

Asymptotic properties of some minor-closed classes of graphs

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- **Theorem [Robertson & Seymour]:** every minor-closed class of graphs can be described by excluding a **finite number** of minors.
- **Examples:** For forests, exclude the 3-cycle. For planar graphs, exclude K_5 and $K_{3,3}$.

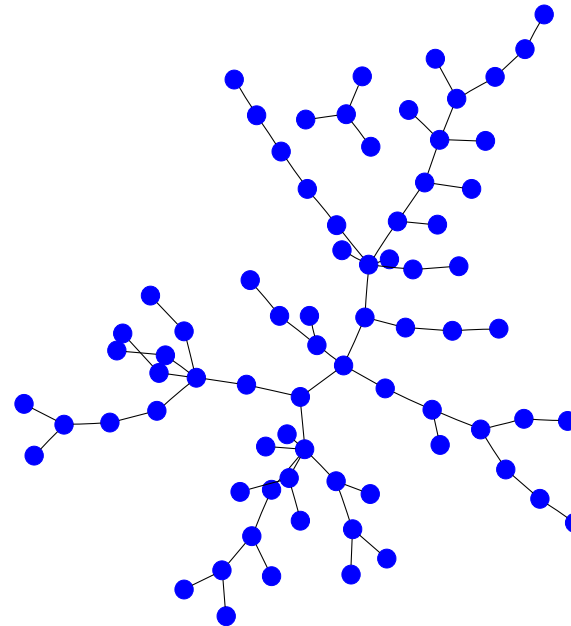
Random graphs

For fixed (and large) n , let \mathcal{A}_n be the set of graphs of the class \mathcal{A} having size n (i.e., n vertices). Let \mathcal{G}_n be a random graph of \mathcal{A}_n , taken uniformly at random: if G has size n ,

$$\mathbb{P}(\mathcal{G}_n = G) = \frac{1}{a(n)}$$

where $a(n) = |\mathcal{A}_n|$.

A uniform random forest
on 75 vertices
(2 components)



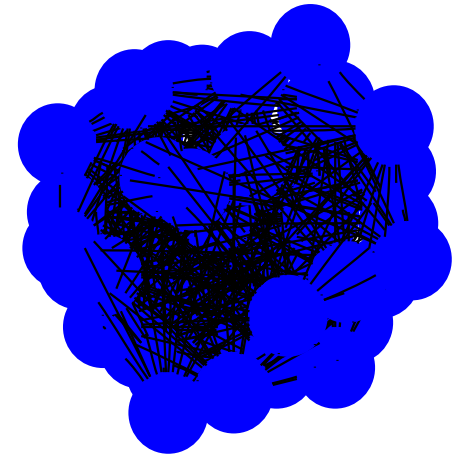
Some properties of (proper) minor-closed classes

- No excluded minor:

$$a(n) = 2^{\binom{n}{2}}.$$

The expected number of edges is quadratic:

$$\mathbb{E}(e(\mathcal{G}_n)) = \frac{n(n-1)}{4}.$$



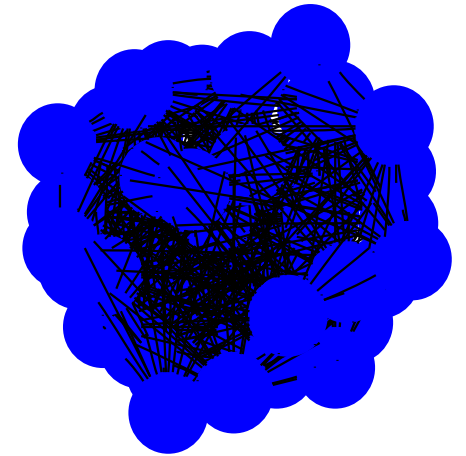
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- At least one excluded minor:

$$a(n) \leq n! \mu^n$$

for some constant μ [Norine, Seymour, Thomas & Wollan 06].

The expected number of edges is at most linear: for *any* graph of \mathcal{A}_n ,

$$e(G) \leq \alpha n$$

for a constant α that depends on the excluded minors.

[Kostochka 82, Thomason 84]

Refined properties: The number and size of components

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Remark: if no minor is excluded, the probability that \mathcal{G}_n is connected tends to 1 as $n \rightarrow \infty$, and these questions have simple answers.

Generating functions

- Let $a(n)$ be the number of graphs of size n in \mathcal{A} , and let

$$A(z) = \sum_{n \geq 0} a(n) \frac{z^n}{n!}$$

be the associated exponential generating function.

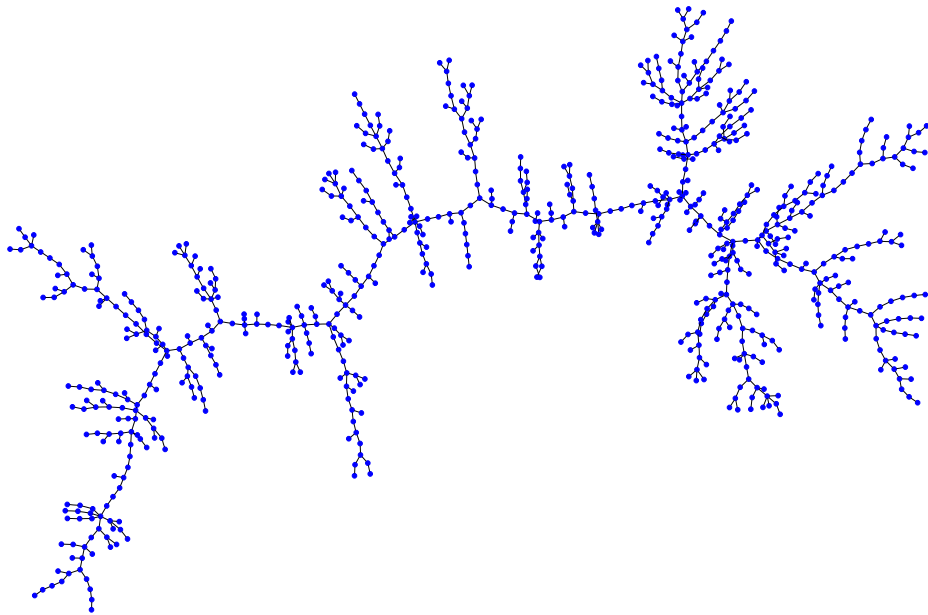
- Use similar notation ($c(n)$ and $C(z)$) for **connected** graphs of \mathcal{A} .
- If all forbidden minors are connected, graphs of \mathcal{A} are arbitrary unions of graphs of \mathcal{C} , and

$$A(z) = \exp(C(z)).$$

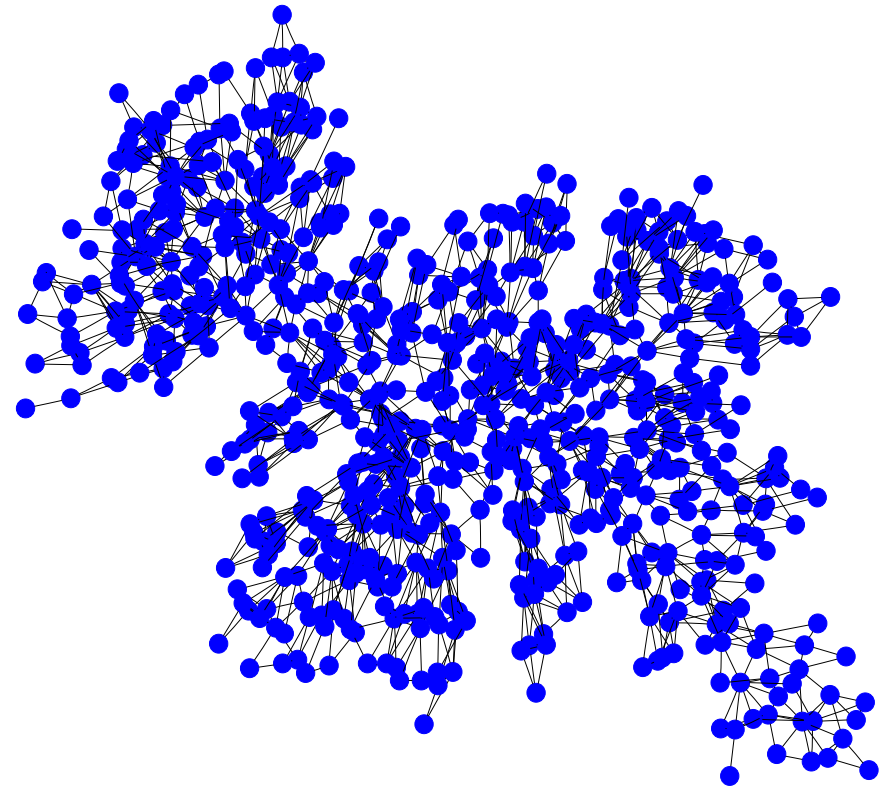
When all excluded minors are 2-connected

Theorem [McDiarmid 09]: If all excluded minors are 2-connected, then

- $C(z)$ and $A(z)$ converge at their (common) radius of convergence ρ
- the probability that \mathcal{G}_n is connected tends to $1/A(\rho)$, which belongs to $[1/\sqrt{e}, 1)$



Forests: $1/A(\rho) = 1/\sqrt{e} = 0.60\dots$



Planar graphs: $1/A(\rho) = 0.97\dots$
[Gimenez & Noy 09]

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- in fact, $N_n - 1$ converges to a Poisson distribution of parameter $C(\rho)$:

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- the probability that \mathcal{G}_n is connected tends to $1/A(\rho)$, which belongs to $[1/\sqrt{e}, 1)$ (*)
- in fact, $N_n - 1$ converges to a Poisson distribution of parameter $C(\rho)$:

$$\mathbb{P}(N_n = i + 1) \rightarrow \frac{C(\rho)^i}{i!A(\rho)}$$

- the root component contains almost all vertices; more precisely,

$$\mathbb{P}(S_n = n - k) \rightarrow \frac{1}{A(\rho)} \frac{a(k)\rho^k}{k!}$$

- the same holds for the largest component.

(*) [Addario Berry, McDiarmid & Reed 11], [Kang & Panagiotou 11]

What if some excluded minors are not 2-connected?

The above properties cannot always hold.

Example: exclude the one-edge graph. Then

- \mathcal{G}_n is never connected (for $n \geq 2$),
- the number of components is maximal: $N_n = n$,
- the largest component has size $L_n = 1$.

More components, of a smaller size

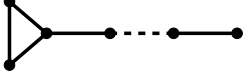






Our results: a collection of examples

For a collection of minor-closed classes avoiding some non-2-connected minors,

- exact determination of $C(z)$ and $A(z)$,
- detailed asymptotics properties of N_n (number of components),
- detailed asymptotics properties of S_n (size of the root component),
- for some classes, detailed asymptotics properties of L_n (size of the largest component)

Key point: behaviour of $C(z)$ near its radius ρ

Note: all forbidden minors are connected, so that $A(z) = \exp(C(z))$

Excluded minors	2-conn. or 	 and 		 or  and 	all conn. graphs of size $k + 1$
$C(\rho)$	$< \infty$	$\log \frac{1}{1-z/\rho}$	$\frac{1}{\sqrt{1-z/\rho}}$	$\frac{1}{1-z/\rho}$	entire
$\mathbb{P}(\text{conn.})$	$\rightarrow p > 0$	$\rightarrow 0$	$\rightarrow 0$	$\rightarrow 0$	$\rightarrow 0$
N_n	$O(1)$ Poisson	$\log n$ gaussian	$n^{1/3}$ gaussian	$n^{1/2}$ gaussian	n/k gaussian (*)
S_n	$n - F_n$ $F_n \rightarrow \text{discr.}$	n $\frac{1}{4}(1-x)^{-3/4}$	$n^{2/3}$ $2\sqrt{\frac{x}{\pi}}e^{-x}$	$n^{1/2}$ xe^{-x}	k Dirac
L_n	$n - F_n$ $F_n \rightarrow \text{discr.}$	n Dickman	?	$n^{1/2} \log n$ Gumbel	k Dirac

(*) [Canfield 77]

Example: forests of paths (no \triangle nor $\begin{array}{c} \diagup \\ \diagdown \end{array}$)

- Exact enumeration:

$$C(z) = \frac{z(2-z)}{2(1-z)}, \quad A(z) = \exp(C(z))$$

- Asymptotic enumeration:

$$c_n = \frac{n!}{2} \quad \text{and} \quad a_n \sim \frac{n! (e/2)^{1/4}}{2 \sqrt{\pi n^{3/4}}} e^{\sqrt{2n}}.$$

In particular, $\frac{c(n)}{a(n)} \rightarrow 0$.

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- The mean and variance of N_n satisfy:

$$\mathbb{E}(N_n) \sim \sqrt{n/2}, \quad \mathbb{V}(N_n) \sim \sqrt{n/8},$$

and the normalized random variable $\frac{N_n - \sqrt{n/2}}{(n/8)^{1/4}}$ converges in law to a standard normal distribution:

$$\mathbb{P} \left(\frac{N_n - \sqrt{n/2}}{(n/8)^{1/4}} \leq y \right) \rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-x^2/2} dx.$$

- The mean and variance of S_n satisfy:

$$\mathbb{E}(S_n) \sim 2\sqrt{2n}, \quad \mathbb{V}(S_n) \sim 4n,$$

and the normalized variable $S_n/\sqrt{2n}$ converges in law to a Gamma(2):

$$\mathbb{P}\left(S_n/\sqrt{2n} \leq y\right) \rightarrow \int_0^y x e^{-x} dx.$$

A local limit law also holds:

$$\sqrt{2n} \mathbb{P}(S_n = \lfloor x\sqrt{2n} \rfloor) \sim x e^{-x}.$$

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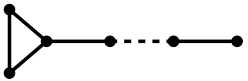





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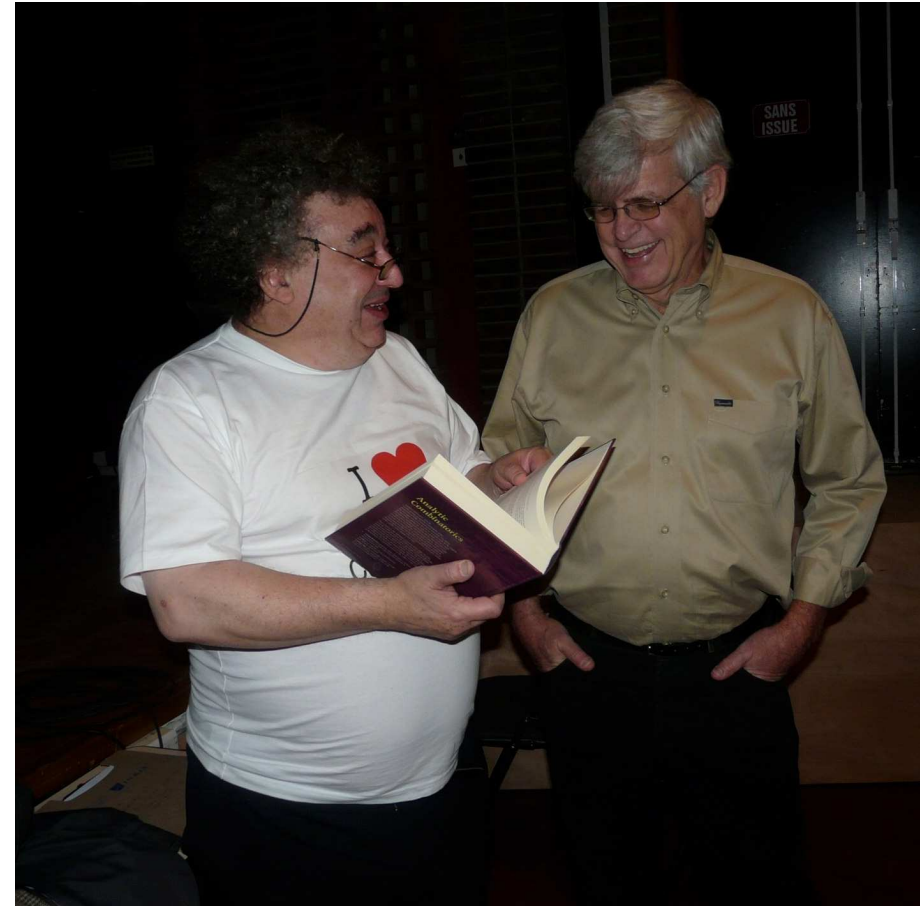
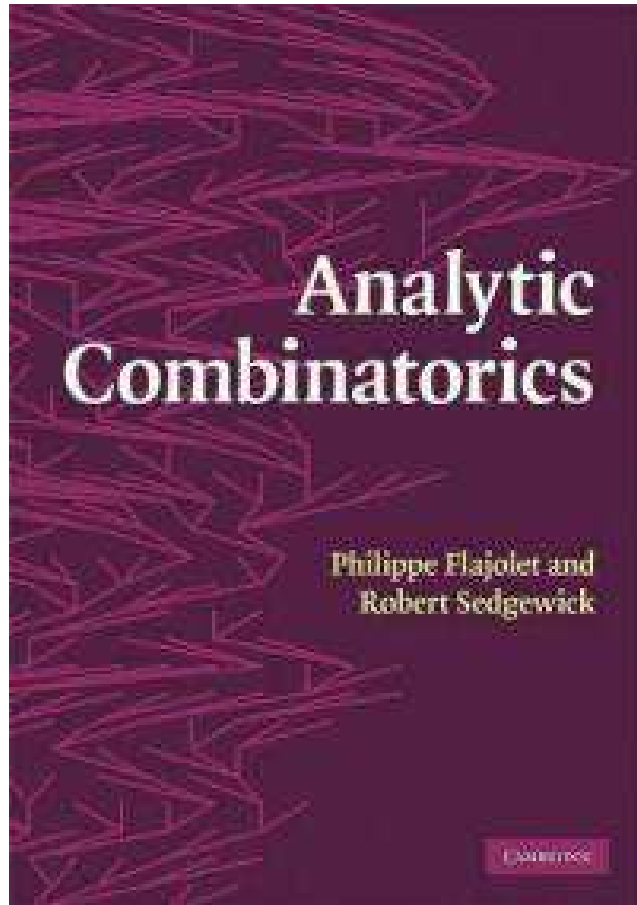
- The normalized variable $\frac{L_n - \sqrt{n/2} \log n}{\sqrt{n/2}}$ converges in distribution to a Gumbel law:

$$\mathbb{P}\left(\frac{L_n - \sqrt{n/2} \log n}{\sqrt{n/2}} \leq y\right) \rightarrow \exp\left(-\frac{e^{-y/2}}{\sqrt{2}}\right).$$

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Techniques: Analytic combinatorics



Philippe Flajolet, Robert Sedgewick

A. Exact enumeration

Write directly equations from the recursive description of the objects

Construction	Numbers	Generating function
Union $A = B \sqcup C$	$a(n) = b(n) + c(n)$	$A(z) = B(z) + C(z)$
Product $A = B \times C$ $ (\beta, \gamma) = \beta + \gamma $	$a(n) = \sum_i \binom{n}{i} b(i) c(n-i)$	$A(z) = B(z) \cdot C(z)$
Sequence $A = \text{Seq}(B)$		$A(z) = \frac{1}{1-B(z)}$
Set $A = \text{Set}(B)$		$A(z) = \exp(B(z))$
Cycle (directed) $A = \text{Cyc}(B)$		$A(z) = \log \frac{1}{1-B(z)}$

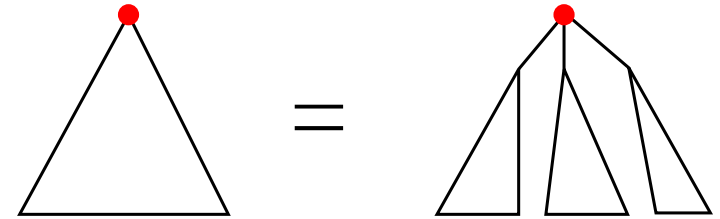
Constructible labelled objects

$$A(z) = \sum_n a(n) \frac{z^n}{n!}$$

A basic example: forests (no Δ)

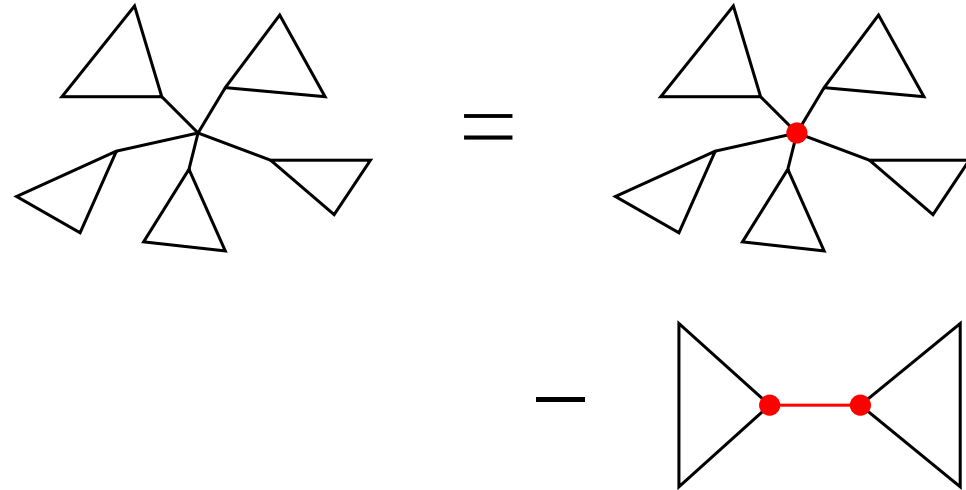
- A (vertex-)rooted tree is a root vertex (z) and a set of (sub-)trees ($T(z)$ for each):

$$T(z) = z \exp(T(z)).$$



- A tree with n nodes has $n - 1$ edges, and edge-rooted trees consist of a pair of vertex-rooted trees:

$$C(z) = T(z) - \frac{1}{2}T(z)^2.$$



- A forest is a set of unrooted trees:

$$A(z) = \exp(C(z)).$$

The number and size of components

- The generating function of forests having k components is

$$\frac{C(z)^k}{k!}.$$

- The number of forests of \mathcal{A} with n vertices and root component of size k :

$$\binom{n-1}{k-1} c(k) a(n-k)$$

$$\Rightarrow \mathbb{P}(S_n = k) = \frac{c(k) a(n-k) \binom{n-1}{k-1}}{a(n)}.$$

B. Asymptotic enumeration and limit laws

How to extract from a series the asymptotic behaviour of its coefficients?

A general correspondence between

the singular expansion of a series $A(z)$ near its dominant singularities

and

the asymptotic expansion of the n th coefficient $a(n)$ of this series

Example: If the dominant singularity has modulus ρ , then the coefficients satisfy

$$\limsup a(n)^{1/n} = 1/\rho.$$

A basic example: forests

- The generating functions of rooted trees, unrooted trees and forests are given by

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- All have radius $\rho = 1/e$, with singular expansions

$$T(z) = 1 - \sqrt{2}(1 - ze)^{1/2} + \frac{2}{3}(1 - ze) - \frac{11}{18\sqrt{2}}(1 - ze)^{3/2} + O((1 - ze)^2),$$

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- By singularity analysis,

$$c(n) \sim n! \frac{e^n}{\sqrt{2\pi n^{5/2}}} \quad \text{and} \quad a(n) \sim e^{1/2} c(n).$$

The probability that \mathcal{G}_n is connected tends to $e^{-1/2}$.

Forests: the number N_n of components

- We have

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this gives

$$C(z)^k = \frac{1}{2^k} + \alpha(1 - ze) + \frac{k}{2^{k-1}} \frac{2\sqrt{2}}{3}(1 - ze)^{3/2} + O((1 - ze)^2),$$

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so that

$$\mathbb{P}(N_n = k) \rightarrow \frac{1}{2^{k-1}} \frac{1}{(k-1)!} \frac{1}{\sqrt{e}}$$

and $N_n - 1$ follows a **Poisson distribution** of parameter $1/2$.

Forests: the size of the root component

We have found

$$c(n) \sim n! \frac{e^n}{\sqrt{2\pi n^{5/2}}} \quad \text{and} \quad a(n) \sim e^{1/2} c(n).$$

Thus for any fixed k ,

$$\mathbb{P}(S_n = n - k) = \frac{c(n - k) a(k) \binom{n-1}{k}}{a(n)} \rightarrow \frac{1}{e^{1/2}} \frac{a(k)}{k! e^k}.$$

With the same tools: when \mathcal{A} is dominated by forests

Definition: the class \mathcal{A} is dominated by forests if

$$C(z) = T(z) - T(z)^2/2 + D(z),$$

where $D(z)$ has radius of convergence strictly larger than $1/e$ (the radius of T).

Proposition: In this case,

$$c(n) \sim n! \frac{e^n}{\sqrt{2\pi n^{5/2}}} \quad \text{and} \quad a(n) \sim A(1/e)c(n),$$

- $N_n - 1$ converges in law to a Poisson distribution of parameter $C(1/e) = 1/2 + D(1/e)$,
- the root component contains almost all vertices, and

$$\mathbb{P}(S_n = n - k) = \frac{1}{A(1/e)} \frac{a(k)e^{-k}}{k!},$$

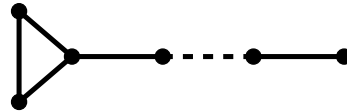
- the same holds for the largest component.

When \mathcal{A} is dominated by forests: examples

Proposition: let $k \geq 1$. Let \mathcal{A} be a minor-closed class of graphs containing all trees, but not the k -spoon. Then \mathcal{A} is dominated by forests:

$$C(z) = T(z) - T(z)^2/2 + D(z),$$

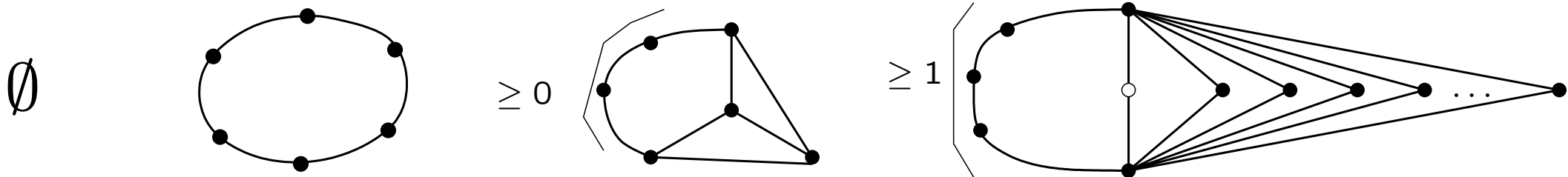
where $\rho(D) > 1/e$.



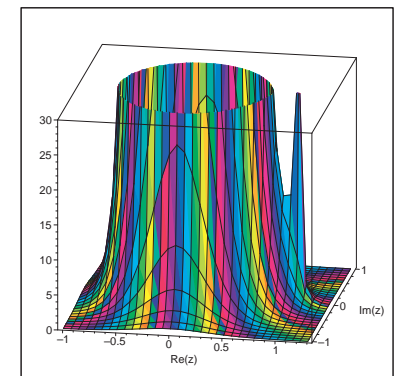
A more complex example: Excluding the bowtie \bowtie

- Analysis of the structure of such graphs, and partition into disjoint sub-classes
- Generating function of connected graphs:

$$C(z) = \frac{T^2 (1 - T + T^2) e^T}{1 - T} + \frac{1}{2} \log \left(\frac{1}{1 - T} \right) + \frac{T (12 - 54T + 18T^2 - 5T^3 - T^4)}{24(1 - T)}.$$



- Asymptotic results and limit laws via the saddle point method



C. Random generation: Boltzmann samplers

A class \mathcal{A} of (labelled) objects that has a “simple” recursive description and a moderate growth (at most exponential) can often be sampled **efficiently** according to the associated Boltzmann distribution: an object G of size n has probability

$$\mathbb{P}(G) = \frac{x^n}{n! A(x)}$$

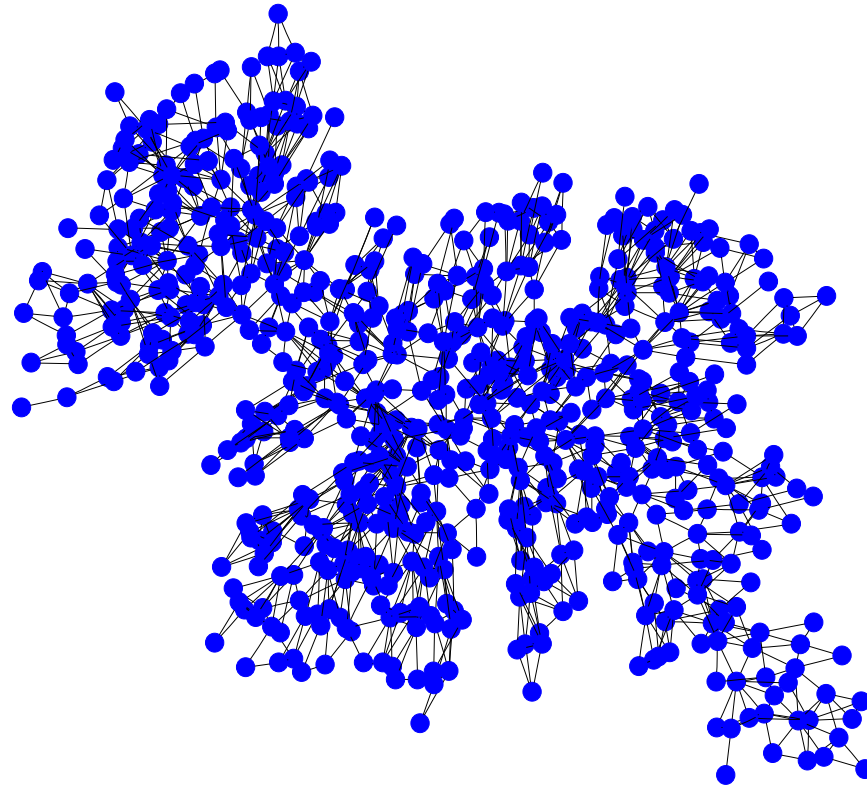
where $A(x) = \sum_n a(n) \frac{x^n}{n!}$ and $x \in (0, \rho]$.

In particular, all objects of size n have the same probability.

[Duchon, Flajolet, Louchard, Schaeffer 04]

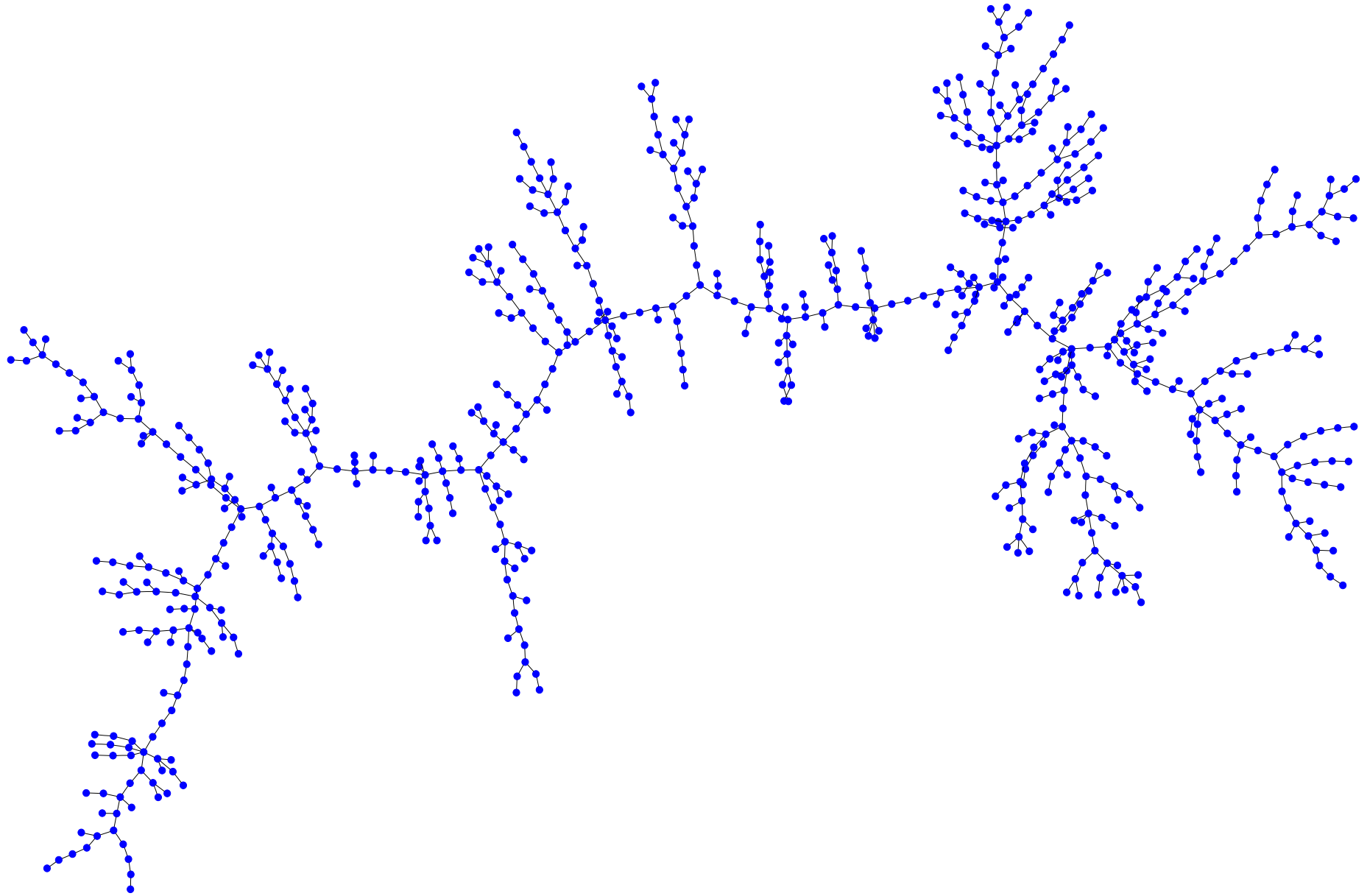
A random planar graph of size 727

[Fusy 09] (100,000 vertices in a few seconds)



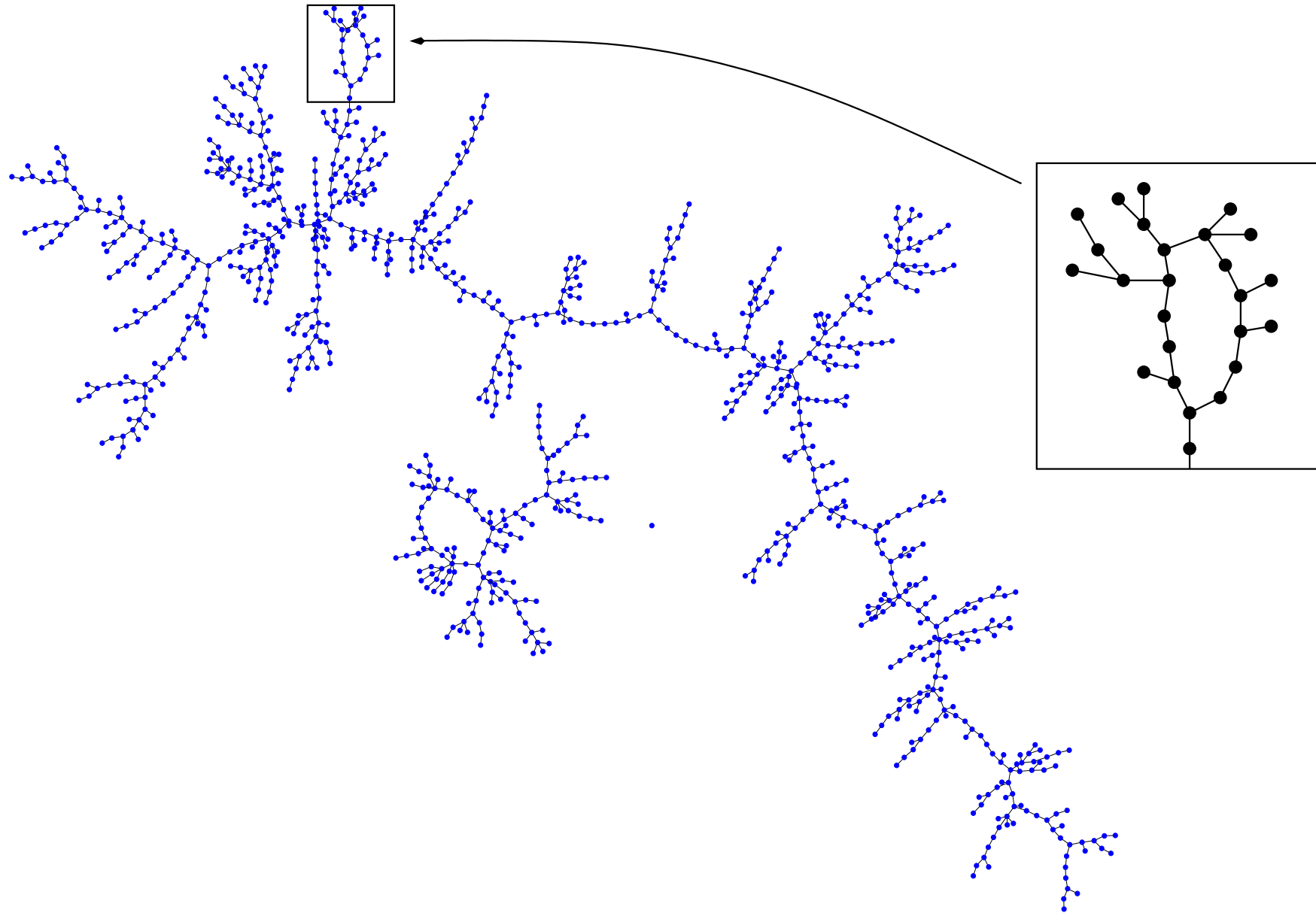
$$\mathbb{E}(N_n) = \Theta(1) \quad \mathbb{E}(S_n) \sim n$$

A random forest of size 745



$$\mathbb{E}(N_n) = \frac{3}{2} \quad \mathbb{E}(S_n) \sim n$$

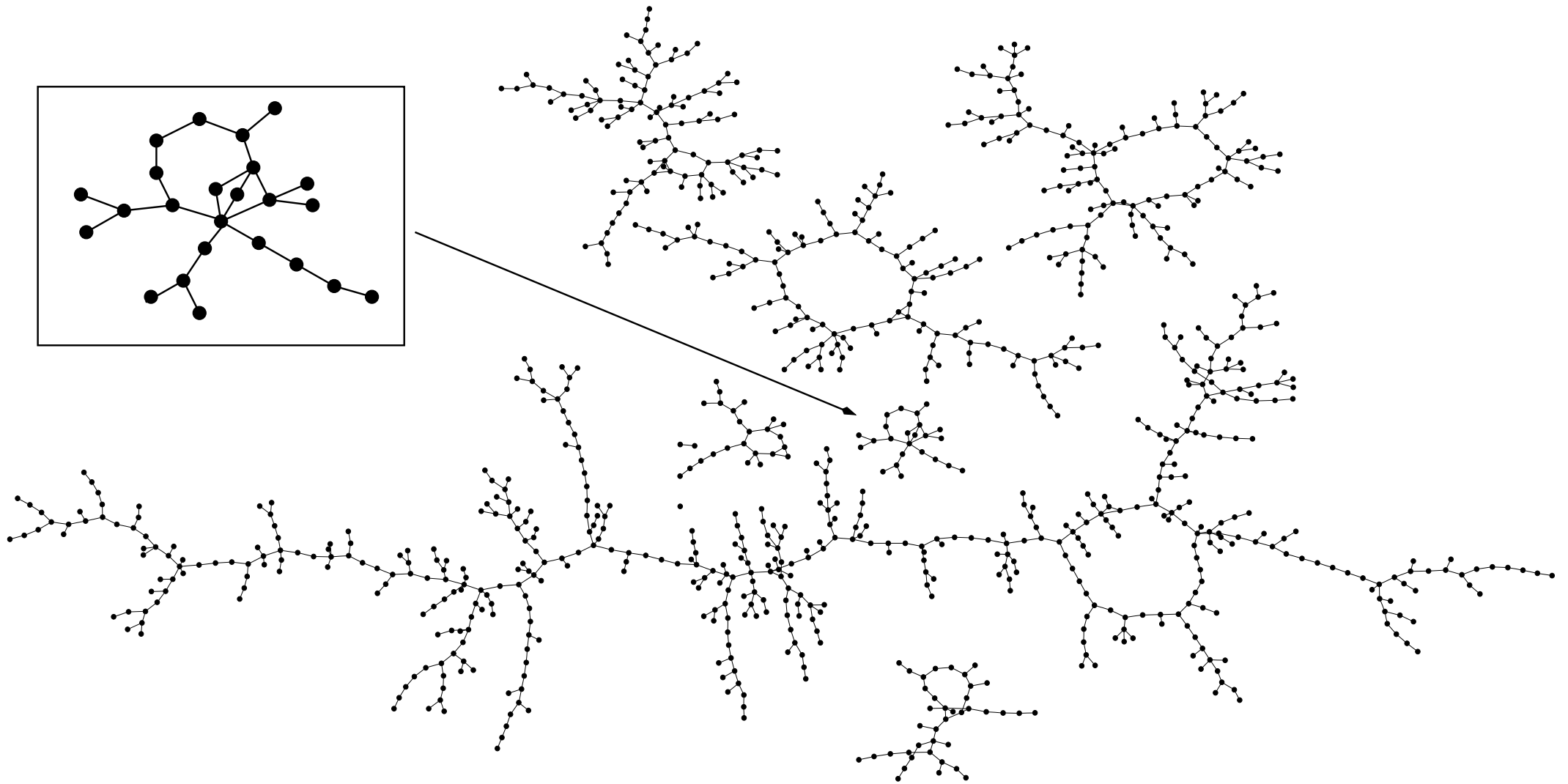
A random graph of size 975 avoiding the diamond and the bowtie



$$\mathbb{E}(N_n) \simeq \log n$$

$$\mathbb{E}(S_n) \simeq n$$

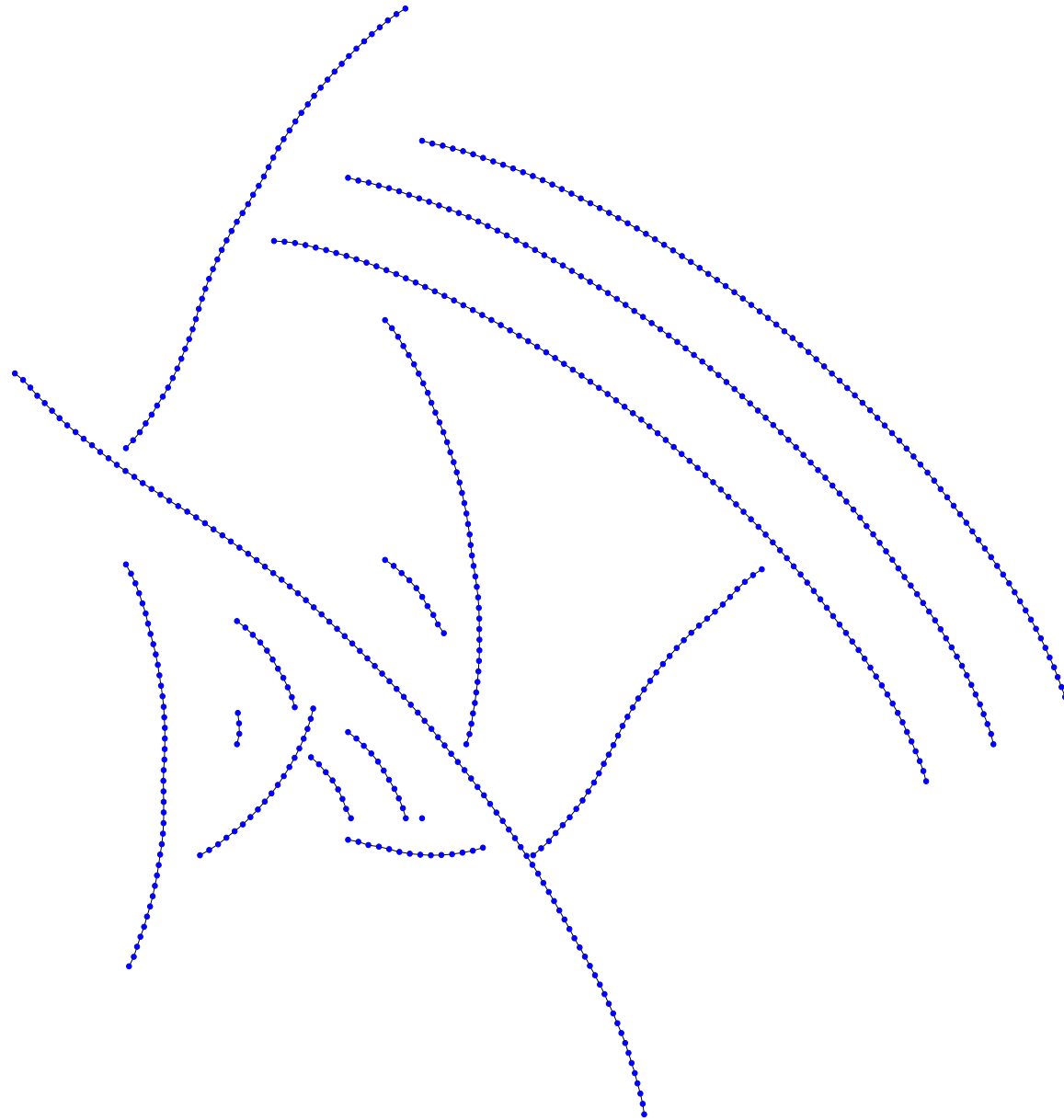
A random graph of size 956 avoiding the bowtie



$$\mathbb{E}(N_n) \simeq n^{1/3}$$

$$\mathbb{E}(S_n) \simeq n^{2/3}$$

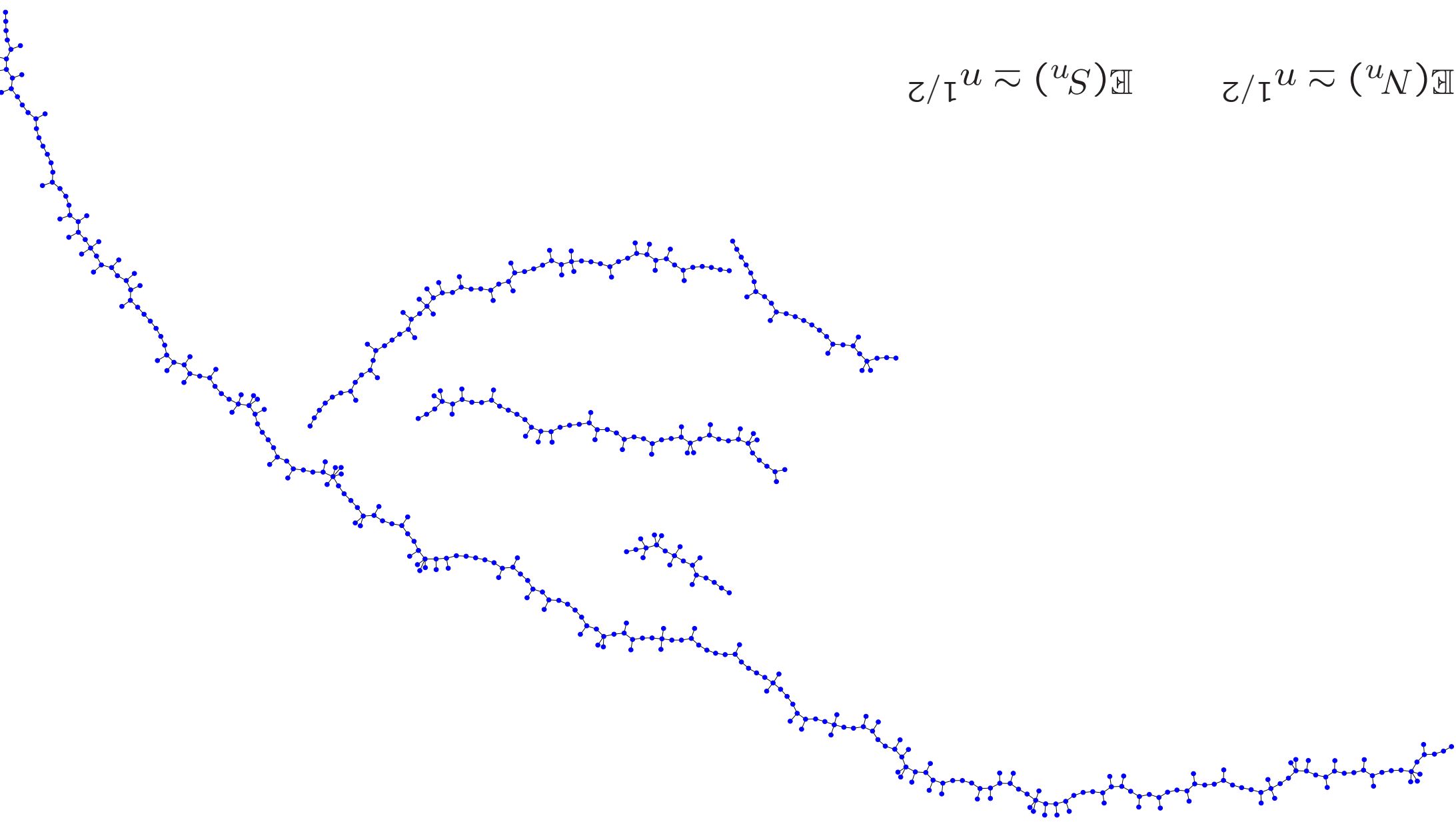
A random forest of paths of size 597 (no \triangle nor \pitchfork)



$$\mathbb{E}(N_n) \simeq n^{1/2}$$

$$\mathbb{E}(S_n) \simeq n^{1/2}$$

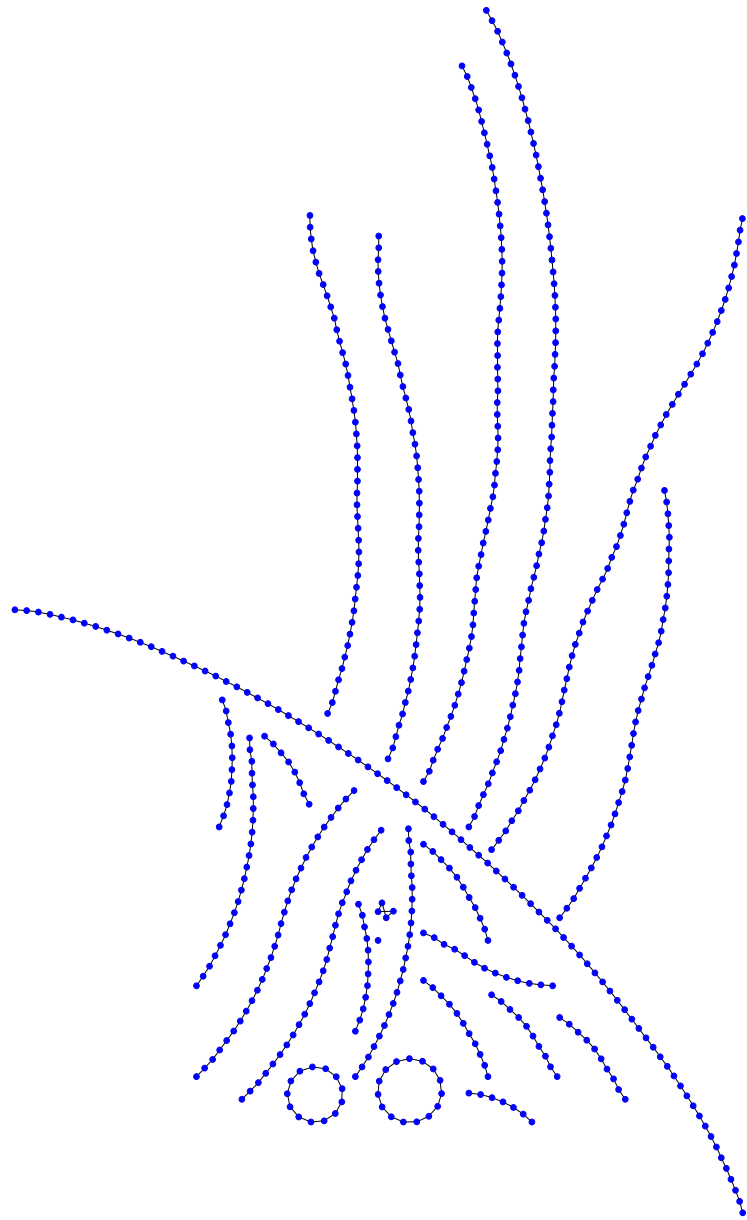
A random forest of caterpillars of size 486 (no ∇ nor \downarrow)



$$\mathbb{E}(S_n) \approx n^{1/2}$$

$$\mathbb{E}(N^n) \approx n^{1/2}$$

A random graph of maximal degree 2 of size 636 (no )

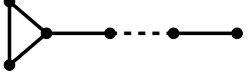








$$\mathbb{E}(N_n) \simeq n^{1/2}$$

$$\mathbb{E}(S_n) \simeq n^{1/2}$$

Final comments

- We do not cover McDiarmid's results (2-connected excluded minors)
- Other classes? (outer-planar graphs with an additional excluded minor...)
- Other parameters? number of edges, distribution of vertex degrees...
- Describe conditions on the excluded minors that determine the nature of the singularities of $C(z)$. For instance,
 - 2-connected minors $\Rightarrow C(\rho)$ finite
 - exclude one spoon but not tree $\Rightarrow C(\rho)$ finite with a singularity in $(1 - ze)^{3/2}$

Excluded minors	2-conn. or 	 and 		 or  and 	all conn. graphs of size $k + 1$
$C(\rho)$	$< \infty$	$\log \frac{1}{1-z/\rho}$	$\frac{1}{\sqrt{1-z/\rho}}$	$\frac{1}{1-z/\rho}$	entire
$\mathbb{P}(\text{conn.})$	$\rightarrow p > 0$	$\rightarrow 0$	$\rightarrow 0$	$\rightarrow 0$	$\rightarrow 0$
N_n	$O(1)$ Poisson	$\log n$ gaussian	$n^{1/3}$ gaussian	$n^{1/2}$ gaussian	n/k gaussian (*)
S_n	$n - F_n$ $F_n \rightarrow \text{discr.}$	n $\frac{1}{4}(1-x)^{-3/4}$	$n^{2/3}$ $2\sqrt{\frac{x}{\pi}}e^{-x}$	$n^{1/2}$ xe^{-x}	k Dirac
L_n	$n - F_n$ $F_n \rightarrow \text{discr.}$	n Dickman	?	$n^{1/2} \log n$ Gumbel	k Dirac

Singularity analysis

Saddle point method