

**The Ising and hard-particle models
on planar maps:
a combinatorial solution**

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The Ising model on a graph

N vertices

spin up

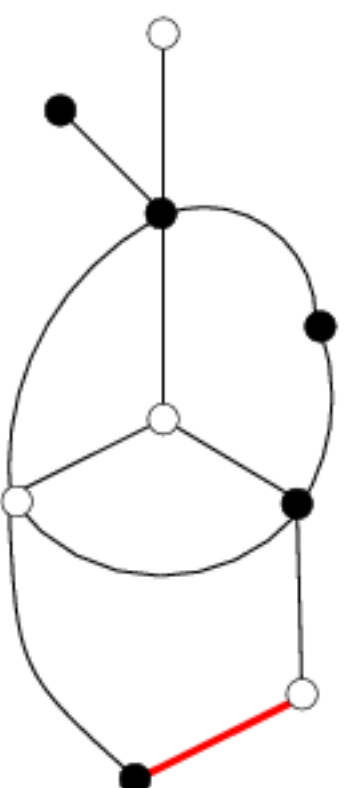


$\sigma_i = 1$

spin down



$\sigma_i = -1$



frustrated edge

- Partition function:

$$Z_I(G) = \sum_{\sigma=(\sigma_1, \dots, \sigma_N)} \exp \left(\frac{J}{kT} \sum_{i-j} \sigma_i \sigma_j + \frac{H}{kT} \sum_i \sigma_i \right) \quad (\text{physics})$$

The Ising model on a graph

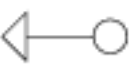
N vertices

spin up

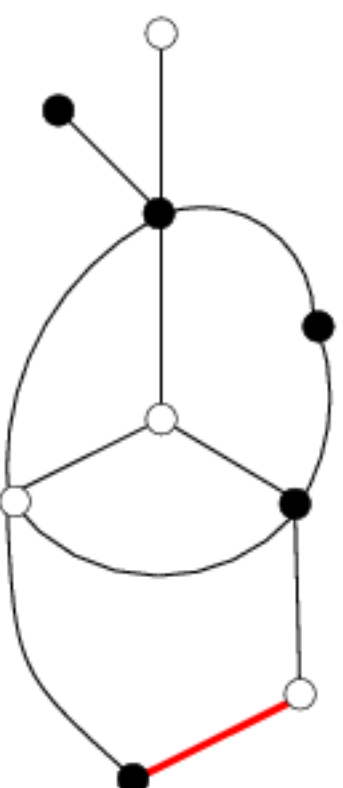


$$\sigma_i = 1$$

spin down



$$\sigma_i = -1$$



frustrated edge

- Partition function:

$$\begin{aligned} Z_I(G) &= \sum_{\sigma=(\sigma_1, \dots, \sigma_N)} \exp\left(\frac{J}{kT} \sum_{i-j} \sigma_i \sigma_j + \frac{H}{kT} \sum_i \sigma_i\right) && \text{(physics)} \\ &= \sum_{\sigma} x^{\circ} y^{\bullet} u^f && \text{(combinatorics)} \end{aligned}$$

with

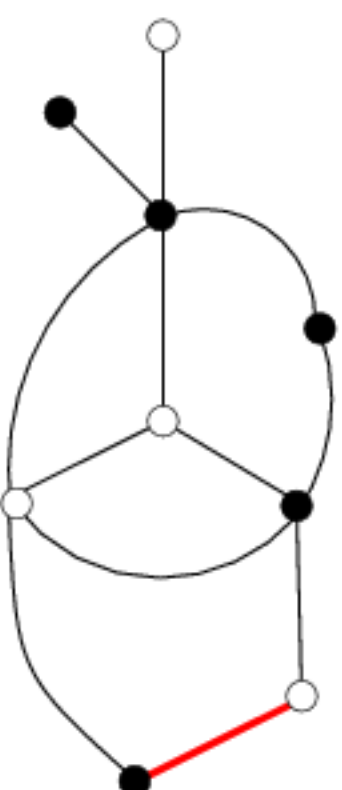
$$x = \exp\left(\frac{J-H}{kT}\right), \quad y = \exp\left(\frac{J+H}{kT}\right), \quad u = \exp\left(\frac{-2J}{kT}\right).$$

The Ising model on a graph

N vertices

spin up $\bullet \uparrow$ $\sigma_i = 1$

spin down $\circ \downarrow$ $\sigma_i = -1$



frustrated edge

- Partition function:

$$Z_I(G) = \sum_{\sigma=(\sigma_1, \dots, \sigma_N)} \exp \left(\frac{J}{kT} \sum_{i \sim j} \sigma_i \sigma_j + \frac{H}{kT} \sum_i \sigma_i \right) \quad (\text{physics})$$

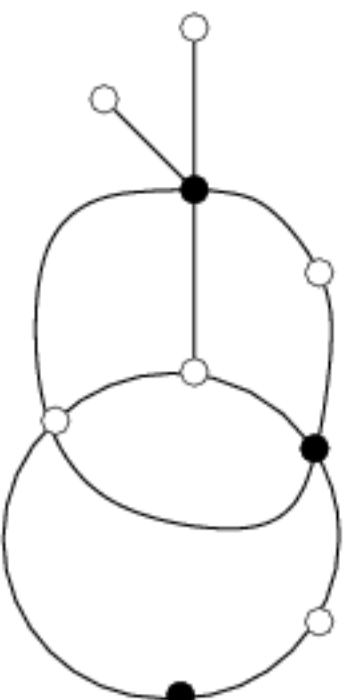
$$= \sum_{\sigma} x^{\circ} y^{\bullet} u^f \quad (\text{combinatorics})$$

- When $H = 0$ (no field), then $x = y$ and

$$Z_I(G) = x^N 2^{k(E)} u^{|E|} (1-u)^{r(E)} \mathcal{T} \left(G; \frac{1+u}{1-u}, u^{-1} \right).$$

The hard-particle model on a graph

- occupied vertex
- vacant vertex

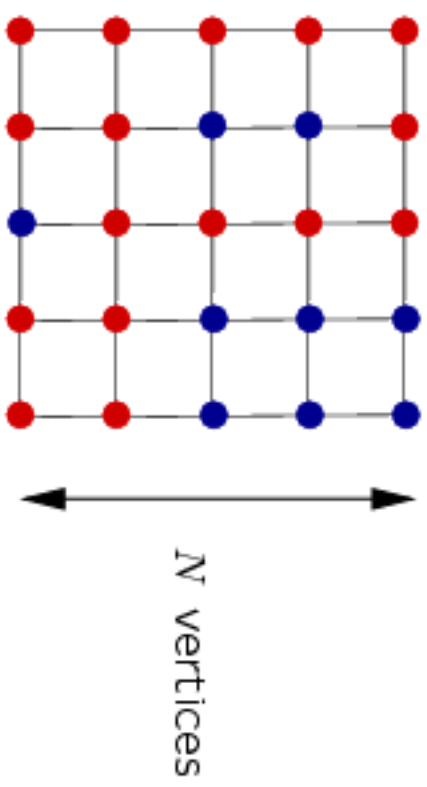
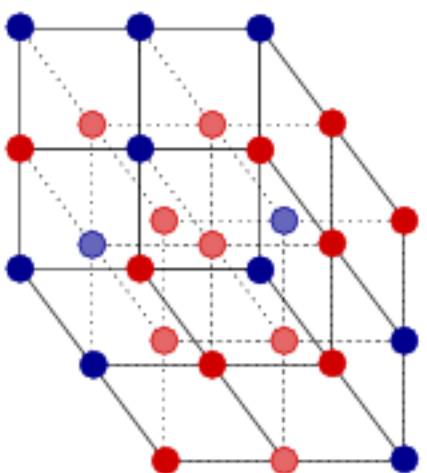


Two adjacent vertices
cannot be both occupied

- Partition function:

$$Z_{HP}(G) = \sum_{\sigma} x^o y^{\bullet}$$

Classical statistical physics



Aim Compute the following sum, over all possible configurations of spins:

$$Z(N) = \sum_{\text{conf.}} x^{\bullet} y^{\bullet} u^{\bullet} \dots$$

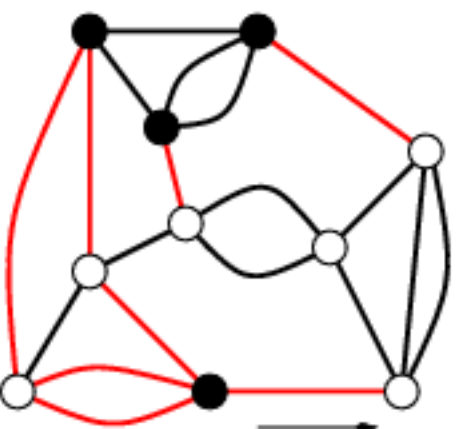
(Only solved in 2D, when $x = y \dots$)

Statistical physics on planar maps

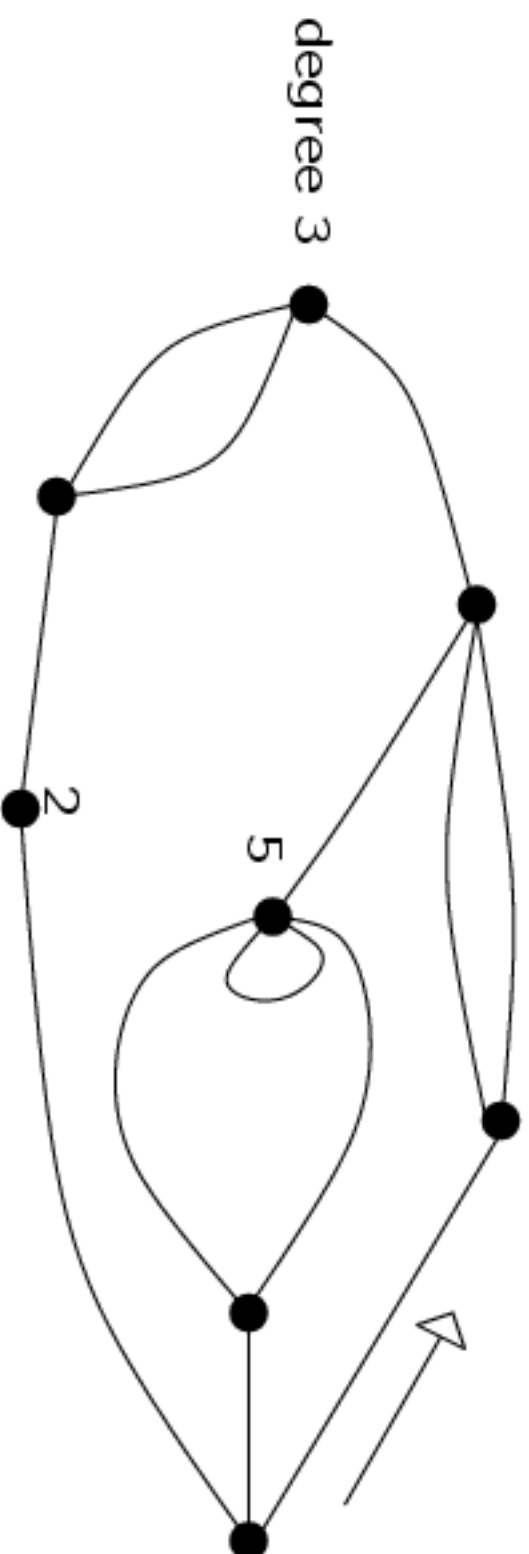
Two-dimensional quantum gravity:

1. replace the regular 2D grid by a random planar graph (or more precisely a random planar **map**)
2. average the partition function over all maps of a given size.

[t'Hooft, Brézin, Itzykson, Parisi, Zuber, Kazakov, Duplantier, Kostov...]
[Knizhnik, Polyakov, Zamolodchikov 88]



Map (rooted, planar): proper embedding of a connected planar graph in the plane. One edge is oriented, with the infinite face on its right.



Note: maps \neq graphs

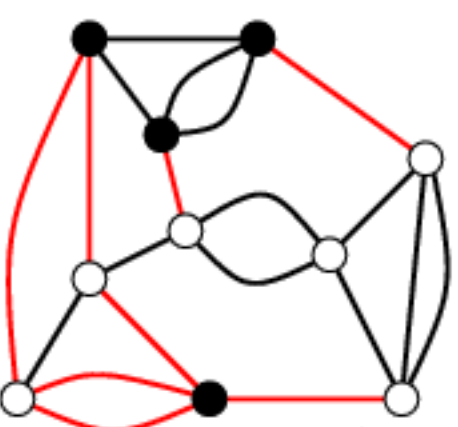


Example: the Ising model on 4-valent maps

Partition function:

$$I(x, y, u) = \sum_M \sum_{\sigma} x^{\circ} y^{\bullet} u^f$$

where f is the number of frustrated edges.



[Kazakov 86, Boulatov-Kazakov 87]: solution via matrix integrals

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} \log \int dH dK e^{N \text{tr} \left(-\frac{H^2 + K^2}{2} + vHK - X \frac{H^4}{4} - Y \frac{K^4}{4} \right)}.$$

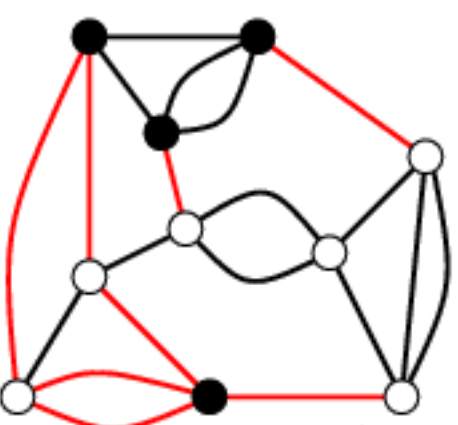
H, K : hermitian $N \times N$ matrices

Example: the Ising model on 4-valent maps

Partition function:

$$I(x, y, u) = \sum_M \sum_{\sigma} x^{\circ} y^{\bullet} u^f$$

where f is the number of **frustrated** edges.



[Kazakov 86, Boulatov-Kazakov 87]: solution via matrix integrals

$\Rightarrow I(x, y, u)$ is an **algebraic** function of x, y, u .

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} \log \int dH dK e^{N \text{tr} \left(-\frac{H^2 + K^2}{2} + vHK - X \frac{H^4}{4} - Y \frac{K^4}{4} \right)}.$$

H, K : hermitian $N \times N$ matrices

Algebraic power series

Def. The generating function $S(x_1, x_2, \dots, x_k) \equiv S(\mathbf{x})$ is **algebraic** if there exists a polynomial Q such that

$$Q(x_1, \dots, x_k; S(\mathbf{x})) = 0.$$

Algebraic power series

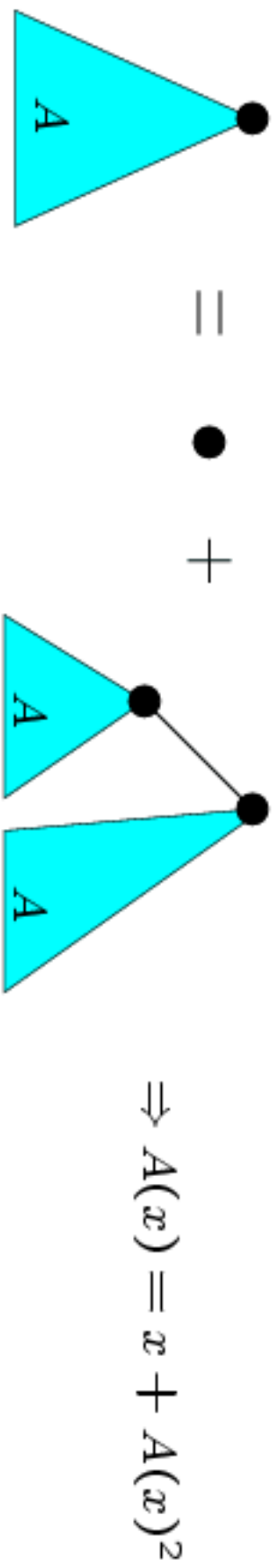
Def. The generating function $S(x_1, x_2, \dots, x_k) \equiv S(\mathbf{x})$ is **algebraic** if there exists a polynomial Q such that

$$Q(x_1, \dots, x_k; S(\mathbf{x})) = 0.$$

Example: the generating function $A(x)$ for rooted trees (embedded in the plane) is algebraic.

$$A(x) = \sum_n a_n x^n$$

where a_n is the number of trees with n vertices.


$$\Rightarrow A(x) = x + A(x)^2$$

Enumeration of planar maps

Arquès Bauer Bédard Bender Bernardi Bessis Bodirsky Bousquet-Mélou
Boulatov Bouttier Brézin Brown Canfield Chauve Cori Di Francesco
Duplantier Fusy Gao Goupil Goulden Guitter t'Hooft Itzykson Jackson
Jacquard Kazakov Kostov Labelle Lehman Leroux Liskovets Liu Machi Mehta
Mullin Parisi Poulalhon Richmond Robinson
Schaeffer Strehl **Tutte** Vainshtein Vauquelin Visentin Walsh Wanless
Wormald Zinn-Justin Zuber Zvonkine...



Planar maps (tend to) have algebraic generating functions

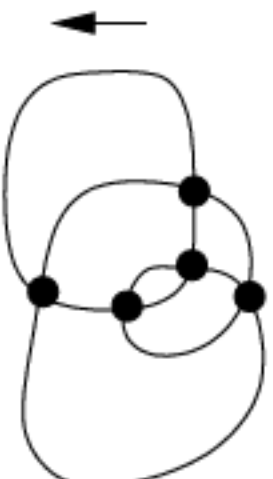
Algebraicity for planar maps of bounded degree

Theorem [Bouttier–Di Francesco–Guitter 02]

For $k \geq 2$, let $M(x_1, x_2, \dots, x_k)$ be the generating function of maps with vertices of degree at most k (x_i counts vertices of degree i).

This series is algebraic.

(Quantum gravity: maps with no matter.)



Example: 4-valent maps (all vertices have degree 4)

$$M(x) = P - xP^3 \quad \text{where} \quad P = 1 + 3xP^2.$$

Three approaches for counting maps

- [Tutte 60's]: recursive decompositions of maps, functional equations, quadratic method
- [Bender, Canfield 94], [MBM, Jehanne 05]
- [Brézin, Itzykson, Parisi, Zuber 78]: matrix integrals
- [Bouttier, Di Francesco, Guitter 02]

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} \log \int dH e^{N \operatorname{tr} \left(-\frac{H^2}{2} + \sum_{i=1}^k \frac{x_i}{i} H^i \right)}.$$

H : hermitian $N \times N$ matrix

- [Schaeffer 97]: bijection with trees
- [BDG 02]

In this talk...

Algebraicity still holds for maps carrying an Ising model (or a hard-particle model)

Theorem [MBM–Schaeffer 02]

For $k \geq 2$, let $I(x_1, x_2, \dots, x_k; y_1, \dots, y_k; u)$ be the Ising generating function of maps with vertices of degree at most k (x_i counts white vertices of degree i , and y_i counts black vertices of degree i).

This series is algebraic.

Its algebraicity is explained combinatorially thanks to a bijection with certain trees.

Not true for the q -state Potts model (equivalent to the Tutte polynomial).

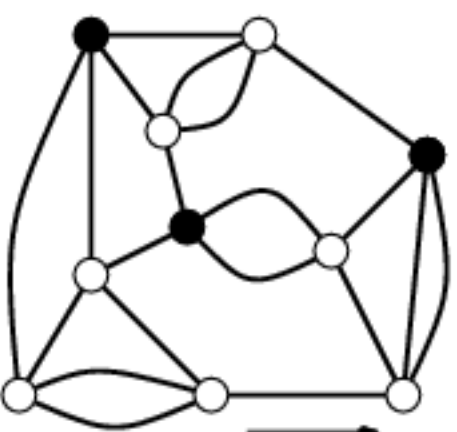
Example: hard particles on tetravalent maps

Let $P \equiv P(x, y)$ be the power series defined by

$$P = 1 + 3xyP^3 + \frac{3xP^2}{(1 - 9xyP^2)^2}.$$

Then the hard-particle generating function of 4-valent maps (rooted at a vacant edge) is

$$H(x, y) = xP^3 + \frac{xP^2(3 - 2P)}{1 - 9xyP^2} - \frac{27x^3yP^6}{(1 - 9xyP^2)^3}.$$

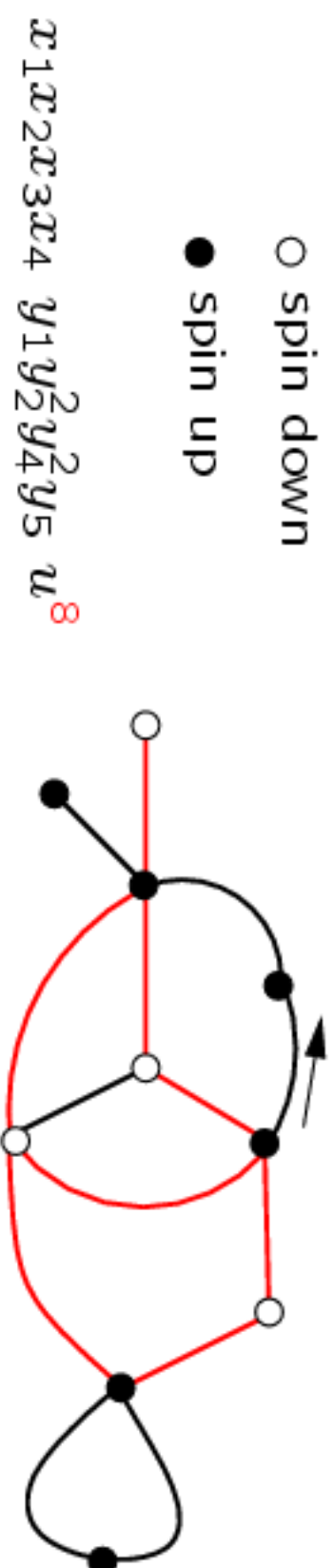


$$H(x, y) = \sum_M \sum_{\sigma} x^{\circ} y^{\bullet}.$$

The Ising and hard-particle models on general planar maps

○ spin down

● spin up



- Ising generating function:

$$\begin{aligned} I(x, y, u) &= \sum_{M, \sigma} x_1^{m_1} x_2^{m_2} \dots y_1^{n_1} y_2^{n_2} \dots u^f \\ &= \sum_{M, \sigma} x^m y^n u^f \end{aligned}$$

where m_i (resp. n_i) is the number of white (resp. black) vertices of M having degree i and f is the number of **frustrated** edges.

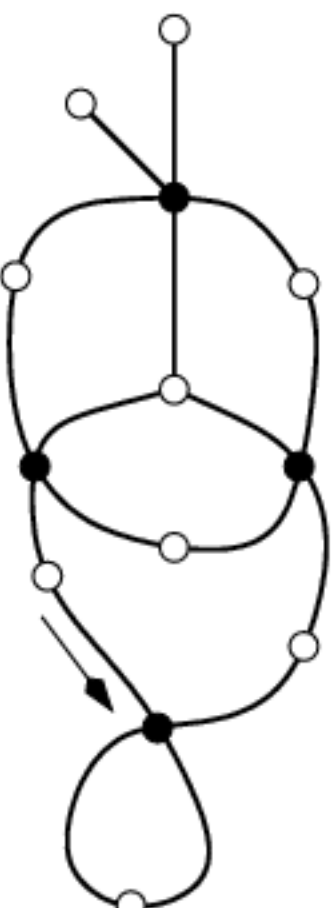
- Hard-particle generating function: sum over hp-configurations only

$$H(x, y) = \sum_M x^m y^n$$

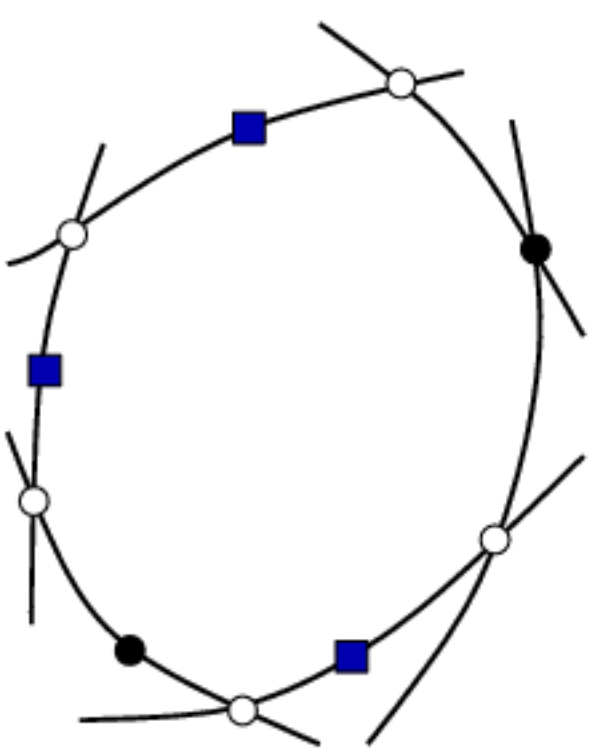
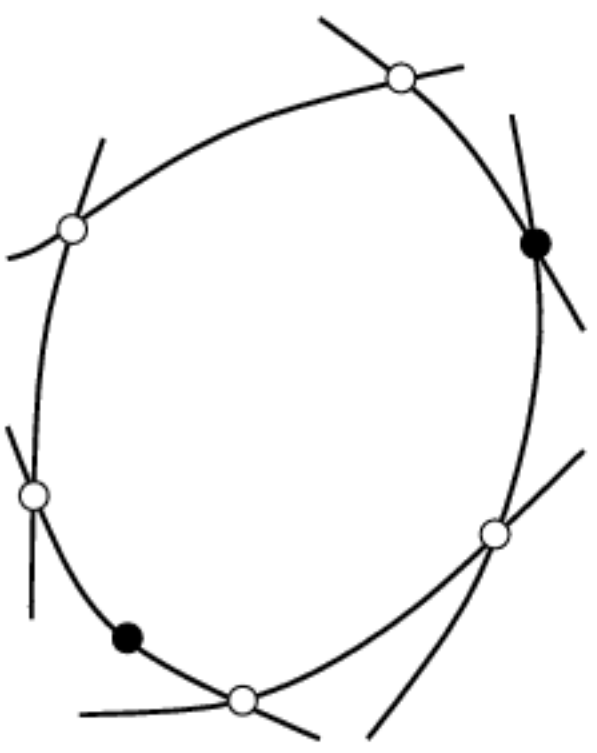
Enumeration of bipartite maps

$$M(\mathbf{x}, \mathbf{y}) = \sum_{M \text{ bip.}} x^m y^n = \sum_M x_1^{m_1} x_2^{m_2} \cdots y_1^{n_1} y_2^{n_2} \cdots$$

$$x_1^2 x_2^6 x_3 y_4^3 y_5$$



From hard-particles to bipartite maps

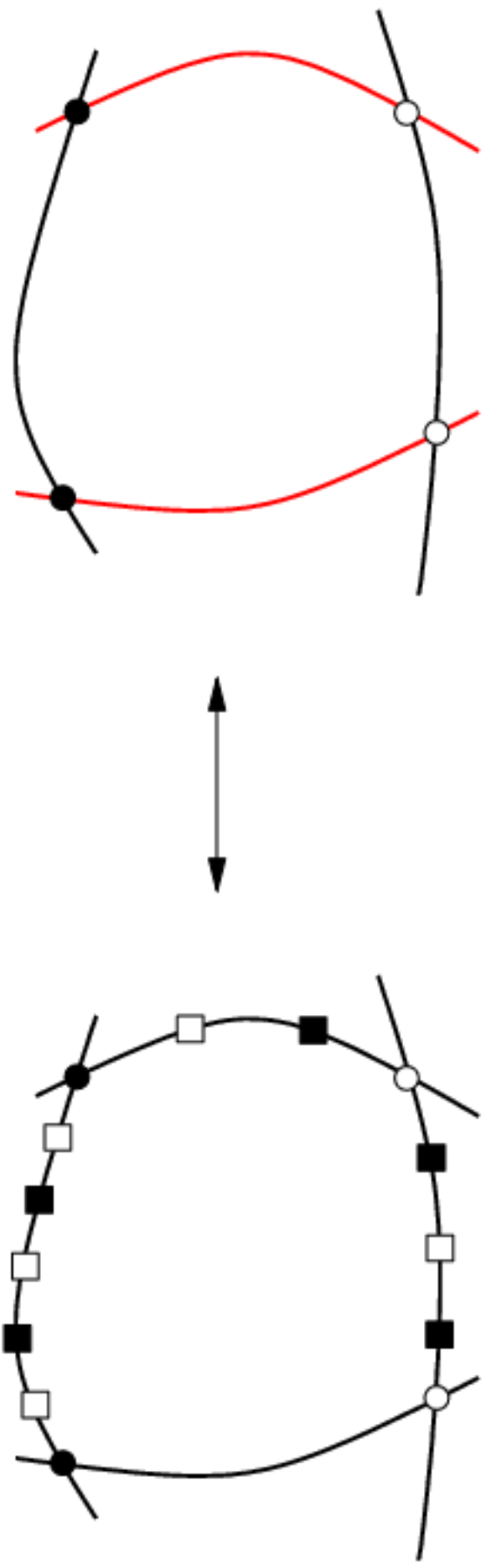


$$H(x_1, x_2, \dots, y_1, y_2, \dots)$$

=

$$M(x_1, x_2, \dots, y_1, 1 + y_2, y_3, \dots)$$

From the Ising model to bipartite maps



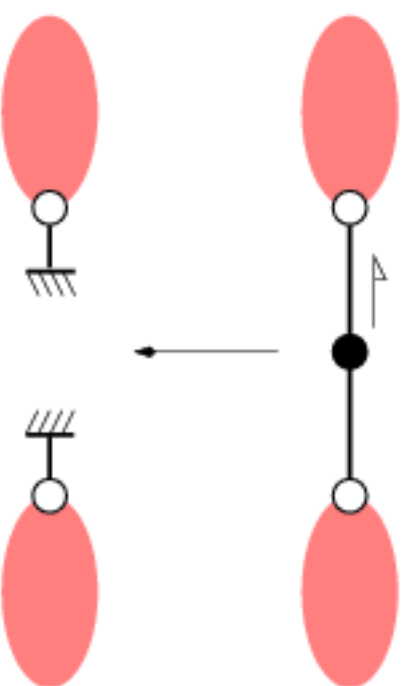
$$I(x_1, x_2, \dots, y_1, y_2, \dots, u) = M(X_1, X_2, \dots, Y_1, Y_2, \dots)$$

with

$$X_k = x_k(u - \bar{u})^{k/2}, \quad Y_k = y_k(u - \bar{u})^{k/2} \quad \text{and} \quad \bar{u} = 1/u.$$

Planar maps with legs

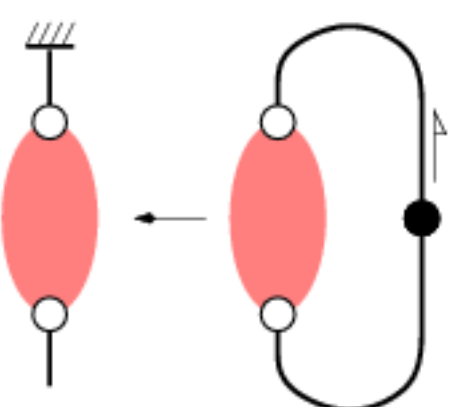
Bipartite maps rooted at a black vertex of degree 2: two cases



Two maps with one leg each



Legs always lie in the infinite face.



One map with two legs

Bijection enumeration of planar maps: which trees hide in which maps?

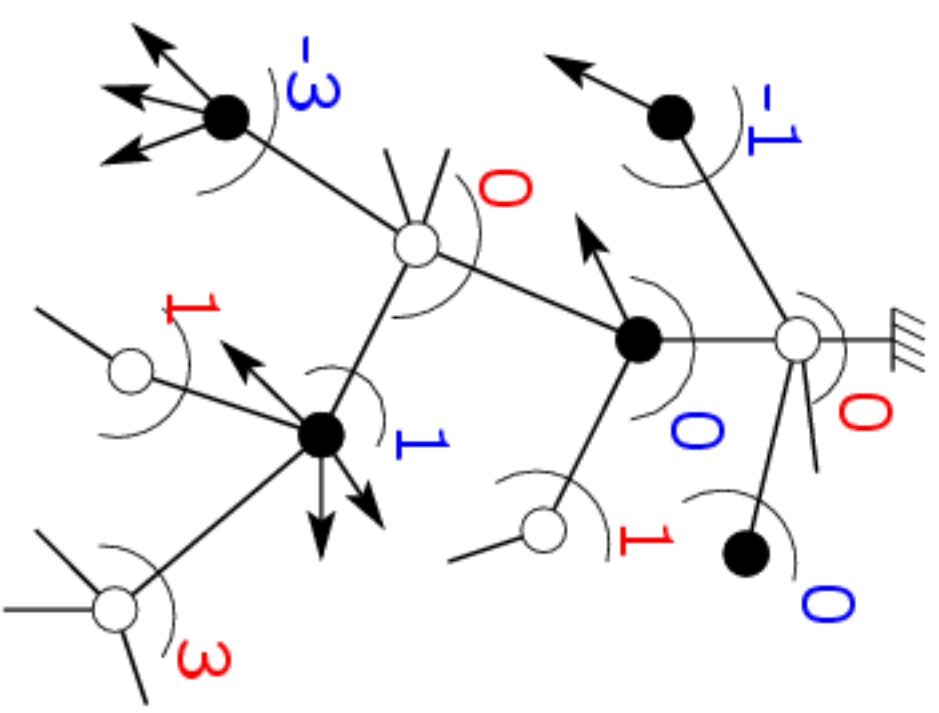
1. Introduce the “right” family of trees (easy to count)
2. Construct a bijection between maps and some of these trees, called **balanced**
3. Count balanced trees

[Schaeffer 97], [MBM–Schaeffer 00], [Poulalhon–Schaeffer 02], [Bouttier–Di
Francesco–Guitter 02] **COMPLETER**

Planar Eulerian maps, constellations, non-separable triangulations, general planar maps...

The “right” family of trees: blossoming trees

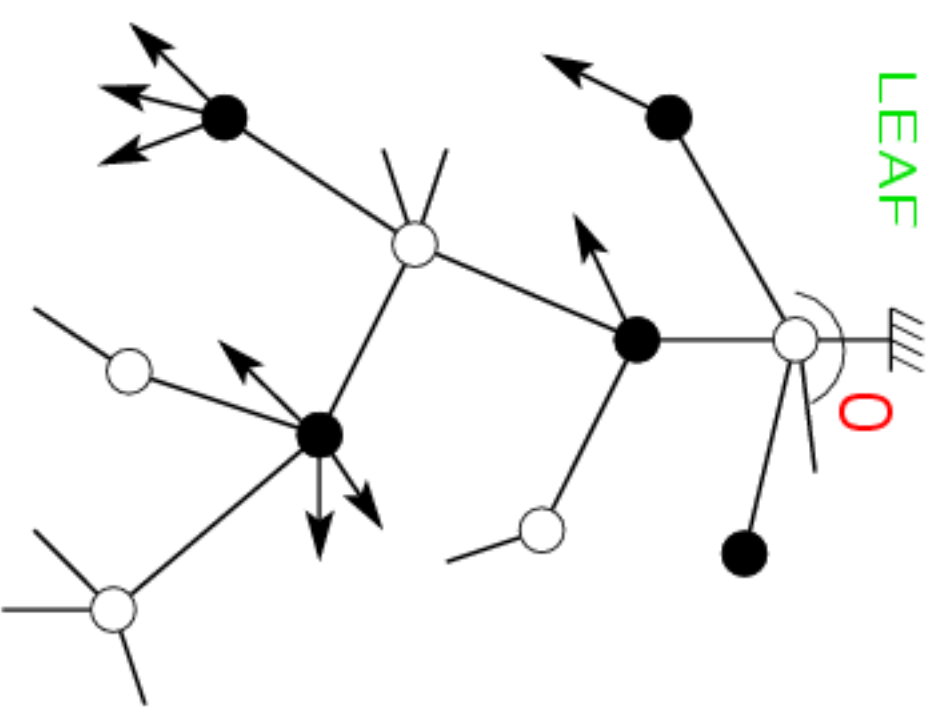
- rooted
- bicolored
- with
 - buds* attached to black vertices
 - leaves* attached to white vertices
- the *charge* at a vertex: $\# \text{ leaves} - \# \text{ buds}$
- the charge at every white vertex is ≥ 0
- the charge at every black vertex is ≤ 1



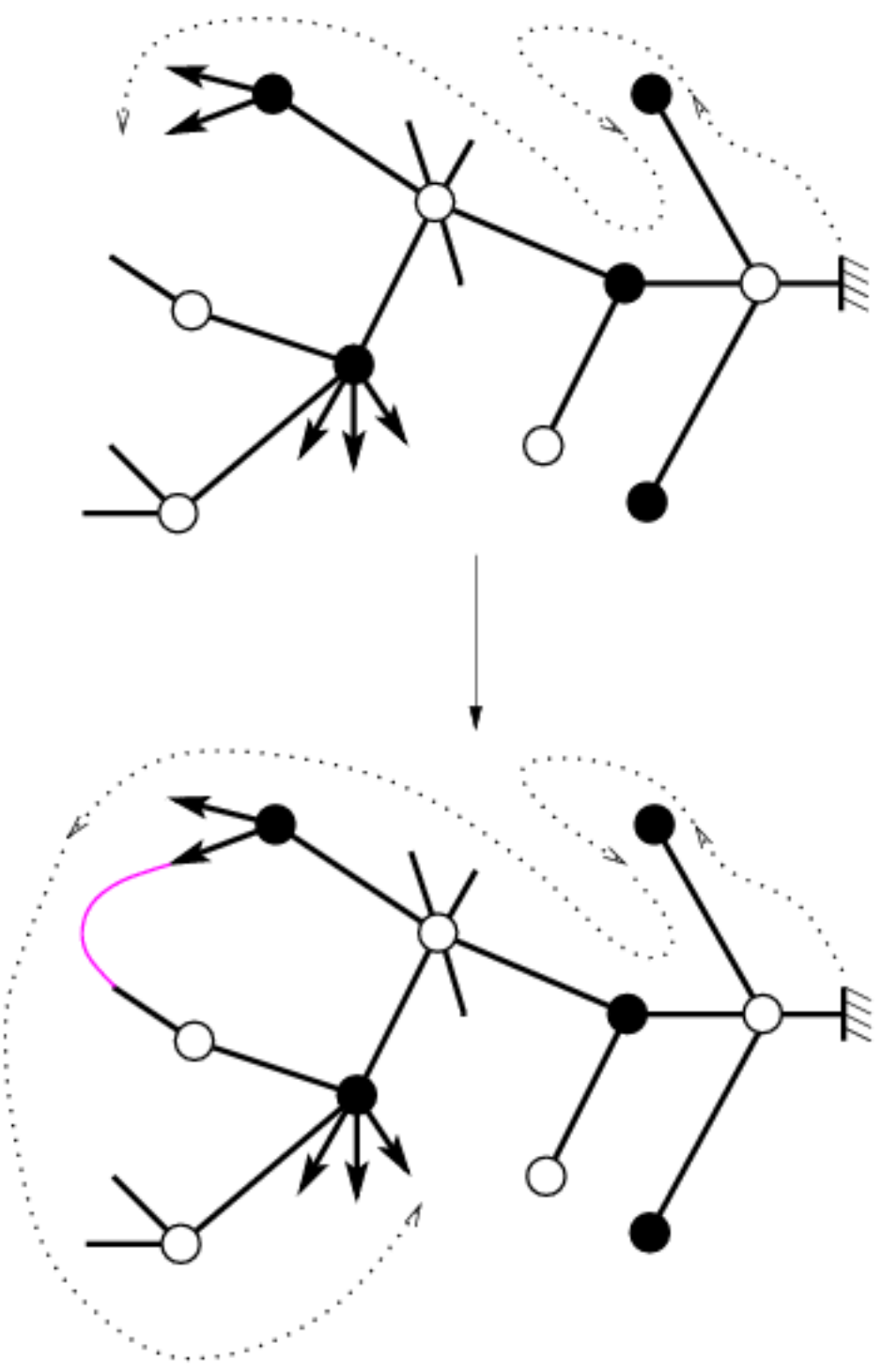
The total charge of a blossoming tree

- rooted
- bicolored
- with
 - buds* attached to black vertices
 - leaves* attached to white vertices
- the *charge* at a vertex: $\# \text{ leaves} - \# \text{ buds}$
- the charge at every white vertex is ≥ 0
- the charge at every black vertex is ≤ 1

$$\text{TOTAL CHARGE} = 0 + 1$$



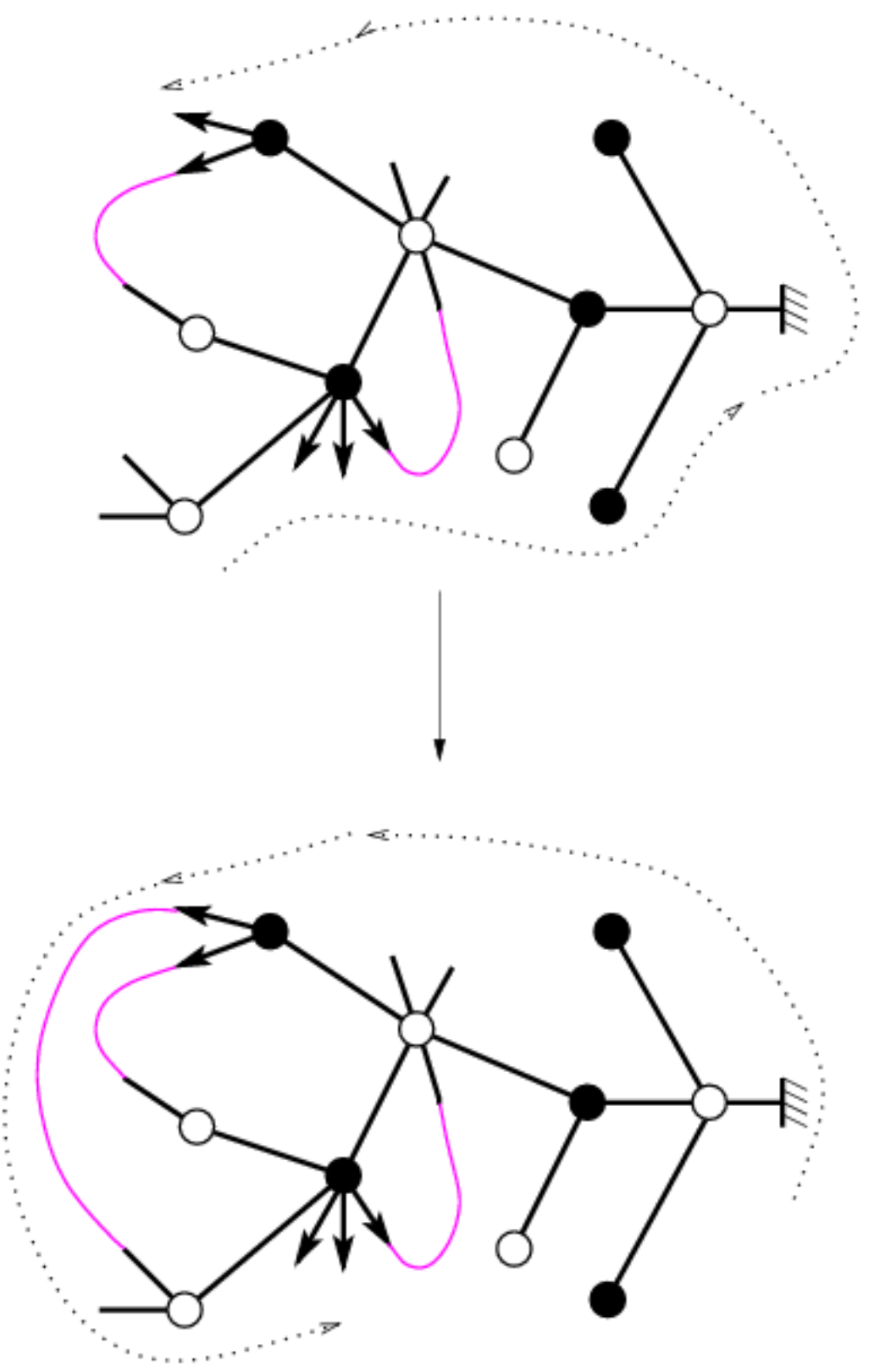
From trees to maps: the closure of a tree



Match buds to leaves

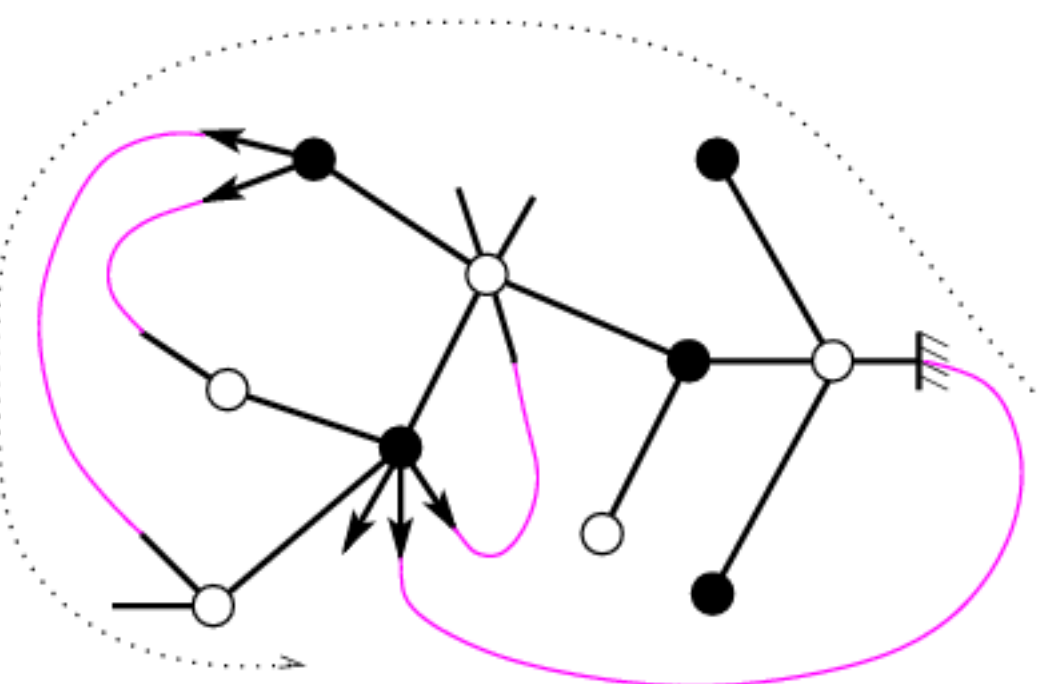
Total charge 2

From trees to maps: the closure of a tree

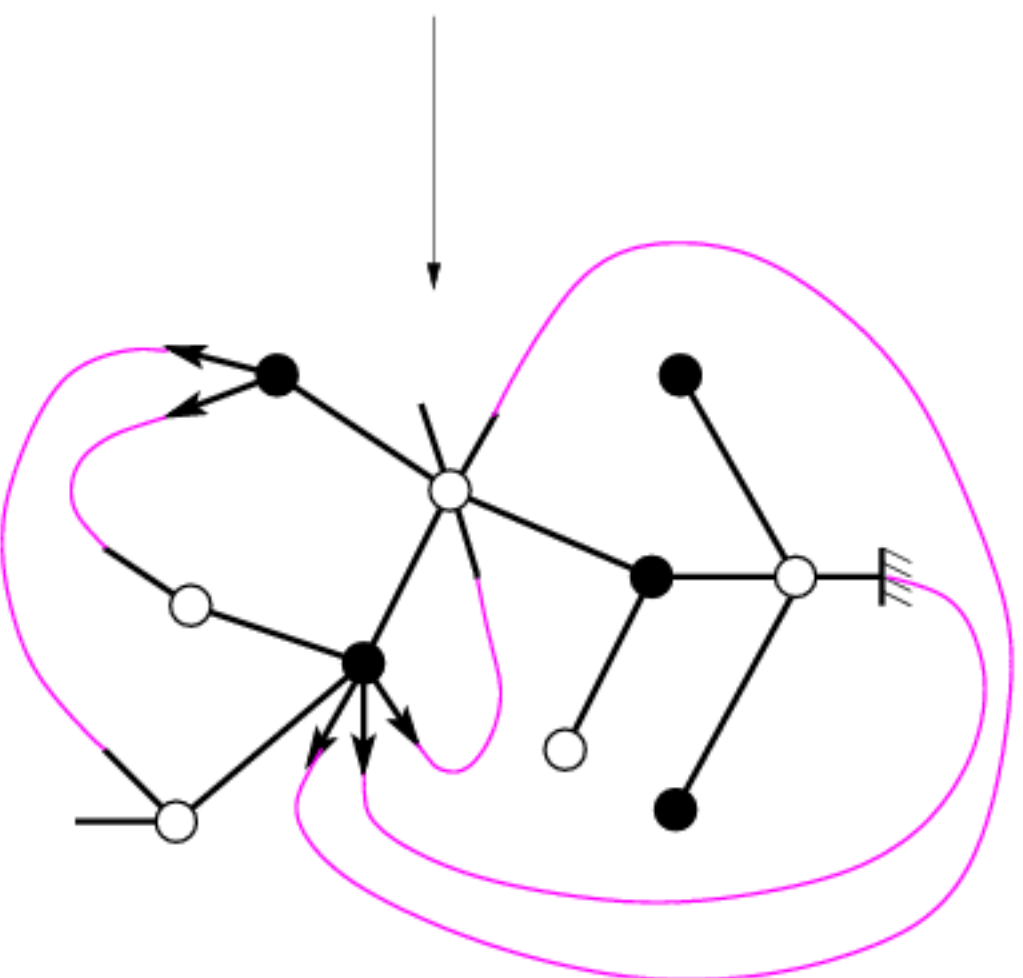


Match
buds to leaves

From trees to maps: the closure of a tree



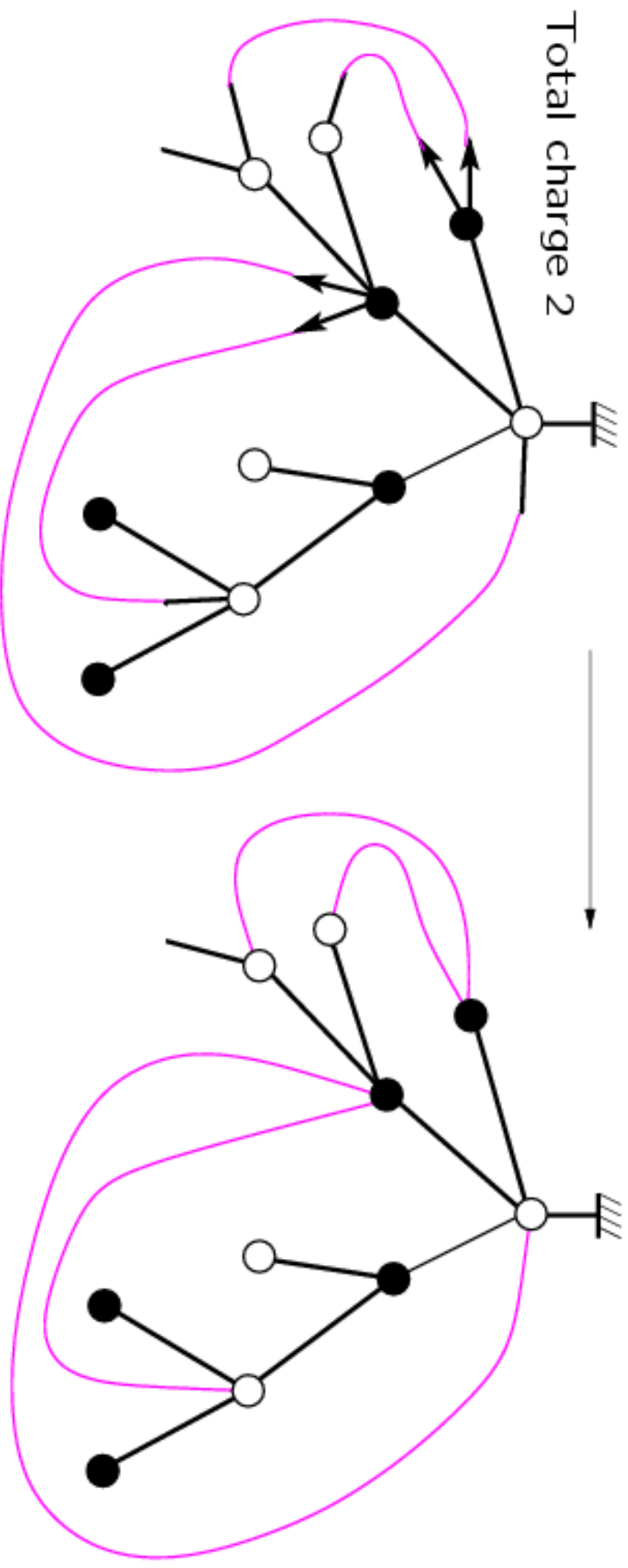
Total charge 2



Two leaves remain unmatched

Balanced blossoming trees

A blossoming tree, rooted at a white vertex, is **balanced** if the root-leaf remains unmatched after the closure of the tree.



Two leaves remain unmatched

Two-leg map

The bijection

Theorem [MBM-Schaeffer 02]

For $k = 1, 2$, the closure is a bijection between:

- balanced blossoming trees of total charge k
- bipartite maps having k legs.

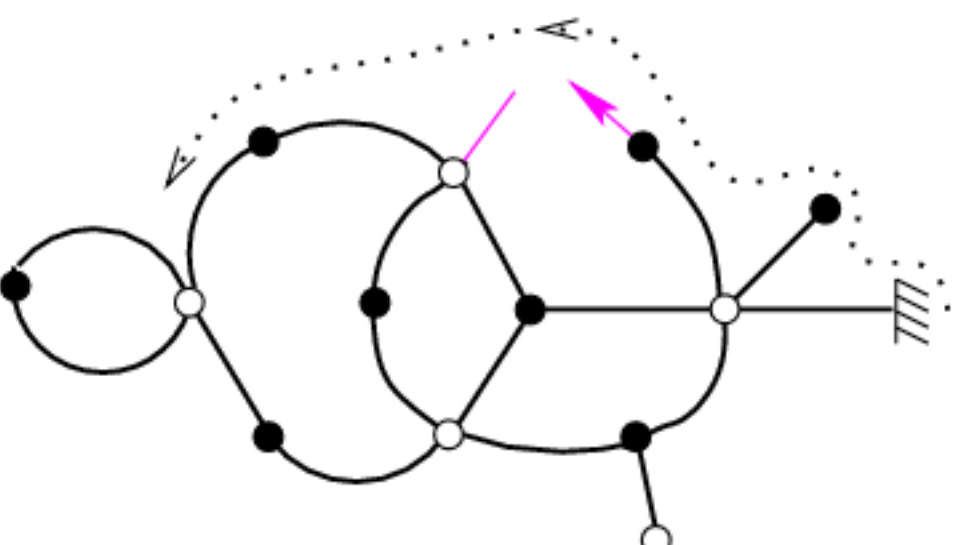
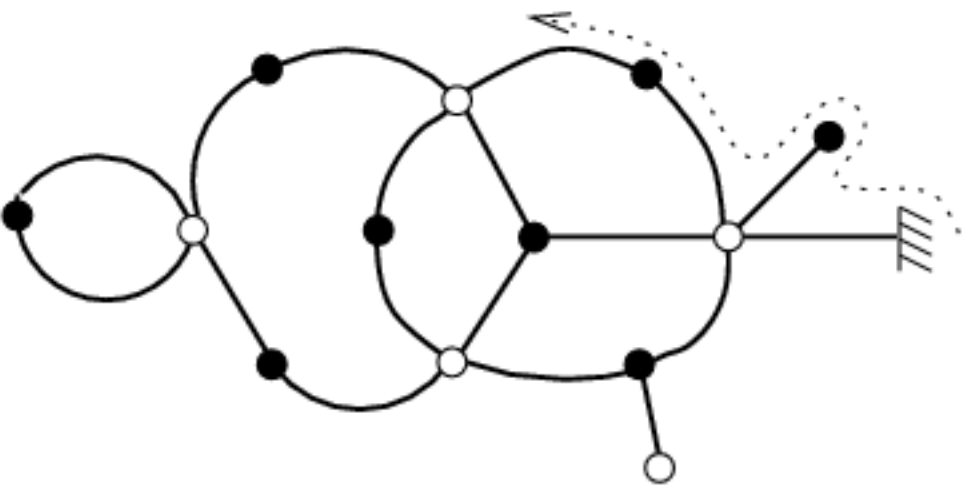
Moreover, this bijection preserves the degrees of black and white vertices.



This theorem includes and extends some former bijections for: maps with even valencies [Schaeffer 97], “constellations” [MBM+Schaeffer 00], maps with arbitrary valencies [Bouttier, Di Francesco, Guitter 02]

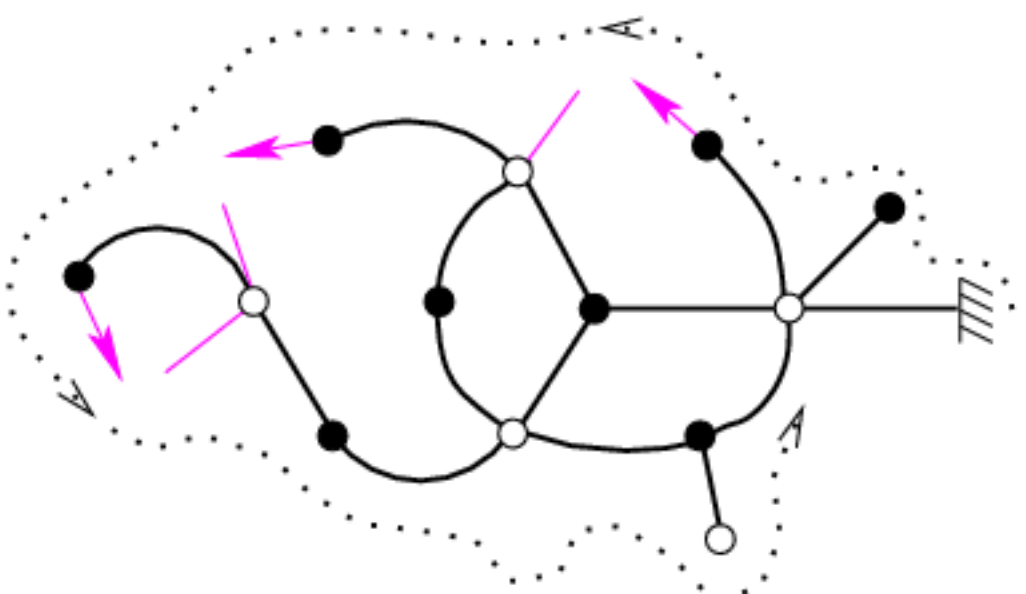
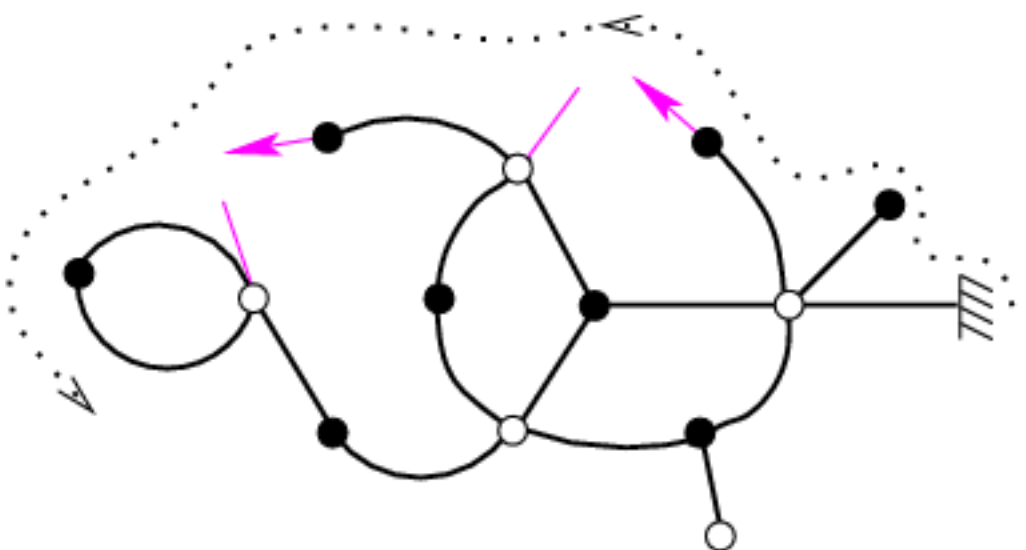
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From maps to balanced blossoming trees



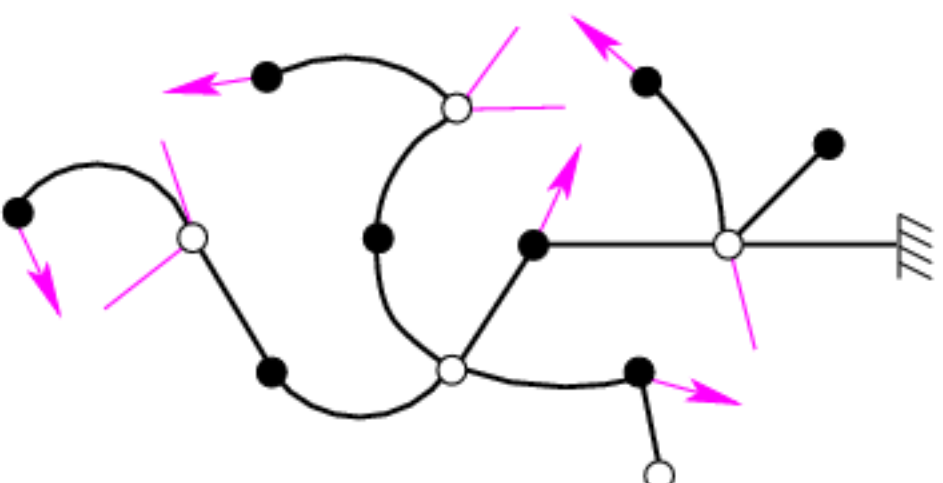
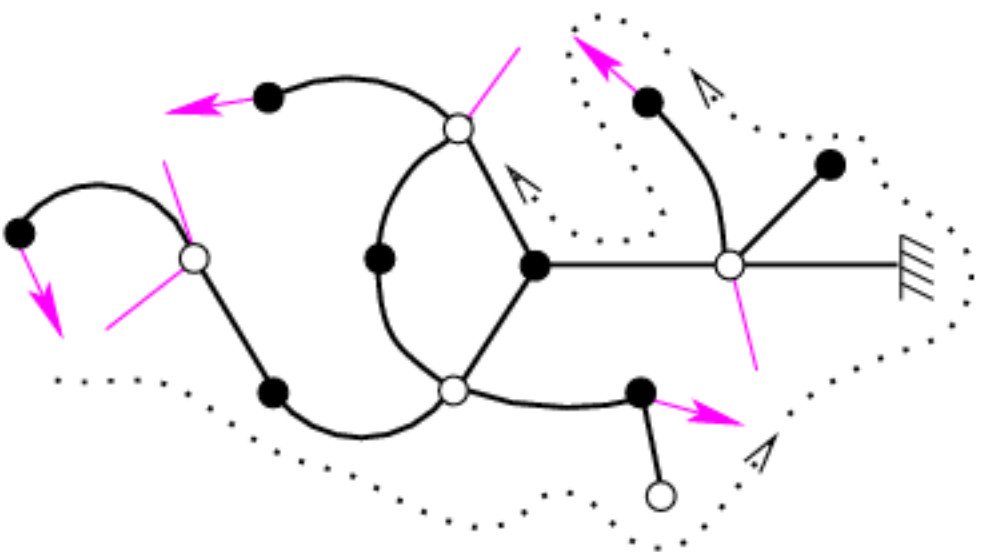
Open
BW edges
when possible

From maps to balanced blossoming trees



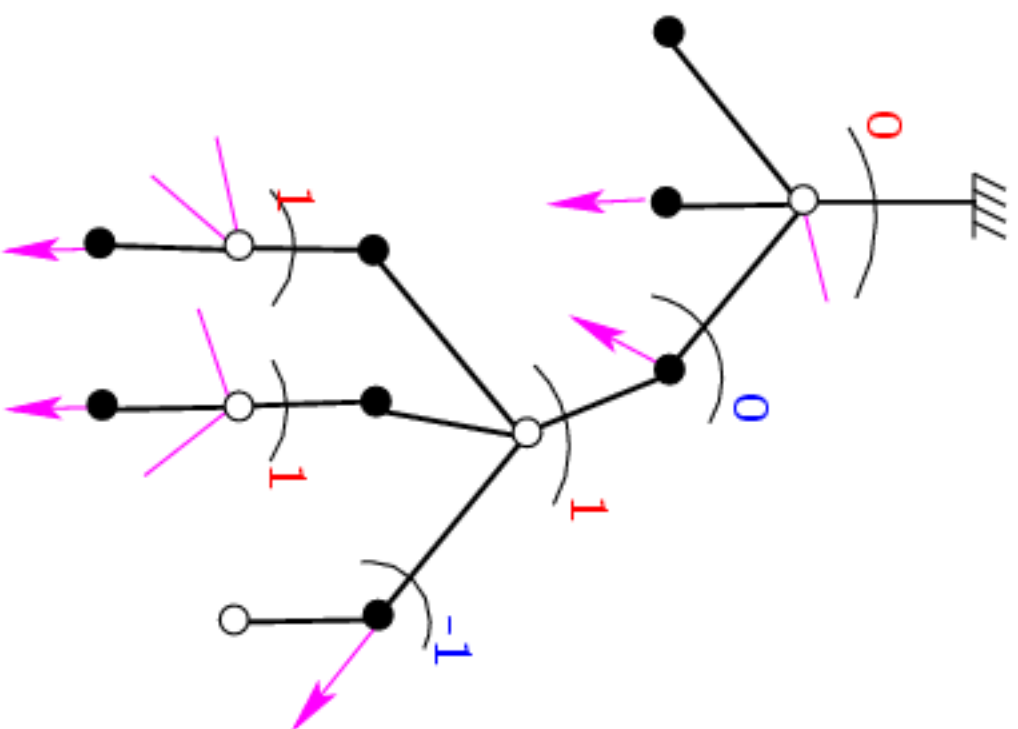
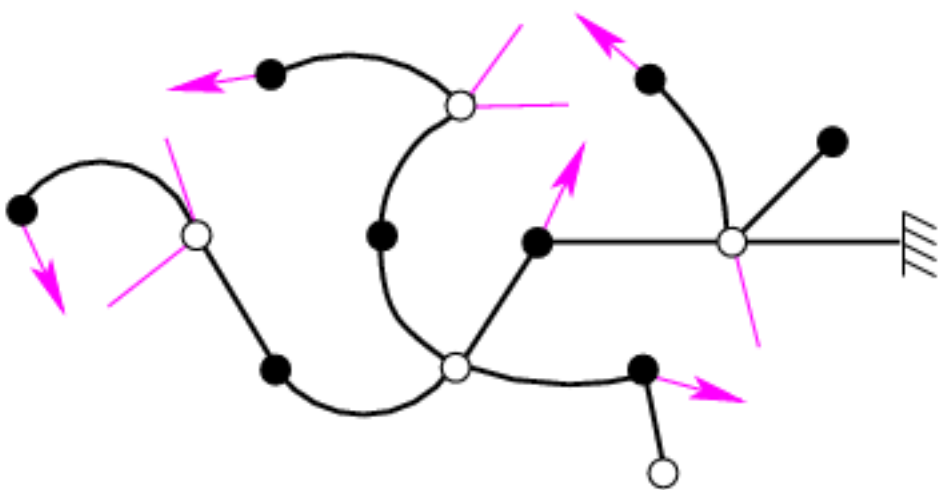
Open
BW edges
when possible

From maps to balanced blossoming trees



Open
BW edges
when possible

From maps to balanced blossoming trees



Counting blossoming trees is easy

For $k \geq 0$, let $W_k(x, y)$ be the generating function of blossoming trees rooted at a **white** vertex of charge k . For $k \leq 1$, let $B_k(x, y)$ be the generating function of blossoming trees rooted at a **black** vertex of charge k .

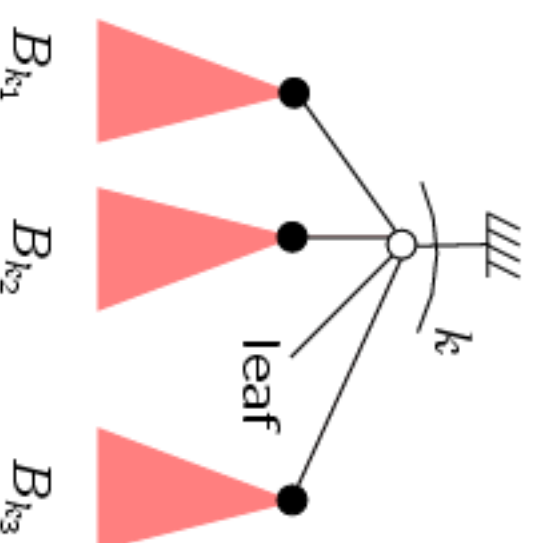
Trees have a simple recursive structure:

white root of degree $i + 1$

charge k

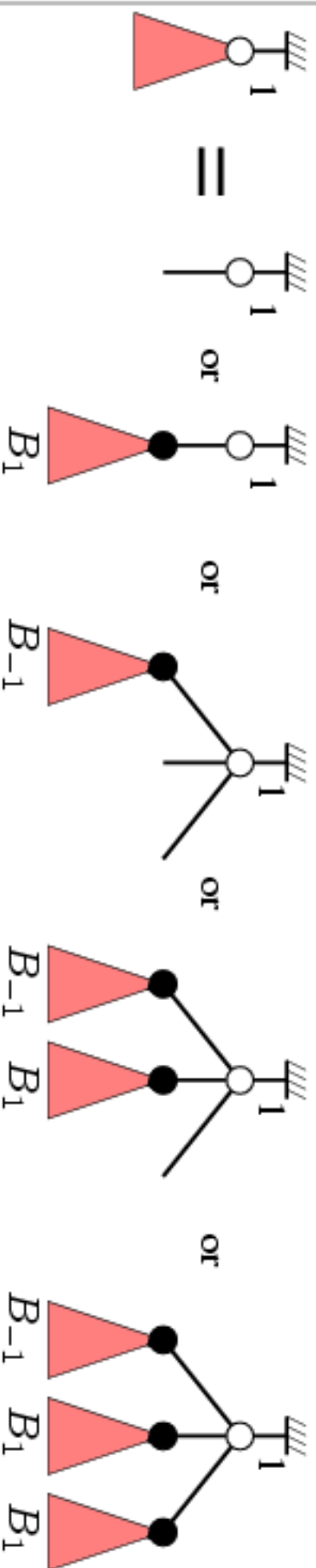
i sons (leaves or black vertices)

$$k_1 + k_2 + k_3 + 1 = k$$



Example: vertices of degree 2 and 4

White root, charge 1



$$W_1 = x_2 + x_2 B_1 + 3x_4 B_{-1} + 6x_4 B_{-1} B_1 + 3x_4 B_{-1} B_1^2$$

$$W_1 = x_2(1 + B_1) + 3x_4 B_{-1}(1 + B_1)^2 \quad B_{-1} = y_2 + 3y_4 W_1$$

$$W_3 = x_4(1 + B_1)^3 \quad B_1 = y_2 W_1 + 3y_4(W_3 + W_1^2).$$

Let $P = 1 + B_1$. Then

$$P = 1 + 3x_4 y_4 P^3 + \frac{P(x_2 + 3x_4 y_2 P)(y_2 + 3y_4 x_2 P)}{(1 - 9x_4 y_4 P^2)^2}.$$

General formulae for the enumeration of blossoming trees

For $k \geq 0$, let $W_k(x, y)$ be the generating function of blossoming trees rooted at a white vertex of charge k . For $k \leq 1$, let $B_k(x, y)$ be the generating function of blossoming trees rooted at a black vertex of charge k . Then

$$W_k = [z^k] \sum_{i \geq 0} x_{i+1} (z + B(z))^i$$

$$B_k = [z^k] \sum_{i \geq 0} y_{i+1} (1/z + W(z))^i$$

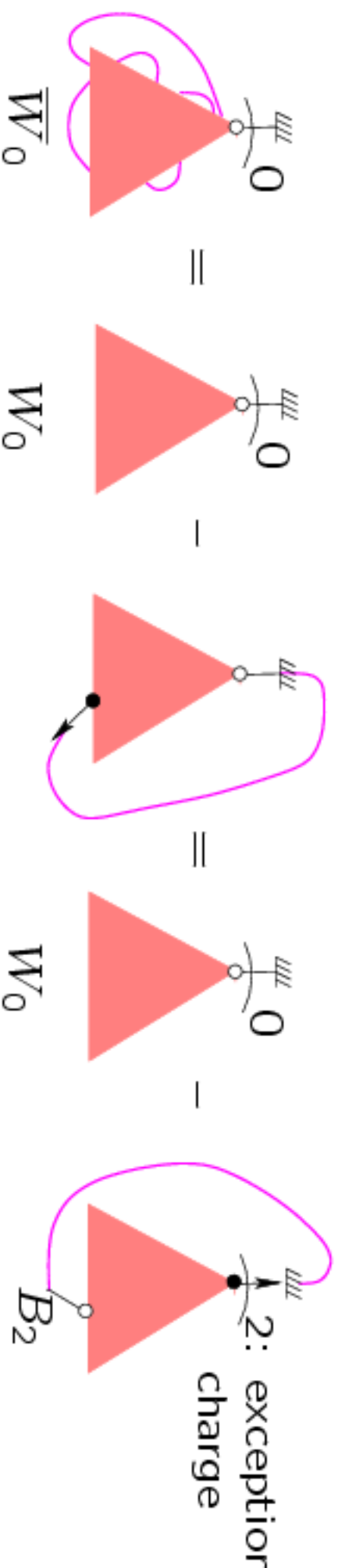
with

$$W(z) = \sum_{k \geq 0} W_k z^k \quad \text{and} \quad B(z) = \sum_{k \leq 1} B_k z^k.$$

If the degree of the trees is bounded, almost all series B_k and W_k are zero, and the others are algebraic.

Balanced blossoming trees of total charge 1

↔ One-leg maps



The GF of balanced blossoming trees of total charge 1 is

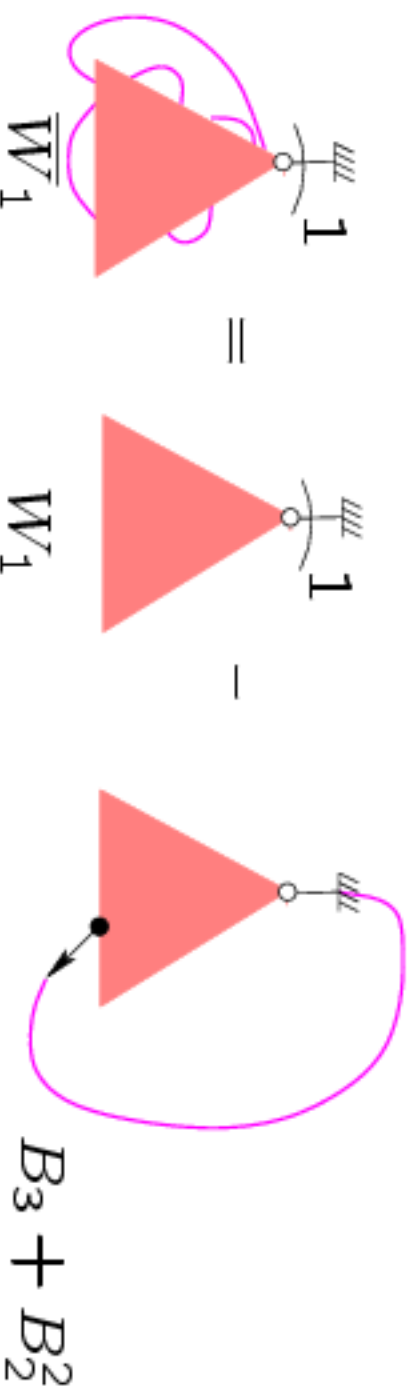
$$\bar{W}_0 = W_0 - B_2$$

with

$$B_2 = [z^2] \sum_{i \geq 0} y_{i+1} (1/z + W(z))^i.$$

Balanced blossoming trees of total charge 2

↔ Two-leg maps



The GF of balanced blossoming trees of total charge 2 is

$$\overline{W}_1 = W_1 - (B_3 + B_2^2)$$

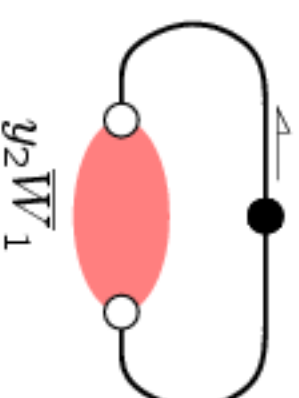
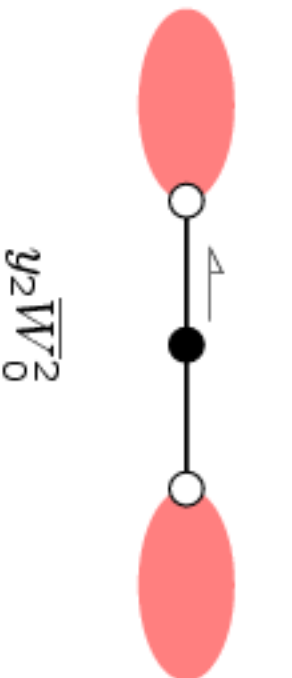
with

$$B_k = [z^k] \sum_{i \geq 0} y_{i+1} (1/z + W(z))^i.$$

Enumeration of bipartite maps

The generating function of bipartite maps (rooted at a black vertex of degree 2), counted by their degree distribution, is

$$M(\mathbf{x}, \mathbf{y}) = y_2 (\overline{W}_0^2 + \overline{W}_1) = y_2 (W_0^2 + W_1 - B_3 - 2W_0 B_2).$$



Applications

The Ising and hard-particle models

- Combinatorial solution
- The partition function is algebraic as soon as the degree is bounded

Former solutions (2-matrix models)

- hard-particles on 3-valent and 4-valent maps [BDG 02]
- Ising on 3-valent and 4-valent maps [Kazakov 86, Boulatov-Kazakov 87]

Corollary: hard particles on tetravalent maps

Let $P \equiv P(x, y)$ be the power series defined by

$$P = 1 + 3xyP^3 + \frac{3xP^2}{(1 - 9xyP^2)^2}.$$

The hard-particle generating function of 4-valent maps rooted at a vacant edge is

$$H(x, y) = xP^3 + \frac{xP^2(3 - 2P)}{1 - 9xyP^2} - \frac{27x^3yP^6}{(1 - 9xyP^2)^3}.$$



[Bouttier–Di Francesco–Guitter 02]: same approach, more general roots (tetravalent case only).

Corollary: Ising on tetravalent maps

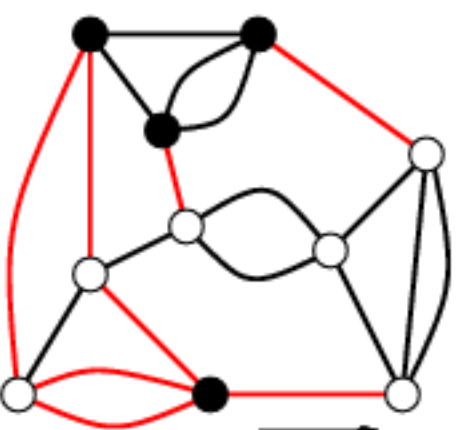
Let $P \equiv P(X, Y, v)$ be the power series defined by

$$P = 1 + 3XY P^3 + v^2 \frac{P(1 + 3XP)(1 + 3YP)}{(1 - 9XY P^2)^2}.$$

Then the Ising generating function of 4-valent maps is

$$I(x, y, u) = \text{Rational}(x, y, u, P),$$

with $X = x(u - \bar{u})^2$, $Y = y(u - \bar{u})^2$, and $v = \bar{u} = 1/u$.



Summary

NEW Enumeration of bipartite planar maps (based on a bijection with trees).

Includes all previous enumeration results based on this principle, except those that deal with maps of higher connectivity [[Poulalhon Schaeffer](#)]**COMPLETER**

Combinatorial explanation for the algebraicity of the generating functions of maps of bounded degree

- bipartite maps
- hard-particle models
- Ising models.

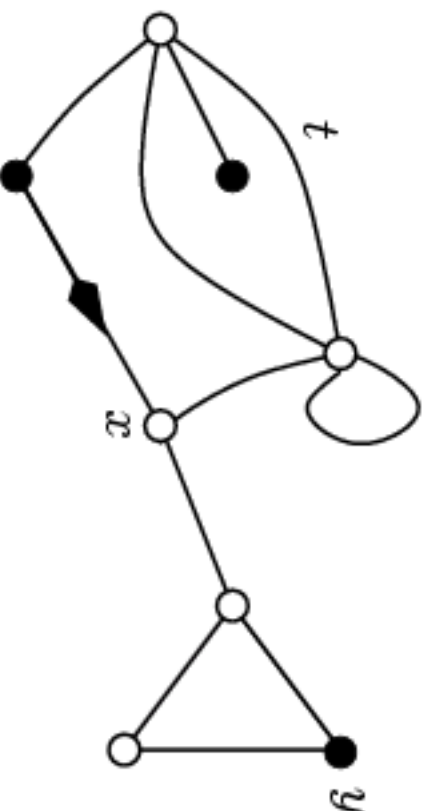
NOT discussed the singularities of the GFs [[Boulatov, Kazakov 87](#)].

<http://xxx.lpthe.jussieu.fr/abs/math.CO/0211070>

Maps with unbounded degree?

Surprisingly little is known...

One exception: The hard-particle model on general planar maps
[MBM-Jehanne 05]



u : degree of the infinite face

Back to Tutte's original approach: functional equations

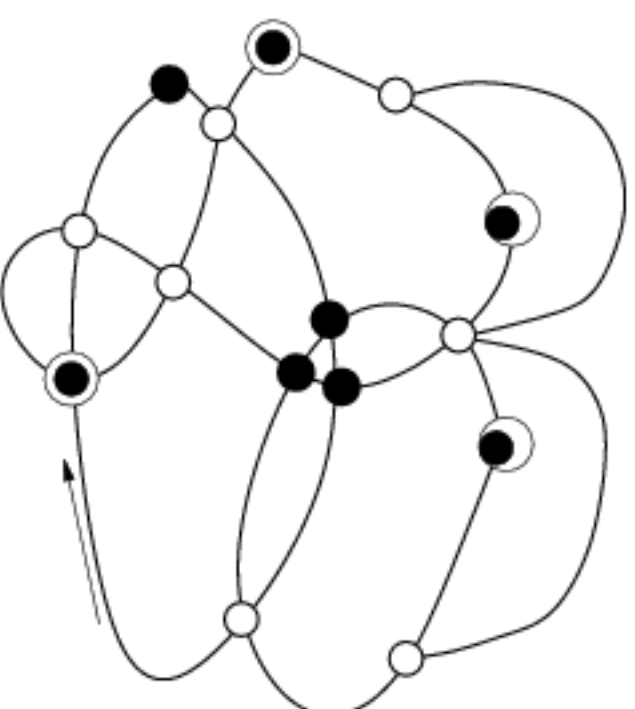
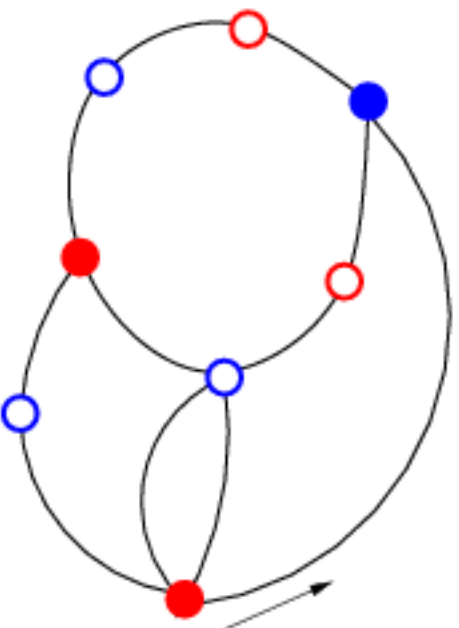
- $F(u) = x + G(u) + tu^2F(u)^2 + \frac{tu(uF(u) - F(1))}{u-1}$
- $G(u) = tyuF(u) + tuF(u)G(u) + \frac{tu(G(u) - G(1))}{u-1}$

$$t^2F(1) = T^2(1-7T+16T^2+T-15T^3+4T^4) \quad \text{with} \quad T(1-2T)(1-3T+3T^2) = t.$$

Extensions – Perspectives

Other models (preferably known to be algebraic)

- Models on general maps
- Hard particles on **bipartite** maps (4-matrix model [BDG]COMPLETER)
- Two-particle models (on bipartite maps: 6-matrix model [BDG])



- 3-state Potts