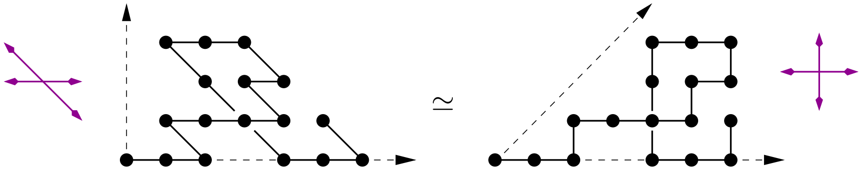
Lattice paths confined to cones

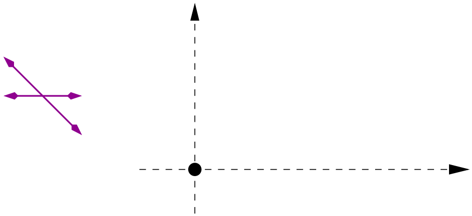
Mireille Bousquet-Mélou, CNRS, LaBRI, Bordeaux



A typical question

Let $\mathcal{S}$ be a finite subset of $\mathbb{Z}^d$ (set of steps) and $p_0 \in \mathbb{Z}^d$ (starting point).

Example. $\mathcal{S} = \{10, \bar{1}0, 1\bar{1}, \bar{1}1\}$, $p_0 = (0,0)$



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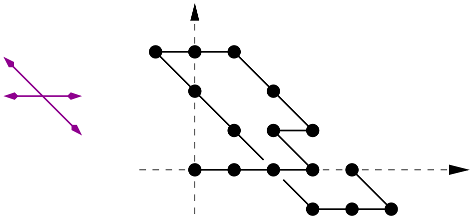

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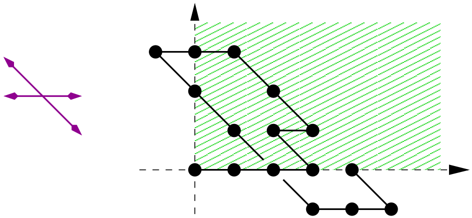
Let C be a cone of $\mathbb{R}^d$ (if $x \in C$ and $r \geq 0$ then $r \cdot x \in C$).

Example. $\mathcal{S} = \{10, \bar{1}0, 1\bar{1}, \bar{1}1\}$, $p_0 = (0,0)$ and $C = \mathbb{R}_+^2$.



A typical question

Questions

- What is the number $a(n)$ of n-step walks starting at p_0 and contained in C?
- For $\boldsymbol{i} = (i_1, \ldots, i_d) \in C$, what is the number $a(\boldsymbol{i}; n)$ of such walks that end at $\boldsymbol{i}$?

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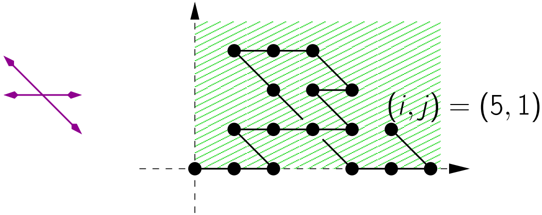

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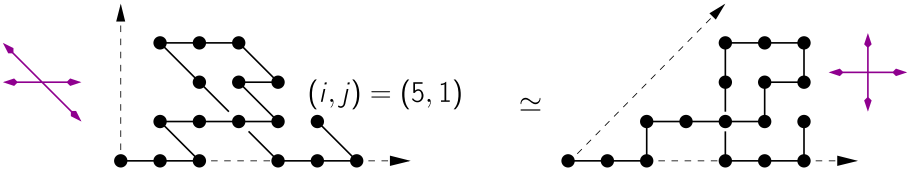

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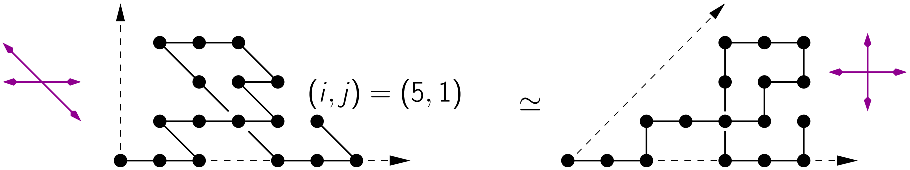

Nice numbers

If $n = 2m + \delta$, with $\delta \in \{0,1\}$,

$$a(n) = \frac{n!(n+1)!}{m!(m+1)!(m+\delta)!(m+\delta+1)!}.$$

Moreover, if $n = 2m + i$,

$$a(i,j;n) = \frac{(i+1)(j+1)(i+j+2)(i+2j+3)n!(n+2)!}{(m-j)!(m+1)!(m+i+2)!(m+i+j+3)!}.$$

Why count walks in cones?

Many discrete objects can be encoded in that way:

- in combinatorics, statistical physics...
- in (discrete) probability theory: random walks, queuing theory...

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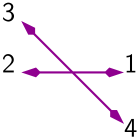

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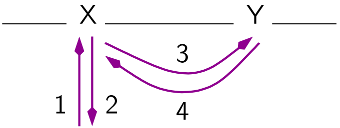

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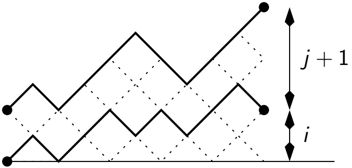

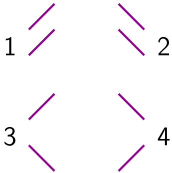

$+$ Young tableaux of height 4 [Gouyou-Beauchamps 89]

Many contributions

Adan, Banderier, Bernardi, Bostan, Cori, Denisov, Duchon, Dulucq, Fayolle, Gessel, Fisher, Flajolet, Gouyou-Beauchamps, Guttmann, Guy, Janse van Rensburg, Johnson, Kauers, Koutschan, Krattenthaler, Kurkova, Kreweras, van Leeuwarden, MacMahon, Melczer, Mishna, Niederhausen, Petkovšek, Prellberg, Raschel, Rechnitzer, Sagan, Salvy, Viennot, Wachtel, Wilf, Yeats, Zeilberger...

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~~Specific question~~
~~Ad hoc solution~~ $\longrightarrow$ Systematic approach

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- Generating functions and their nature

$$A(t) = \sum_{n \geq 0} a(n) t^n, \qquad A(x_1, \ldots, x_d; t) = \sum_{\boldsymbol{i}, n} a(\boldsymbol{i}; n) \boldsymbol{x}^{\boldsymbol{i}} t^n$$

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Remarks

- $A(1, \dots, 1; t) = A(t)$
- if $C \subset \mathbb{R}_+^d$, then $A(0, \dots, 0; t)$ counts walks ending at $(0, \dots, 0)$
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Can one express these series? Are they rational? algebraic? D-finite?

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A hierarchy of formal power series

- The formal power series $A(t)$ is rational if it can be written

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◦ Nice closure properties + asymptotics of the coefficients
◦ Extension to several variables (D-finite: one DE per variable)

A (very) basic cone: the full space

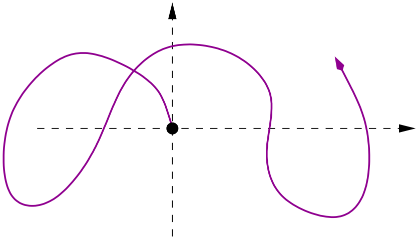
Rational series

If $\mathcal{S} \subset \mathbb{Z}^d$ is finite and $C = \mathbb{R}^d$, then $A(\boldsymbol{x}; t)$ is rational:

$$a(n) = |\mathcal{S}|^n \quad \Leftrightarrow \quad A(t) = \sum_{n \geq 0} a(n) t^n = \frac{1}{1 - |\mathcal{S}|\, t}$$

More generally:

$$A(x; t) = \frac{1}{1 - t \sum_{s \in \mathcal{S}} \boldsymbol{x}^{\boldsymbol{s}}}.$$

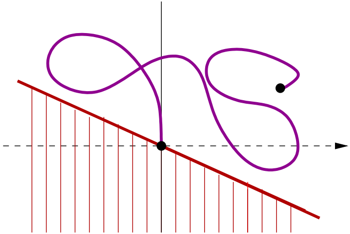


Also well-known: a (rational) half-space

Algebraic series

If $\mathcal{S} \subset \mathbb{Z}^d$ is finite and C is a rational half-space, then $A(x; t)$ is algebraic, given by an explicit system of polynomial equations.

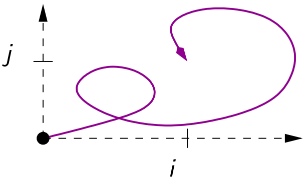
[mbm-Petkovšek 00]; [Gessel 80], [Duchon 00]...



The "next" case: intersection of two half-spaces

- The quarter plane, with $p_0 = (0, 0)$

$$A(x, y; t) = \sum_{i,j,n \geq 0} a(i, j; n) x^i y^j t^n = \ ?$$



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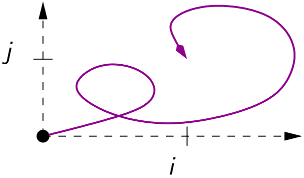
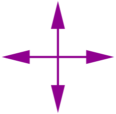
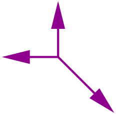




D–finite



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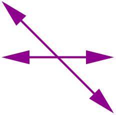

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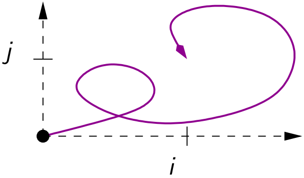

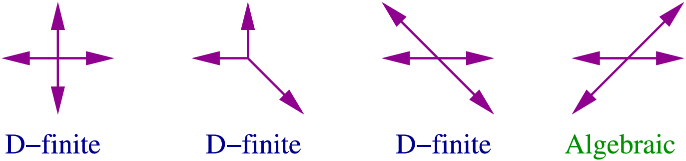

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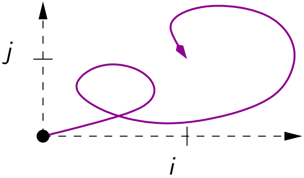

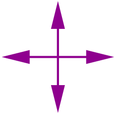

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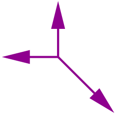

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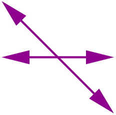

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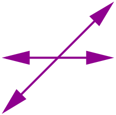
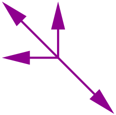


Algebraic



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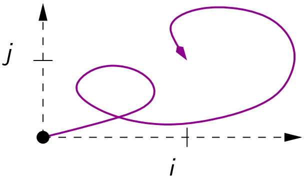

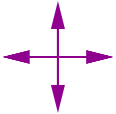

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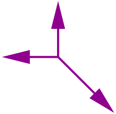

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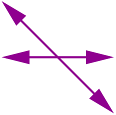

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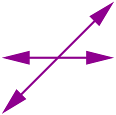

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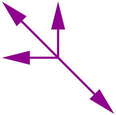

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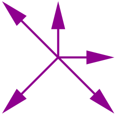

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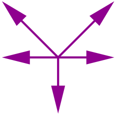


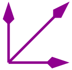
D–finite

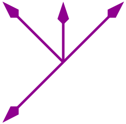
Quadrant walks with *small* steps: classification

- $\mathcal{S} \subset \{\bar{1}, 0, 1\} \setminus \{00\} \Rightarrow 2^8 = 256$ step sets (or: models)

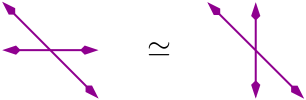
Quadrant walks with *small* steps: classification

- $\mathcal{S} \subset \{\bar{1}, 0, 1\} \setminus \{00\} \Rightarrow 2^8 = 256$ step sets (or: models)
- However, some models are equivalent:
  - to a model of walks in the full or half-plane ($\Rightarrow$ algebraic)

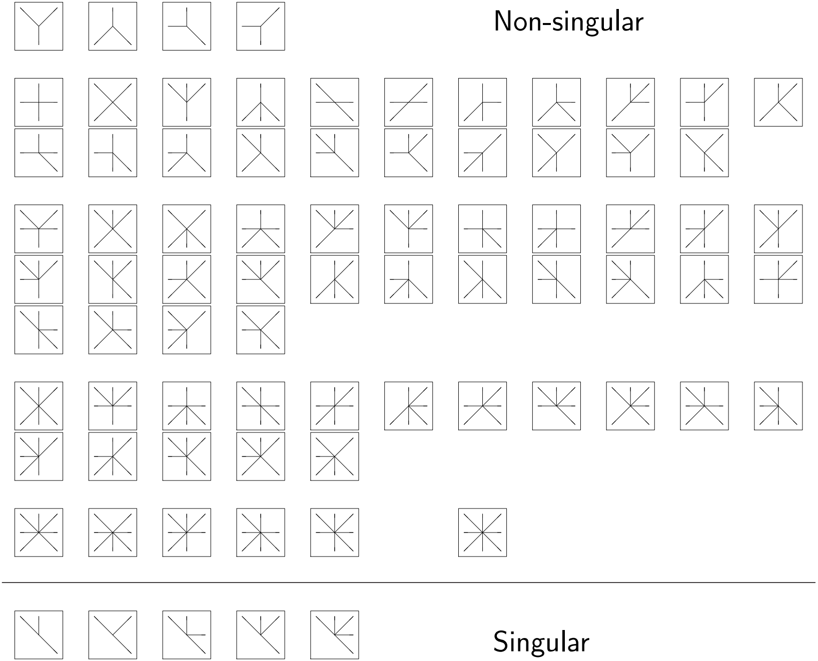


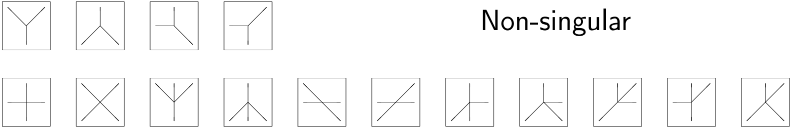


  - to another model in the collection (diagonal symmetry)



- One is left with 79 interesting distinct models.


Non-singular
Singular



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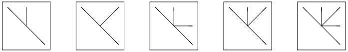

Singular

Classification

- What is the nature of $A(x, y; t)$?
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Preview:

The series $A(x, y; t)$ is D-finite
iff a certain group associated with $\mathcal{S}$ is finite.

The group of the model

Example. Take $\mathcal{S} = \{\bar{1}0, 01, 1\bar{1}\}$, with step polynomial

$$S(x, y) = \frac{1}{x} + y + x \cdot \frac{1}{y} = \bar{x} + y + x\bar{y}$$

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Observation: $S(x,y)$ is left unchanged by the rational transformations

$$\Phi : (x,y) \mapsto (\bar{x}y, y) \quad \text{and} \quad \Psi : (x,y) \mapsto (x, x\bar{y})\,.$$

The group of the model

Example. Take $\mathcal{S} = \{\bar{1}0, 01, 1\bar{1}\}$, with step polynomial

$$S(x,y) = \frac{1}{x} + y + x \cdot \frac{1}{y} = \bar{x} + y + x\bar{y}$$

Observation: $S(x,y)$ is left unchanged by the rational transformations

$$\Phi : (x,y) \mapsto (\bar{x}y, y) \quad \text{and} \quad \Psi : (x,y) \mapsto (x, x\bar{y}) \, .$$

They are involutions, and generate a finite dihedral group G:

$$\begin{tikzcd}[arrows=-]
 & (\bar{x}y, y) \arrow[r, "\Psi"] & (\bar{x}y, \bar{x}) \arrow[dr, "\Phi"] & \\
(x,y) \arrow[ur, "\Phi"] \arrow[dr, "\Psi"'] & & & (\bar{y}, \bar{x}) \\
 & (x, x\bar{y}) \arrow[r, "\Phi"'] & (\bar{y}, x\bar{y}) \arrow[ur, "\Psi"'] &
\end{tikzcd}$$

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 & (x, x\bar{y}) \arrow[r, dash, "\Phi"'] & (\bar{y}, x\bar{y}) \arrow[ur, dash, "\Psi"'] &
\end{tikzcd}$$

Remark. G can be defined for any quadrant model with small steps

The group is not always finite

- If $\mathcal{S} = \{0\bar{1}, \bar{1}\bar{1}, \bar{1}0, 11\}$, then $S(x,y) = \bar{x}(1+\bar{y}) + \bar{y} + xy$ and

$$\Phi : (x,y) \mapsto (\bar{x}\bar{y}(1+\bar{y}), y) \quad \text{and} \quad \Psi : (x,y) \mapsto (x, \bar{x}\bar{y}(1+\bar{x}))$$

generate an infinite group:

$$\begin{array}{lllllll}
 & \overset{\Phi}{\diagup} & (\bar{x}\bar{y}(1+\bar{y}), y) & \xrightarrow{\Psi} \cdots & \xrightarrow{\Phi} \cdots & \xrightarrow{\Psi} \cdots \\
(x,y) & & & & & \\
 & \underset{\Psi}{\diagdown} & (x, \bar{x}\bar{y}(1+\bar{x})) & \xrightarrow[\Phi]{} \cdots & \xrightarrow[\Psi]{} \cdots & \xrightarrow[\Phi]{} \cdots
\end{array}$$

When G is finite: the orbit sum

Example. If $\mathcal{S} = \{01, \bar{1}0, 1\bar{1}\}$, the orbit of (x, y) is

$$
\begin{tikzcd}
 & (\bar{x}y, y) \arrow[r, dash, "\Psi"] & (\bar{x}y, \bar{x}) \arrow[dr, dash, "\Phi"] & \\
(x, y) \arrow[ur, dash, "\Phi"] \arrow[dr, dash, "\Psi"'] & & & (\bar{y}, \bar{x}) \\
 & (x, x\bar{y}) \arrow[r, dash, "\Phi"'] & (\bar{y}, x\bar{y}) \arrow[ur, dash, "\Psi"'] &
\end{tikzcd}
$$

and the (alternating) orbit sum is

$$OS = xy - \bar{x}y^2 + \bar{x}^2 y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}$$

Classification of quadrant walks with small steps

Theorem

The series $A(x, y; t)$ is D-finite iff the group G is finite.
It is algebraic iff, in addition, the orbit sum is zero.

[mbm-Mishna 10], [Bostan-Kauers 10] D-finite
[Kurkova-Raschel 12] non-singular non-D-finite
[Mishna-Rechnitzer 07], [Melczer-Mishna 13] singular non-D-finite

quadrant models: 79

$|G| < \infty$: 23

$|G| = \infty$: 56

OS$\neq$0: 19

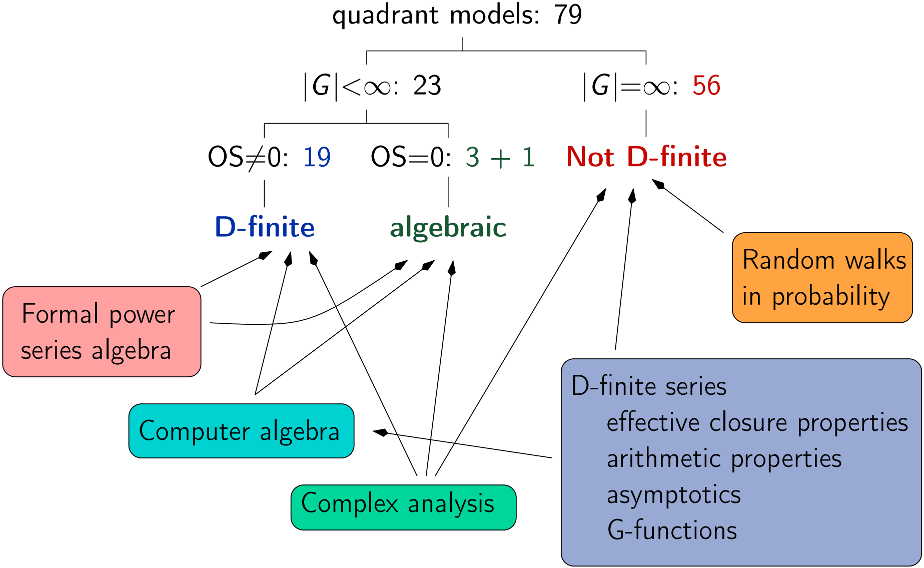
OS$=$0: $3 + 1$

Not D-finite

D-finite

algebraic

Classification of quadrant walks with small steps



Exact enumeration

- the kernel method
- computer algebra
- an approach using complex analysis

Starting point: recurrence relation / functional equation

A recurrence relation...

The numbers $a(i,j;n)$ satisfy

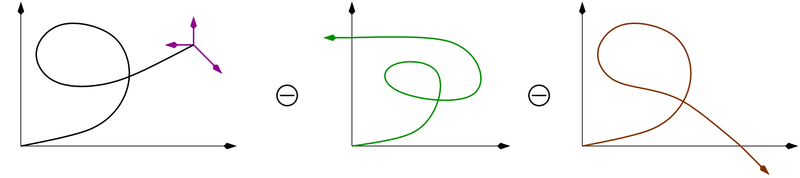
$$a(i,j;n) = \begin{cases} 0 & \text{if } i<0 \text{ or } j<0 \text{ or } n<0, \\ \mathbb{1}_{i=j=0} & \text{if } n=0, \\ \displaystyle\sum_{i'j'\in\mathcal{S}} a(i-i',j-j';n-1) & \text{otherwise.} \end{cases}$$

$\Rightarrow$ Compute $a(i,j;n)$ for n "small" (less than a few thousands) and try to guess algebraic or differential equations ($\rightarrow$ Gfun package of Maple).

... and the corresponding functional equation

Example: $\mathcal{S} = \{01, \bar{1}0, 1\bar{1}\}$

$$A(x, y; t) \equiv A(x, y) = 1 + t(y + \bar{x} + x\bar{y})A(x, y) - t\bar{x}A(0, y) - tx\bar{y}A(x, 0)$$



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or

$$\big(1 - t(y + \bar{x} + x\bar{y})\big)A(x, y) = 1 - t\bar{x}A(0, y) - tx\bar{y}A(x, 0),$$

... and the corresponding functional equation

Example: $\mathcal{S} = \{01, \bar{1}0, 1\bar{1}\}$

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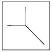
or

$$\boxed{\big(1 - t(y + \bar{x} + x\bar{y})\big)xyA(x,y) = xy - tyA(0,y) - tx^2A(x,0)}$$

- The polynomial $1 - t(y + \bar{x} + x\bar{y})$ is the kernel of this equation

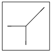
- The equation is linear, with two catalytic variables x and y (tautological at $x = 0$ or $y = 0$)

79 models, and 79 functional equations...



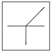
D-finite transcendental

$$\left(1 - t(y + \bar{x} + x\bar{y})\right)xyA(x, y) = xy - tyA(0, y) - tx^2A(x, 0)$$



Algebraic

$$\left(1 - t(\bar{x} + \bar{y} + xy)\right)xyA(x, y) = xy - tyA(0, y) - txA(x, 0)$$



Not D-finite

$$\left(1 - t(x + \bar{x} + \bar{y} + xy)\right)xyA(x, y) = xy - tyA(0, y) - txA(x, 0)$$

But why?

The kernel method

- The equation reads (with $K(x,y) = 1 - t(y + \bar{x} + x\bar{y})$):

$$K(x,y)xyA(x,y) = xy - tx^2A(x,0) - tyA(0,y)$$

- The orbit of (x,y) under G is

$$(x,y) \overset{\Phi}{\longleftrightarrow} (\bar{x}y, y) \overset{\Psi}{\longleftrightarrow} (\bar{x}y, \bar{x}) \overset{\Phi}{\longleftrightarrow} (\bar{y}, \bar{x}) \overset{\Psi}{\longleftrightarrow} (\bar{y}, x\bar{y}) \overset{\Phi}{\longleftrightarrow} (x, x\bar{y}) \overset{\Psi}{\longleftrightarrow} (x,y).$$

- All transformations of G leave $K(x,y)$ invariant. Hence

$$\begin{array}{rclcccc}
K(x,y)\, xyA(x,y) & = & xy & - & tx^2A(x,0) & - & tyA(0,y) \\
K(x,y)\, \bar{x}y^2A(\bar{x}y,y) & = & \bar{x}y^2 & - & t\bar{x}^2y^2A(\bar{x}y,0) & - & tyA(0,y) \\
K(x,y)\, \bar{x}^2yA(\bar{x}y,\bar{x}) & = & \bar{x}^2y & - & t\bar{x}^2y^2A(\bar{x}y,0) & - & t\bar{x}A(0,\bar{x}) \\
\cdots & = & & & \cdots & & \\
K(x,y)\, x^2\bar{y}A(x,x\bar{y}) & = & x^2\bar{y} & - & tx^2A(x,0) & - & tx\bar{y}A(0,x\bar{y}).
\end{array}$$

The kernel method

$\Rightarrow$ Form the alternating sum of the equation over all elements of the orbit:

$$K(x,y)\Big(xyA(x,y) - \bar{x}y^2A(\bar{x}y,y) + \bar{x}^2yA(\bar{x}y,\bar{x}) - \bar{x}\bar{y}A(\bar{y},\bar{x}) + x\bar{y}^2A(\bar{y},x\bar{y}) - x^2\bar{y}A(x,x\bar{y})\Big) =$$

$$xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}$$

(the orbit sum).

Why is this interesting?

$$xyA(x,y) - \bar{x}y^2A(\bar{x}y,y) + \bar{x}^2yA(\bar{x}y,\bar{x}) \\ - \bar{x}\bar{y}A(\bar{y},\bar{x}) + x\bar{y}^2A(\bar{y},x\bar{y}) - x^2\bar{y}A(x,x\bar{y}) = \\ \frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}$$

- Both sides are power series in t, with coefficients in $\mathbb{Q}[x,\bar{x},y,\bar{y}]$.

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$$xyA(x,y) - \bar{x}y^2A(\bar{x}y,y) + \bar{x}^2yA(\bar{x}y,\bar{x}) \\ - \bar{x}\bar{y}A(\bar{y},\bar{x}) + x\bar{y}^2A(\bar{y},x\bar{y}) - x^2\bar{y}A(x,x\bar{y}) = \\ \frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}$$

- Both sides are power series in t, with coefficients in $\mathbb{Q}[x,\bar{x},y,\bar{y}]$.
- Extract the part with positive powers of x and y:

$$xyA(x,y) = [x^{>0}y^{>0}]\ \frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}$$

is a D-finite series.
[Lipshitz 88]

The kernel method in general (finite groups)

- For all models with a finite group,

$$\sum_{g\in G} \text{sign}(g) g(xyA(x,y;t)) = \frac{1}{K(x,y;t)} \sum_{g\in G} \text{sign}(g) g(xy) = \frac{OS}{K(x,y;t)},$$

where $g(A(x,y)) := A(g(x,y))$.

- The right-hand side is an explicit rational series.

[mbm-Mishna 10]

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where $g(A(x,y)) := A(g(x,y))$.

- The right-hand side is an explicit rational series.

- For the 19 models where the orbit sum is non-zero,

$$xyA(x,y;t) = [x^{>0}y^{>0}] \frac{OS}{K(x,y;t)}$$

is a D-finite series.
[mbm-Mishna 10]

Classification of quadrant walks with small steps

- quadrant models: 79
  - $|G|<\infty$: 23
    - OS$\neq$0: 19
      - **D-finite**
        - Kernel method
    - OS=0: 3 + 1
      - **algebraic**
  - $|G|=\infty$: 56
    - **Not D-finite**

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    - OS=0: 3 + 1
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        - Half-orbit sum
        - computer algebra
  - $|G|=\infty$: 56
    - **Not D-finite**

A computer algebra approach

Example. When $\mathcal{S} = \{\bar{1}0, 0\bar{1}, 11\}$, the equation reads

$$(1 - t(\bar{x} + \bar{y} + xy))xyA(x, y; t) = xy - tyA(0, y; t) - txA(x, 0; t).$$

Naïve route: guess and check!

- Guess a polynomial equation Pol satisfied by $A(x, y; t)$ (degrees $[18, 18, 17, 12]$ in x, y, t, A)
- Let $F(x, y; t)$ be the solution of Pol that coincides with $A(x, y; t)$ up to high order (in t)
- Prove that $F(x, y; t)$ is a formal power series in t with polynomial coefficients in x and y $\Rightarrow$ $F(x, 0; t)$ and $F(0, y; t)$ are well-defined
- By taking resultants, prove that $F(x, y; t)$ satisfies the above functional equation.

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A better route...

Work with $A(x, 0; t) = A(0, x; t)$ only

A computer algebra approach: climax

Algebraicity of Gessel's model [Bostan-Kauers 10]

- When $\mathcal{S} = \{10, \bar{1}0, 11, \bar{1}\bar{1}\}$, the series $A(x, y; t)$ is algebraic (degree 72).

A computer algebra approach: climax

Algebraicity of Gessel's model [Bostan-Kauers 10]

• When $\mathcal{S} = \{10, \bar{1}0, 11, \bar{1}\bar{1}\}$, the series $A(x, y; t)$ is algebraic (degree 72).

• In particular, the series $A(0, 0; t)$, which counts loops, has degree 8, and the following expansion:

$$A(0,0;t) = \sum_{n\geq 0} 16^n \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} \, t^{2n}, \tag{1}$$

with $(i)_n = i(i+1)\cdots(i+n-1)$.

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(??): Conjectured around 2001, first proof by [Kauers, Koutschan & Zeilberger 09] using computer algebra.

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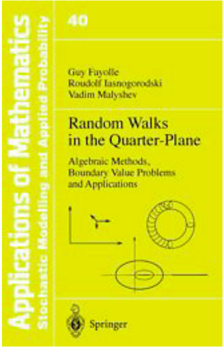
The algebraicity of $A(x, y; t)$ has just been re-proved using a complex analysis approach [Bostan, Kurkova & Raschel 13(a)]

A complex analysis approach

Markov chains with small steps in the quadrant: stationary distribution(s) [Malyshev 71+]

Le petit livre jaune
[Fayolle, Iasnogorodski & Malyshev 99]



$\Rightarrow$ Reduction to a boundary value problem of the Riemann-Carleman type

A complex analysis approach

An expression of Q for any non-singular model $\mathcal{S}$

$$\tilde{K}(x,0;t)A(x,0;t) - \tilde{K}(0,0;t)A(0,0;t) = xY_0(x;t) + \frac{1}{2i\pi}\int_{x_1(t)}^{x_2(t)} u\left[Y_0(u;t) - Y_1(u;t)\right]\left[\frac{\partial_u w(u;t)}{w(u;t)-w(x;t)} - \frac{\partial_u w(u;t)}{w(u;t)-w(0;t)}\right]\mathrm{d}u$$

where Y_0, Y_1, x_1 and x_2 are explicit algebraic series and w is explicit/ very well understood.

[Raschel 12] + Fayolle, Kurkova

+ other formulas that complete the characterization of $A(x,y;t)$

A complex analysis approach

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where Y_0, Y_1, x_1 and x_2 are explicit algebraic series and w is explicit/ very well understood.
In particular, w is D-finite (in fact, algebraic!) iff the group is finite.

[Raschel 12] + Fayolle, Kurkova

+ other formulas that complete the characterization of $A(x,y;t)$

A complex analysis approach: climax(es)

Theorems

• If $\mathcal{S}$ has an infinite group and is not singular, then $A(x, y; t)$ is not D-finite in x ($\equiv$ no differential equation with respect to x)

[Kurkova & Raschel 12]

• A new proof of the algebraicity of Gessel's model

[Bostan, Kurkova & Raschel 13(a)]

Asymptotics

- Random walks in a cone
- Asymptotics of coefficients of D-finite series

$\Rightarrow A(0,0;t)$ is not D-finite if $\mathcal{S}$ has an infinite group and is not singular

Random walks in a cone

For loops in the quadrant [Denisov & Wachtel 12(a)]

For a non-singular models $\mathcal{S}$, the number of n-step loops satisfies

$$a(0,0;n) \sim \kappa \mu^n n^{-\gamma}$$

where

$$\gamma = \frac{\pi}{\arccos(-c)} + 1,$$

with c an algebraic number that can be described in terms of $\mathcal{S}$.

Random walks in a cone + asymptotics of D-finite series

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with c an algebraic number that can be described in terms of $\mathcal{S}$.

Possible asymptotic behaviours

If $B(t) = \sum_n b(n) t^n$ is D-finite with integer coefficients and

$$b(n) \sim \kappa \mu^n n^{-\gamma},$$

then γ is rational.
(connection with G-functions [André 89], [Chudnovsky^2 85], [Katz 70])

Random walks in a cone + asymptotics of D-finite series

Strategy for proving non-D-finiteness of $A(0,0;t)$

Prove that γ is irrational, that is, that $\arccos(-c)$ is not a rational multiple of π.

For any of the 51 non-singular models with an infinite group, $A(0,0;t)$ is not D-finite.

[Bostan, Raschel & Salvy 14]

Perspectives

- larger steps
- intersections of three half-spaces: walks in the 3D octant
- and more...

Larger steps

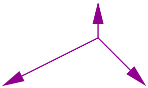
- Define (and use) a group G for models with larger steps?

Larger steps

- Define (and use) a group G for models with larger steps?

- Example: When $\mathcal{S} = \{01, 1\bar{1}, \bar{2}\bar{1}\}$, there is an underlying group that is finite and

$$xyA(x, y; t) = [x^{>0}y^{>0}]\frac{(x - 2\bar{x}^2)(y - (x - \bar{x}^2)\bar{y})}{1 - t(x\bar{y} + y + \bar{x}^2\bar{y})}$$



[Bostan, mbm & Melczer]

Three-dimensional walks in the positive octant

• Take $\mathcal{S} \subset \{\bar{1}, 0, 1\}^3 \setminus \{000\}$, $p_0 = (0, 0, 0)$ and study walks confined to the positive octant $\mathbb{R}_+^3$

• Problem: there are 11 074 225 distinct interesting models ⇒ Focus on those of cardinality at most 6

[Bostan, mbm, Kauers, Melczer 14(a)]

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Example. The model $\{111, \bar{1}00, 0\bar{1}0, 00\bar{1}\}$ has a finite group of order 24. The orbit sum vanishes. Is it D-finite?

[Bostan, mbm, Kauers, Melczer 14(a)]

Et encore...

- Non D-finiteness of $A(1,1;t)$ (which counts all quadrant walks by length), via probabilistic results [Denisov-Wachtel 12] or [Duraj 14]
  Bostan, Raschel, Salvy...
- Exact asymptotics for D-finite cases (using [Pemantle & Wilson 13], asymptotics of coefficients of multivariate rational series)
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