Counting plane lattice walks avoiding a quadrant

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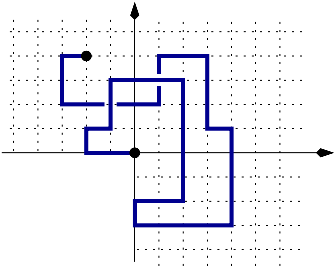
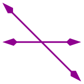


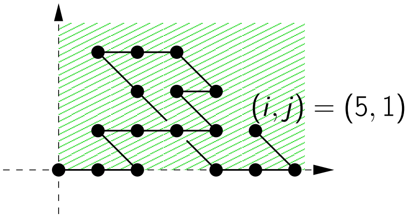
Counting walks in (rational) cones

Take a starting point p_0 in $\mathbb{Z}^2$, a (finite) step set $\mathcal{S} \subset \mathbb{Z}^2$ and a cone $\mathcal{C}$.

Questions

- What is the number $c(n)$ of n-step walks starting at p_0, taking their steps in $\mathcal{S}$ and contained in $\mathcal{C}$?
- For $(i, j) \in \mathcal{C}$, what is the number $c(i, j; n)$ of such walks that end at (i, j)?





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- Generating function:

$$C(x,y;t) = \sum_{i,j,n} c(i,j;n) x^i y^j t^n$$
$$= \sum_{w \text{ walk}} x^{i(w)} y^{j(w)} t^{|w|}$$

What is the value/nature of this series?

A hierarchy of formal power series

- Rational series

$$A(t) = \frac{P(t)}{Q(t)}$$

- Algebraic series

$$P(t, A(t)) = 0$$

- Differentially finite series (D-finite)

$$\sum_{i=0}^{d} P_i(t) A^{(i)}(t) = 0$$

- D-algebraic series

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Multi-variate series: one DE per variable

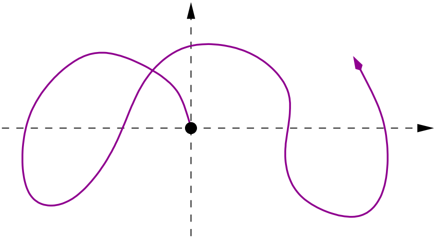
Normalizing the cone: 4 cases

- The full space: rational series

$$C(x,y;t) = \frac{1}{1-tS(x,y)} = \sum_{n\geq 0} t^n S(x,y)^n,$$

where $S(x,y)$ is the step polynomial:

$$S(x,y) = \sum_{(i,j)\in\mathcal{S}} x^i y^j.$$



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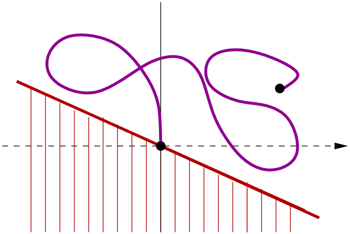
- A half-space: algebraic series
  [Gessel 80]; [mbm-Petkovšek 00], [Duchon 00], [Banderier & Flajolet 02]...



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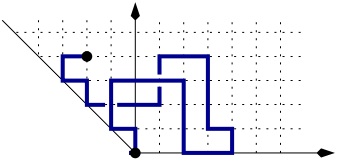

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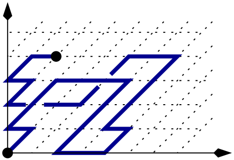
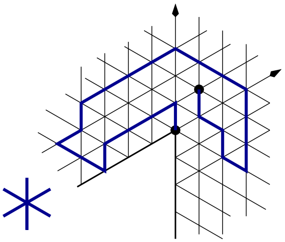


Normalizing the cone: 4 cases

- The full space: rational series
- A half-space: algebraic series
  [Gessel 80]; [mbm-Petkovšek 00], [Duchon 00], [Banderier & Flajolet 02]...
- A convex cone → walks in the non-negative quadrant: $Q(x, y; t)$
- A non-convex cone → walks avoiding the negative quadrant: $C(x, y; t)$



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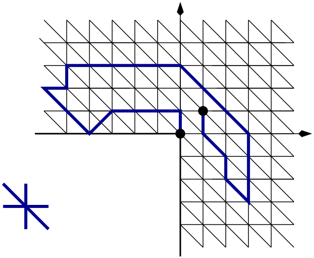
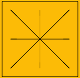


Walks with *small* steps



- $\mathcal{S} \subset \{\bar{1}, 0, 1\}^2 \setminus \{(0,0)\} \Rightarrow 2^8 = 256$ step sets (or: models)

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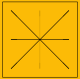

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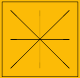

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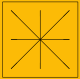

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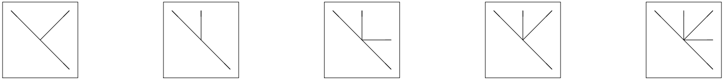
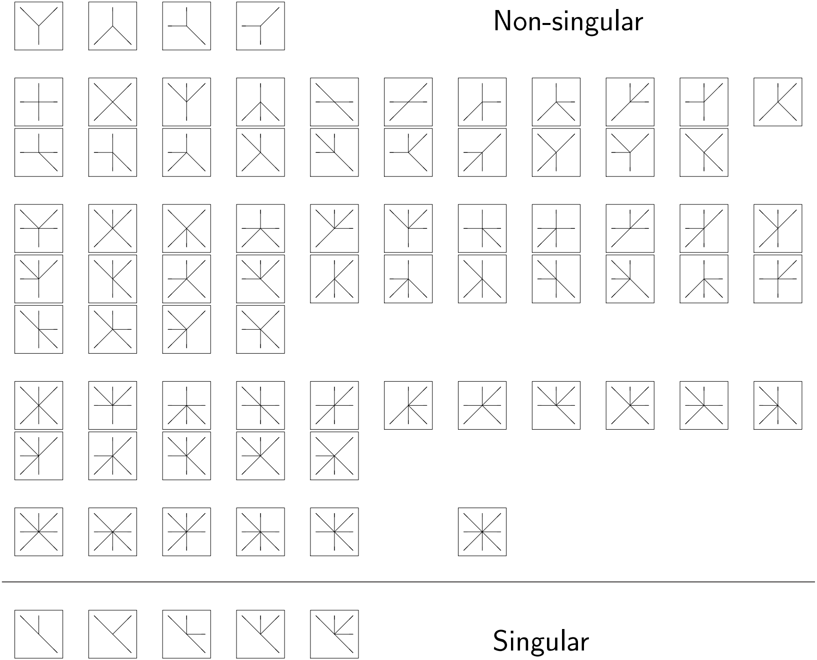
- However, some models are equivalent to a half-space problem (hence algebraic) and/or to another model (diagonal symmetry).

- On the quadrant, one is left with 79 interesting distinct models [mbm-Mishna 09].

- On the three-quadrant cone, one is left with 74 interesting distinct models: the 5 "singular" models on the quadrant become trivial.

Singular models

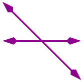


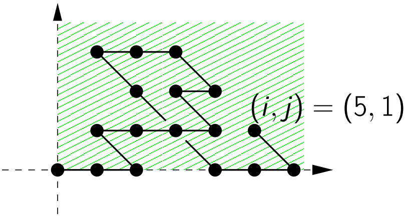


Non-singular
Singular

I. Functional equations

A step by step construction of walks

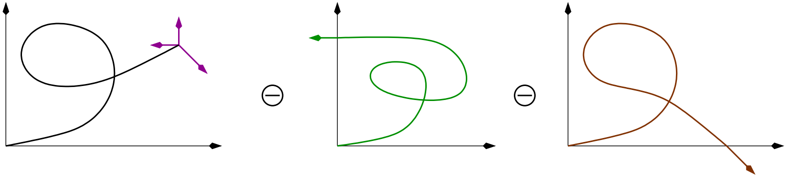




In the quadrant

Example: $\mathcal{S} = \{01, \bar{1}0, 1\bar{1}\}$, with $\bar{x} := 1/x$ and $\bar{y} := 1/y$

$$Q(x,y;t) \equiv Q(x,y) = 1 + t(y + \bar{x} + x\bar{y})Q(x,y) - t\bar{x}Q(0,y) - tx\bar{y}Q(x,0)$$



$$Q(x,y;t) = \sum_{i,j,n\geq 0} q(i,j;n)x^i y^j t^n$$

In the quadrant

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or

$$\big(1 - t(y + \bar{x} + x\bar{y})\big)Q(x, y) = 1 - t\bar{x}Q(0, y) - tx\bar{y}Q(x, 0),$$

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or

$$\boxed{\big(1 - t(y + \bar{x} + x\bar{y})\big)xyQ(x,y) = xy - tyQ(0,y) - tx^2Q(x,0)}$$

- The right-hand side is decoupled in x/y.
- The polynomial $1 - t(y + \bar{x} + x\bar{y})$ is the kernel of this equation
- The equation involves two catalytic variables x and y (tautological at $x = 0$ or $y = 0$)

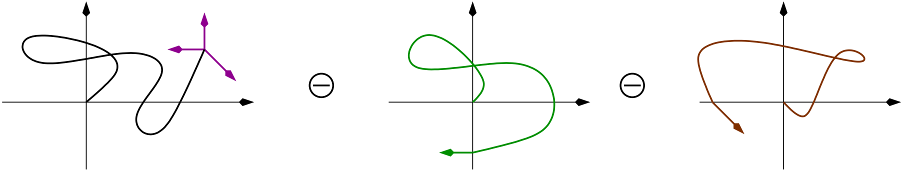
In three quadrants

Step by step construction:

$$C(x,y;t) \equiv C(x,y) = 1 + t(y + \bar{x} + x\bar{y})C(x,y) - t\bar{x}C_{0,-}(\bar{y}) - tx\bar{y}C_{-,0}(\bar{x})$$

with

$$C_{0,-}(\bar{y}) = \sum_{j<0, n\geq 0} c(0,j;n) y^j t^n, \qquad C_{-,0}(\bar{x}) = \sum_{i<0, n\geq 0} c(i,0;n) x^i t^n.$$



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Step by step construction:

$$C(x,y;t) \equiv C(x,y) = 1 + t(y+\bar{x}+x\bar{y})C(x,y) - t\bar{x}C_{0,-}(\bar{y}) - tx\bar{y}C_{-,0}(\bar{x})$$

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A comparison

- First quadrant:

$$\big(1 - t(y + \bar{x} + x\bar{y})\big)xyQ(x,y) = xy - tyQ(0,y) - tx^2 Q(x,0)$$

- Three quadrants:

$$\big(1 - t(y + \bar{x} + x\bar{y})\big)xyC(x,y) = xy - tyC_{0,-}(\bar{y}) - tx^2 C_{-,0}(\bar{x})$$

with

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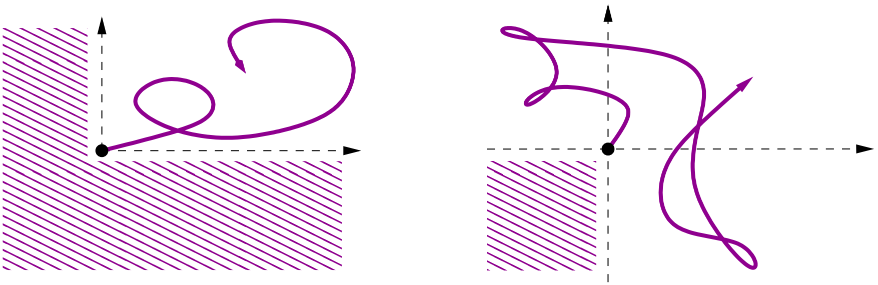
- A similar form... but $C(x,y)$ involves negative powers of x and y (Laurent polynomials)

II. The group of the model and the orbit sum



The group of the model

Example. Take $\mathcal{S} = \{\bar{1}0, 01, 1\bar{1}\}$, with step polynomial

$$S(x, y) = \frac{1}{x} + y + \frac{x}{y} = \bar{x} + y + x\bar{y}$$

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$$S(x, y) = \frac{1}{x} + y + \frac{x}{y} = \bar{x} + y + x\bar{y}$$

Observation: $S(x, y)$ is left unchanged by the rational transformations

$$\Phi : (x, y) \mapsto (\bar{x}y, y) \quad \text{and} \quad \Psi : (x, y) \mapsto (x, x\bar{y})\,.$$

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They are involutions, and generate a finite dihedral group G:

$$\begin{tikzcd}
 & (\bar{x}y, y) \arrow[rr, dash, "\Psi"] & & (\bar{x}y, \bar{x}) \arrow[dr, dash, "\Phi"] & \\
(x,y) \arrow[ur, dash, "\Phi"] \arrow[dr, dash, "\Psi"'] & & & & (\bar{y}, \bar{x}) \\
 & (x, x\bar{y}) \arrow[rr, dash, "\Phi"'] & & (\bar{y}, x\bar{y}) \arrow[ur, dash, "\Psi"'] &
\end{tikzcd}$$

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\end{tikzcd}$$

The group G can be defined for any model with small steps

The group is not always finite

- If $\mathcal{S} = \{0\bar{1}, \bar{1}\bar{1}, \bar{1}0, 11\}$, then $S(x, y) = \bar{x}(1+\bar{y}) + \bar{y} + xy$ and

$$\Phi : (x, y) \mapsto (\bar{x}\bar{y}(1+\bar{y}), y) \quad \text{and} \quad \Psi : (x, y) \mapsto (x, \bar{x}\bar{y}(1+\bar{x}))$$

generate an infinite group:

$$\begin{tikzcd}[column sep=small]
 & (\bar{x}\bar{y}(1+\bar{y}), y) \arrow[r, dash, "\Psi"] & \cdots \arrow[r, dash, "\Phi"] & \cdots \arrow[r, dash, "\Psi"] & \cdots \\
(x, y) \arrow[ur, dash, "\Phi"] \arrow[dr, dash, "\Psi"'] & & & & \\
 & (x, \bar{x}\bar{y}(1+\bar{x})) \arrow[r, dash, "\Phi"'] & \cdots \arrow[r, dash, "\Psi"'] & \cdots \arrow[r, dash, "\Phi"'] & \cdots
\end{tikzcd}$$

The orbit sum

- The quadrant equation reads (with $K(x,y)=1-t(y+\bar{x}+x\bar{y})$):

$$K(x,y)xyQ(x,y)=xy-tx^2Q(x,0)-tyQ(0,y)$$

- The orbit of (x,y) under G is

$$(x,y)\overset{\Phi}{\longleftrightarrow}(\bar{x}y,y)\overset{\Psi}{\longleftrightarrow}(\bar{x}y,\bar{x})\overset{\Phi}{\longleftrightarrow}(\bar{y},\bar{x})\overset{\Psi}{\longleftrightarrow}(\bar{y},x\bar{y})\overset{\Phi}{\longleftrightarrow}(x,x\bar{y})\overset{\Psi}{\longleftrightarrow}(x,y).$$

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- All transformations of G leave $K(x,y)$ invariant. Hence

$$\begin{array}{rcccccc}
K(x,y)\ xyQ(x,y) & = & xy & - & tx^2Q(x,0) & - & tyQ(0,y) \\
K(x,y)\ \bar{x}y^2Q(\bar{x}y,y) & = & \bar{x}y^2 & - & t\bar{x}^2y^2Q(\bar{x}y,0) & - & tyQ(0,y) \\
K(x,y)\ \bar{x}^2yQ(\bar{x}y,\bar{x}) & = & \bar{x}^2y & - & t\bar{x}^2y^2Q(\bar{x}y,0) & - & t\bar{x}Q(0,\bar{x}) \\
\cdots & = & & & \cdots & & \\
K(x,y)\ x^2\bar{y}Q(x,x\bar{y}) & = & x^2\bar{y} & - & tx^2Q(x,0) & - & tx\bar{y}Q(0,x\bar{y}).
\end{array}$$

The orbit sum

$\Rightarrow$ Form the alternating sum of the equation over all elements of the orbit:

$$K(x,y)\Big(xyQ(x,y)-\bar{x}y^2Q(\bar{x}y,y)+\bar{x}^2yQ(\bar{x}y,\bar{x})$$

$$-\bar{x}\bar{y}Q(\bar{y},\bar{x})+x\bar{y}^2Q(\bar{y},x\bar{y})-x^2\bar{y}Q(x,x\bar{y})\Big)=$$

$$xy-\bar{x}y^2+\bar{x}^2y-\bar{x}\bar{y}+x\bar{y}^2-x^2\bar{y}$$

the orbit sum.

The orbit sum 

$\Rightarrow$ Form the alternating sum of the equation over all elements of the orbit:

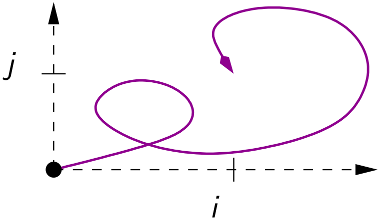
$$K(x,y)\Big(xyQ(x,y) - \bar{x}y^2Q(\bar{x}y,y) + \bar{x}^2yQ(\bar{x}y,\bar{x}) - \bar{x}\bar{y}Q(\bar{y},\bar{x}) + x\bar{y}^2Q(\bar{y},x\bar{y}) - x^2\bar{y}Q(x,x\bar{y})\Big) = xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}$$

the orbit sum.

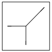
Similarly, for walks avoiding a quadrant:

$$K(x,y)\Big(xyC(x,y) - \bar{x}y^2C(\bar{x}y,y) + \bar{x}^2yC(\bar{x}y,\bar{x}) - \bar{x}\bar{y}C(\bar{y},\bar{x}) + x\bar{y}^2C(\bar{y},x\bar{y}) - x^2\bar{y}C(x,x\bar{y})\Big) = xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}$$

III. Classification of walks in the first quadrant

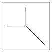


Some examples



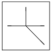
Algebraic [Kreweras 65, Gessel 86]

$$(1 - t(\bar{x} + \bar{y} + xy))xyQ(x,y) = xy - tyQ(0,y) - txQ(x,0)$$



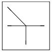
D-finite, but transcendental [Gessel 90]

$$(1 - t(y + \bar{x} + x\bar{y}))xyQ(x,y) = xy - tyQ(0,y) - tx^2Q(x,0)$$



D-algebraic, but not D-finite [Bernardi, mbm & Raschel 17]

$$(1 - t(x + \bar{x} + y + x\bar{y}))xyQ(x,y) = xy - tyQ(0,y) - tx^2Q(x,0)$$



Not D-algebraic [Dreyfus, Hardouin, Roques & Singer 17]

$$(1 - t(x\bar{y} + \bar{x} + \bar{y} + y))xyQ(x,y) = xy - tyQ(0,y) - tx(1+x)Q(x,0)$$

Classification of quadrant walks

- quadrant models: 79
  - $|G|<\infty$: 23
    - **D-finite**
      - OS$=$0: 4
        - **algebraic**
      - OS$\neq$0: 19
        - **DF transc.**
  - $|G|=\infty$: 56
    - **Not D-finite**
      - decoupled: 9
        - **D-alg.**
      - not decoupled: 47
        - **not D-alg.**

The finite group case (23 models)

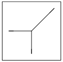
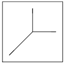
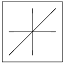
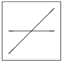
- For the 19 models for which the orbit sum is non-zero:

$$xyQ(x,y;t) = [x^{>0}y^{>0}]\frac{OS}{K(x,y;t)}$$

is a D-finite series. [mbm-Mishna 10]

$\triangleleft \quad \lhd \quad \diamond \quad \rhd \quad \triangleright$

- 4 models have a vanishing orbit sum:

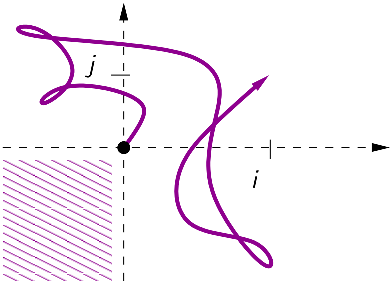
They all have an algebraic generating function $Q(x,y;t)$

[Gessel 86, Mishna 08, mbm-Mishna 10, Bostan & Kauers 10], and more!

IV. Three quadrants

A solution for some models

- relating $C(x,y)$ to the quadrant series $Q(x,y)$
- ...
- do three-quadrant equations with orbit sum zero have algebraic solutions?



A conjectural classification

Walks with small steps avoiding the negative quadrant:

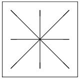
- models: 74
  - $|G|<\infty$: 23
    - OS$\neq$0: 19
      - **D-finite?**
    - OS=0: 4
      - **algebraic?**
  - $|G|=\infty$: 51
    - **Not D-finite**

- Non-D-finiteness is proved by an asymptotic argument [Mustapha 19]

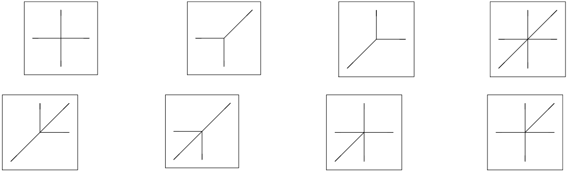
Some references

- Explicit D-finite expression for 2 step sets [mbm 16], [Budd 17(a)]

- Explicit D-finite expression for the king's walk [mbm & Wallner 19+]



- Integral expressions for some symmetric models [Raschel-Trotignon 18(a)]



D-finite

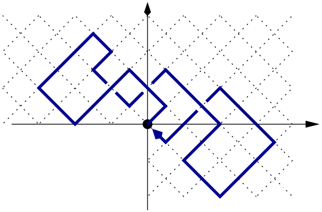
Not D-finite

Explicit expressions for numbers

Example. The number of walks of length $2n$ on the diagonal square lattice starting and ending at $(0,0)$ and avoiding the negative quadrant is

$$c(0,0;2n) = \frac{16^n}{9}\left(3\,\frac{(1/2)_n^2}{(2)_n^2} + 8\,\frac{(1/2)_n(7/6)_n}{(2)_n(4/3)_n} - 2\,\frac{(1/2)_n(5/6)_n}{(2)_n(5/3)_n}\right)$$

with $(a)_n = a(a+1)\cdots(a+n-1)$.



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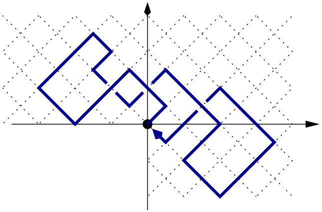

- The first term is $\frac{1}{3}q(0,0;2n)$.
- The generating functions of the other two terms are algebraic.

Explicit expressions for series

- The quadrant series $Q(x, y) \equiv Q(x, y; t)$ is the non-negative part in x and y of an explicit rational series (hence D-finite).

Explicit expressions for series



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A structured series [mbm 15(a), mbm & Wallner 19+]

The series $C(x,y;t) \equiv C(x,y)$ counting walks that avoid the negative quadrant is

$$\frac{1}{3}\left(Q(x,y) - \bar{x}^2 Q(\bar{x},y) - \bar{y}^2 Q(x,\bar{y})\right) + A(x,y)$$

where $\bar{x} = 1/x$, $\bar{y} = 1/y$ and $A(x,y)$ is algebraic. This series satisfies

$$(1 - t(x+\bar{x}+y+\bar{y}))\, A(x,y) = (2+\bar{x}^2+\bar{y}^2)/3 - t\bar{y}A_-(\bar{x}) - t\bar{x}A_-(\bar{y}),$$

where $A_-(x)$ is a series in t with coefficients in $\mathbb{Q}[x]$, algebraic of degree 24/24/72, given explicitly with its intermediate extensions.

Two points

The solution:

$$C(x,y) = \frac{1}{3}\left(Q(x,y) - \bar{x}^2 Q(\bar{x},y) - \bar{y}^2 Q(x,\bar{y})\right) + A(x,y)$$

where $A(x,y)$ is algebraic.

- Where the series $A(x,y)$ comes from
- Polynomial equations with one catalytic variable have algebraic solutions

Where the series $A(x, y)$ comes from

- The group: $(x, y), (\bar{x}, y), (\bar{x}, \bar{y}), (x, \bar{y})$

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$$\big(1 - t(x + \bar{x} + y + \bar{y})\big)\,(xyC(x, y) - \bar{x}yC(\bar{x}, y) + \bar{x}\bar{y}C(\bar{x}, \bar{y}) - x\bar{y}C(x, \bar{y}))$$
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$xyC(x, y), \quad xyQ(x, y),$

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- Define $A(x,y)$ by

$$xyC(x,y) := xyA(x,y) + \frac{1}{3}\left(xyQ(x,y) - \bar{x}yQ(\bar{x},y) - x\bar{y}Q(x,\bar{y})\right).$$

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- Then $A(x,y)$ has orbit sum zero and satisfies

$$\boxed{(1-t(x+\bar{x}+y+\bar{y}))xyA(x,y)=\frac{2xy+\bar{x}y+x\bar{y}}{3}-tyA_-(\bar{y})-txA_-(\bar{x})}$$

Three-quadrant equations with vanishing orbit sum: algebraicity?

- The series $A(x,y)$ has orbit sum zero and satisfies

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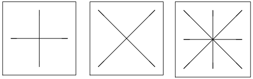
Thm. The series $A(x,y)$ is algebraic for the three models.



[mbm 16, mbm & Wallner 19+]

Three-quadrant equations with vanishing orbit sum: algebraicity?

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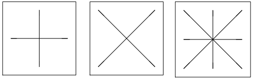


[mbm 16, mbm & Wallner 19+]

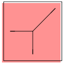
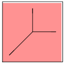
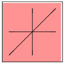
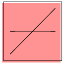
• Another indication: for three-quadrant walks on the square lattice starting at $(-1,0)$, with equation

$$(1-t(x+\bar x+y+\bar y))xyC(x,y)=y-tyC_{0,-}(\bar y)-txC_{-,0}(x).$$

The orbit sum is $y-y+\bar y-\bar y=0$ and $C(x,y)$ is algebraic [mbm 16].

A general structure? Some conjectures

- The 4 models with orbit sum zero have an algebraic generating function:

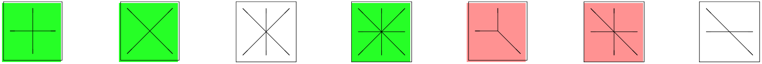
   

- The 4 models with orbit sum zero have an algebraic generating function:

- For any "reflection" model,

$$xyC(x,y) = xyA(x,y) + \frac{1}{|G|-1} \sum_{g \in G, g \neq g_{\max}} \mathrm{sign}(g) g(xyQ(x,y))$$

where $A(x,y)$ is algebraic.



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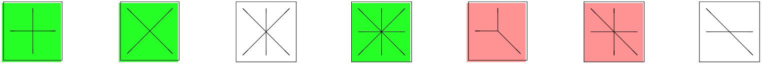
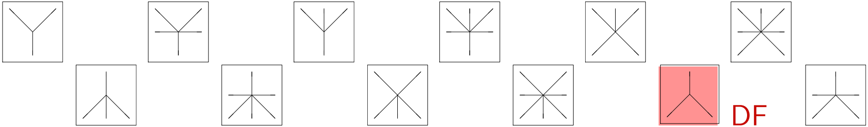


- What about the following vertically symmetric models?

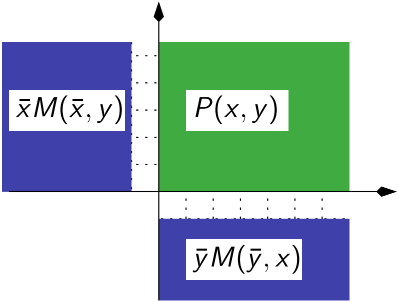


- The equation for $A(x,y)$ (with $K(x,y) = 1 - t(x + \bar{x} + y + \bar{y})$):

$$K(x,y)A(x,y) = (2 + \bar{x}^2 + \bar{y}^2)/3 - t\bar{y}A_-(\bar{x}) - t\bar{x}A_-(\bar{y})$$

- Split $A(x,y)$ into three parts:

$$A(x,y) = P(x,y) + \bar{x}M(\bar{x},y) + \bar{y}M(\bar{y},x)$$



$\Rightarrow$ Equations for $P(x,y)$ and $M(x,y)$

The equation for $M(x, y)$ is (almost) a quadrant equation

- $M(x, y)$ is a series in t with coefficients in $\mathbb{Q}[x, y]$, satisfying

$$(1 - t(x + \bar{x} + y + \bar{y}))\,(2M(x, y) - M(0, y)) =$$
$$2x/3 - 2t\bar{y}M(x, 0) + t(x - \bar{x})M(0, y) + t\bar{y}M(y, 0).$$

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- Proof that $M(x,y)$ is algebraic:
  - [...]
  - equation for $M(0,x)$: let $S(x)=txM(0,x)$, then

$$((1-t(x+\bar{x}))^2-4t^2)\left(S(x)^3+\frac{2x+1}{x+1}S(x)^2+\frac{S(x)}{\bar{x}+1}\right) = \left(t^2(x+\bar{x})-A_2\right)(S(x)+1)+t^2A_1-\frac{2t^2A_1-A_2}{x+1}-t^2\bar{x},$$

  where A_1 and A_2 are series in t only.

A polynomial equation with one catalytic variable (x) only.

Polynomial equations with one catalytic variable

Theorem [mbm-Jehanne 06]

Let $P(t, x, S(x;t), A_1(t), \ldots, A_k(t))$ be a proper polynomial equation in one catalytic variable y (it defines uniquely $S(x;t), A_1(t), \ldots, A_k(t)$ as formal power series in t/x). Then each of these series is algebraic.

The proof is constructive.

Example:

$$((1-t(x+\bar{x}))^2-4t^2)\left(S(x)^3+\frac{2x+1}{x+1}S(x)^2+\frac{S(x)}{\bar{x}+1}\right)=$$

$$\left(t^2(x+\bar{x})-A_2\right)\left(S(x)+1\right)+t^2A_1-\frac{2t^2A_1-A_2}{x+1}-t^2\bar{x}.$$

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$$\left(t^2(x+\bar{x}) - A_2\right)(S(x)+1) + t^2 A_1 - \frac{2t^2 A_1 - A_2}{x+1} - t^2\bar{x}.$$

◁ ◁ ◇ ▷ ▷

Algebraicity follows from (a special case of) an Artin approximation theorem with "nested" conditions [Popescu 86, Swan 98]

The king's walk: technical difficulties

Computer algebra to the rescue!

- The final equation with one catalytic variable

$$P(t, x, S(x; t), A_1(t), \ldots, A_k(t)) = 0$$

  involve $k = 4$ series A_i (instead of $k = 2$ for the square lattice)
- they are given by a big system of 4 polynomial equations, solved (partly) by guess & check
- each A_i ends up having degree 24 (instead of 8)
- their common genus is 2 instead of 0 ($\Rightarrow$ normal hyperelliptic forms)

[mbm & Wallner 19+]

An example

- The generating function of walks ending at $(-1,0)$ is algebraic of degree 24:

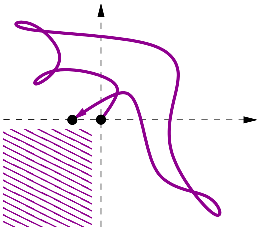
$$\sum_n c(-1,0;n)t^n = \frac{1}{2t}\left(\frac{1+2V}{2V^3-4V-1}W-1\right)$$

where

$$U = t(1+18U^2-27U^4)+t^2(1+U)(1-3U)^3,$$

$$V = U\,\frac{1+3V-V^3}{1+V+V^2},$$

$$W^2 = 1+4V-4V^3-4V^4.$$



Further questions

- Complete the classification...
- First for models such that an (algebraic?) series $A(x, y)$ can be defined by

$$xyC(x,y) = xyA(x,y) + \frac{1}{|G|-1} \sum_{g \in G, g \neq g_{\max}} \mathrm{sign}(g) g(xyQ(x,y))$$

- What about the other models with a finite group?
- Are there D-algebraic (but not D-finite) models?

models: 74

$|G|<\infty$: 23 $\qquad$ $|G|=\infty$: 51

OS$\neq$0: 19 $\qquad$ OS$=$0: 4 $\qquad$ **Not D-finite**

D-finite? $\qquad$ **algebraic?**

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- Are there D-algebraic (but not D-finite) models?

models: 74

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OS$\neq$0: 19 — OS$=$0: 4 — **Not D-finite**

D-finite? — **algebraic?**

Further questions

- Complete the classification...
- First for models such that an (algebraic?) series $A(x, y)$ can be defined by

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