

CONNECTIVITY CHECK IN 3-CONNECTED PLANAR GRAPHS WITH OBSTACLES

M. M. KANTÉ¹ B. COURCELLE¹ C. GAVOILLE¹ A. TWIGG²

¹Université Bordeaux 1, LaBRI, CNRS.

²Computer Laboratory, Cambridge University.

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FORBIDDEN-SET ROUTING

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How to check the connectivity of u and v locally ?
- [Solution] Assign each node a label that must be computed once.
- But we cannot represent the whole graph in each node (space constraints).
- [Optimistic] We want “short” labels (say of (poly)logarithmic size) in each node.

1 PRELIMINARIES

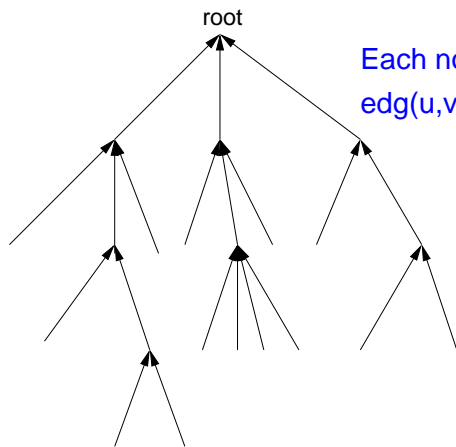
2 Plane Graphs

3 Representation by Unary Functions

LABELING SCHEME

- Let $P(x_1, \dots, x_p, Y_1, \dots, Y_q)$ be a graph property.
- An $f(n)$ -labeling scheme for property P in a class \mathcal{C} of n -vertex graphs is a pair of algorithms $(\mathcal{A}, \mathcal{B})$ such that:
 - ▶ For all $G \in \mathcal{C}$, \mathcal{A} constructs a labeling $J: V(G) \rightarrow \{0, 1\}^*$ such that $|J(x)| \leq f(n)$ for each $x \in V(G)$.
 - ▶ \mathcal{B} checks whether G satisfies $P(a_1, \dots, a_p, U_1, \dots, U_q)$ by using $J(a_1), \dots, J(a_p), J(U_1), \dots, J(U_q)$ where $J(U) = \{J(x) \mid x \in U\}$.

- A $2 \log(n)$ -labeling scheme for adjacency in the class of forests.



Each node u store its father $p(u)$
 $\text{edg}(u,v)$ iff $u=p(v)$ or $v = p(u)$

- The construction can be extended to planar graphs (union of 3 forests).

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- We let $d(x, y, X)$ denote the distance between x and y in the graph $G \setminus X$ (subgraph of G induced by $V(G) - X$).
- Courcelle and Twigg have proved that $d(x, y, X)$ admits an $O(\log^2(n))$ -labeling scheme in the class of graphs of bounded clique-width (STACS'07).
- Any property expressible in monadic second order logic admits an $O(\log(n))$ -labeling scheme in the class of graphs of bounded clique-width (Courcelle and Vanicat 2003).
- Any property expressible in first order logic with set arguments admits an $O(\log(n))$ -labeling scheme in the class of graphs that are locally tree-decomposable (Courcelle et al. 2008).

CONNECTIVITY QUERY

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- For $u, v \in V(G)$, $X \subseteq V(G) - \{u, v\}$ we let $Conn(u, v, X)$ mean:
 u and v are in the same connected component in $G \setminus X$.

MAIN THEOREM

There exists an $O(\log(n))$ -labeling scheme for $Conn(u, v, X)$ in the class of 3-connected planar graphs.

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REMARK

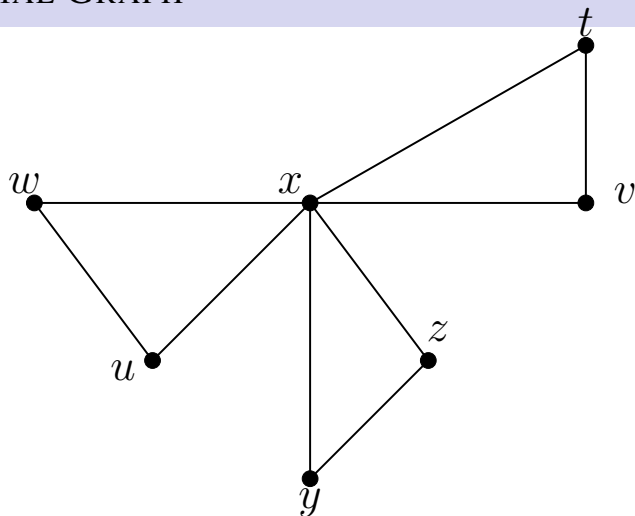
We have extended the main theorem to **all planar graphs**, but we will only give the proof for 3-connected planar graphs.

1 Preliminaries

2 **PLANE GRAPHS**

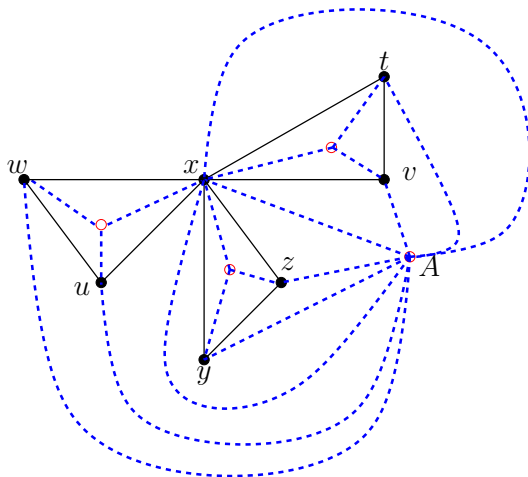
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RADIAL GRAPH



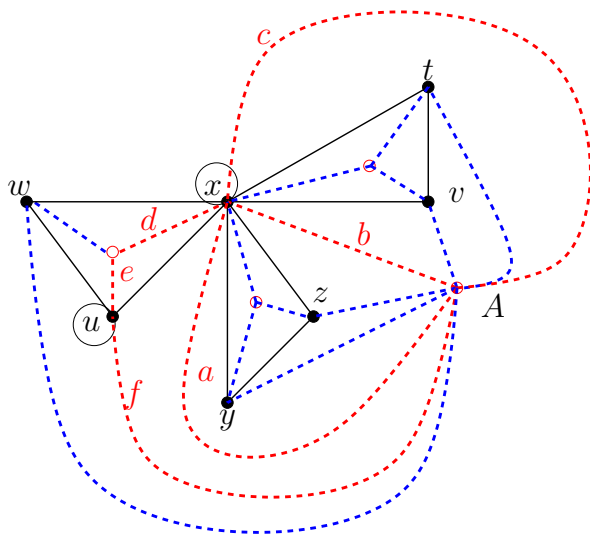
Plane Graph $G = (V, E, F)$

RADIAL GRAPH



$$G^+ = (V \cup F, \underbrace{E \cup \{ \{f, x\} \mid x \in V, f \in F \text{ and } x \text{ incident with } f \}}_{E'})$$

BARRIER WITH RESPECT TO EMBEDDING



$X = \{x, u\}$, $Bar(X, \mathcal{E}^+) = \text{Red Lines}$

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Let $\mathcal{E}^+ = (p, s)$ be an embedding of G^+ . We let

$$\text{Bar}(X, \mathcal{E}^+) = \{s(e) \mid e \in \text{Bar}(X)\}$$

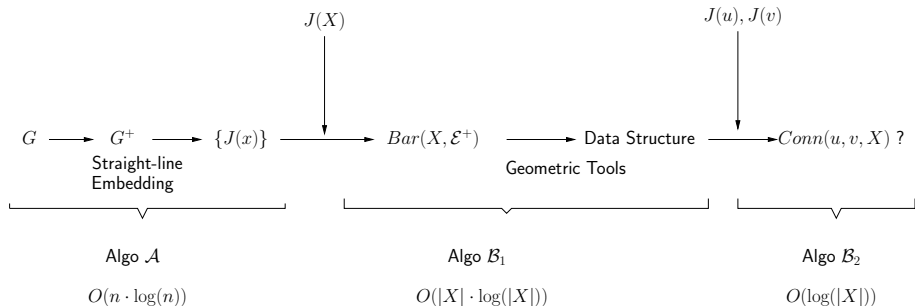
where $\text{Bar}(X)$ is the set

$$\left\{ \{f, x\} \in E' \mid x \in X \text{ and there exists } y \in X \text{ such that } \{f, y\} \in E' \right\}.$$

REMARK

In the sequel, We let \mathcal{E}^+ be a straight-line embedding in order to use geometric tools.

OVERVIEW OF THE ALGORITHMS

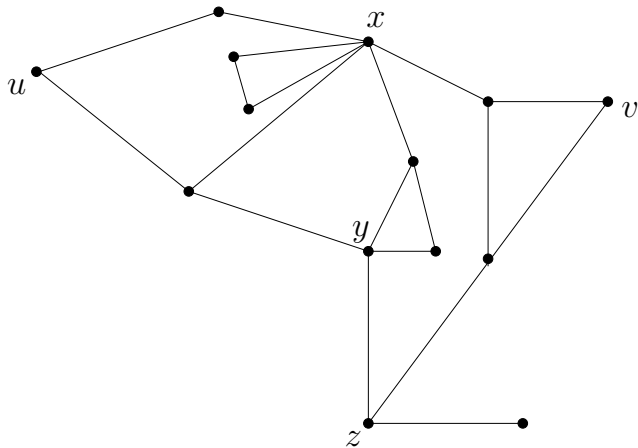


PROPOSITION 1

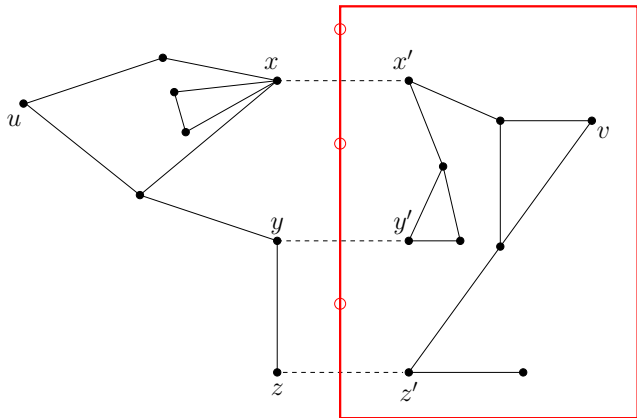
PROPOSITION 1

Let $\mathcal{E}^+ = (p, s)$ be an embedding of G^+ . For every $X \subseteq V$ and $u, v \in V - X$, u and v are separated by X if and only if $p(u)$ and $p(v)$ are separated by $\text{Bar}(X, \mathcal{E}^+)$.

PROOF SKETCH

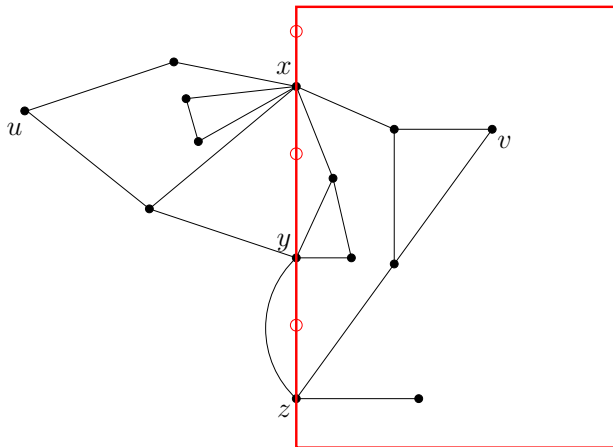


PROOF SKETCH



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Red lines = $\text{Bar}(X, \mathcal{E}^+)$, $X = \{x, y, z\}$ and separates u and v



WHY LABELS OF VERTICES IN X

- Each connectivity request is composed of two vertices $x, y \in V(G)$ and of a set of vertices $X \subseteq V(G)$ to be removed.
- By **Proposition 1**, x and y are separated by X iff $p(x)$ and $p(y)$ are separated by $Bar(X, \mathcal{E}^+)$.
- We cannot store in the labels of each vertex all the possible barriers.
- To have a chance to get an $O(\log(n))$ -labeling scheme, we will specify $Bar(X, \mathcal{E}^+)$ in the labels of vertices in X .

1 Preliminaries

2 Plane Graphs

3 REPRESENTATION BY UNARY FUNCTIONS

REPRESENTATION OF ADJACENCY BY UNARY FUNCTIONS

LEMMA 1 (FOLKLORE)

There exists a $4 \log(n)$ -labeling scheme for adjacency in planar graphs.

PROOF.

- Edge set $E(G)$ can be partitioned into 3 forests E_1, E_2 and E_3 .
- For each forest E_i there exists a $2 \log(n)$ -labeling scheme for adjacency (each vertex x has label $(x, g_i(x))$) Forests).
- for each x we let $J(x) = (x, g_1(x), g_2(x), g_3(x))$ (of size $4 \log(n)$).
- Then x and y are adjacent in G iff

$$x \neq y \wedge \left(\bigvee_{1 \leq i \leq 3} x = g_i(y) \vee y = g_i(x) \right).$$

SAME FACE PROPERTY

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Let $G = (V, E, F)$ be a plane graph. Two vertices x and y **satisfy the same face property** if there exists at least one face incident with x and y .

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PROPOSITION 2

There exists a $15 \log(n)$ -labeling scheme for same face property in connected plane graphs.

PROOF OF PROPOSITION 2(1)

- We let g_i^+ for $i = 1, 2, 3$ be the 3 partial functions representing adjacency in G^+ .
- x and y satisfy the same-face property iff:

$$\bigvee_{1 \leq i, j \leq 3} g_i^+(x) = g_j^+(y) \in F \quad (1a)$$

$$\vee \bigvee_{1 \leq i, j \leq 3} g_i^+(x) \in F \wedge g_j^+(g_i^+(x)) = y \quad (1b)$$

$$\vee \bigvee_{1 \leq i, j \leq 3} g_i^+(y) \in F \wedge g_j^+(g_i^+(y)) = x \quad (1c)$$

$$\vee \exists f \in F. \left(\bigvee_{1 \leq i, j \leq 3} g_i^+(f) = x \wedge g_j^+(f) = y \right). \quad (1d)$$

PROOF OF PROPOSITION 2(2)

CONDITION (1A)

$g_i^+(x) \in F$ is replaced by

$g'_i(x) = \text{if } g_i^+(x) \in F \text{ then } g_i^+(x) \text{ else undefined.}$

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CONDITION (1B)-(1C)

$g_i^+(x) \in F \wedge g_j^+(g_i^+(x)) = y$ is replaced by:

$g'_{i,j} = \text{if } g_i^+(x) \in F \text{ and } g_j^+(g_i^+(x)) \text{ is defined then } g_j^+(g_i^+(x))$
else undefined.

PROOF OF PROPOSITION 2(3) \Rightarrow (1D)

CONDITION (1D)

$$\exists f \in F \left(\bigvee_{1 \leq i, j \leq 3} g_i^+(f) = x \wedge g_j^+(f) = y \right)$$

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ELIMINATION OF THE EXISTENTIAL QUANTIFICATION

- Let $H = (V(G), E(H))$ where $xy \in E(H)$ iff $g_i^+(f) = x$ and $g_j^+(f) = y$ for some $i, j \in [3]$.
- H is planar because we can draw such an edge in a drawing of G by adding to each face of G at most 3 edges.
- Let h_1, h_2 and h_3 be the adjacency functions of H (Lemma 1).
- It is clear that (1d) can be replaced by

$$\bigvee_{1 \leq i \leq 3} h_i(x) = y \vee h_i(y) = x.$$

LABELING EACH VERTEX

LABEL OF x

- For each x we let

$$J(x) = \left((g'_i(x))_{i \in [3]}, (h_i(x))_{i \in [3]}, (g'_{i,j}(x))_{i,j \in [3]} \right).$$

- It is clear that $|J(x)| \leq 15 \log(n)$.
- By [Equation \(1\)](#) we can answer same face property between x and y just by looking at labels $J(x)$ and $J(y)$.

LABELING SCHEME FOR FACE SELECTION

FACE SELECTION PROBLEM

Let $G = (V, E, F)$ be a 3-connected plane graph. For every $x, y \in V$ we let $Faces(x, y)$ mean the **selection** of the at most **two faces incident with x and y** . We call it the **face selection problem**.

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PROPOSITION 3

There exists a $27 \log(n)$ -labeling scheme for face selection problem in 3-connected plane graphs.

PROOF OF PROPOSITION 3(1)

- We let g_i^+ for $i = 1, 2, 3$ be the 3 partial functions representing adjacency in G^+ .
- x and y satisfy the same-face property iff:

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PROOF OF PROPOSITION 3(2)

- 1 Either the two faces incident with x and y verify conditions (1a)-(1c).
- 2 Then we are done since we can select them by [Proposition 2](#) with labels of size $15 \log(n)$.

PROOF OF PROPOSITION 3(2)

- 1 Either the two faces incident with x and y verify conditions (1a)-(1c).
- 2 Then we are done since we can select them by Proposition 2 with labels of size $15 \log(n)$.
- 3 Or at least one verifies condition (1d).
- 4 Hence either $h_i(x) = y$ or $h_i(y) = x$ (6 cases mutually exclusive).
We let:

$h_{i,j}^+(x) = \text{Face}_j(x, h_i(x))$ if $h_i(x)$ and $\text{Face}_j(x, h_i(x))$ are defined,

$h_{i,j}^-(x) = \text{Face}_j(h_i(x), x)$ similarly.

PROOF OF PROPOSITION 3(3)

- It is clear that a face f satisfy (1d) if and only if

$$\bigvee_{i,j} f = h_{i,j}^+(x) \wedge y = h_i(x) \vee \bigvee_{i,j} f = h_{i,j}^-(y) \wedge x = h_i(y).$$

- For each x we let $J(x)$ be the label computed by Proposition 2. We let

$$L(x) = \left(J(x), h_{i,j}^+(x), h_{i,j}^-(x) \right).$$

- It is clear that $L(x)$ and $L(y)$ allow to select the at most two faces incident with x and y .

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- For each x we let:

$$D(x) = (p(x), L(x))$$

CORRECTNESS

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By using geometric tools.

POINT LOCATION PROBLEM

What we want is:

Given $Bar(X, \mathcal{E}^+)$, a set of straight-line segments, say if $p(u)$ and $p(v)$ are separated by $Bar(X, \mathcal{E}^+)$.

POINT LOCATION PROBLEM

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THEOREM 2 (TARJAN ET AL.)

Let Y be a set of m straight-line segments. One can construct in time $O(m \log(m))$ a data structure of size $O(m)$ and given any points u and v in the plane test in $O(\log(m))$ if u and v are separated by $\cup Y$.

CONCLUSION

- 1 We can extend the construction to deletions of edges by subdividing each edge of G and by letting for each edge $e = uv$ $D(e) = (D(u), D(v))$.
- 2 **Proposition 3** can be extended to graphs such that each pair of vertices is incident with a bounded number of faces.
- 3 By using logical tools from Courcelle and Vanicat- **$O(\log(n))$ -labeling scheme for MSO queries in trees**- we can construct an $O(\log(n))$ -labeling scheme for $Conn(u, v, X)$ on planar graphs.
- 4 Can we do the same for bounded-genus graphs ?
- 5 What about solving $d(u, v, X)$ (distance with failures) in planar graphs ?