On the shape of binary trees

Mireille Bousquet-Mélou, CNRS, LaBRI, Bordeaux

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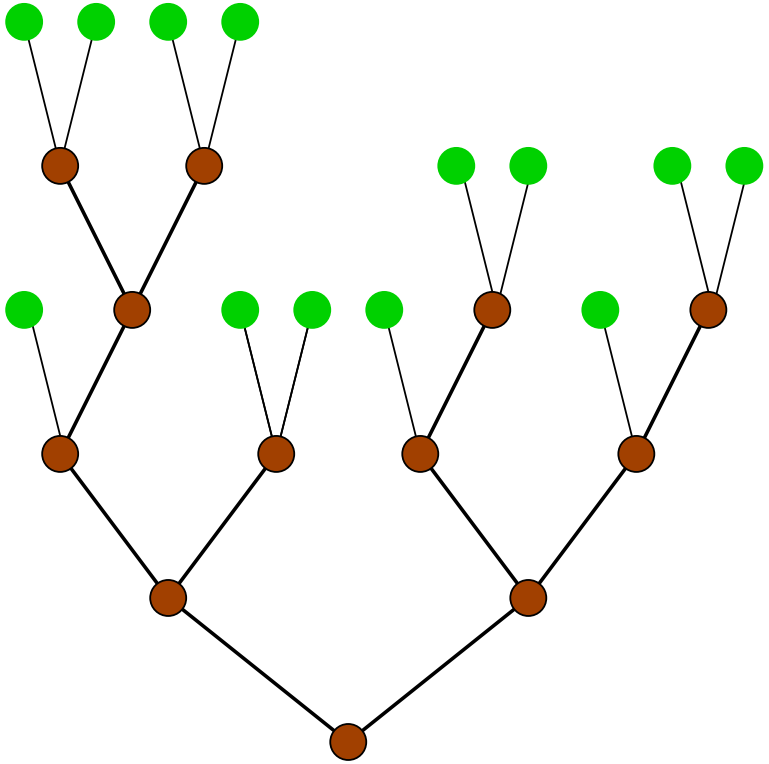
http://www.labri.fr/∼bousquet

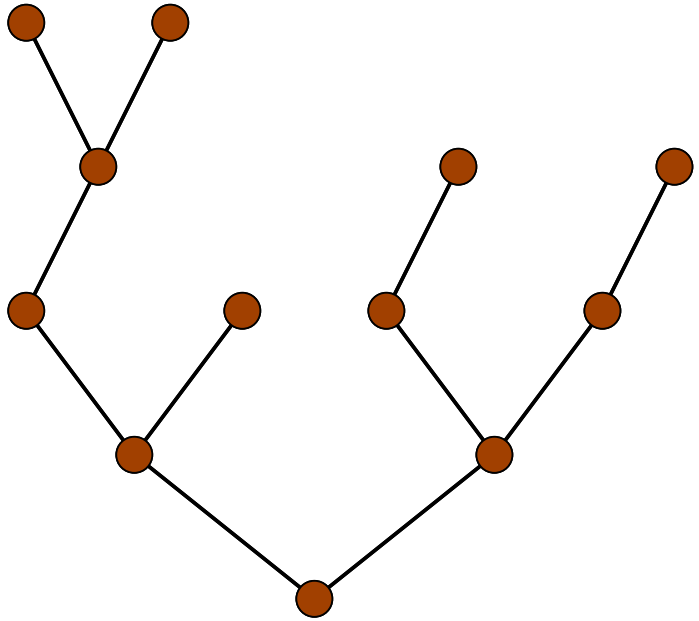
A complete binary tree



n internal vertices, called nodes (size n)

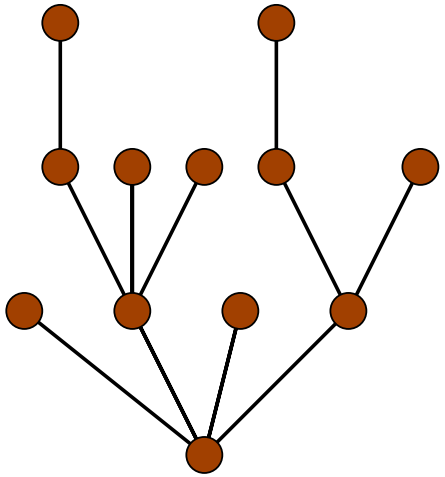
$n + 1$ external vertices, called leaves

An (incomplete) binary tree



n nodes (size n)

A plane tree

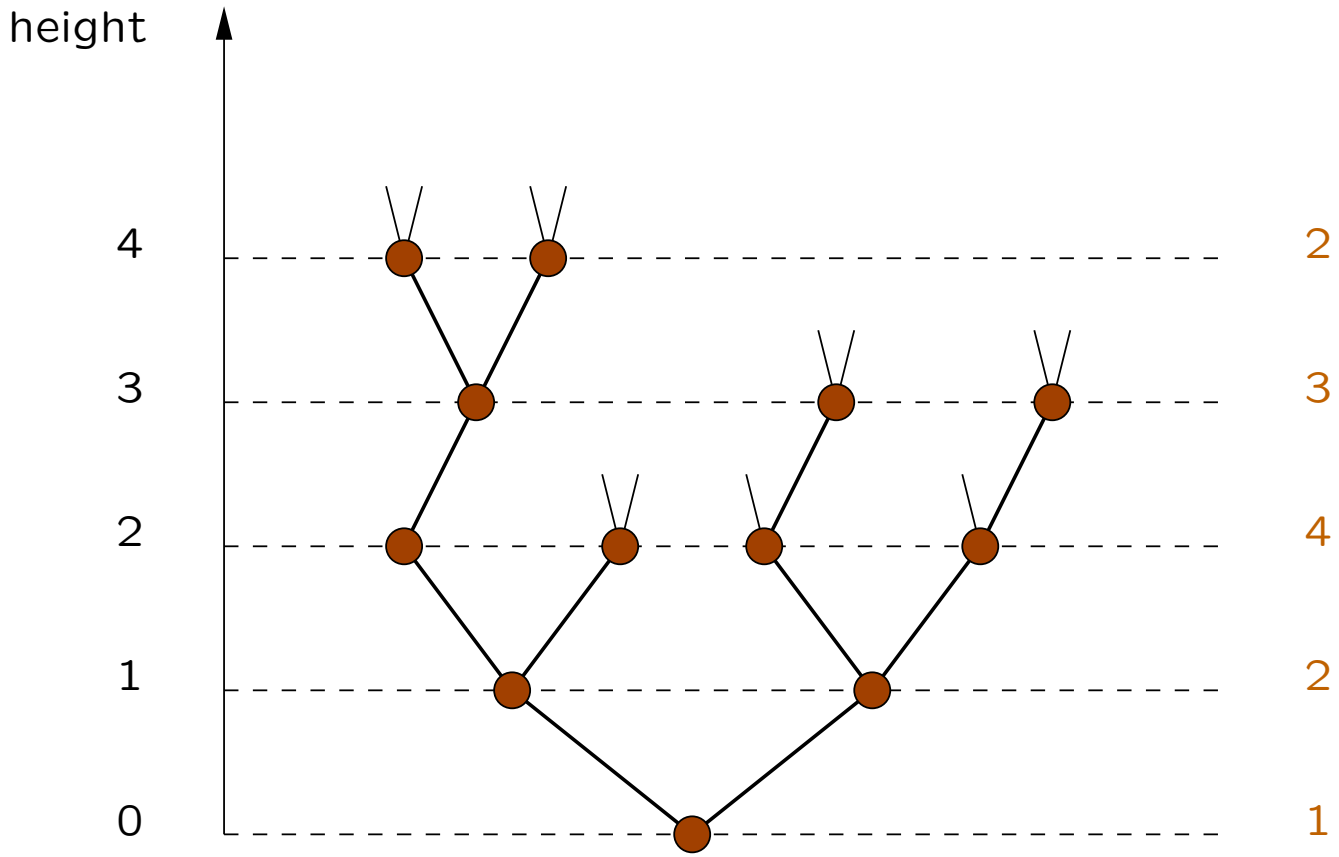


Each node has a (possibly empty) ordered sequence of children

n edges, $n+1$ nodes (size n)

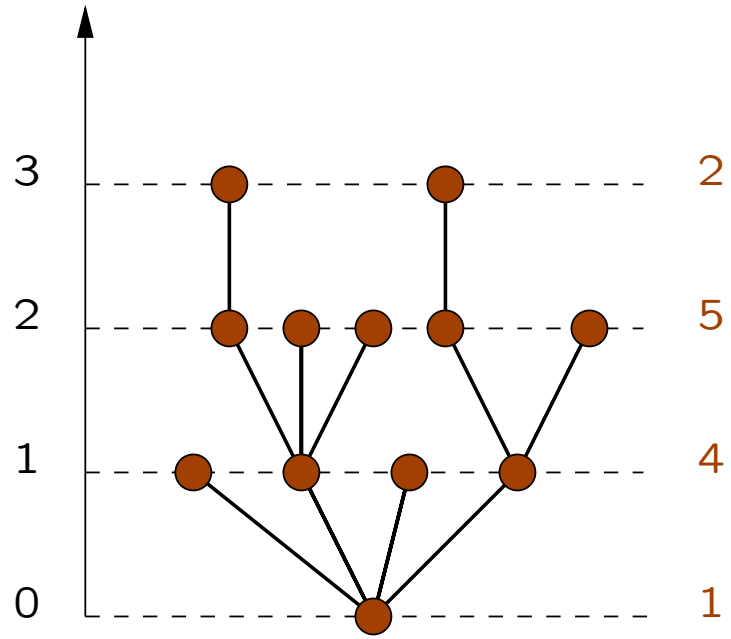
What is the shape of a (large) *random* binary tree?

The (horizontal) profile of a binary tree



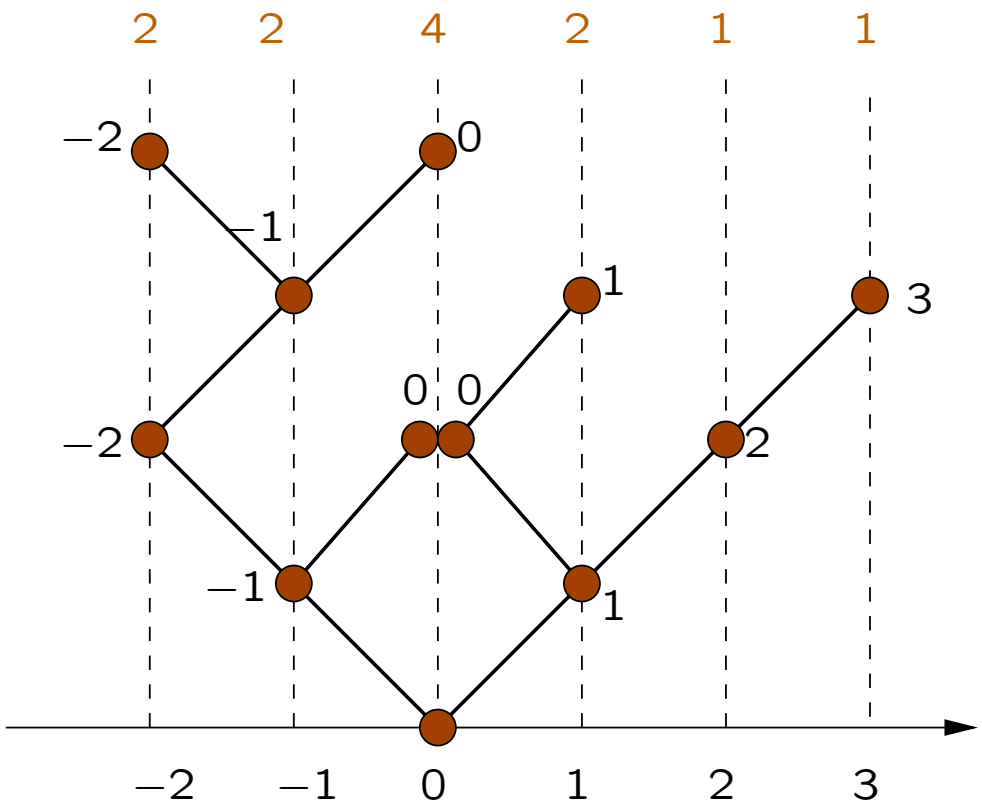
A binary tree of height 4 (or 5...)

The horizontal profile of a plane tree



A plane tree of height 3.

NEW! The *vertical* profile of a binary tree



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A binary tree of right width 3, and vertical profile $[2, 2; 4, 2, 1, 1]$.

Why study the vertical profile of trees?

Why study the vertical profile of trees?

Wait and see!

Limit results on the shape of trees: general approach

• **Enumerative combinatorics** (decompositions of trees, recurrence relations, generating functions...)

• **Singularity analysis** [Flajolet-Odlyzko 90]: extract from the generating functions the asymptotic behaviour of their coefficients

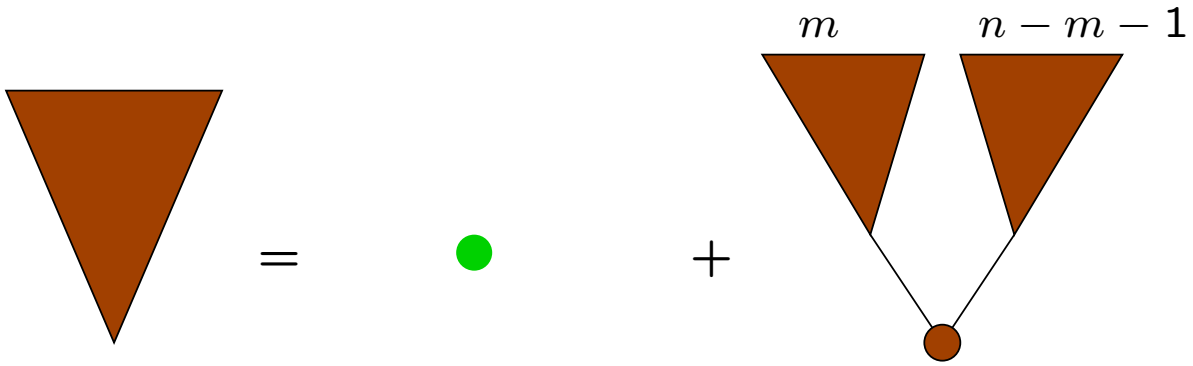
Decomposition and enumeration of binary trees

Let a_n be the number of binary trees with n nodes (internal vertices):

$$\begin{cases} a_0 &= 1 \\ a_n &= \displaystyle\sum_{m=0}^{n-1} a_m\, a_{n-m-1} \end{cases}$$

Let $A(t) := \displaystyle\sum_{n\geqslant 0} a_n t^n$ be the associated generating function:

$$A(t) = 1 + tA(t)^2.$$



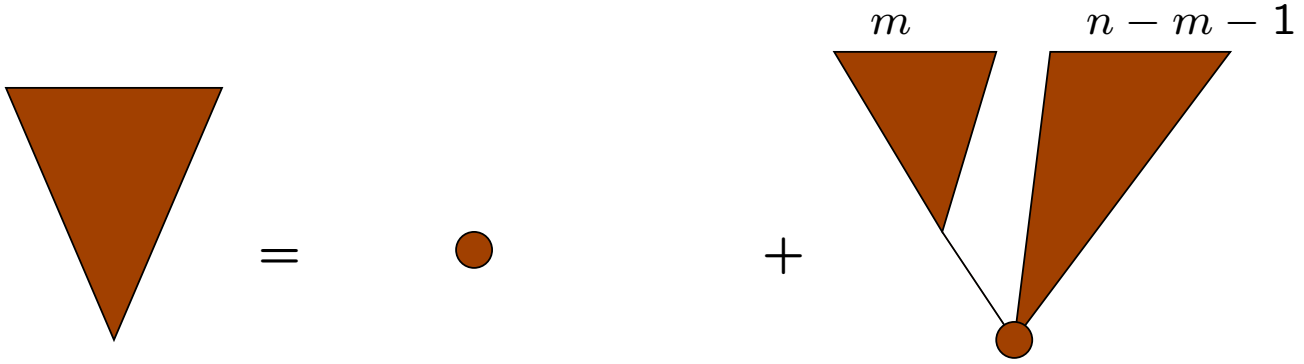
Decomposition and enumeration of plane trees

Let a_n^* be the number of plane trees with n edges ($n+1$ vertices):

$$\begin{cases} a_0^* &= 1 \\ a_n^* &= \displaystyle\sum_{m=0}^{n-1} a_m^* \, a_{n-m-1}^* \end{cases}$$

Let $A^*(t) := \displaystyle\sum_{n \geqslant 0} a_n^* t^n$ be the associated generating function:

$$A^*(t) = 1 + tA^*(t)^2.$$



The Catalan numbers

There are as many binary trees with n nodes as plane trees with n edges:

$$a_n = a_n^*$$

The associated generating function is

$$A(t) = A^*(t) = \sum_n a_n t^n$$

It satisfies

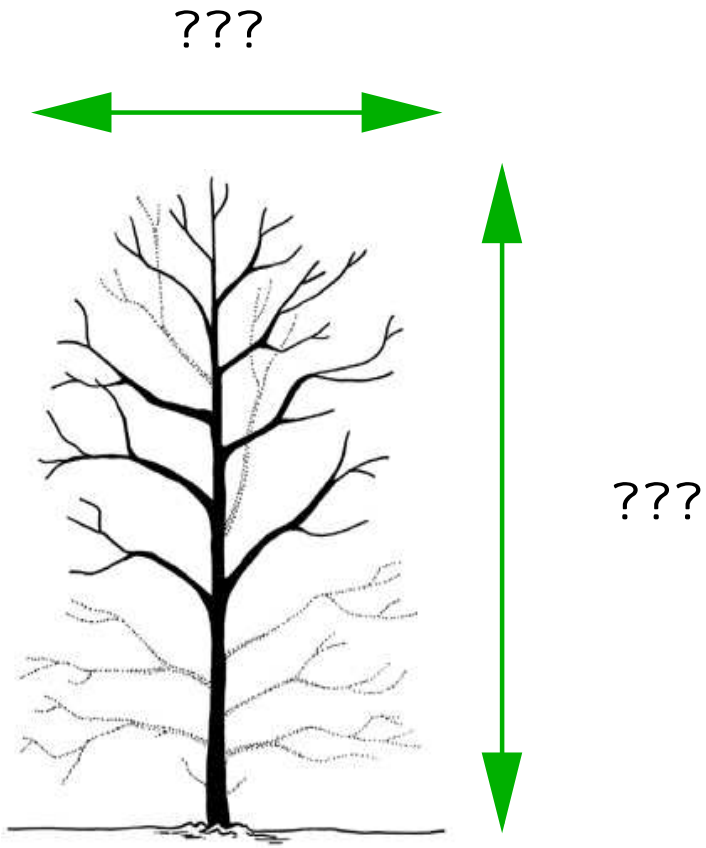
$$A(t) = 1 + tA(t)^2 \Rightarrow A(t) = \frac{1 - \sqrt{1-4t}}{2t} = \sum_{n \geqslant 0} \frac{1}{n+1} \binom{2n}{n} t^n.$$

Hence

$$a_n = a_n^* = \frac{1}{n+1} \binom{2n}{n}$$

is the nth Catalan number.

Part 1. Height and width of binary trees



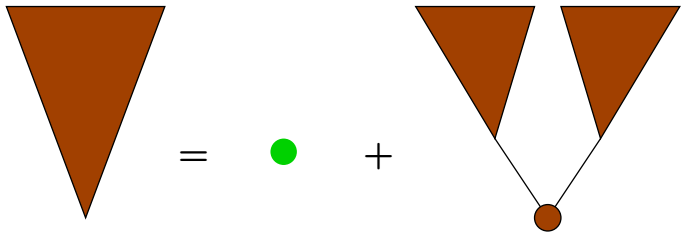
Counting trees of bounded height

Let $H_{\leqslant j}(t) \equiv H_{\leqslant j}$ be the generating function of binary trees of height at most j:

$$H_{\leqslant j} = \sum_{n \geqslant 0} h_{n,\leqslant j} t^n$$

$$H_{\leqslant 0} = 1$$

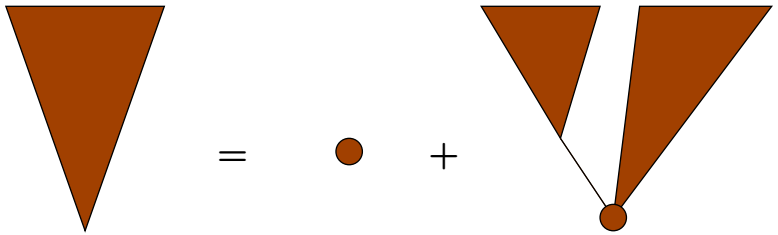
$$H_{\leqslant j} = 1 + tH_{\leqslant j-1}^2$$



Let $H^*_{\leqslant j}(t) \equiv H^*_{\leqslant j}$ be the generating function of plane trees of height at most j, where t counts edges.

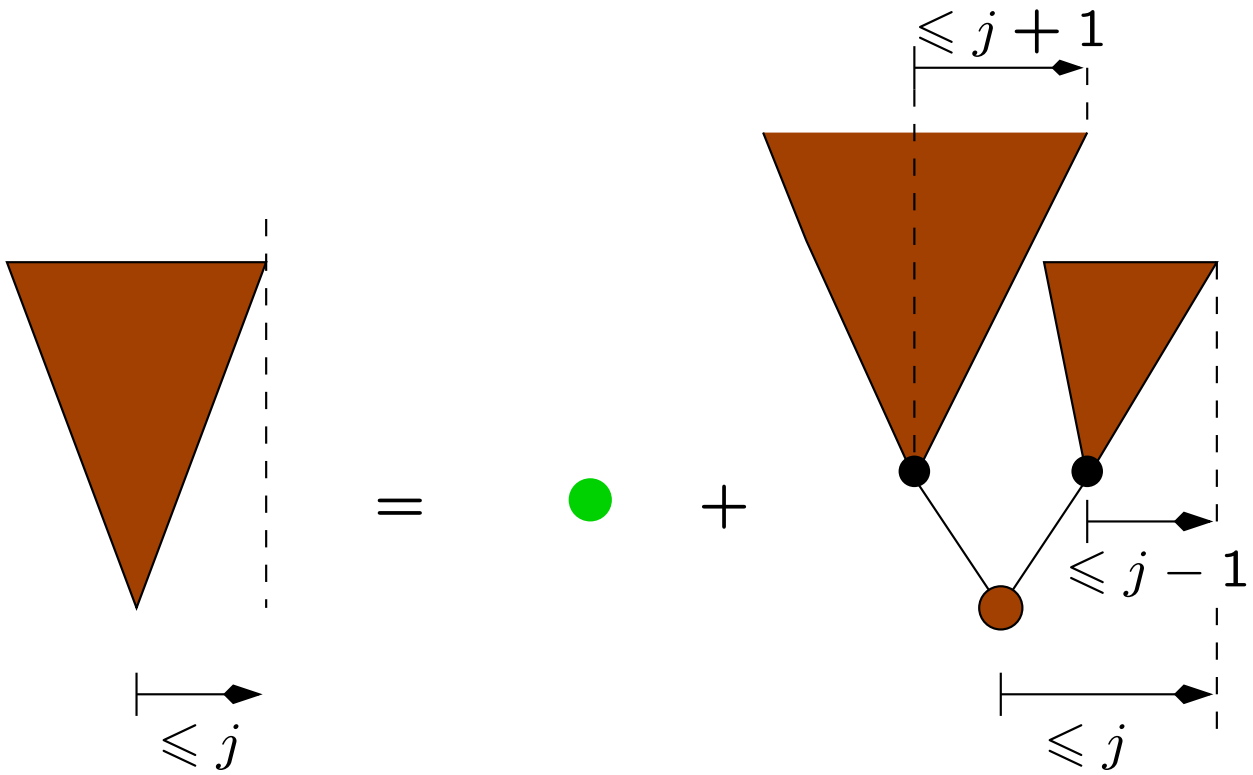
$$H^*_{\leqslant 0} = 1$$

$$H^*_{\leqslant j} = 1 + tH^*_{\leqslant j-1} H^*_{\leqslant j}$$



Counting trees of bounded (right) width

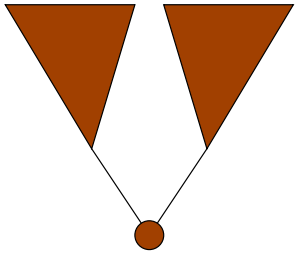
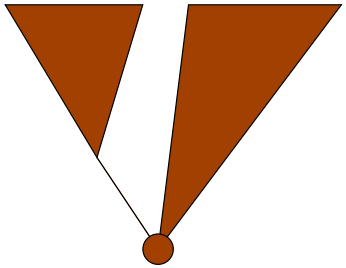
Let $W_{\leqslant j}(t) \equiv W_{\leqslant j}$ be the generating function of binary trees of right width at most j.



$$W_{\leqslant -1} = 1$$

For $j \geqslant 0$, $\quad W_{\leqslant j} = 1 + tW_{\leqslant j+1}W_{\leqslant j-1}$

Trees of bounded height or bounded width: functional equations

Family of trees	Bounded height	Bounded (right) width
	$H_{\leqslant 0} = 1$ $H_{\leqslant j} = 1 + tH^2_{\leqslant j-1}$	$W_{\leqslant -1} = 1$ $W_{\leqslant j} = 1 + tW_{\leqslant j+1}W_{\leqslant j-1}$ $(j \geqslant 0)$
	$H^*_{\leqslant 0} = 1$ $H^*_{\leqslant j} = 1 + tH^*_{\leqslant j-1}H^*_{\leqslant j}$	

Generating functions for trees of bounded height/width

Proposition: The generating function of plane trees of height $\leqslant j$ is:

$$H^*_{\leqslant j} = A\,\frac{1-Q^{j+1}}{1-Q^{j+2}}$$

where A counts plane trees and $Q = A - 1$ [de Bruijn, Knuth, Rice 72].

Generating functions for trees of bounded height/width

Proposition: The generating function of plane trees of height $\leqslant j$ is:

$$H^*_{\leqslant j} = A\,\frac{1-Q^{j+1}}{1-Q^{j+2}}$$

where A counts plane trees and $Q = A - 1$ [de Bruijn, Knuth, Rice 72].

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Proposition [mbm 06]: The generating function of binary trees of right width $\leqslant j$ is:

$$W_{\leqslant j} = A\,\frac{(1-Z^{j+2})(1-Z^{j+7})}{(1-Z^{j+4})(1-Z^{j+5})},$$

where A counts binary trees and $Z \equiv Z(t)$ is the unique series in t such that

$$Z = t\,\frac{\left(1+Z^2\right)^2}{1-Z+Z^2} \qquad \text{and} \qquad Z(0) = 0.$$

But. . . w h y ?

Limit results on the shape of trees: general approach

• Enumerative combinatorics: Using a recursive decomposition of trees, write equations for the relevant generating functions and solve them...

• Use the results (or the technique) of **singularity analysis** [Flajolet-Odlyzko 90] to extract from these series the asymptotic behaviour of their coefficients

– Results: "If $B(t) = \sum_n b_n t^n$ behaves like this in the neighborhood of its dominant singularity t_s, then its coefficients b_n behave asymptotically like that."

– Technique: Cauchy's formula

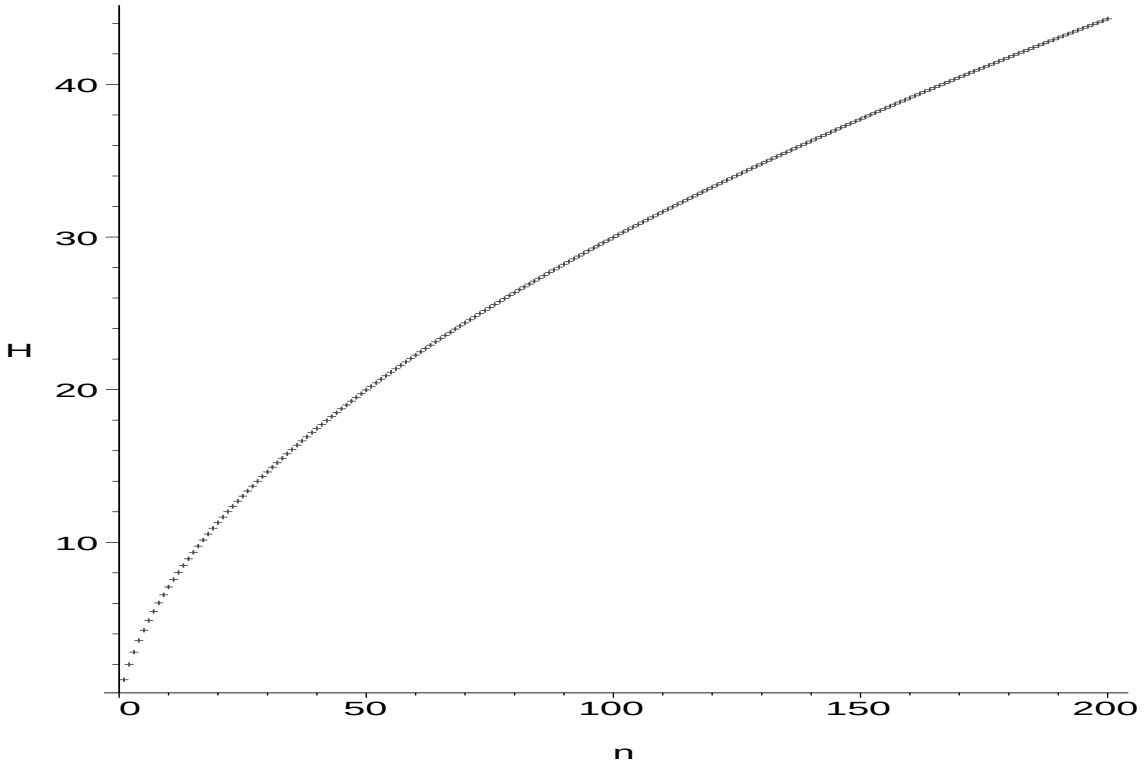
$$b_n = \frac{1}{2i\pi} \int_{\mathcal{C}_n} B(t) \frac{dt}{t^{n+1}}$$

for a carefully chosen contour $\mathcal{C}_n$.

The average height of binary trees: experimental approach

Consider the average height of plane trees of size n:

$$\mathbb{E}(\mathsf{H}_n^*) := \frac{1}{a_n} \sum_{|\tau|=n} h(\tau) = \frac{1}{a_n} \sum_{j \geqslant 0} j \ (h^*_{n,\leqslant j} - h^*_{n,\leqslant j-1}).$$



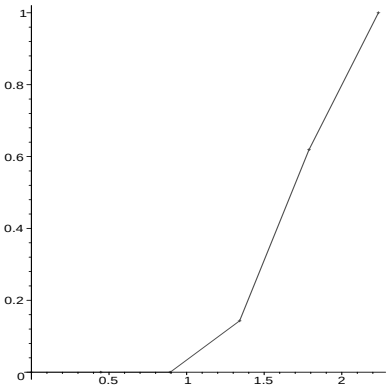
Convergence in law of the height: experimental approach

The number of trees of size n and height at most j is $h^*_{n,\leqslant j}$. For $x \geqslant 0$, consider

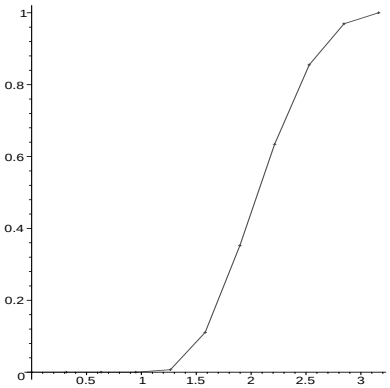
$$F_n(x) := \mathbb{P}\left(\frac{\mathsf{H}^*_n}{\sqrt{n}} \leqslant x\right) = \frac{1}{a_n} h^*_{n,\leqslant x\sqrt{n}},$$

the proportion of trees of size n having height at most $x\sqrt{n}$. It is the probability that a random tree of size n has height at most $x\sqrt{n}$.

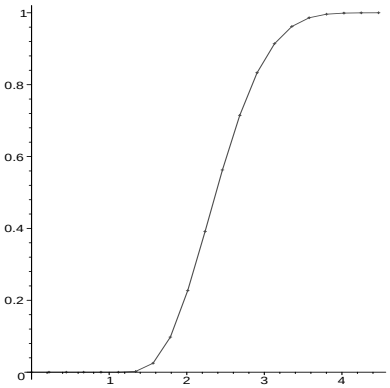
Graph of $F_n(x)$:



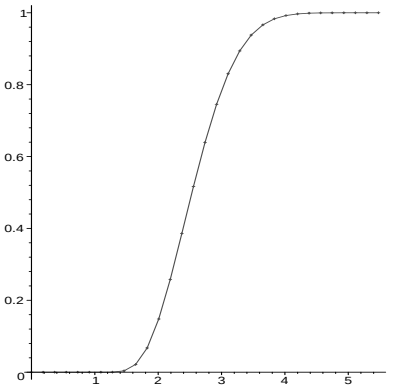
$n = 5$



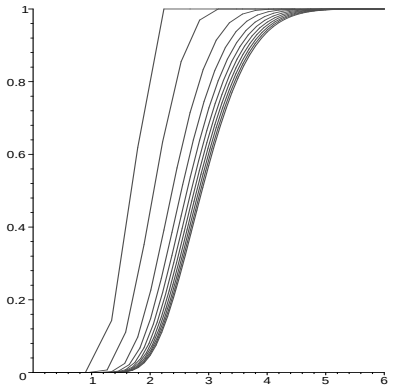
$n = 10$



$n = 20$



$n = 30$



$5 \leqslant n \leqslant 100.$

Limit results for the height of trees

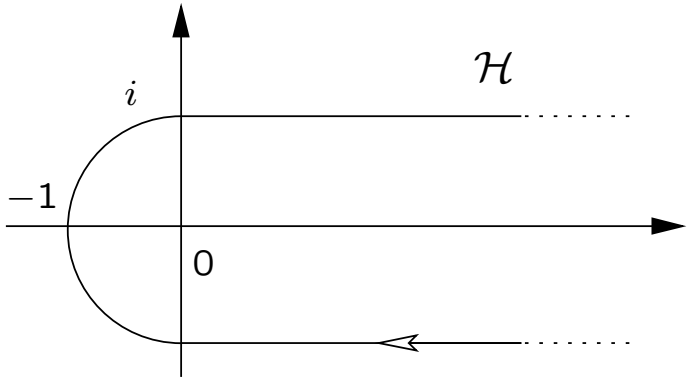
Convergence in law: For all $x \geqslant 0$,

$$\mathbb{P}\left(\frac{\mathsf{H}_n^*}{\sqrt{n}} \leqslant x\right) \to F(x),$$

where

$$F(x) = -\frac{1}{i\sqrt{\pi}} \int_{\mathcal{H}} \coth(x\sqrt{-z})\sqrt{-z}e^{-z}dz$$

$$= \sum_{k=-\infty}^{+\infty} e^{-k^2x^2}(1-2k^2x^2)$$



Convergence of the moments: Let H_n^* be the (random) height of a plane tree with n edges. As $n \to \infty$,

$$\frac{\mathbb{E}(\mathsf{H}_n^*)}{\sqrt{n}} \to \sqrt{\pi}, \qquad \mathbb{E}\left(\left(\frac{\mathsf{H}_n^*}{\sqrt{n}}\right)^k\right) \to k(k-1)\Gamma(k/2)\zeta(k) \qquad \text{for } k \geqslant 2.$$

[Flajolet-Odlyzko 82], [Brown-Schubert 84]

$+$ similar results for binary trees

Limit results for the right width of binary trees [mbm 06]

Convergence of the moments: Let W_n be the (random) right width of a binary tree with n nodes. Then, as $n \to \infty$,

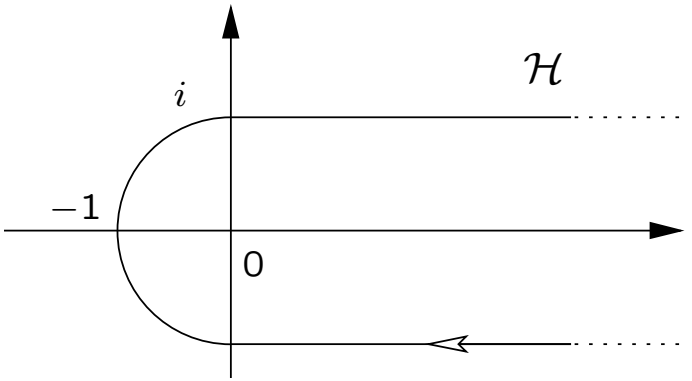
$$\frac{\mathbb{E}(W_n)}{n^{1/4}} \to \frac{3\sqrt{\pi}}{\sqrt{2}\Gamma(3/4)}, \qquad \mathbb{E}\left(\left(\frac{W_n}{n^{1/4}}\right)^2\right) \to 6\sqrt{\pi},$$

$$\mathbb{E}\left(\left(\frac{W_n}{n^{1/4}}\right)^k\right) = \frac{24\sqrt{\pi}k!\zeta(k-1)}{\sqrt{2}^k\Gamma((k-2)/4)} \quad \text{for } k \geqslant 3.$$

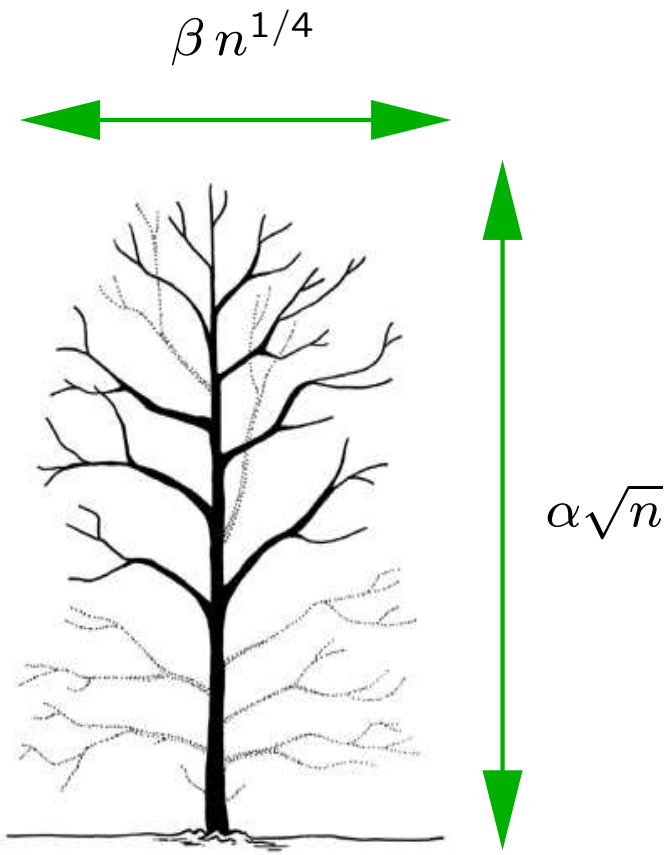
Convergence in law:

$$\mathbb{P}\left(\frac{W_n}{n^{1/4}} \geqslant x\right) \to G(x)$$

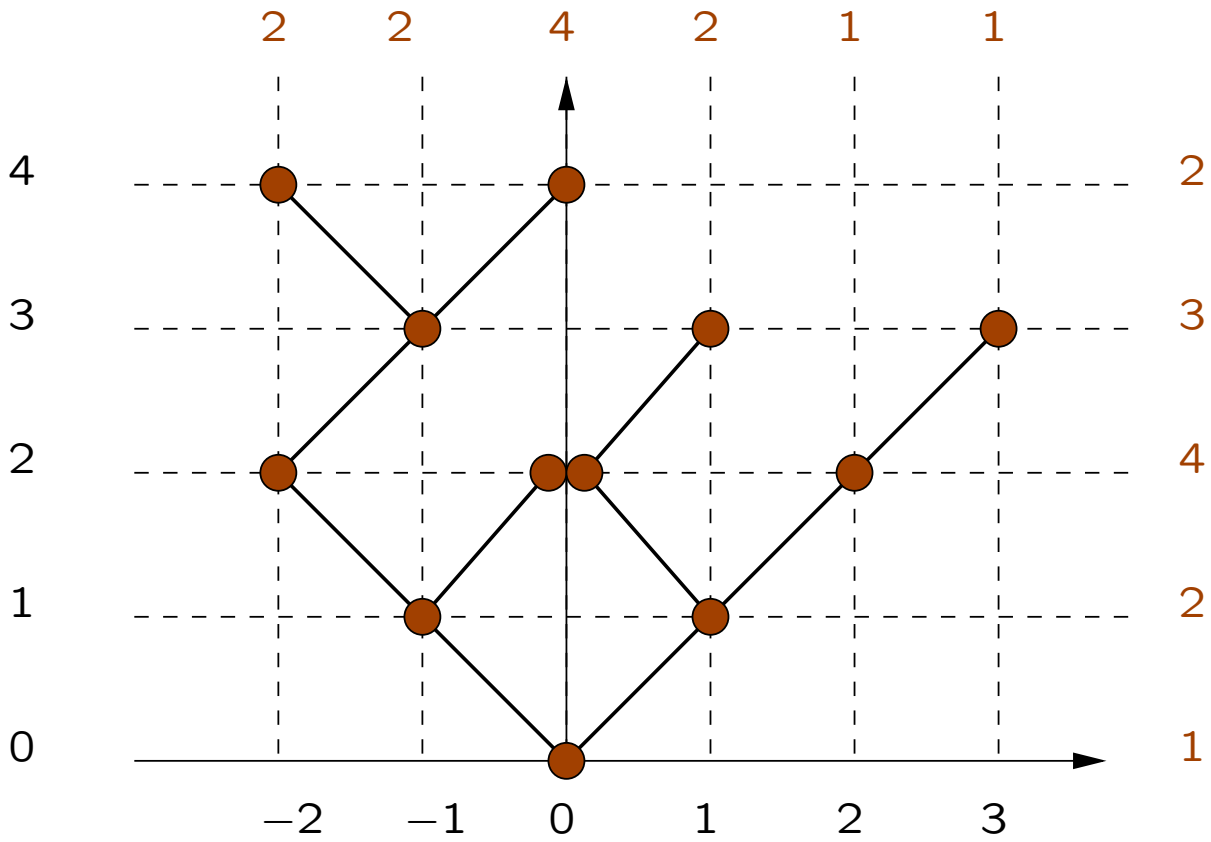
$$G(x) = \frac{3}{i\sqrt{\pi}} \int_{\mathcal{H}} \frac{\sqrt{-z}e^{-z}}{\sinh^2(x(-z^{1/4}/\sqrt{2}))} dz$$



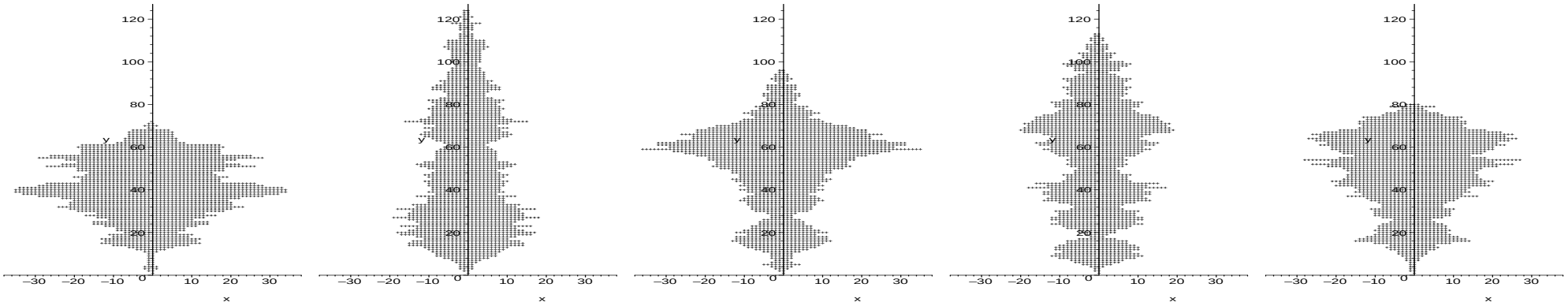
Height and width of binary trees



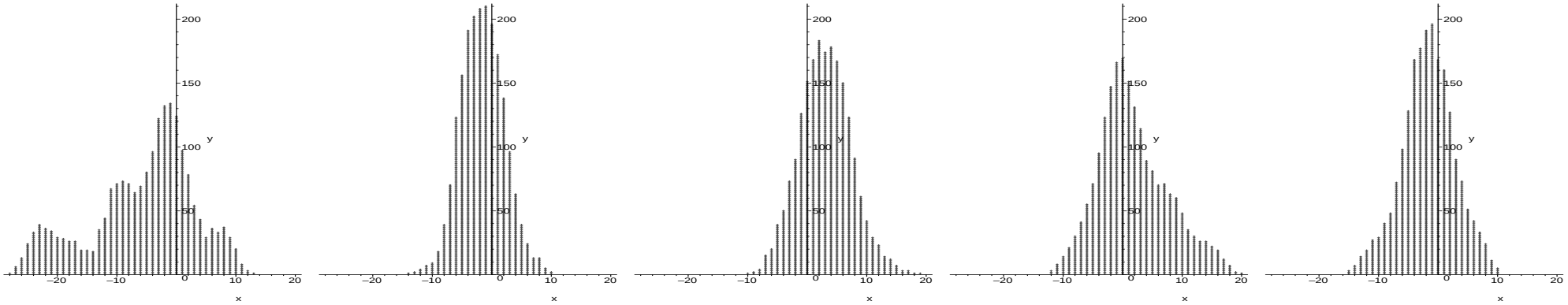
Part 2. What about the profiles?



Gallery of (horizontal and vertical) profiles



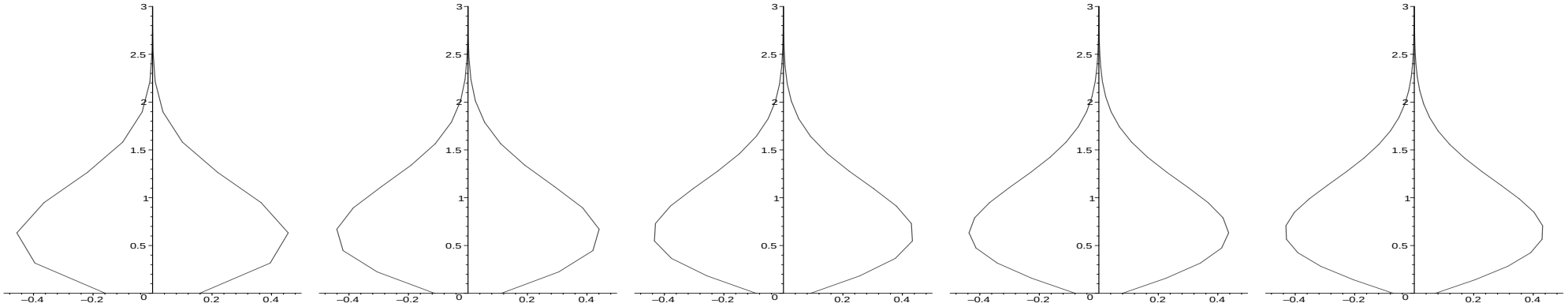
Horizontal profiles of random binary trees with 1000 nodes.



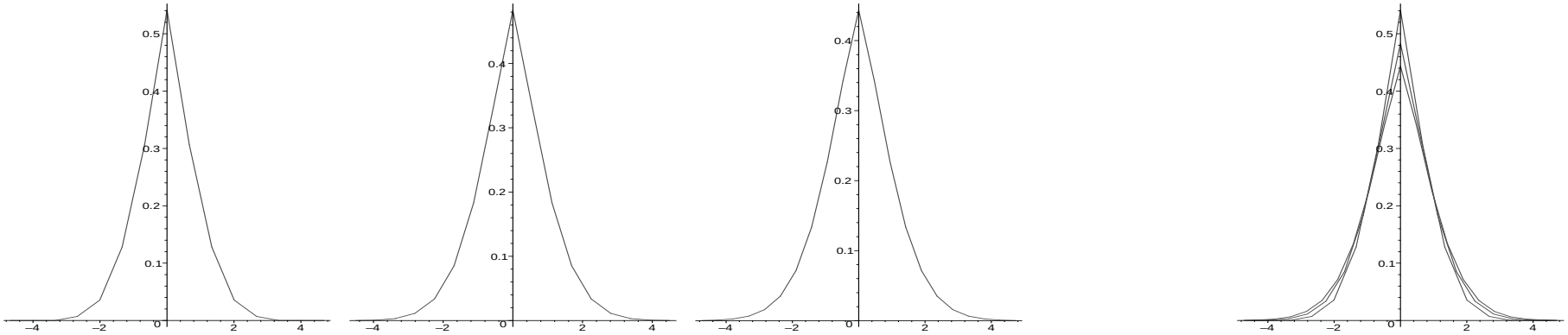
Vertical profiles of random binary trees with 1000 nodes.

The average (horizontal and vertical) profiles

Average horizontal profile of plane trees of size 10, 20, 30, 40, 50:



Average vertical profile of binary trees of size 5, 10, 20:



One new ingredient: bivariate generating functions

Before: Given j, how many plane trees of size n have height at most j?

$$\Rightarrow \text{series} \quad H^*_{\leqslant j}(t) = \sum_n h_{n,\leqslant j}\ t^n$$

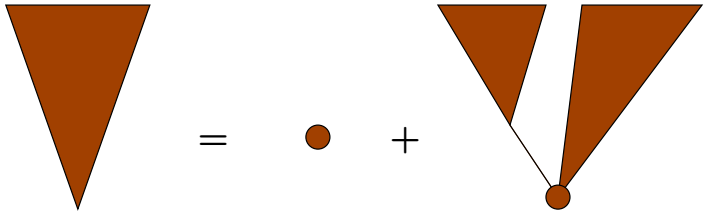
Now: Given j, how many plane trees of size n have exactly k nodes at height j?

$$\Rightarrow \text{series} \quad P^*_j(t,u) = \sum_{n,k} p_{n,k,j}\ t^n u^k$$

The number of nodes at height j (plane trees)

Let $P_j^* \equiv P_j^*(t,u)$ be the generating function of plane trees, counted by edges (variable t) and by the number of nodes at height j (variable u):

$$P_0^* = uA(t)$$
$$P_j^* = 1 + tP_{j-1}^* P_j^*$$



Proposition [???]:

$$P_j^* = A\,\frac{1-MQ^j}{1-MQ^{j+1}}$$

where A counts plane trees, $Q = A - 1$ and

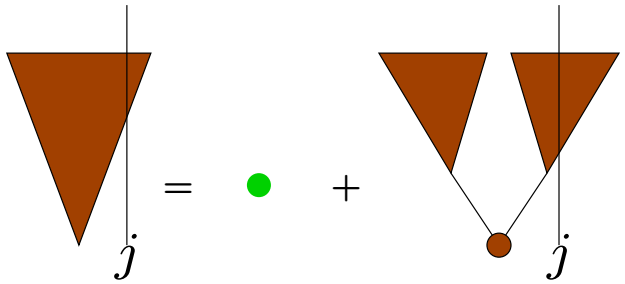
$$M = \frac{A - u - tuA^2}{u + A(1-u) + tuA^2(1-A)}.$$

The number of nodes at abscissa j (binary trees)

Let $V_j \equiv V_j(t,u)$ be the generating function of binary trees, counted by nodes (t) and by the number of nodes at abscissa j (variable u).

$$V_0 = 1 + utV_1^2$$

For $j \geqslant 1$, $\quad V_j = 1 + tV_{j+1}V_{j-1}$



Proposition [mbm 06]: the series V_j are algebraic. Moreover,

$$V_j = A\,\frac{(1+MZ^j)(1+MZ^{j+5})}{(1+MZ^{j+2})(1+MZ^{j+3})}$$

where A counts binary trees,

$$Z = t\,\frac{\left(1+Z^2\right)^2}{1-Z+Z^2},$$

and $M \equiv M(t,u)$ is the unique power series in t such that

$$M = (u-1)\frac{Z(1+MZ)^2(1+MZ^2)(1+MZ^6)}{(1+Z)^2(1+Z+Z^2)(1-Z)^3(1-M^2Z^5)}.$$

The horizontal profile

A tree of size n has, on average, height $O(\sqrt{n})$. How many nodes are there at a given (horizontal) level?

???

The horizontal profile

A tree of size n has, on average, height $O(\sqrt{n})$. How many nodes are there at a given (horizontal) level?

About $\sqrt{n}$.

The horizontal profile

A tree of size n has, on average, height $O(\sqrt{n})$. How many nodes are there at a given (horizontal) level?

About $\sqrt{n}$.

Let $\mathsf{X}_n^*(j)$ be the number of nodes located at height j in a random plane tree with n edges.

We study the quantity

$$\frac{\mathsf{X}_n^*(\lfloor \lambda\sqrt{n} \rfloor)}{\sqrt{n}}.$$

The vertical profile

A binary tree of size n has, on average, width $O(n^{1/4})$. How many nodes are there on a given (vertical) layer?

About $n^{3/4}$.

Let $\mathsf{Y}_n(j)$ be the number of nodes located at abscissa j in a random binary tree with n nodes.

We study the quantity

$$\frac{\mathsf{Y}_n(\lfloor \lambda n^{1/4} \rfloor)}{n^{3/4}}.$$

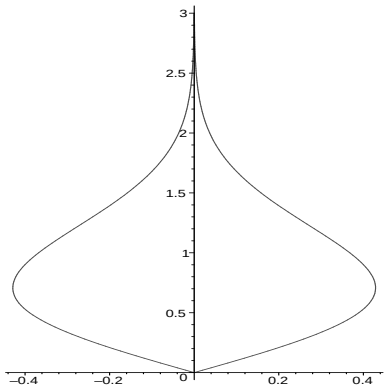
The horizontal profile of plane trees

- The sequence $\frac{X_n^*(\lfloor \lambda\sqrt{n} \rfloor)}{\sqrt{n}}$ converges in law for each λ (and as a process [Drmota-Gittenberger 97])

- The first moment:

$$\mathbb{E}\left(\frac{X_n^*(\lfloor \lambda\sqrt{n} \rfloor)}{\sqrt{n}}\right) \to 2\lambda e^{-\lambda^2}.$$

On average, there are about $2\lambda e^{-\lambda^2}\sqrt{n}$ nodes at height $\lfloor \lambda\sqrt{n} \rfloor$ in a plane tree having n edges.



In other words, the height of a random node in a random tree, once normalized by $\sqrt{n}$, follows a law of density $2\lambda e^{-\lambda^2}$.

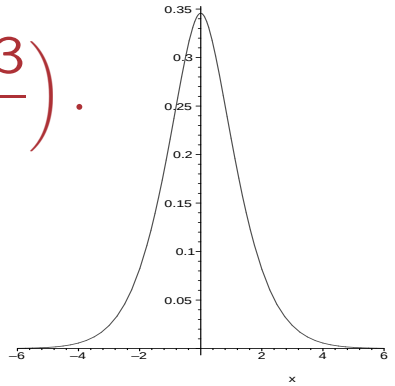
The vertical profile of binary trees

Let $Y_n(j)$ be the (random) number of nodes at abscissa j in a binary tree having n nodes.

• The sequence $\frac{Y_n(\lfloor \lambda n^{1/4} \rfloor)}{n^{3/4}}$ converges in law to a random variable $Y(\lambda)$ described explicitly by its Laplace transform for every λ (and as a process [mbm-Janson 06])

• Moreover,

$$\mathbb{E}\left(\frac{Y_n(\lfloor \lambda n^{1/4} \rfloor)}{n^{3/4}}\right) \to \frac{1}{\sqrt{2\pi}} \sum_{m \geqslant 0} \frac{(-\sqrt{2}|\lambda|)^m}{m!} \cos\frac{(m+1)\pi}{4} \Gamma\left(\frac{m+3}{4}\right).$$



This gives the average number of nodes at abscissa $\lfloor \lambda n^{1/4} \rfloor$ in a random binary tree having n nodes.

[mbm 06]

An example: nodes at abscissa 0

- The random variable

$$\frac{\mathrm{Y}_n(0)}{n^{3/4}},$$

which gives the (normalized) number of nodes at abscissa 0, converges in law to a variable Y(0) such that

$$E\left(Y(0)^k\right) = \left(\frac{\sqrt{2}}{3}\right)^k \frac{\Gamma(1+3k/4)}{\Gamma(1+k/2)}.$$

Hence (...) $Y(0) = \frac{\sqrt{2}}{\sqrt[3]{T_{2/3}}}$, where $T_{2/3}$ follows a unilateral stable law of parameter 2/3.

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Merci Alain Rouault !

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Hence (...) $Y(0) = \frac{\sqrt{2}}{3\sqrt{T_{2/3}}}$, where $T_{2/3}$ follows a unilateral stable law of parameter 2/3.

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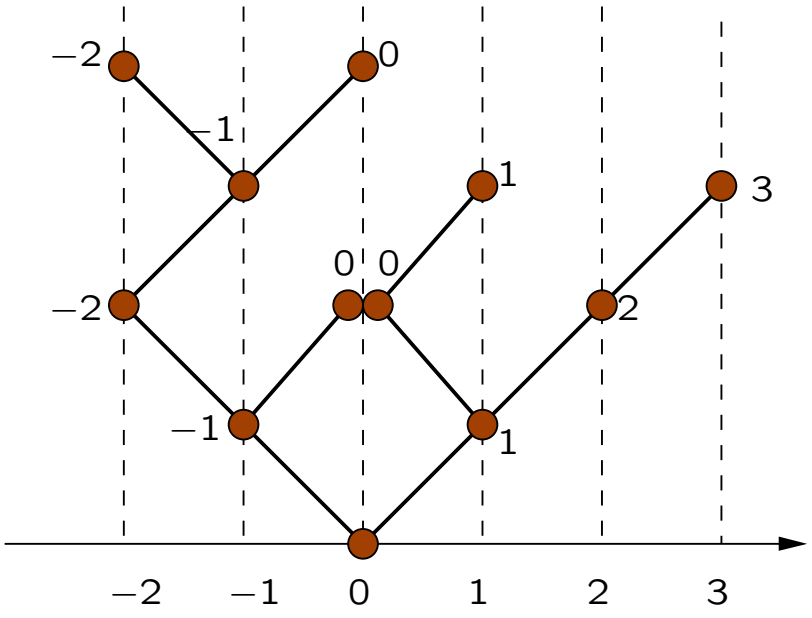
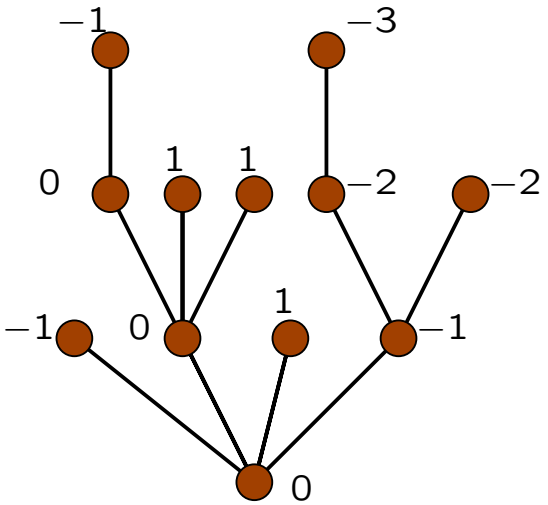
• The number of nodes lying at a positive abscissa, normalized by n, converges to $U(0,1)$ [Aldous].

Part 3. Why study the vertical profile?

• Binary trees (drawn in a canonical way) form a family of embedded trees.
There are many other families!





Random increments $0, \pm 1$ along edges

• The same questions can be asked for any class of embedded trees:
- what is the largest label? (width)
- How many nodes have label j? (profile)

Why study embedded trees?

- They have interesting combinatorial properties (mysterious algebraic series)

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The GF of embedded plane trees with increments ± 1 having largest label at most j is

$$T_j = T\,\frac{(1-Z^{j+1})(1-Z^{j+5})}{(1-Z^{j+2})(1-Z^{j+4})},$$

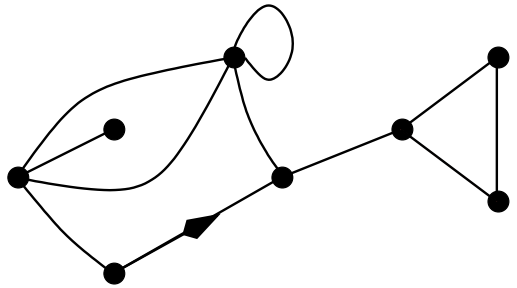
where $T = 1 + 2tT^2$ and

$$Z = t\,\frac{(1+Z)^4}{1+Z^2}.$$

Why study embedded trees?

• They have interesting combinatorial properties (mysterious algebraic series)

• Embedded trees with nonnegative labels are related bijectively to planar maps [Cori-Vauquelin 81], [Chassaing-Schaeffer 04], [Del Lungo, Del Ristoro, Penaud 00], [Bouttier-Di Francesco-Guitter 03]



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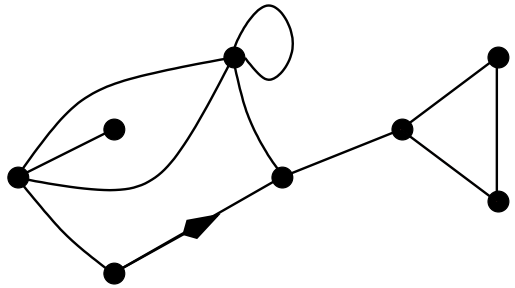
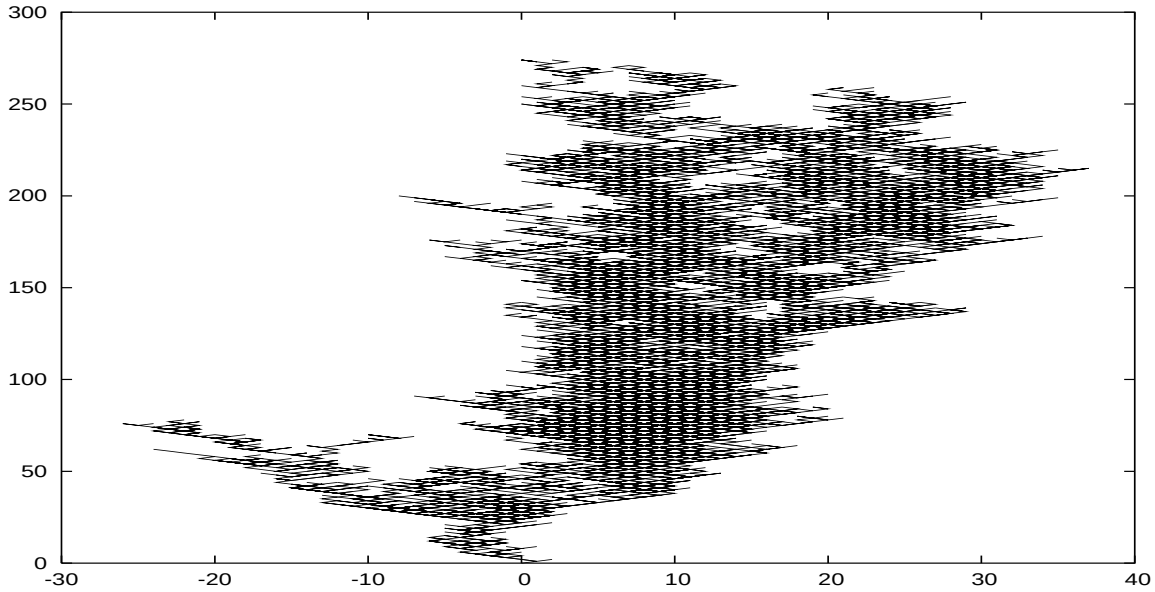


• They are related to ISE (the Integrated SuperBrownian Excursion) [Aldous 93], [Borgs et al. 99]

The Integrated SuperBrownian Excursion

The ISE is a (random) probability distribution on $\mathbb{R}^d$ that occurs (almost) everywhere. At least, as soon as a branching structure (tree) is combined with an embedding of the nodes in the space. The ISE describes how the space is occupied by the nodes [Aldous 93], [Marckert-Mokkadem 03].

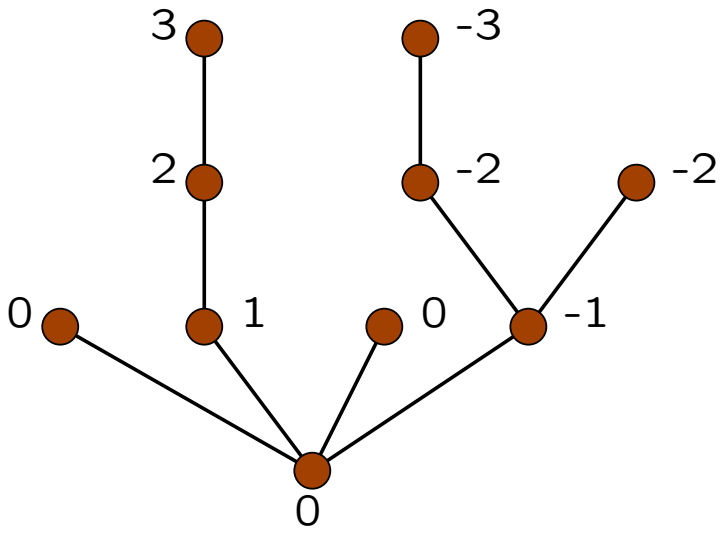


Embedded trees and ISE

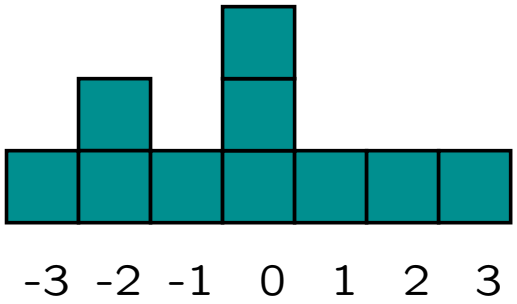
• Let T_n denote a random embedded tree with n nodes. Let μ_n be the occupation measure of T_n:

$$\mu_n = \sum_{v \in T_n} \delta(\ell(v) \qquad),$$

where $\ell(v)$ denotes the label (position) of the node v, and $\delta(x)$ is the Dirac measure at x.



Histogram of μ_n



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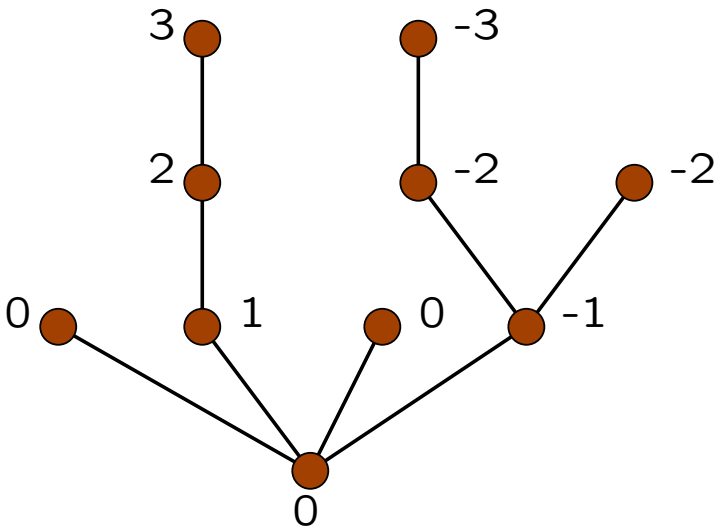

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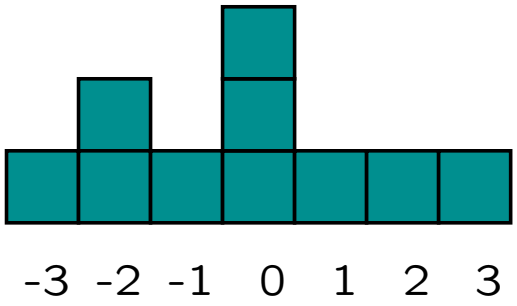

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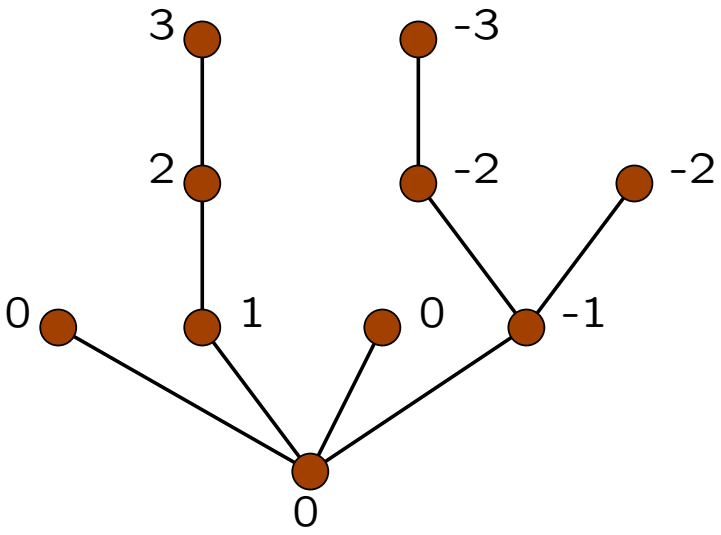
• Let T_n denote a random embedded tree with n nodes. Let μ_n be the occupation measure of T_n:

$$\mu_n = \frac{1}{n} \sum_{v \in T_n} \delta(\ell(v) n^{-1/4}),$$

where $\ell(v)$ denotes the label (position) of the node v, and $\delta(x)$ is the Dirac measure at x. Then μ_n is a probability distribution (total weight 1).
It is random since it depends on the random tree T_n.



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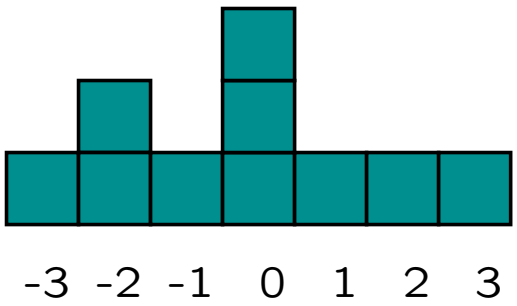

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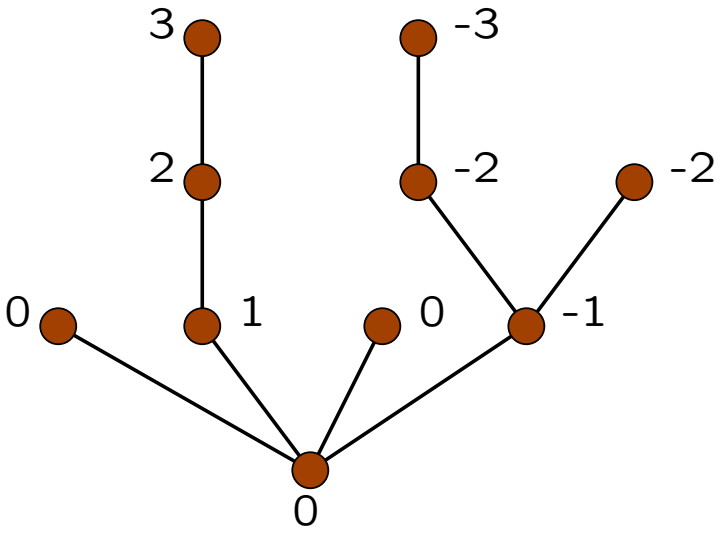
• **Thm.** As n grows, $\mu_n \to \mu_{\text{ise}}$, where μ_{ise} is the ISE.

[Aldous 93, Borgs et al. 99, Janson-Marckert 04]
This holds for "many" families of random embedded trees!



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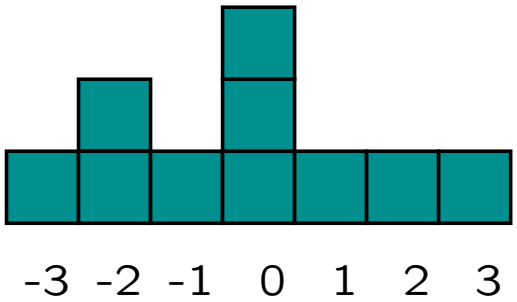
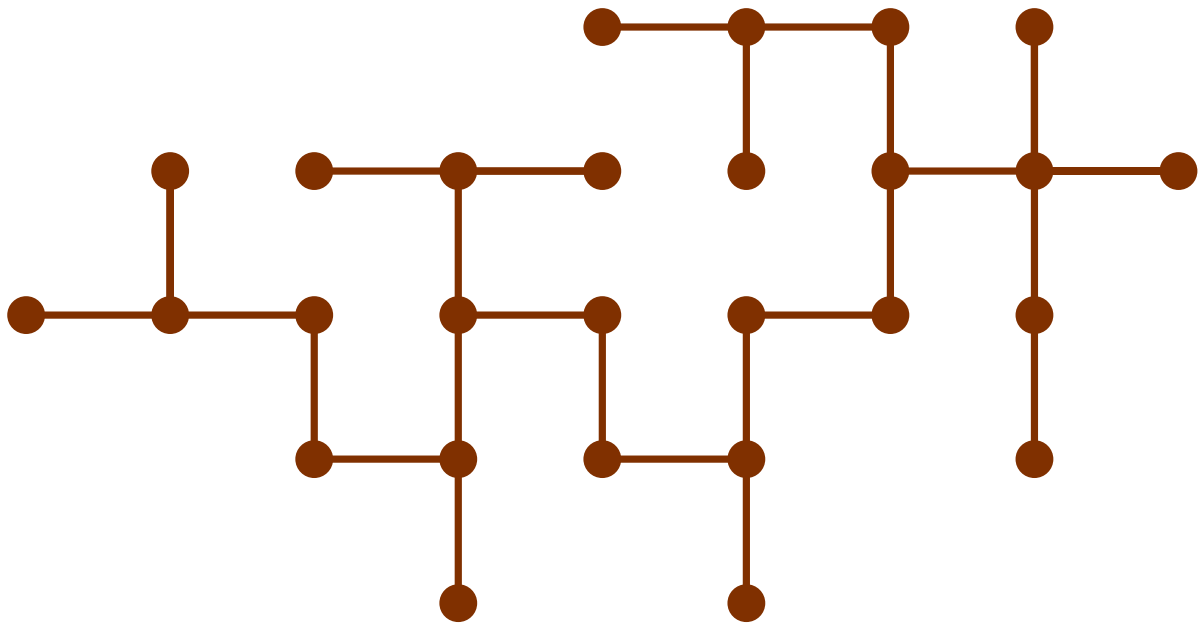
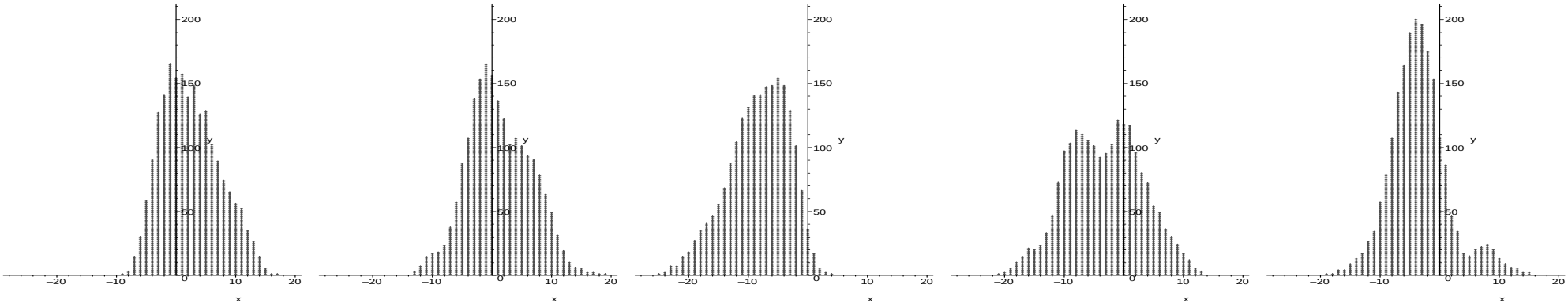


Another occurrence of ISE: Properly embedded trees



In dimension $d \geqslant 8$, the occupation measure of properly embedded trees converges also to the d-dimensional ISE.
[Derbez-Slade 98]

Main objective: study ISE (in 1D) via embedded trees



- Largest label $\longrightarrow$ law of the maximum of the support of the ISE

- Number of nodes at abscissa $j = \lambda n^{1/4} \longrightarrow$ law of the density of the ISE at a fixed point λ

- Number of nodes at abscissa $\leqslant j = \lambda n^{1/4} \longrightarrow$ law of the distribution function of the ISE at a fixed point λ

A beautiful slide for Wjcch.

- Let M_i be the ith moment of the ISE (M_i is a random variable!)

$$M_i = \lim_n \frac{1}{n} \sum_{v \in T_n} \left(\ell(v) n^{-1/4} \right)^i$$

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- Then $\mathbb{E}\left(M_1^{2k}\right) = 0$ and $\mathbb{E}\left(M_1^{2k}\right) = \dfrac{a_k \Gamma(1/2)}{2^{k/2}\Gamma((5k-1)/2)}$, where $a_0 = -2$ and for $k > 0$,

$$4\, a_k = \sum_{i=1}^{k-1} \binom{2k}{2i} a_i a_{k-i} + k(2k-1)(5k-4)(5k-6)a_{k-1}$$

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- Also,

$$\mathbb{E}(M_1^{2k} M_2^{\ell}) = \frac{a_{k,\ell}\Gamma(1/2)}{2^{(k+\ell)/2}\Gamma((5k+3\ell-1)/2)},$$

where $a_{0,0} = -2$ and the $a_{k,\ell}$ are determined by induction on $k+\ell$:

$$a_{k,\ell} = \frac{1}{4} \sum_{(0,0)<(i,j)<(k,\ell)} \binom{2k}{2i}\binom{\ell}{j} a_{i,j} a_{k-i,\ell-j} + 2\ell(\ell-1)a_{k+1,\ell-2}$$

$$+\frac{1}{4}k(2k-1)(5k+3\ell-4)(5k+3\ell-6)a_{k-1,\ell} + \frac{1}{2}(4k+1)\ell(5k+3\ell-4)a_{k,\ell-1}.$$

That's it!

The Laplace transform of $Y(\lambda)$

Let $\lambda \geqslant 0$. The sequence $Y_n(\lfloor \lambda n^{1/4} \rfloor)$ converges in distribution to a non-negative random variable $Y(\lambda)$ whose Laplace transform is given, for $|a| < 4/\sqrt{3}$, by

$$\mathbb{E}\left(e^{aY(\lambda)}\right) = L(\lambda, a)$$

where

$$L(\lambda, a) = 1 + \frac{48}{i\sqrt{\pi}} \int_\Gamma \frac{A(a/v^3)e^{-2\lambda v}}{(1 + A(a/v^3)e^{-2\lambda v})^2} v^5 e^{v^4} dv,$$

$A(x) \equiv A$ is the unique solution of

$$A = \frac{x}{24} \frac{(1+A)^3}{1-A}$$

satisfying $A(0) = 0$, and the integral is taken over

$$\Gamma = \{1 - te^{-i\pi/4}, t \in (\infty, 0]\} \cup \{1 + te^{-i\pi/4}, t \in [0, \infty)\}.$$

Did you say "ISE"?