Automata based graph algorithms for logically defined problems

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(includes current work with Irène Durand)

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References :

B.C. & J. Engelfriet *: Graph structure and monadic second-order logic,* Cambridge University Press, 2012.

BC & I. Durand : *Automata for the verification of monadic second-order graph properties*, J. Applied Logic, 10 (2012) 368-409

BC & I. Durand : *Computing by fly-automata beyond MSOL,* Preprint, Abstract in Proceedings of CAI 2013 (Conference on Algebraic Informatics).

Topics of lectures

Algorithmic meta-theorems : existence and construction of (relatively) efficient graph algorithms from logical descriptions of the problems.

These lectures : meta-theorems based :

on problem descriptions in (extensions of) MSO (Monadic Second-Order) logic,

on hierarchical decompositions of graphs

and on automata, possibly with *infinitely many states.*

A kind of *theory* of *dynamic programming.*

Summary of 3 lectures

Part 1

First example : construction of a finite automaton for the 2-colorability of series-parallel graphs.

Graph decompositions expressed by algebraic terms : tree-width and clique-width, parameters for FPT and XP graph algorithms.

Automata based algorithms : the general scheme.

Difficulty : the size of automata; the example of connectedness.

Fly-automata : definitions.

3 types of fly-automata : P, FPT and XP.

Part 2

Monadic Second-Order logic : definitions, examples.

The main construction : from MSO formulas to automata

(accepting clique-width terms).

Existential quantifications and nondeterministic automata.

Example : colorability problems.

Part 3 (Recent work)

Beyond MSO logic for graph properties and functions.

A fly-automaton for regularity of graphs (not an MSO property).

Boolean and first-order constructions of properties and functions, and their interpretations in terms of fly-automata

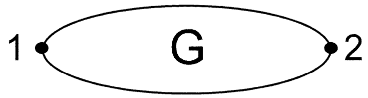
Monadic-second order constructions; spectra.

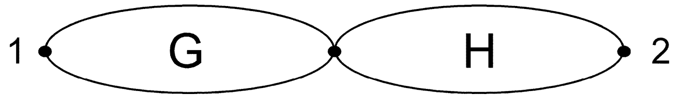
Implementation (in AUTOGRAPH) and tests.

Conclusions and call for interesting problems to handle in this way

2-colorability of Series-Parallel (SP) graphs

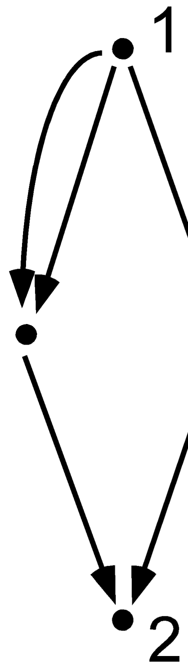
Graphs with distinguished vertices marked 1 and 2, generated from $e = 1 \rightarrow 2$ by the operations of *parallel-composition* // and *series-composition* •

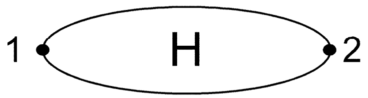


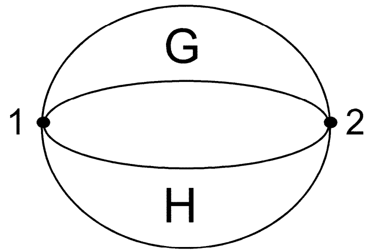


G • H

((e // e) • e) // (e • e)







G//H

The defining equation is $S = S \mathbin{//} S \ \cup\ S \bullet S \ \cup\ \{ e \}$

Inductive computation : Test of 2-colorability for SP graphs

Not all series-parallel graphs are 2-colorable (see K_3)

G and H 2-colorable does not imply that $G//H$ is 2-colorable (see $K_3 = P_3//e$).

One can check 2-colorability with 2 auxiliary properties :

Same(G) = G is 2-colorable with sources of the same color,

Diff(G) = G is 2-colorable with sources of different colors

by using the rules :

$$\mathrm{Diff}(e) = \mathrm{True} \ ; \ \mathrm{Same}(e) = \mathrm{False}$$

$$\mathrm{Same}(G//H) \Leftrightarrow \mathrm{Same}(G) \wedge \mathrm{Same}(H)$$

$$\mathrm{Diff}(G//H) \Leftrightarrow \mathrm{Diff}(G) \wedge \mathrm{Diff}(H)$$

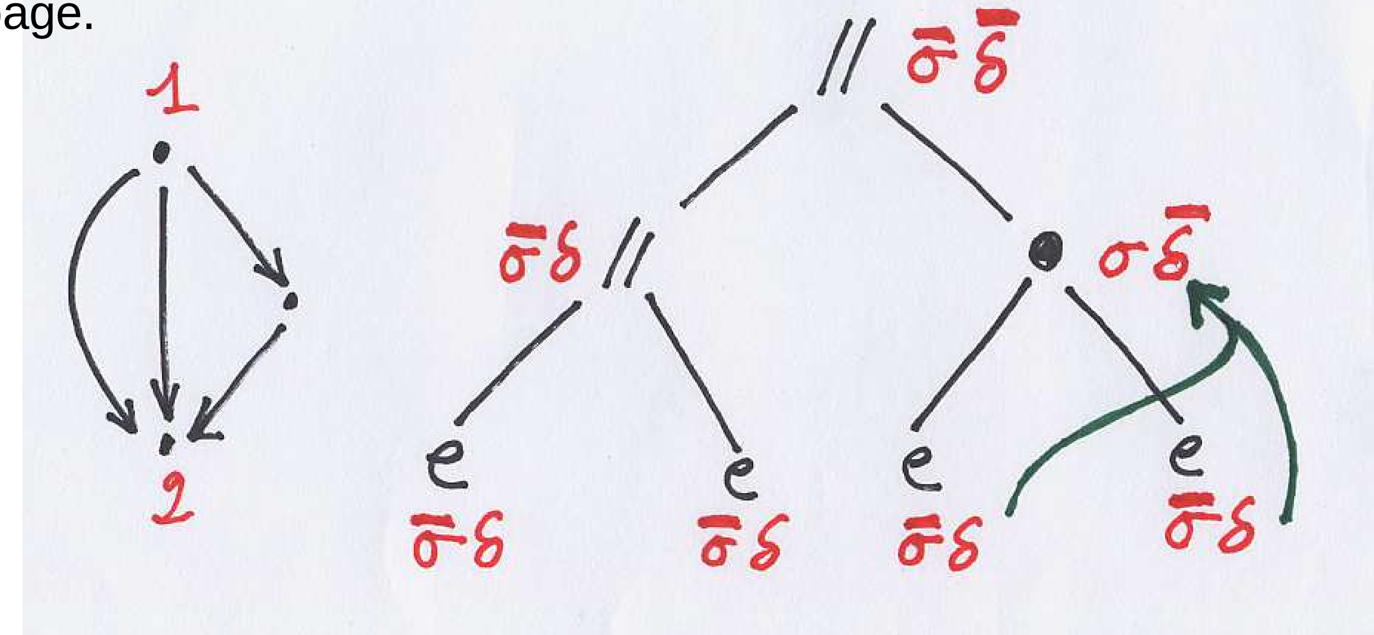
$$\mathrm{Same}(G \bullet H) \Leftrightarrow (\mathrm{Same}(G) \wedge \mathrm{Same}(H)) \vee (\mathrm{Diff}(G) \wedge \mathrm{Diff}(H))$$

$$\mathrm{Diff}(G \bullet H) \Leftrightarrow (\mathrm{Same}(G) \wedge \mathrm{Diff}(H)) \vee (\mathrm{Diff}(G) \wedge \mathrm{Same}(H))$$

Application : An algorithm based on a finite bottom-up automaton

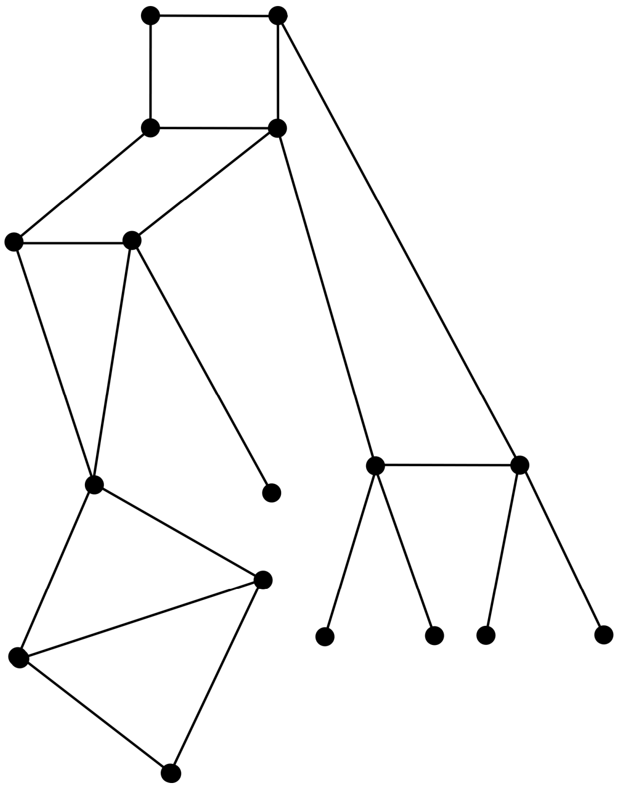
For every term t, we can compute, by running a finite deterministic bottom-up automaton on t, the pair of Boolean values (Same(G(t)) , Diff(G(t))), where G(t) is the graph value of t. We get the answer whether G(t) is 2-colorable.

Example : σ at node u means that Same(G(t /u)) is true, $\overline{\sigma}$ that it is false, δ that Diff (G(t /u)) is true, etc… Computation is *done bottom-up* with the rules of previous page.

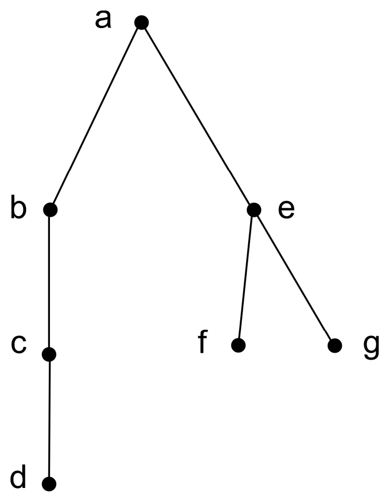


Answer : the graph is not 2-colorable.

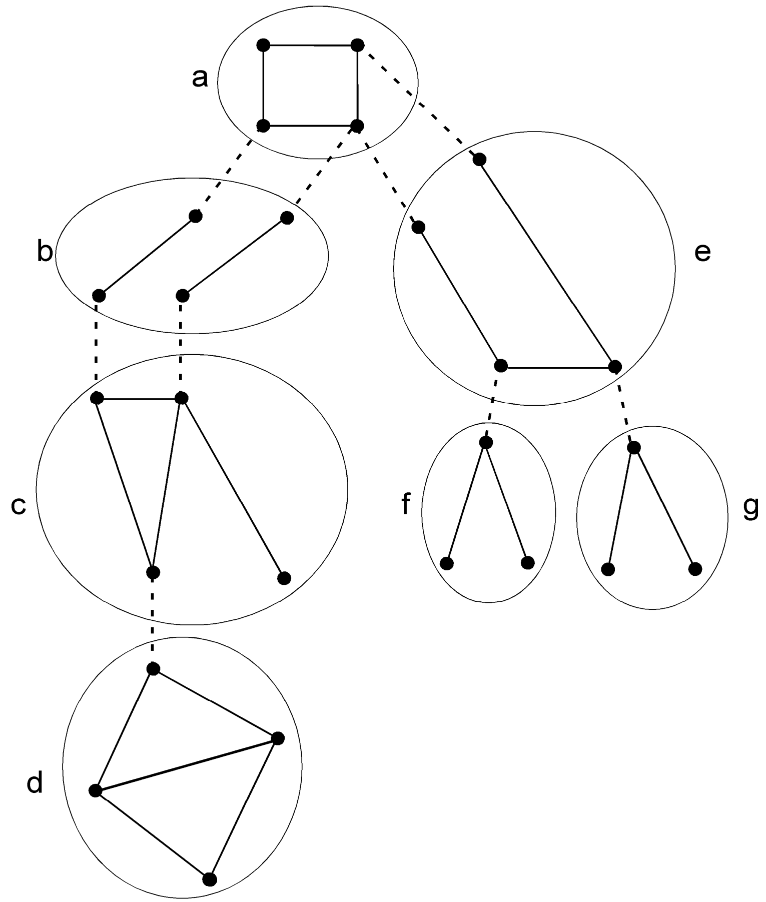
Algebraic view of tree-decompositions



Graph G



Tree T



Tree-decomposition of G

Dotted lines - - - - link *copies* of a same vertex.

Width = max. size of a box -1. Tree-width = minimal width of a tree-decomposition

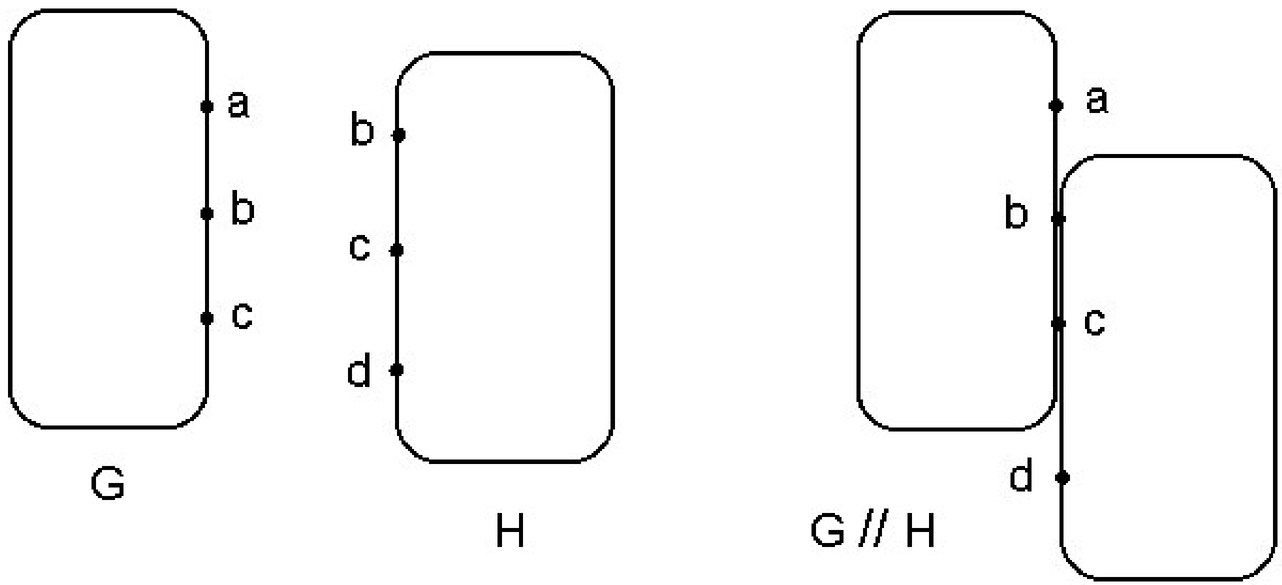
Graph operations and terms for tree-decompositions

Graphs have distinguished vertices called *sources,* (or terminals or boundary vertices) pointed to by source labels from $\{a, b, c, \ldots, d\}$.

Binary operation : *Parallel composition*

G // H is the disjoint union of G and H and sources with same label are fused.

(If G and H are not disjoint, one first makes a copy of H disjoint from G).



Unary operations :

Forget a source label

$Forget_a(G)$ is G without any *a*-source: the source is no longer distinguished (it is made "internal").

Source renaming :

$Ren_{a \leftrightarrow b}(G)$ exchanges source labels *a* and *b*

(replaces *a* by *b* if *b* is not the label of any source)

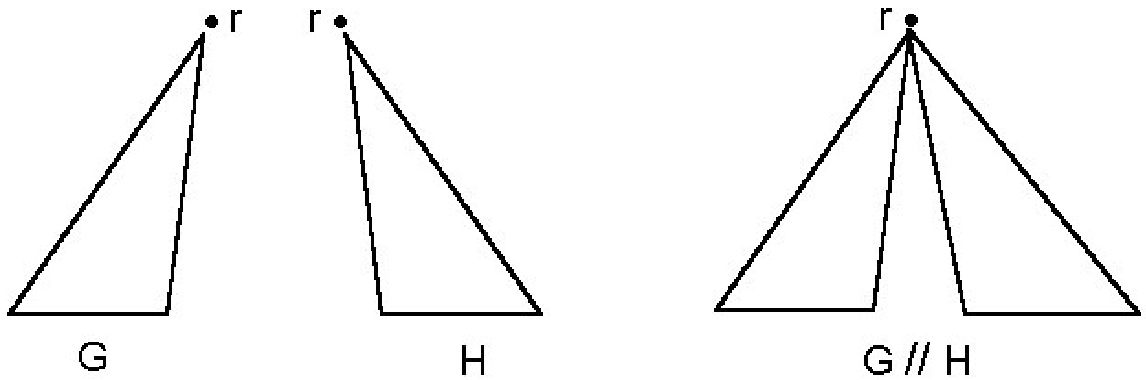
Nullary operations denote *basic graphs* : edge graphs, isolated vertices.

Terms over these operations *define* (or *denote*) graphs (with or without sources)

Example : Trees

Constructed with two source labels, *r* (root) and *n* (new root).

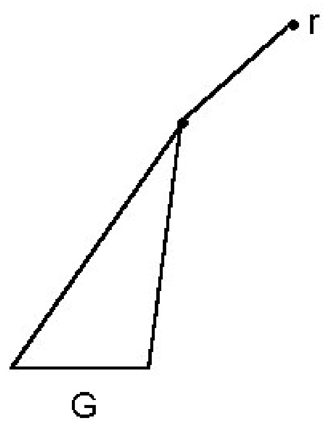
Fusion of two trees at their roots :



Extension of a tree by parallel composition with a new edge, forgetting the old root, making the "new root" as current root :

$$e = \boldsymbol{r} \; \bullet\!\!-\!\!-\!\!-\!\!\bullet \; \boldsymbol{n}$$

*Ren**n*** ***r*** (*Forget**r*** (G // e))



Defining equation *: T = T // T ∪ extension(T) ∪* ***r***

Series-parallel graphs have tree-width 2.

Proposition: A graph has tree-width $\leq$ k

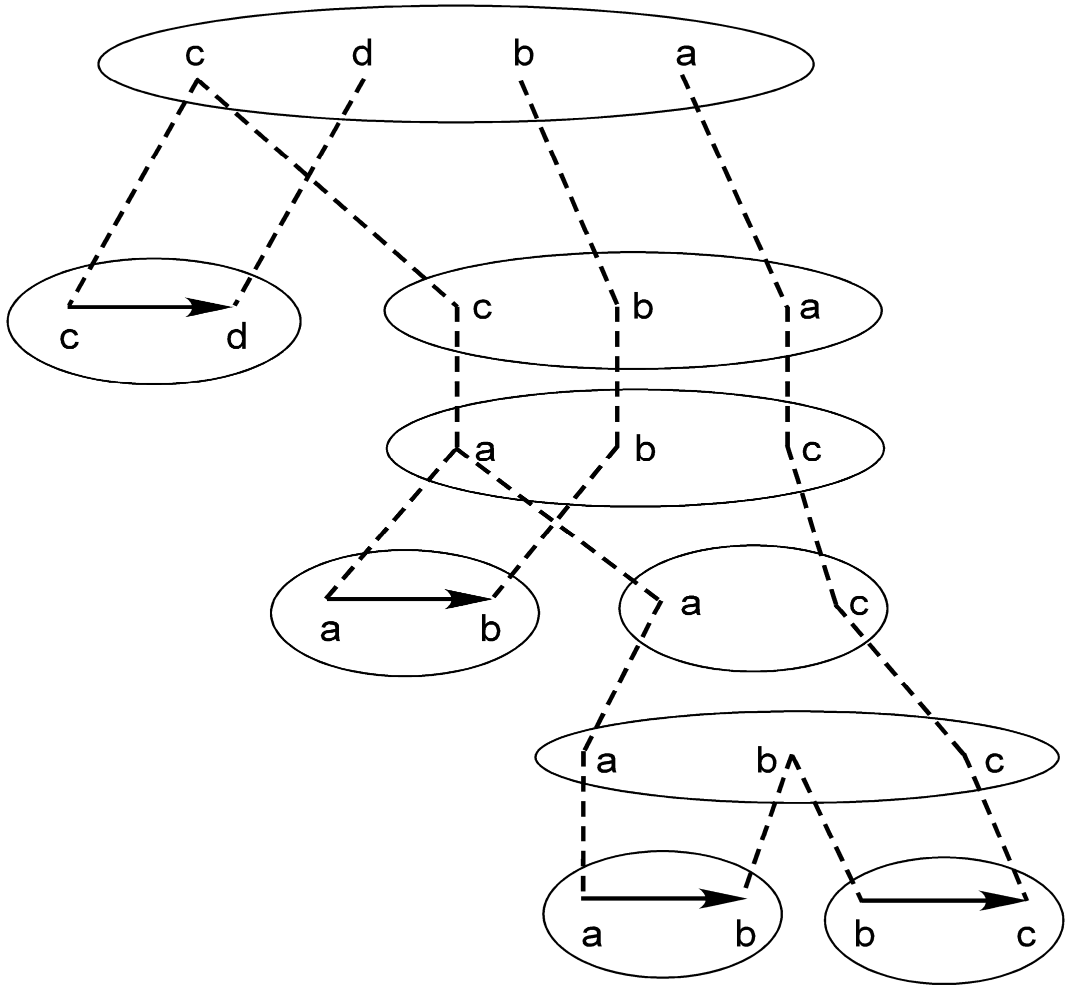
if and only if it can be constructed from edges by using the operations //, *Ren* $_{a \leftrightarrow b}$ and *Forget* $_a$ with $\leq$ k+1 labels a,b, ….

Consequences :

- Representation of tree-decompositions by terms.
- Algebraic characterization of tree-width.
- Terms as inputs to graph algorithms

From an algebraic expression to a tree-decomposition

Example : cd // $Ren_{a \leftrightarrow c}$ (ab // $Forget_b$(ab // bc)) (ab denotes an edge from a to b)



Graph operations for *defining* clique-width

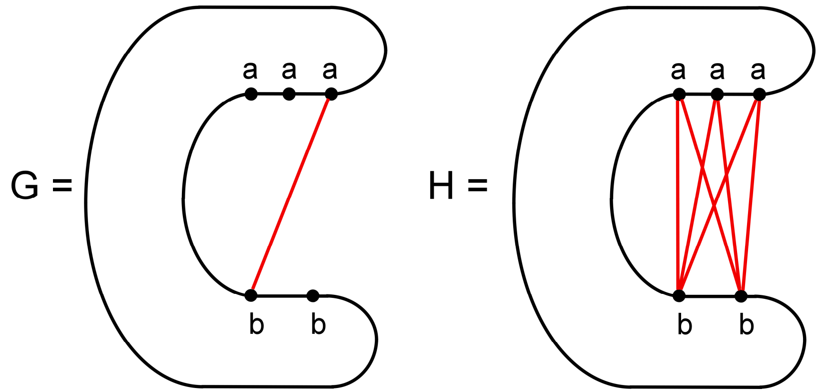
Graphs are simple, directed or not, and labelled by *a , b , c, ...* .

A vertex labelled by *a* is called an *a-vertex*.

One binary operation: *disjoint union* : $\oplus$

Unary operations: edge addition denoted by $Add_{a,b}$

$Add_{a,b}$(G) is G augmented
with directed or undirected edges
from every *a*-vertex to every *b*-vertex.
The number of added edges depends
on the argument graph.



H = $Add_{a,b}$(G) ; only 5 new edges added vertex relabellings :

$Relab_{a \longrightarrow b}(G)$ is G with every *a*-vertex is made into a *b*-vertex

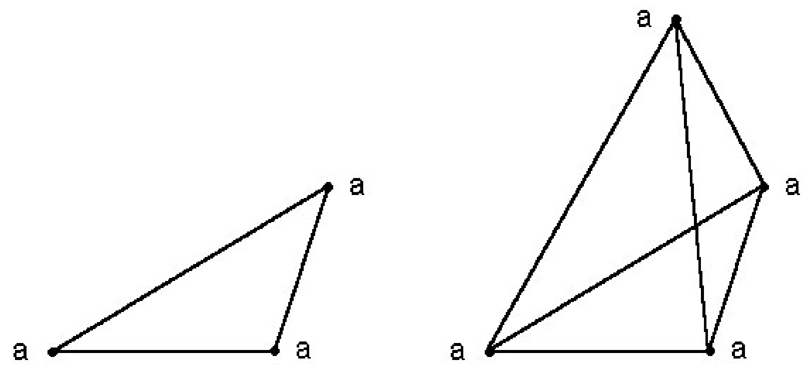
Basic graphs : those with a single vertex.

Definition: A graph G has clique-width $\leq$ k $\Leftrightarrow$ G=G(t) is defined by a term t using $\leq$ k labels.

Example : Cliques have clique-width 2.

K_n is defined by t_n where t_{n+1} =

$Relab_{b \longrightarrow a}(Add_{a,b} (t_n \oplus \mathbf{b}))$



Tree-width and clique-width

Proposition : (1) Bounded tree-width implies bounded clique-width ($cwd(G) \leq 2^{2twd(G)+1}$ for G directed), but not conversely.

(2) Unlike tree-width, clique-width is sensible to edge directions : Cliques have clique-width 2, tournaments have unbounded clique-width.

Classes of unbounded tree-width and bounded clique-width:

Distance hereditary graphs (3),

Graphs without $\{P_5, \mathbf{1}\otimes P_4\}$ (5), or $\{\mathbf{1}\oplus P_4, \mathbf{1}\otimes P_4\}$ (16)

as induced subgraphs.

Classes of unbounded clique-width :

Planar graphs of degree 3, Tournaments, Interval graphs.

Graphs without induced P_5. (P_n = path with n vertices)

Exercises

1) Complete the proof of the proposition page 14: transform a tree-decomposition of width k into a term that defines the same graph and uses $k+1$ source labels.
2) Prove that this proposition holds without the source renaming operations.
3) What is the maximal clique-width of a SP graph ?
4) Give upper-bounds to the tree-width and the clique-width of the rectangular $n \times m$ grids.
5) Give an upper bound to the clique-width of a graph whose biconnected components have clique-width at most k.

The parsing problem: construction of decompositions

Automata take terms as inputs, not graphs : the parsing must be done before. (Graph automata do not exist in a satisfactory way).

A difficult problem : deciding $twd(G) \leq k$ and $cwd(G) \leq k$ (for input (G,k)) are NP-complete problems.

There are FPT approximation algorithms, taking time $f(k).n^a$, that output the following for given k and G with n vertices:

(i) either the answer that $wd(G) > k$,

(ii) or a term witnessing that $wd(G) \leq g(k)$.

Hence from an algorithm taking as input a term t in $T(F_k)$ (F_k : the operations for terms of width $\leq k$) and whose computation time is $h(k).n^b$, we get (by trying $k = 1, 2, \ldots$ until we reach Case (ii)) an FPT algorithm for given G with computation time $\leq m(wd(G)).n^{\max(a,b)}$

Algorithms : for tree-width : see Bodlaender et al., Information and Computation 2010 and 2011, ACM Trans. Algos 2012).

For clique-width : approximation algorithms based on articles by Oum, Seymour, Hlineny, Kanté, 2005-2013).

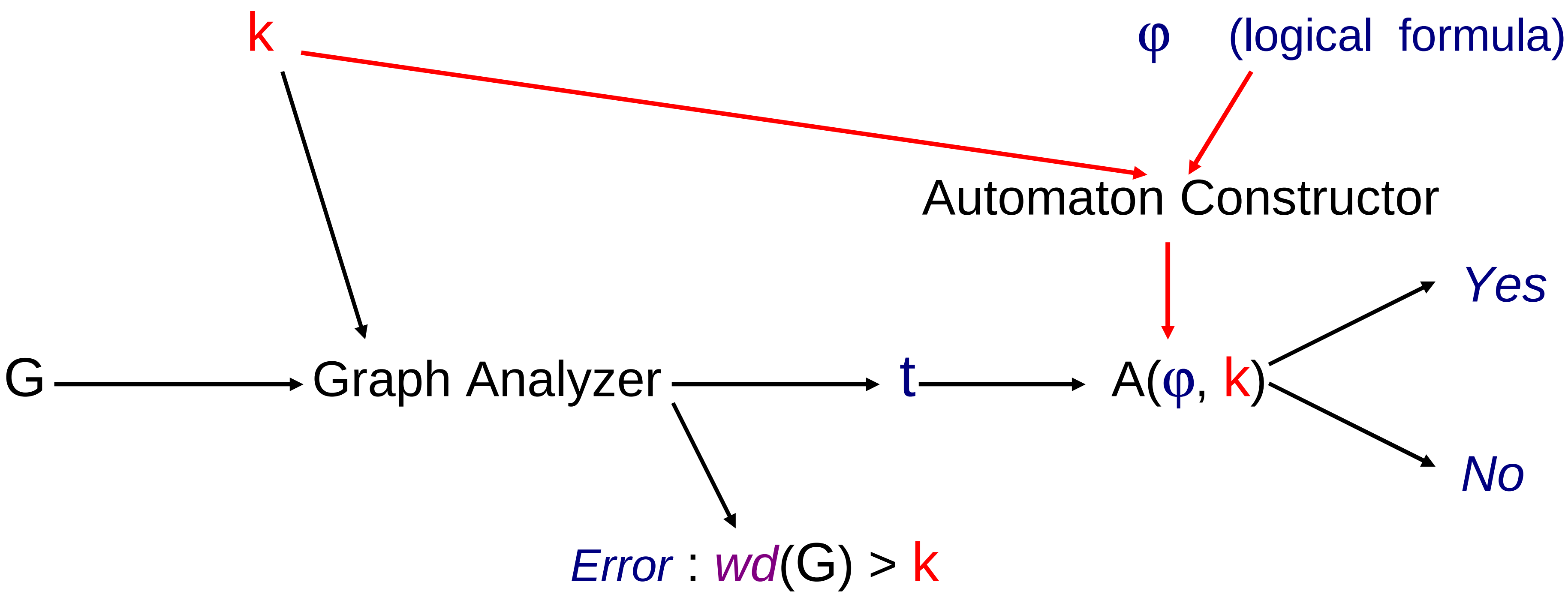
However, graphs arising from concrete problems are not random. They may have "natural" hierarchical decompositions from which terms of small tree-width or clique-width are not hard to find.

Compilation : flow-graphs of structured programs have

tree-width $\leq$ 6.

In linguistics and chemistry: graphs of tree-width $\leq$ 3.

Algorithmic meta-theorems through automata: the general scheme



Steps ⟶ are done "once for all", independently of G

A(φ,k): finite automaton on terms t

wd = tree-width or clique-width or equivalent notion.

Automata on terms that check graph properties

Terms are seen as labelled trees. We want to check a property $P(G)$, for $G = G(t)$, t in $T(F)$.

For each *labelled* graph G, we define some piece of information $q(G)$ consisting of properties of G and of values attached to G, with:

(i) inductive behaviour of q : for f in F and graphs G,H:

$$q(f(G,H)) = f^{q}(q(G), q(H))$$

for some computable function f^{q} .

(ii) $P(G)$ can be decided from $q(G)$.

Recall the 2-colorability of SP graphs, page 8.

Then $q(G(t/u))$ is computed bottom-up in a term t, for each node u. This information is relative to the graph $G(t/u)$ defined by the subterm t/u of t issued from u.

$q(G(t/u))$ is a state of a *finite or infinite deterministic bottom-up automaton*.

These automata formalize some form of dynamic programming.

In the sequel we only consider clique-width: the automata are simpler to build and they can be adapted to bounded tree-width as bounded tree-width implies bounded clique-width.

Now an example.

The deterministic automaton for connectedness.

The state at node u is the *set of types (sets of labels)* of the connected components of the graph G(t/u). For k labels (k = bound on clique-width), the set of states has size $\leq 2^{\wedge}(2^{\wedge}k)$.

Proved lower bound : $2^{\wedge}(2^{\wedge}k/2)$.

→ Impossible to "compile" the automaton (i.e., to list the transitions) .

Example of a state : q = { {a}, {a,b}, {b,c,d}, {b,d,f } }, (a,b,c,d,f : labels).

Some transitions :

$Add_{a,c}$: $q \longrightarrow$ { {a,b,c,d}, {b,d,f } },

$Relab_{a \to b}$: $q \longrightarrow$ { {b}, {b,c,d}, {b,d,f } }

Transitions for ⊕ : union of sets of types.

Note *: Also state (p,p) if G(t/u) has ≥ 2 connected components, all of type p.*

In a *fly-automaton* : the states and transitions are *computed* and not tabulated.

We allow fly-automata with *infinitely* many states and with *outputs* : numbers, finite sets of tuples of numbers, etc.

Example continued : For computing the number of connected components, we use states such as :

q = { ({a}, 4), ({a,b}, 2), ({b,c,d},2), ({b,d,f },3) },

where 4, 2, 2, 3 are the numbers of connected components of respective types {a}, {a,b}, {b,c,d}, {b,d,f }.

Fly-automaton (FA)

Definition : A = < F, Q, δ, Out >

F : finite or countable (effective) signature (set of operations),

Q : finite or countable (effective) set of states (integers, pairs of integers, finite sets of integers: states can be encoded as finite words, integers in binary),

Out : Q $\rightarrow$ D (an effective domain, i.e., set of finite words), computable.

δ : computable (bottom-up) transition function

Nondeterministic case : δ is *finitely multi-valued*.

This automaton defines a computable function : T(F) → *D*

(or : T(F) → *P*(*D*) if it is not deterministic)

If *D* = { *True, False* }, it defines a decidable property, equivalently,

a decidable subset of T(F).

Deterministic computation of a nondeterministic FA :

bottom-up computation of *finite* sets of states (classical simulation of the determinized automaton): these states are the useful ones of the *determinized automaton*; these sets are *finite* because the transition function is finitely multivalued.

Fly-automata are "implicitly determinized" and they run deterministically

Computation time of a fly-automaton

F : all graph operations, F_k : those using k labels.

On term $t \in T(F_k)$ defining $G(t)$ with n vertices, if a fly-automaton

takes time bounded by :

$(k + n)^c$ → it is a P-FA (a polynomial-time FA),

$f(k).n^c$ → it is an FPT-FA,

$a.n^{g(k)}$ → it is an XP-FA.

The associated algorithm is polynomial-time, FPT or XP for clique-width as parameter.

Proposition : Every polynomial-time computable function : T(F) → *D* is computable by a fly-automaton whose computation time is polynomial.

Nothing new ! : Our concern is to have easy and uniform *constructions* of FA's from *logical and combinatorial* descriptions of functions and properties.

Theorem : Every graph property expressible in monadic second-order (MS) logic can be checked by a fly-automaton whose restriction to each subsignature F_k has finitely many states.

Hence, it is a *linear* FPT-FA.

Linear : its computation-time is bounded by f(k).n