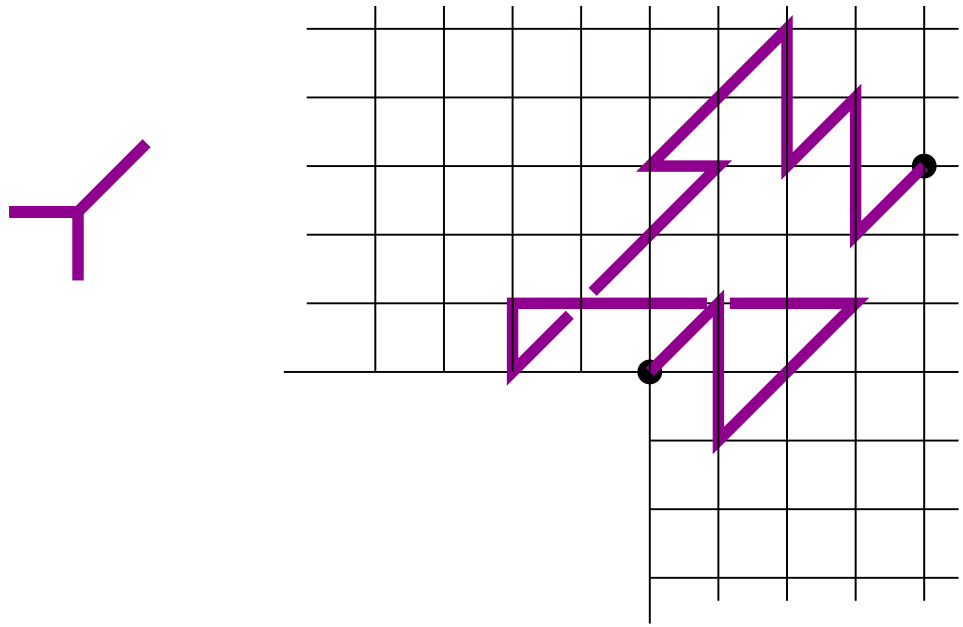
Invariants for three-quadrant walks: the Kreweras trilogy

Mireille Bousquet-Mélou, CNRS, Université de Bordeaux, France

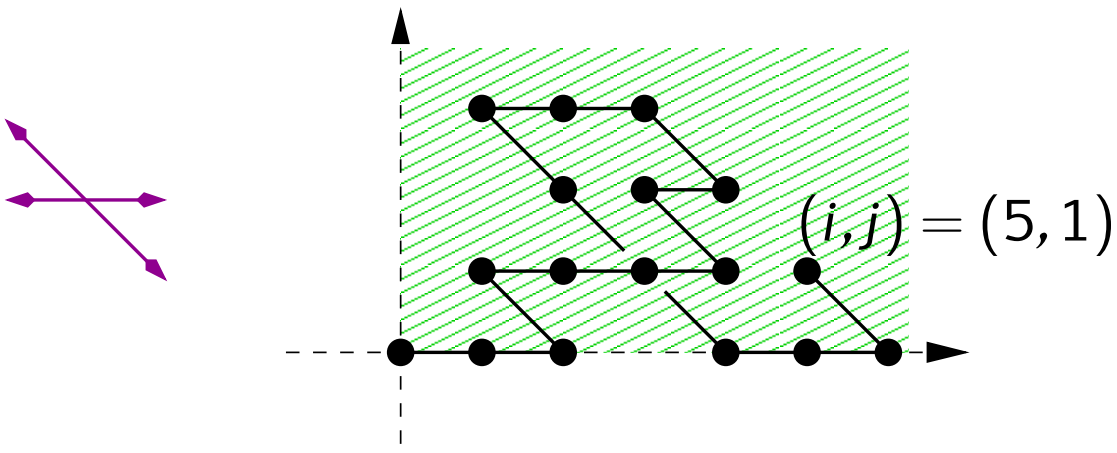


Counting walks in (rational) cones

Take a starting point p_0 in $\mathbb{Z}^2$, a (finite) step set $\mathcal{S} \subset \mathbb{Z}^2$ and a cone $\mathcal{C}$.

Questions

- What is the number $c(n)$ of n-step walks starting at p_0, taking their steps in $\mathcal{S}$ and contained in $\mathcal{C}$?
- For $(i,j) \in \mathcal{C}$, what is the number $c_{i,j}(n)$ of such walks that end at (i,j)?



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- Generating function:

$$C(x,y;t) = \sum_{i,j,n} c_{i,j}(n) x^i y^j t^n$$
$$= \sum_{w \text{ walk}} x^{i(w)} y^{j(w)} t^{|w|}$$

What is the value/nature of this series?

A hierarchy of formal power series

- Rational series

$$A(t) = \frac{P(t)}{Q(t)}$$

- Algebraic series

$$P(t, A(t)) = 0$$

- Differentially finite series (D-finite)

$$\sum_{i=0}^{d} P_i(t) A^{(i)}(t) = 0$$

- D-algebraic series

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Multi-variate series: one DE per variable

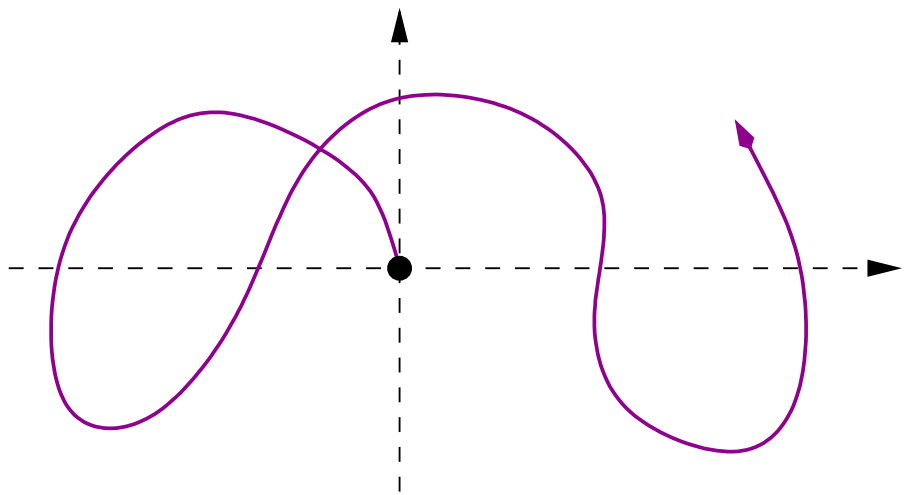
Normalizing the cone: 4 cases

- The full space: rational series

$$C(x,y;t) = \frac{1}{1 - tS(x,y)} = \sum_{n \geq 0} t^n S(x,y)^n,$$

where $S(x,y)$ is the step polynomial:

$$S(x,y) = \sum_{(i,j) \in \mathcal{S}} x^i y^j.$$



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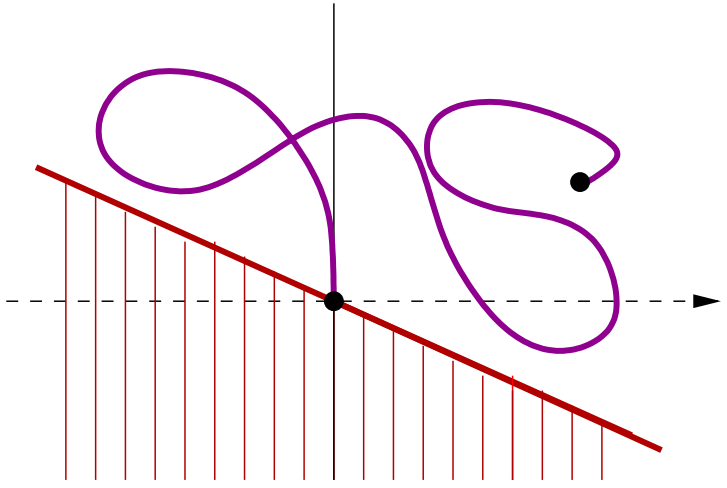

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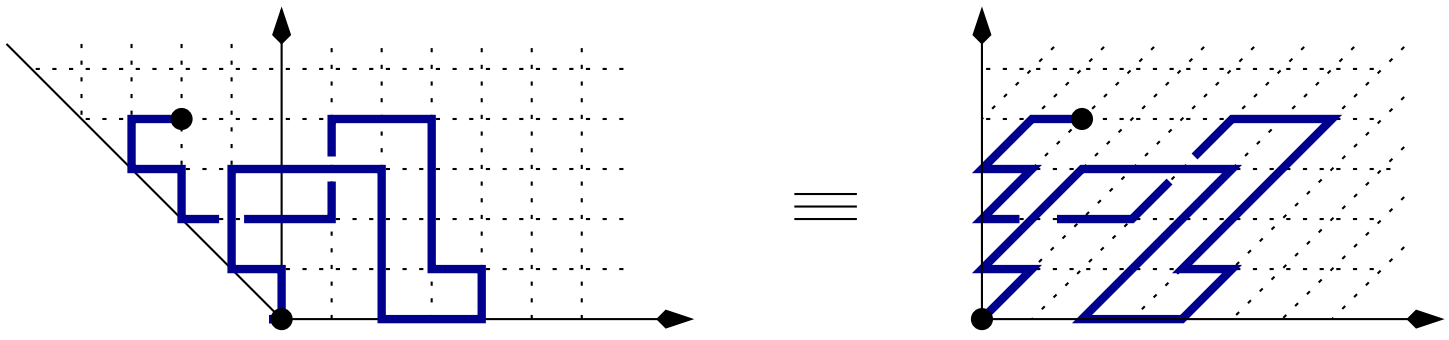

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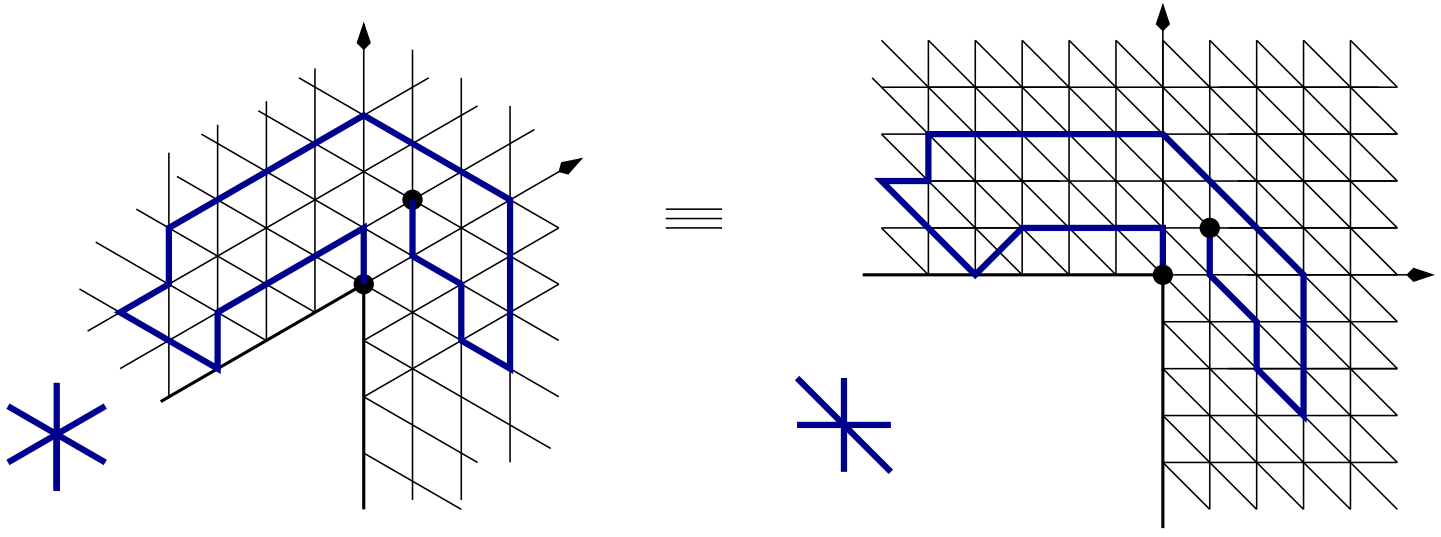
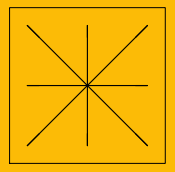
- A non-convex cone $\longrightarrow$ walks avoiding the negative quadrant: $C(x, y; t)$



Walks with *small* steps



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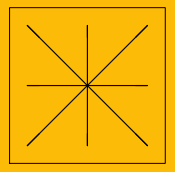

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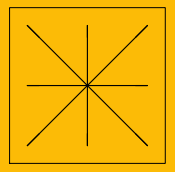

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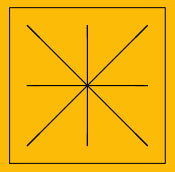

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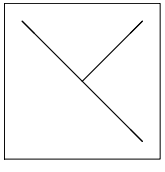
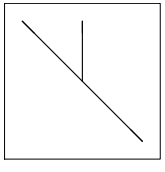
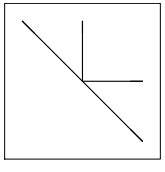
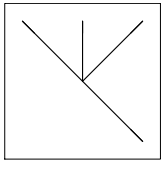
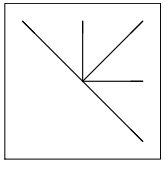
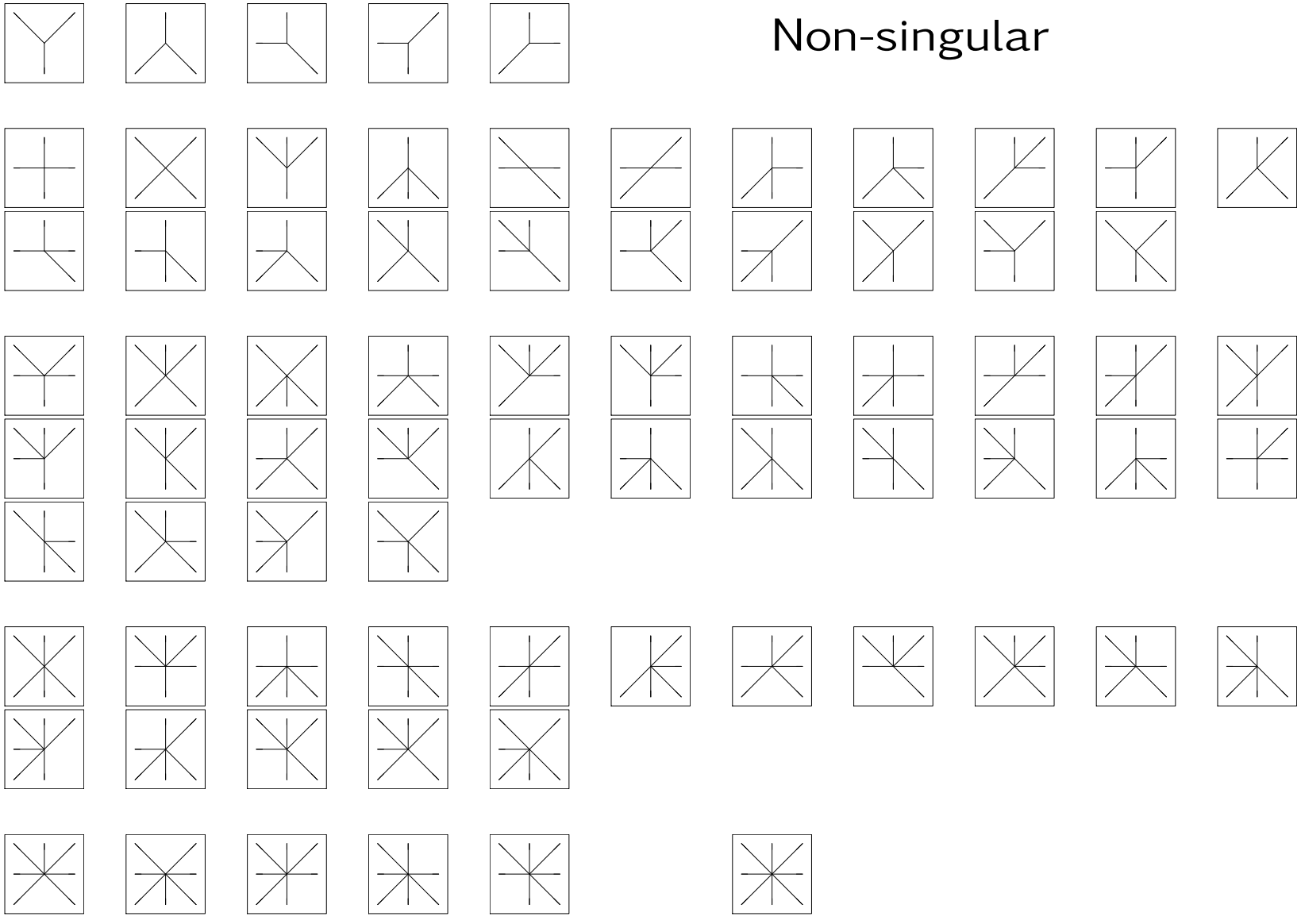
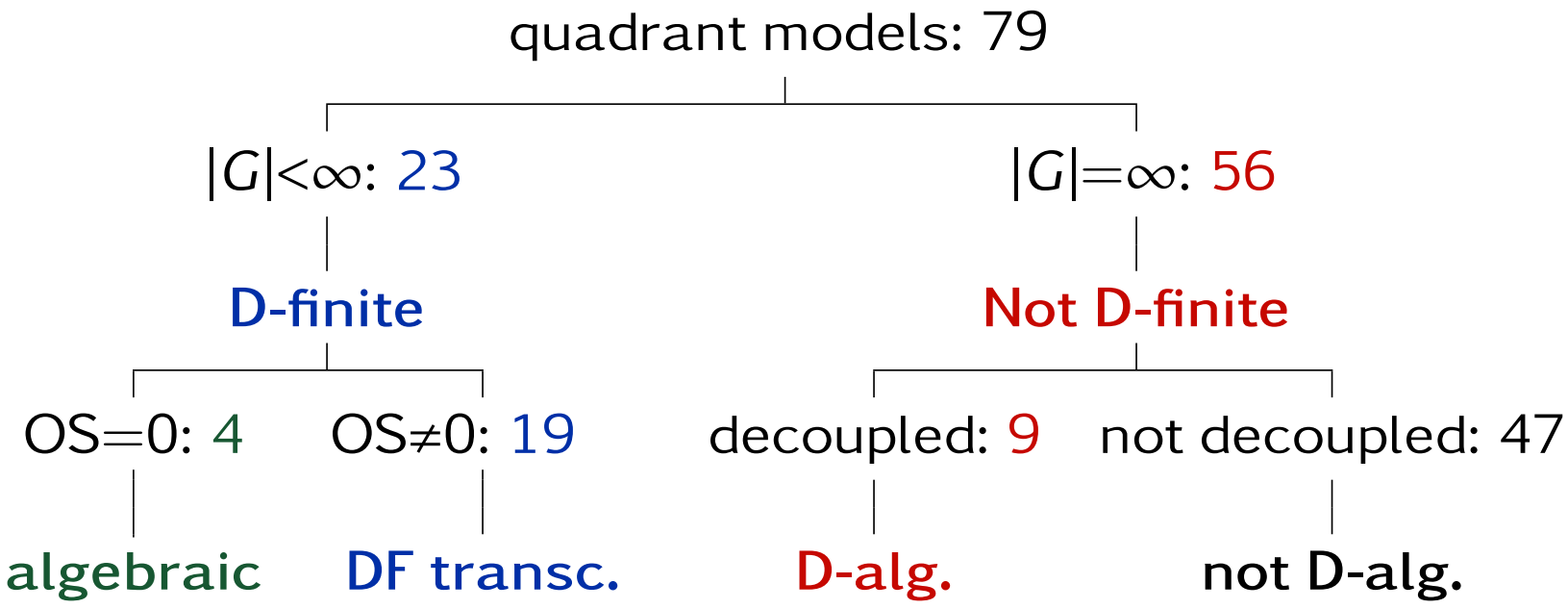
- In the three-quadrant cone, one is left with only 74 interesting distinct models: the 5 "singular" models become trivial.

Singular models

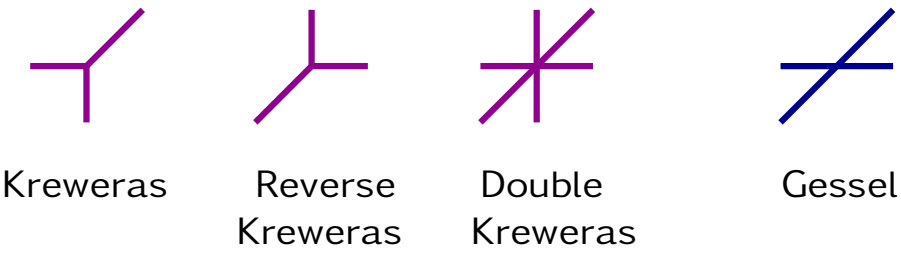







Non-singular
Singular

Classification of quadrant walks with small steps



The 4 algebraic models:



What about walks in the three-quadrant cone?

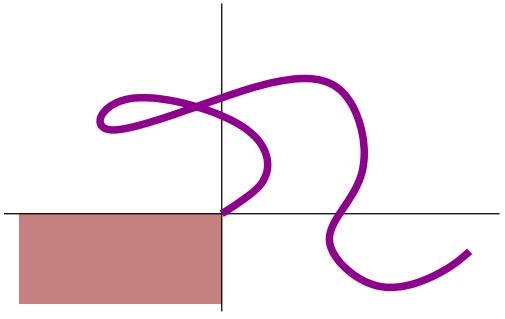
Outline

I. Kreweras' walks in the quadrant

- Functional equation
- A solution via invariants
  [Bernardi, mbm, Raschel 17(a)]

II. Kreweras' walks in three quadrants

- Functional equation
- A solution via invariants



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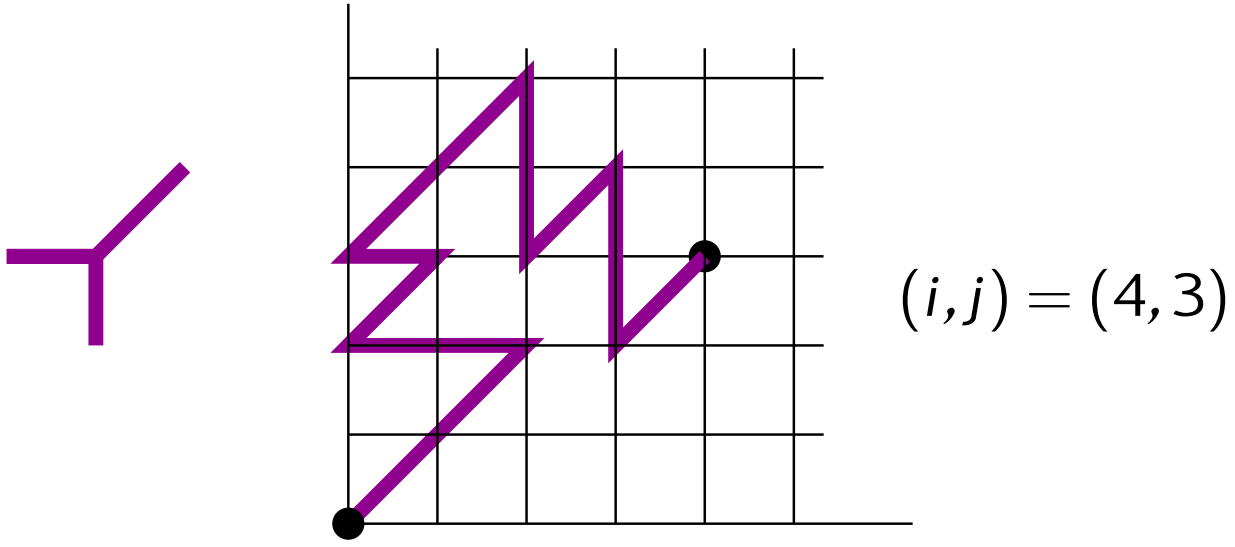
Invariants: a method introduced by W. Tutte to count properly coloured planar triangulations (1973-1984)

Kreweras′ walks in the quadrant

- Generating function:

$$Q(x,y) \equiv Q(x,y;t) = \sum_{i,j,n\geq 0} q_{i,j}(n) x^i y^j t^n$$

where $q_{i,j}(n)$ is the number of walks of length n from $(0,0)$ to (i,j) in the quadrant.



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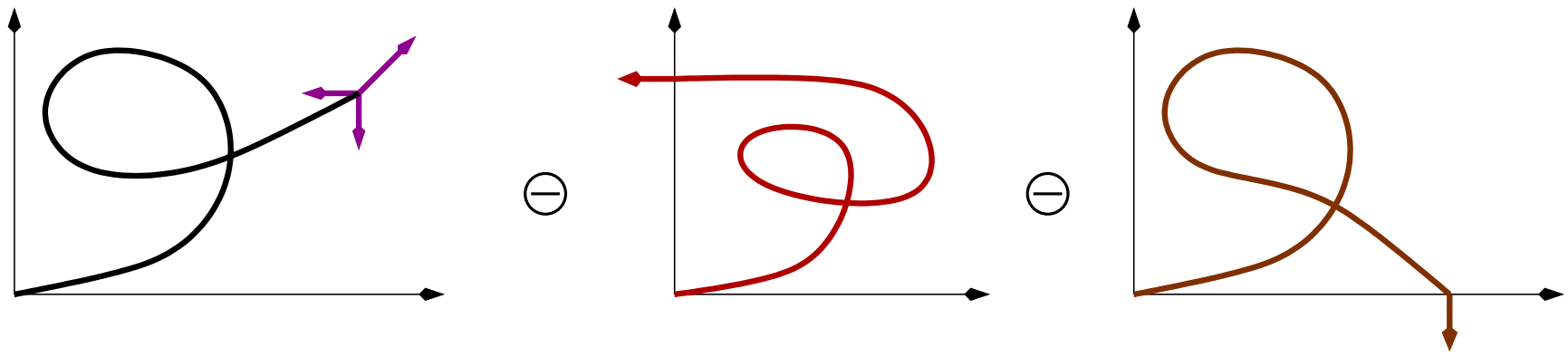
- Functional equation (with $\bar{x} = 1/x$, $\bar{y} = 1/y$):

$$Q(x,y) = 1 + t(xy + \bar{x} + \bar{y})Q(x,y) - t\bar{x}Q(0,y) - t\bar{y}Q(x,0).$$



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Equivalently,

$$\boxed{xy(1 - t(xy + \bar{x} + \bar{y}))Q(x,y) = xy - tyQ(0,y) - txQ(x,0),}$$

where

$$K(x,y) := xy\,(1 - t(xy + \bar{x} + \bar{y}))$$

is the kernel (a polynomial in t, x, y). Two catalytic variables: x and y.

What are invariants?

A pair of series $(I(x), J(y))$ in t with coefficients in $\mathbb{Q}(x)$ and $\mathbb{Q}(y)$ (respectively) is a pair of invariants if $I(x) - J(y)$ is divisible by $K(x, y)$.

- Isn't *any* series divisible by $K(x, y)$? After all,

$$\frac{1}{K(x,y)} = \frac{1}{xy(1 - t(xy + \bar{x} + \bar{y}))} = \bar{x}\bar{y} \sum_{n \geq 0} t^n (xy + \bar{x} + \bar{y})^n$$

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- We want more: the coefficients (of t^n, $n \geq 0$) in the ratio

$$H(x,y) = \frac{I(x)-J(y)}{K(x,y)},$$

which are rational functions of the form $p(x,y)/(d(x)d'(y))$, must have poles of bounded order at $x = 0$ and $y = 0$.
Equivalently, there exists i, j such that the coefficients of $x^i y^j H(x,y)$ have no pole at $x = 0$ nor $y = 0$.

Do such pairs exist?

- Trivial invariants: take $I(x) = J(y) \in \mathbb{Q}((t))$

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- Basic rational invariants: let

$$I_0(x) = \bar{x}^2 - \bar{x}/t - x, \qquad J_0(y) = I_0(y).$$

Then

$$K(x,y) \cdot \frac{x-y}{x^2y^2} \cdot \frac{1}{t} = I_0(x) - J_0(y),$$

and the series $H_0(x,y) = (x-y)/(tx^2y^2)$ has poles of bounded order at $x = 0$ and $y = 0$.

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New invariants from old ones

The componentwise sum (resp. product) of two pairs of invariants $(I_0(x), J_0(y)), (I_1(x), J_1(y))$ is another pair of invariants.

$$(I_0(x) + I_1(x), J_0(y) + J_1(y)), \qquad (I_0(x)I_1(x), J_0(y)J_1(y))$$

What are invariants good for?

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1/ Build a new pair of invariants:

$$\begin{aligned} K(x,y)H_1(x,y) &= F(x) + G(y) - tyQ(0,y) - txQ(x,0) \\ &= I_1(x) - J_1(y) \end{aligned}$$

with

$$I_1(x) = F(x) - txQ(x,0), \qquad J_1(y) = -G(y) + tyQ(0,y).$$

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2/ Relate this pair to the basic rational invariants $(I_0(x), J_0(y))$ using:

The invariant lemma

If $K(x,y)H(x,y) = I(x) - J(y)$ and the coefficients of $H(x,y)$ (in t) have no pole at $x = 0$ nor at $y = 0$, then

$$H(x,y) = 0 \qquad \text{and} \qquad I(x) = J(y) \in \mathbb{Q}((t)).$$

Step 1: decoupling

Functional equation:

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Easy! The term xy *decouples* modulo $K(x,y)$:

$$-\frac{1}{txy}K(x,y) = xy + \bar{x} + \bar{y} - 1/t.$$

So we have a new pair of invariants:

$$I_1(x) = txQ(x,0) + \bar{x} - 1/(2t), \qquad J_1(y) = -I_1(y).$$

Step 2: relate (I_1, J_1) to the basic invariants (I_0, J_0)

We have two pairs of invariants:

$$I_0(x) = \bar{x}^2 - \bar{x}/t - x, \qquad J_0(y) = I_0(y),$$
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Observation: $I_1(x)^2 - I_0(x)$ has no pole! Define

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where

$$H(x,y) = \frac{x-y}{tx^2y^2} + \left(Q(x,y) + \frac{1}{xyt}\right)(I_1(x) + J_1(y))$$

has no pole at $x = 0$ nor $y = 0$.

Applying the invariant lemma

Hence

$$I(x) = I_1(x)^2 - I_0(x) = (txQ(x,0) - 1/(2t))^2 + 2tQ(x,0) + x$$

is a series of $\mathbb{Q}((t))$, equal to its value at $x = 0$:

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$\Rightarrow$ systematic solution (via Brown's quadratic method, or [mbm-Jehanne 06])

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$$I(x) = I_1(x)^2 - I_0(x) = (txQ(x,0) - 1/(2t))^2 + 2tQ(x,0) + x$$

is a series of $\mathbb{Q}((t))$, equal to its value at $x = 0$:

$$\boxed{(txQ(x,0) - 1/(2t))^2 + 2tQ(x,0) + x = 1/(2t)^2 + 2tQ(0,0).}$$

We have obtained an equation in *one* catalytic variable
$\Rightarrow$ systematic solution (via Brown's quadratic method, or [mbm-Jehanne 06])

GF of Kreweras' walks in the quadrant

Let $Z \equiv Z(t)$ be the only series in t such that $Z = t(2 + Z^3)$. Then

$$Q(x,0) = \frac{1}{tx}\left(\frac{1}{2t} - \frac{1}{x} - \left(\frac{1}{Z} - \frac{1}{x}\right)\sqrt{1 - xZ^2}\right).$$

[Kreweras 65], [Gessel 86], [mbm 05]...

Other quadrant models with small steps

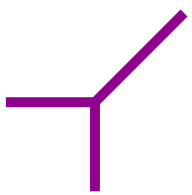
Two ingredients:

- A pair $(I_0(x), J_0(y))$ of rational invariants $\Rightarrow$ 23 models with a finite group
- A decoupling identity of the form

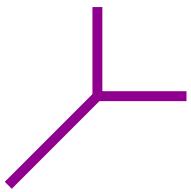
$$xy = F(x) + G(y) + K(x,y)H(x,y)$$

$\Rightarrow$ 4 of the 23 models

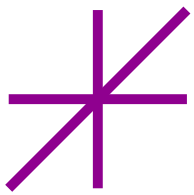
The use of invariants solves these four models, proving algebraicity.



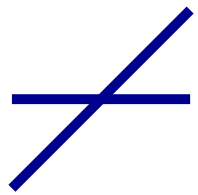
Kreweras



Reverse Kreweras



Double Kreweras



Gessel

[Bernardi, mbm, Raschel 17(a)]

Reverse Kreweras walks

- Basic rational invariants

$$\mathscr{K}(x,y)\frac{x-y}{txy} = \mathscr{I}_0(x) - \mathscr{J}_0(y)$$

with

$$\mathscr{I}_0(x) = \bar{x} + x/t - x^2, \qquad \mathscr{J}_0(y) = \mathscr{I}_0(y).$$

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- Invariants related to $\mathscr{Q}(x,y)$:

$$-\frac{1}{ty}\mathscr{K}(x,y)(1+ty\mathscr{Q}(x,y)) = \mathscr{I}_1(x) - \mathscr{J}_1(y)$$

with

$$\mathscr{I}_1(x) = t\mathscr{Q}(x,0) - x/t + x^2, \qquad \mathscr{J}_1(y) = -t\mathscr{Q}(0,y) - \bar{y} + t\mathscr{Q}(0,0)$$

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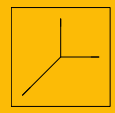

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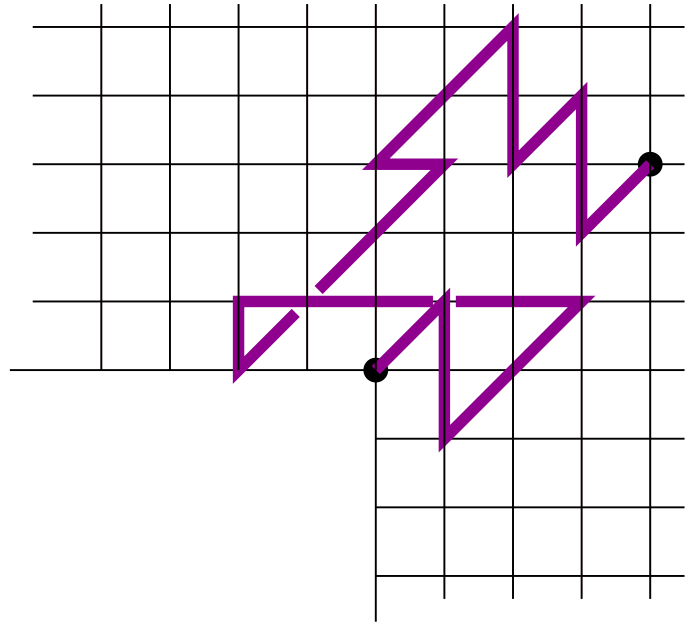
GF of reverse Kreweras walks [Mishna 09, mbm-Mishna 10]

$$\mathscr{Q}(x,0) = \frac{Z\left(4-Z^3\right)}{16t} - \frac{t-x^2+tx^3}{2xt^2} - \frac{\left(2x^2-xZ^2-Z\right)\sqrt{1-xZ(Z^3+4)/4+x^2Z^2/4}}{2txZ}.$$

II. Kreweras′ walks in three quadrants



$$C(x,y) \equiv C(x,y;t) = \sum_{i,j,n\geq 0} c_{i,j}(n) x^i y^j t^n$$

where $c_{i,j}(n)$ is the number of walks of length n from $(0,0)$ to (i,j) avoiding the negative quadrant.

First functional equation

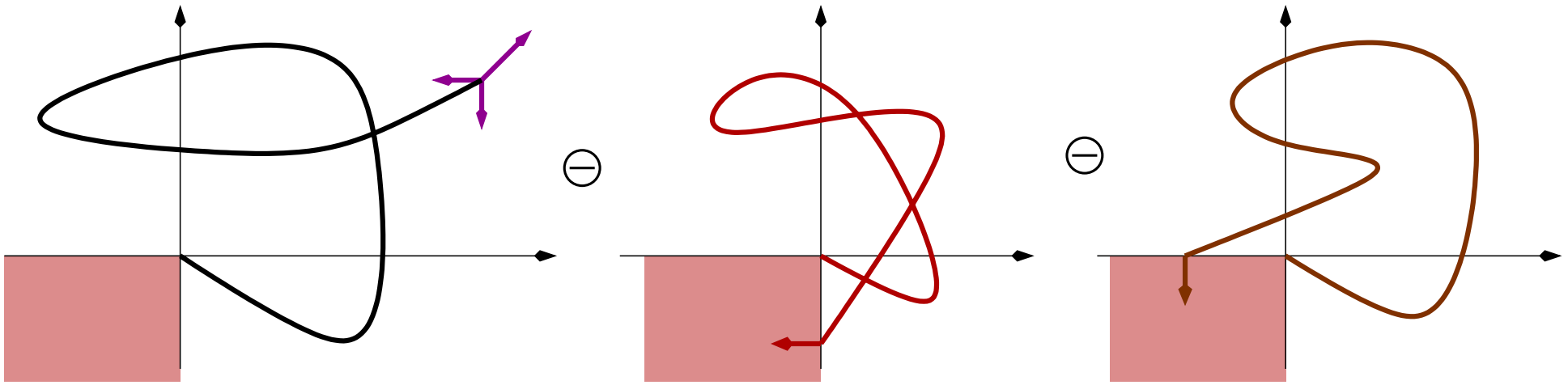
Functional equation (with $\bar{x} = 1/x, \bar{y} = 1/y$):

$$C(x,y) = 1 + t(xy + \bar{x} + \bar{y})C(x,y) - t\bar{x}C_-(\bar{y}) - t\bar{y}C_-(\bar{x}),$$

where

$$C_-(\bar{x}) = \sum_{i<0,n\geq 0} c_{-i,0}(n)x^i t^n$$

counts walks ending on the negative x-axis.



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counts walks ending on the negative x-axis.

Equivalently,

$$\boxed{xy(1 - t(xy + \bar{x} + \bar{y}))C(x,y) = xy - tyC_-(\bar{y}) - txC_-(\bar{x}).}$$

- The coefficients of $C(x,y)$ have poles of arbitrarily large order at $x = 0$ and $y = 0$.

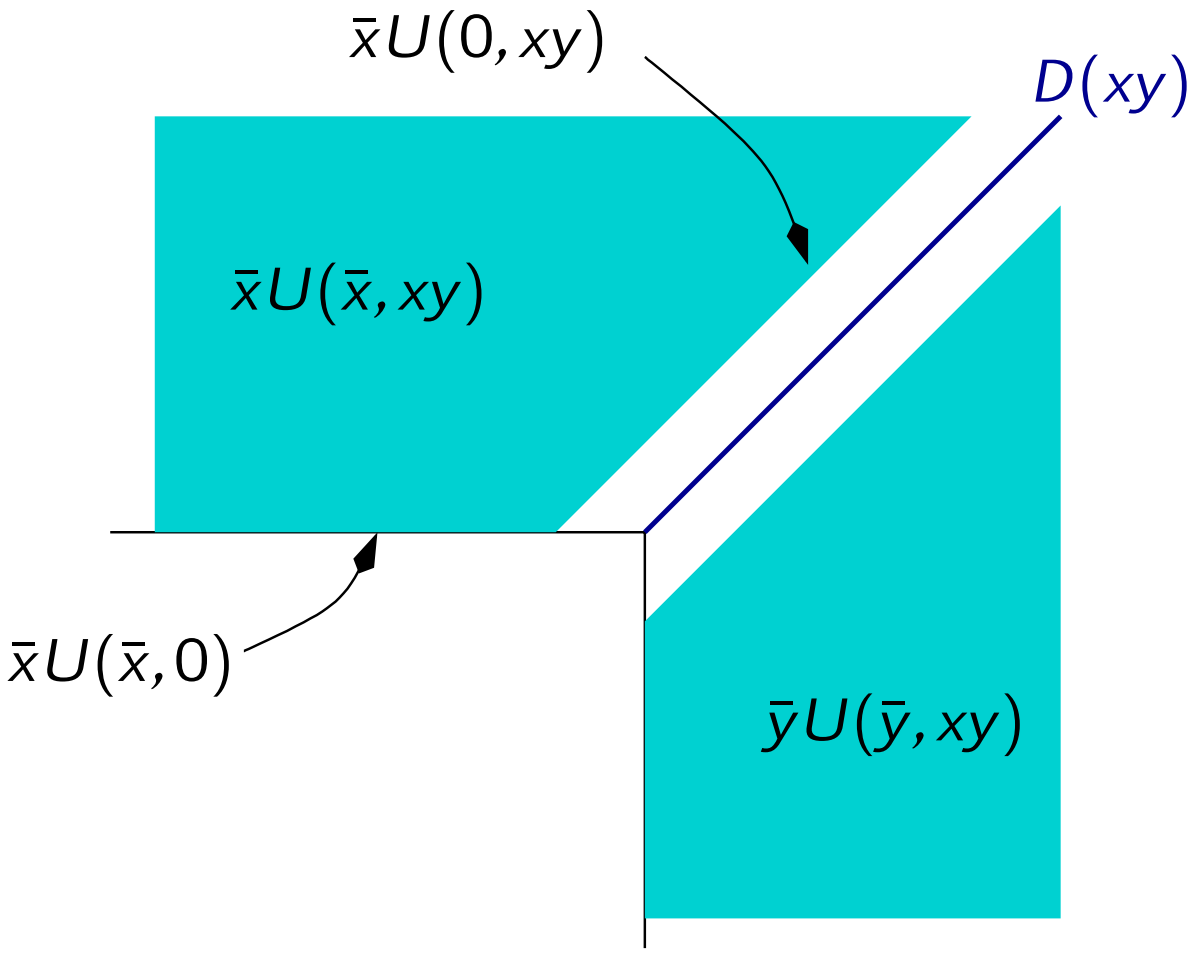
- It suffices to determine $C_-(\bar{x})$, the GF of walks ending on the negative x-axis.

Alternative: Split the cone in two convex cones

- À la Raschel-Trotignon (2019):

$$C(x,y) = \bar{x}U(\bar{x},xy) + D(xy) + \bar{y}U(\bar{y},xy),$$

where $U(x,y) \in \mathbb{Q}[x,y][[t]]$ and $D(x) \in \mathbb{Q}[x][[t]]$.



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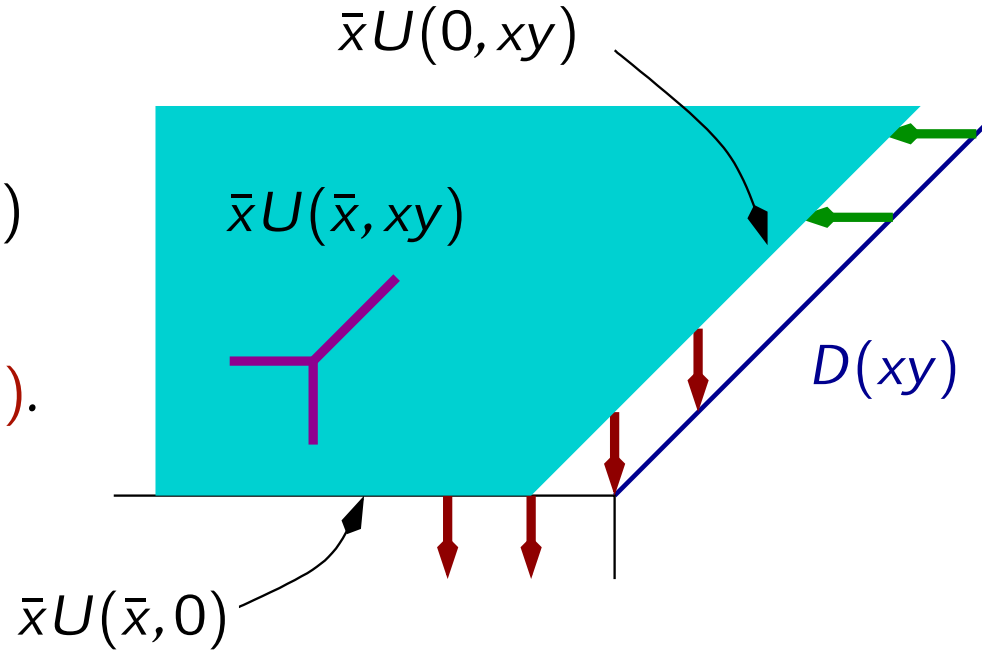
- Step-by-step construction:

$$\bar{x}U(\bar{x},xy) = t(xy+\bar{x}+\bar{y})\bar{x}U(\bar{x},xy) + t\bar{x}D(xy) - t\bar{y}\,(\bar{x}U(\bar{x},0) + \bar{x}U(0,xy) - \bar{x}U(0,0)).$$



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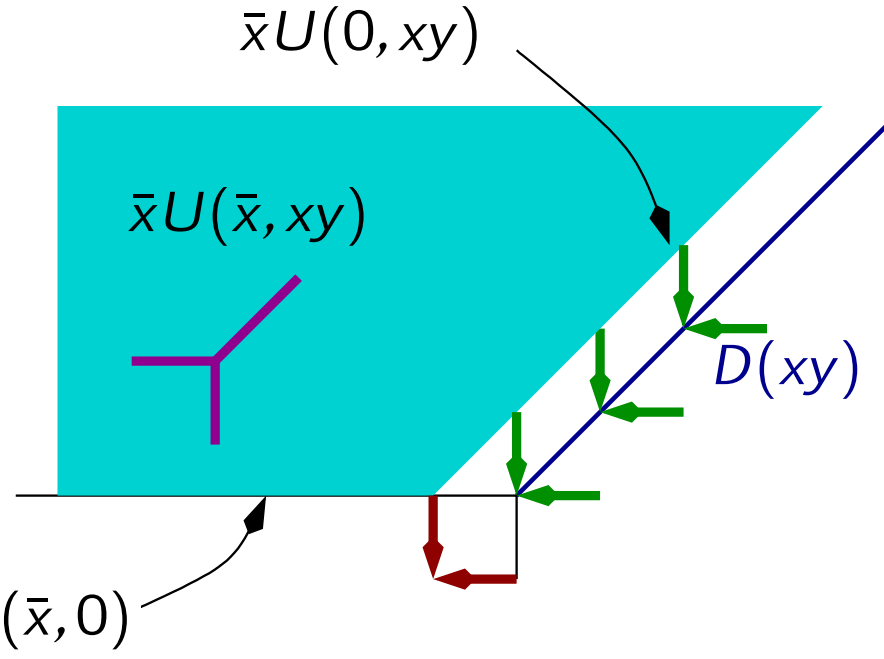
$$\begin{aligned}\bar{x}U(\bar{x},xy) = {} & t(xy+\bar{x}+\bar{y})\bar{x}U(\bar{x},xy) \\ & + t\bar{x}D(xy) \\ & - t\bar{y}\,(\bar{x}U(\bar{x},0) + \bar{x}U(0,xy) - \bar{x}U(0,0)).\end{aligned}$$



Analogously,

$$D(xy) = 1 + txyD(xy) + 2t\bar{y}\,(\bar{x}U(0,xy) - \bar{x}U(0,0)).$$

Split the cone in two convex cones

- Two equations:

$$\bar{x}\,(1 - t(xy + \bar{x} + \bar{y}))\,U(\bar{x}, xy) = -t\bar{y}\bar{x}U(\bar{x}, 0) + t\bar{x}D(xy) \\ - t\bar{y}\bar{x}U(0, xy) + t\bar{y}\bar{x}U(0, 0),$$

$$(1 - txy)D(xy) = 1 + 2t\bar{y}\bar{x}U(0, xy) - 2t\bar{y}\bar{x}U(0, 0).$$

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- Straightening: $(\bar{x},xy)\mapsto(x,y)$

$$\begin{aligned}x\,(1-t(y+x+\bar{x}\bar{y}))\,U(x,y) &= -t\bar{y}U(x,0)+txD(y)\\ &\quad -t\bar{y}U(0,y)+t\bar{y}U(0,0),\\ (1-ty)D(y) &= 1+2t\bar{y}U(0,y)-2t\bar{y}U(0,0).\end{aligned}$$

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- Linear combination:

$$\boxed{2\mathscr{K}(x,y)U(x,y)=y+(ty+2tx-1)yD(y)-2tU(x,0),}$$

where now $\mathscr{K}(x,y)=xy(1-t(x+y+\bar{x}\bar{y}))$ is the kernel of reverse Kreweras′ walks.

We want to construct $\mathscr{K}$-invariants...

$$2\mathscr{K}(x,y)U(x,y) = y + (ty + 2tx - 1)yD(y) - 2tU(x,0)$$

Two new difficulties:

- one needs to "decouple" y, in some sense,
- the term $(ty + 2tx - 1)$ depends on x (and y).

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Two good news:

- Decoupling:

$$y = (ty + 2tx - 1)G(y) - F(x)$$

with $G(y) = 1/t$ and $F(x) = 2x - 1/t$.

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Two good news:

- Decoupling:

  $$y = (ty + 2tx - 1)G(y) - F(x)$$

  with $G(y) = 1/t$ and $F(x) = 2x - 1/t$.
- The square of $(ty + 2tx - 1)$ is a function of y (modulo $\mathscr{K}(x,y)$):

  $$(ty + 2tx - 1)^2 = \Delta(y) - 4t\bar{y}\mathscr{K}(x,y)$$

  where $\Delta(y) = (1 - ty)^2 - 4t\bar{y}$.

A new pair of $\mathscr{K}$-invariants

$$2\mathscr{K}(x,y)U(x,y) = y + (ty + 2tx - 1)yD(y) - 2tU(x,0)$$

- Decoupling: since $y = (ty + 2tx - 1)/t - 2x + 1/t$,

$$2\mathscr{K}(x,y)U(x,y) = (ty + 2tx - 1)\Big(yD(y) + 1/t\Big) - \Big(2tU(x,0) + 2x - 1/t\Big)$$

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- We have found a new pair of $\mathscr{K}$-invariants:

$$\boxed{\mathscr{I}_2(x) = \left(2tU(x,0) + 2x - \frac{1}{t}\right)^2, \qquad \mathscr{J}_2(y) = \Delta(y)\left(yD(y) + \frac{1}{t}\right)^2}$$

with $\Delta(y) = (1 - ty)^2 - 4t\bar{y}$.

Two known pairs of $\mathscr{K}$-invariants (from Part I)

- Recall:

$$\mathscr{I}_0(x) = \bar{x} + x/t - x^2, \qquad \mathscr{J}_0(x) = \mathscr{I}_0(y),$$
$$\mathscr{I}_1(x) = t\mathscr{Q}(x,0) - x/t + x^2, \qquad \mathscr{J}_1(y) = -\bar{y} - t\mathscr{Q}(0,y) + t\mathscr{Q}(0,0).$$

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- We have just found another pair:

$$\mathscr{I}_2(x) = \left(2tU(x,0) + 2x - \frac{1}{t}\right)^2 = \mathcal{O}(x^0),$$
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- Define

$$\mathscr{I}(x) = \mathscr{S}_2(x) - 4\mathscr{S}_1(x), \qquad \mathscr{J}(y) = \mathscr{J}_2(y) - 4\mathscr{J}_1(y).$$

Then $(\mathscr{I}(x), \mathscr{J}(y))$ is a pair of invariants with no pole at 0.

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$$\mathscr{K}(x,y)\mathscr{H}(x,y) = \mathscr{I}(x) - \mathscr{J}(y)$$

where $\mathscr{H}(x,y)$ has no pole at 0 either

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Then $(\mathscr{I}(x), \mathscr{J}(y))$ is a pair of invariants with no pole at 0. Moreover,

$$\mathscr{K}(x,y)\mathscr{H}(x,y) = \mathscr{I}(x) - \mathscr{J}(y)$$

where $\mathscr{H}(x,y)$ has no pole at 0 either $\Rightarrow \mathscr{I}(x)$ and $\mathscr{J}(y)$ are trivial.

Trivial invariants for reverse Kreweras′ steps

Conclusion:

$$\mathscr{I}_2(x) = \left(2tU(x,0) + 2x - \frac{1}{t}\right)^2 = 4\left(t\mathscr{Q}(x,0) - x/t + x^2\right) + cst,$$

$$\mathscr{J}_2(y) = \Delta(y)\left(yD(y) + \frac{1}{t}\right)^2 = 4\left(-\bar{y} - t\mathscr{Q}(0,y) + t\mathscr{Q}(0,0)\right) + cst,$$

with $\Delta(y) = (1-ty)^2 - 4t\bar{y}$.

The constant can be determined by specializing y to the unique root of $\Delta(y)$ that is a power series in t.

The GF of Kreweras walks in three quadrants [mbm]

- Walks ending on the negative x-axis: series $U(x,0)$, with

$$\frac{1}{2}\left(2tU(x,0)+2x-\frac{1}{t}\right)^2=\frac{(1-Z^3)^{3/2}}{Z^2}+(1-xZ)^2\left(\frac{1}{Z^2}-\frac{1}{x}\right)$$
$$+\left(\bar{x}+Z-\frac{2x}{Z}\right)\sqrt{1-Z\frac{4+Z^3}{4}x+\frac{Z^2}{4}x^2}.$$

- Walks ending on the diagonal: series $D(x)$, with

$$\frac{\Delta(x)}{2}\left(xD(x)+\frac{1}{t}\right)^2=\frac{(1-Z^3)^{3/2}}{Z^2}+(1-xZ)^2\left(\frac{1}{Z^2}-\frac{1}{x}\right)$$
$$-\left(\bar{x}+Z-\frac{2x}{Z}\right)\sqrt{1-Z\frac{4+Z^3}{4}x+\frac{Z^2}{4}x^2}.$$

where $\Delta(x)=(1-tx)^2-4t\bar{x}$ and $Z=t(2+Z^3)$.

The GF of Kreweras walks in three quadrants [mbm]

- Walks ending on the negative x-axis: series $U(x,0)$, with

$$\frac{1}{2}\left(2tU(x,0)+2x-\frac{1}{t}\right)^2=\frac{(1-Z^3)^{3/2}}{Z^2}+(1-xZ)^2\left(\frac{1}{Z^2}-\frac{1}{x}\right)$$
$$+\left(\bar{x}+Z-\frac{2x}{Z}\right)\sqrt{1-Z\frac{4+Z^3}{4}x+\frac{Z^2}{4}x^2}.$$

- Walks ending on the diagonal: series $D(x)$, with

$$\frac{\Delta(x)}{2}\left(xD(x)+\frac{1}{t}\right)^2=\frac{(1-Z^3)^{3/2}}{Z^2}+(1-xZ)^2\left(\frac{1}{Z^2}-\frac{1}{x}\right)$$
$$-\left(\bar{x}+Z-\frac{2x}{Z}\right)\sqrt{1-Z\frac{4+Z^3}{4}x+\frac{Z^2}{4}x^2}.$$

where $\Delta(x)=(1-tx)^2-4t\bar{x}$ and $Z=t(2+Z^3)$.

- All walks in three quadrants:

$$xy(1-t(\bar{x}+\bar{y}+xy))C(x,y)=xy-tU(\bar{x},0)-tU(\bar{y},0).$$

(Algebraicity of excursions proved by [Elvey Price, FPSAC 20])

Asymptotics and harmonic function

[mbm]

- Number of n-step walks ending at (i,j) in the three quadrant plane:

$$c_{i,j}(n) \sim -\frac{H_{i,j}}{\Gamma(-3/4)}\, 3^n n^{-7/4} \qquad (\text{for } n+i+j \equiv 0 \mod 3)$$

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$$\mathscr{H}(x,y) := \sum_{j\geq 0, i\leq j} H_{i,j} x^{j-i} y^j,$$

is algebraic of degree 16, given by

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[Trotignon 19(a)]

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Other three-quadrant models with small steps

Two new ingredients:

- x/y-symmetry of the step set (for the functional equation)
- A decoupling identity of the form

$$y = p(x,y)G(y) + F(x) + K(x,y)H(x,y)$$

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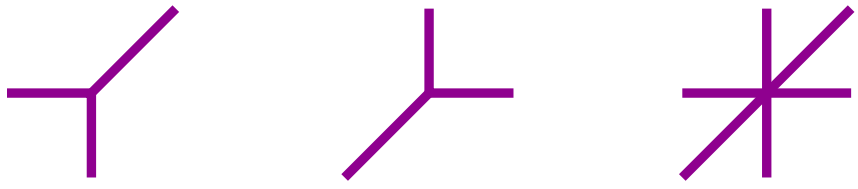
The use of invariants solves these 3 models, proving algebraicity:



Kreweras | Reverse Kreweras | Double Kreweras

D-finite: [Raschel-Trotignon 19]

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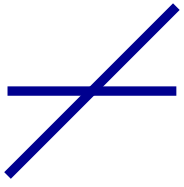
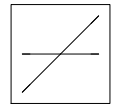


Gessel
TO DO

What's next?

- Gessel's step set (algebraicity of excursions [Budd 20])



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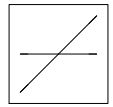

- [Beyond finite groups] Another x/y-symmetric model with an infinite group but a "weak" invariant

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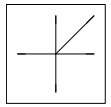

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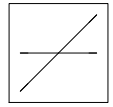

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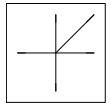
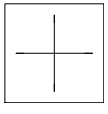
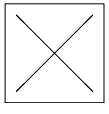
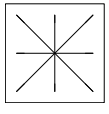


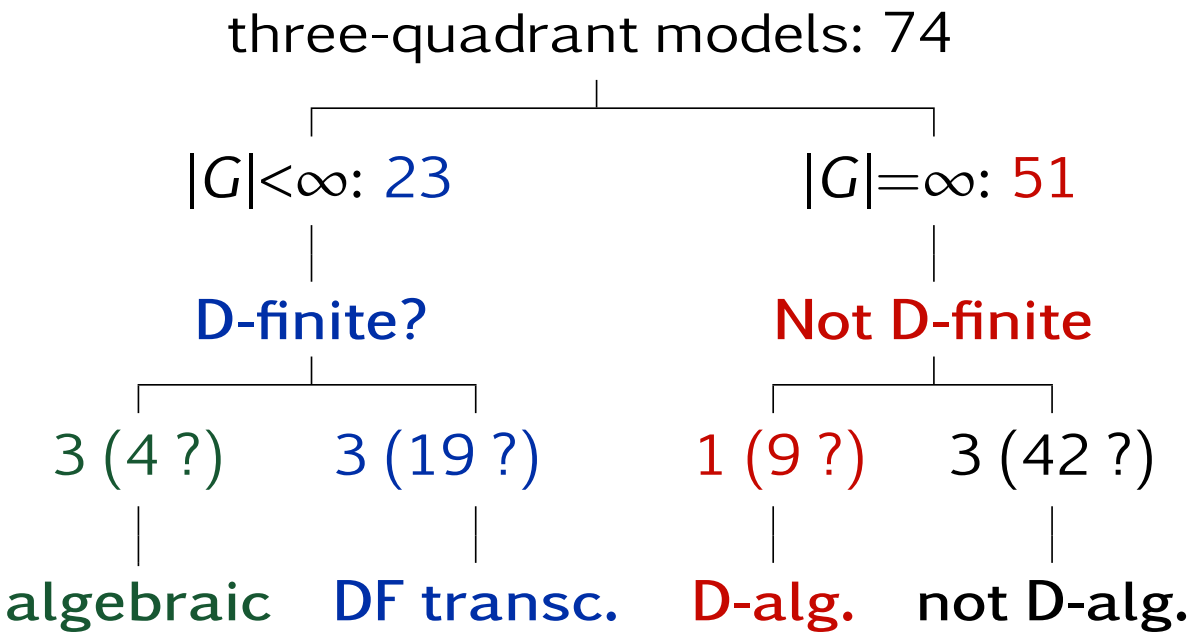
- [Beyond purely algebraic solutions] Revisit some solved D-finite cases, e.g. NSEW walks, where the crux is to prove the algebraicity of the series $A(x,y)$ defined by:

$$xy(1-t(x+\bar{x}+y+\bar{y}))A(x,y) = \frac{1}{3}(2xy+\bar{x}y+x\bar{y}) - txA_{-}(\bar{x}) - tyA_{-}(\bar{y}).$$

[mbm 16], [mbm-Wallner 21]

Partial classification of three quadrant walks



+ results for excursions [Budd 20, Elvey Price 20]

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algebraic [mbm]
DF transc. [mbm 16, Raschel-Trotignon 19, mbm-Wallner 21]
Non-D-finite [Mustapha 19]
D-alg./not D-alg [Dreyfus-Trotignon 20(a)]