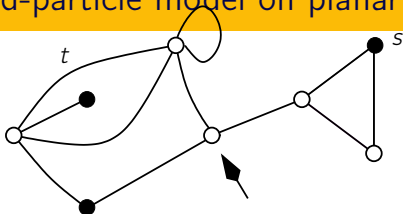
- More efficient ways to compute with equations in one catalytic variable
- Effective construction of invariants — or prove that there are not any
- Prove more algebraicity results with them (e.g. lattice walks confined to/avoiding a quadrant)

Some questions

- More efficient ways to compute with equations in one catalytic variable
- Effective construction of invariants — or prove that there are not any
- Prove more algebraicity results with them (e.g. lattice walks confined to/avoiding a quadrant)



Example: the hard-particle model on planar maps



- $F(y) = 1 + G(y) - s + ty^2F(y)^2 + ty \frac{yF(y) - F_1}{y-1}$
- $G(y) = s + tyF(y)G(y) + ty \frac{G(y) - G_1}{y-1}$

Then $F(1)$ is an explicit rational function of s and A , where

$$A = t \frac{1 - (2 - s)A}{(1 - A)(1 - 2A)(1 - 3A + 3A^2)}$$

Intermediate steps: Y_1 and Y_2 have degree 10. Numerous intermediate factorisations... some of the factors are enormous.