Counting planar maps,

coloured or uncoloured

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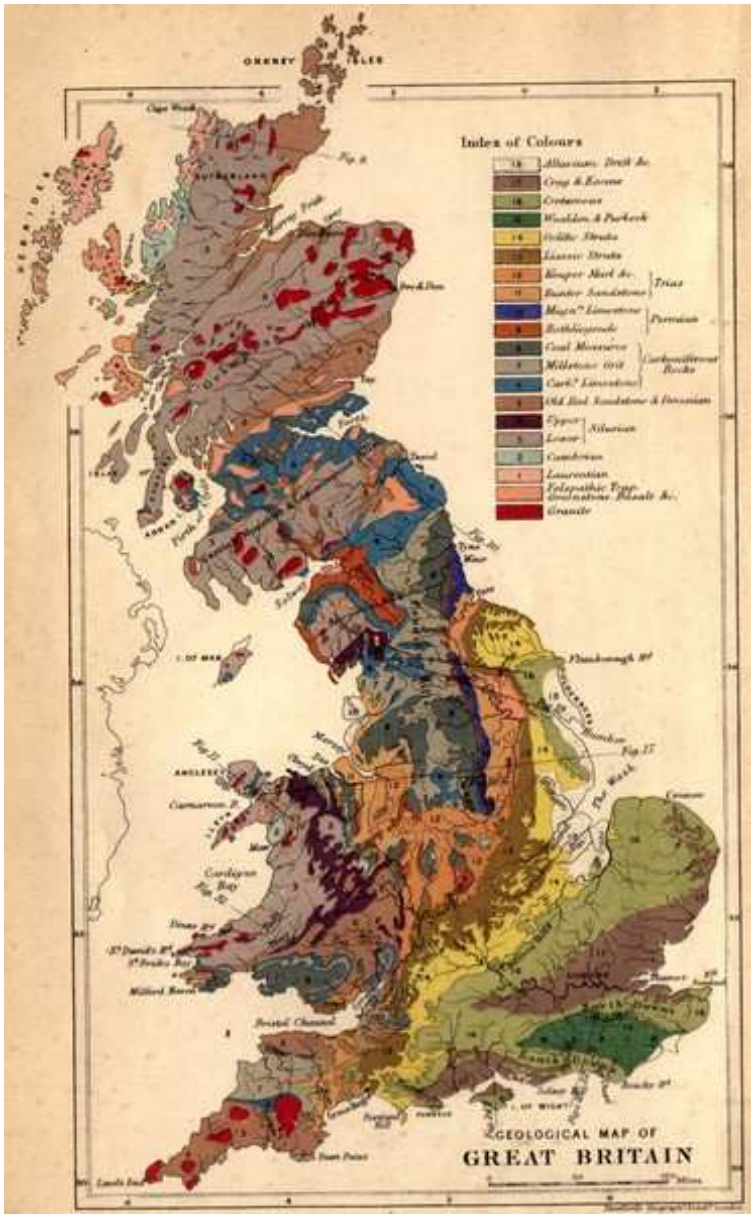
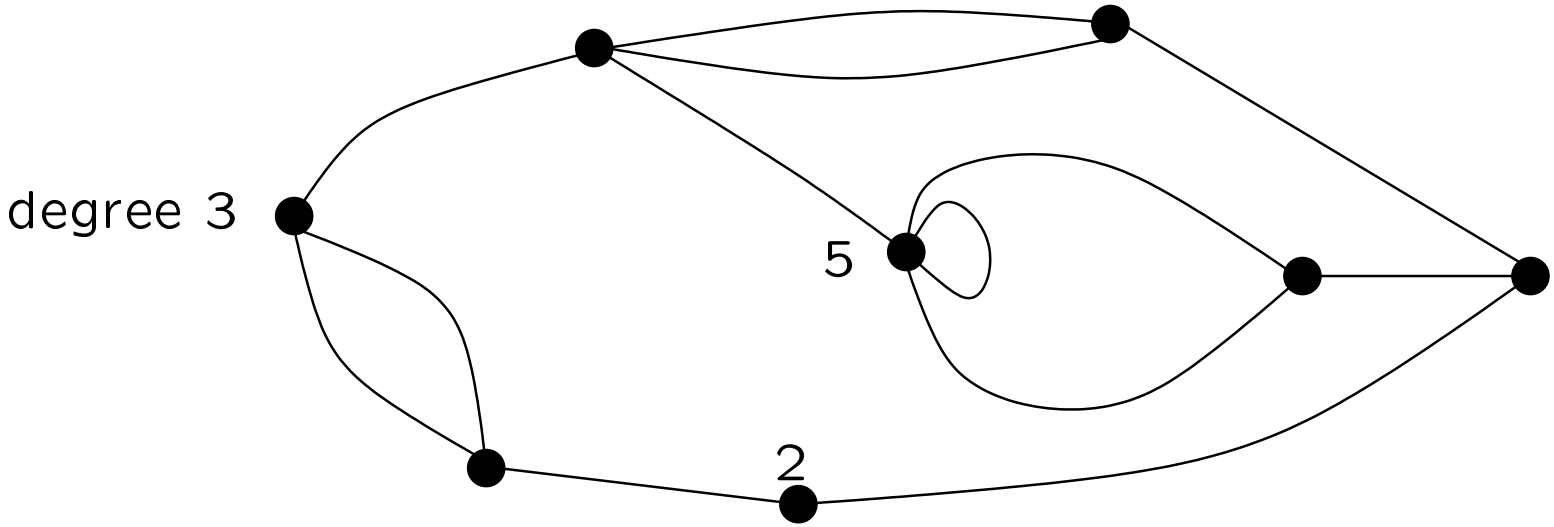
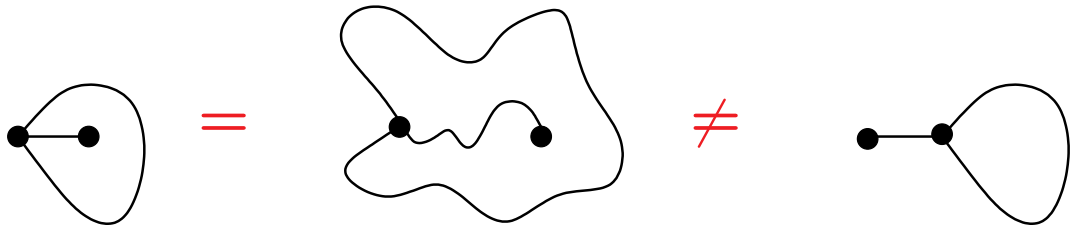
A planar map

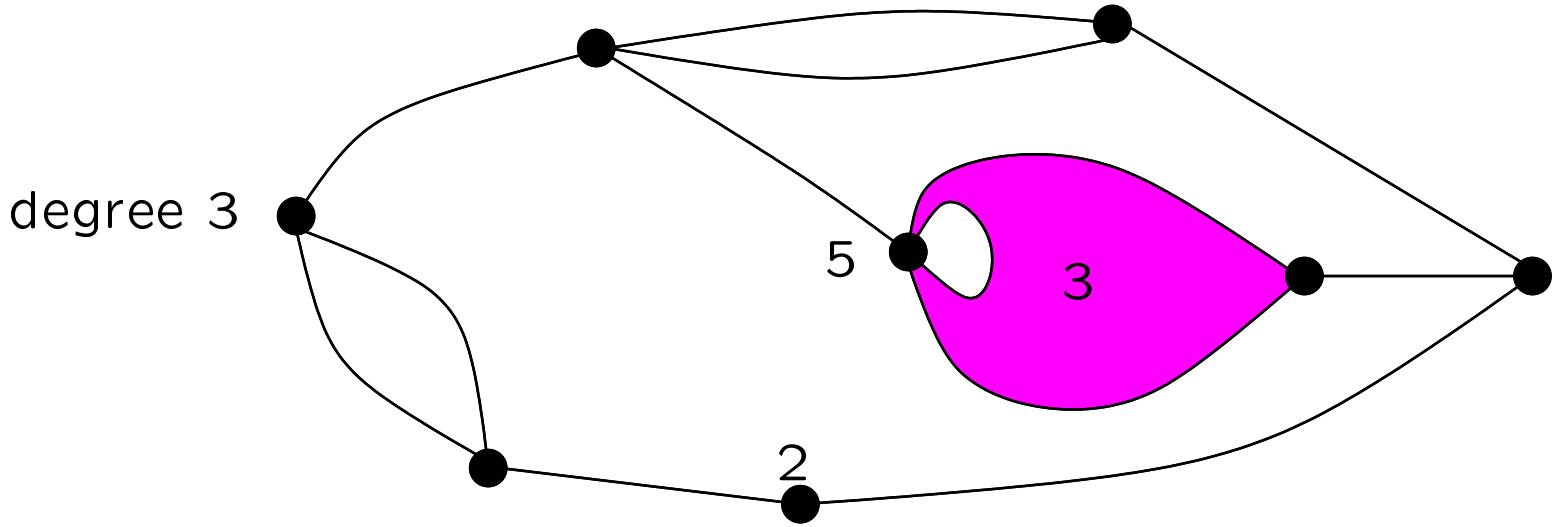

Index of Colours
Trias
Permian
Carboniferous Rocks
Silurian
GEOLOGICAL MAP OF
GREAT BRITAIN

Planar maps



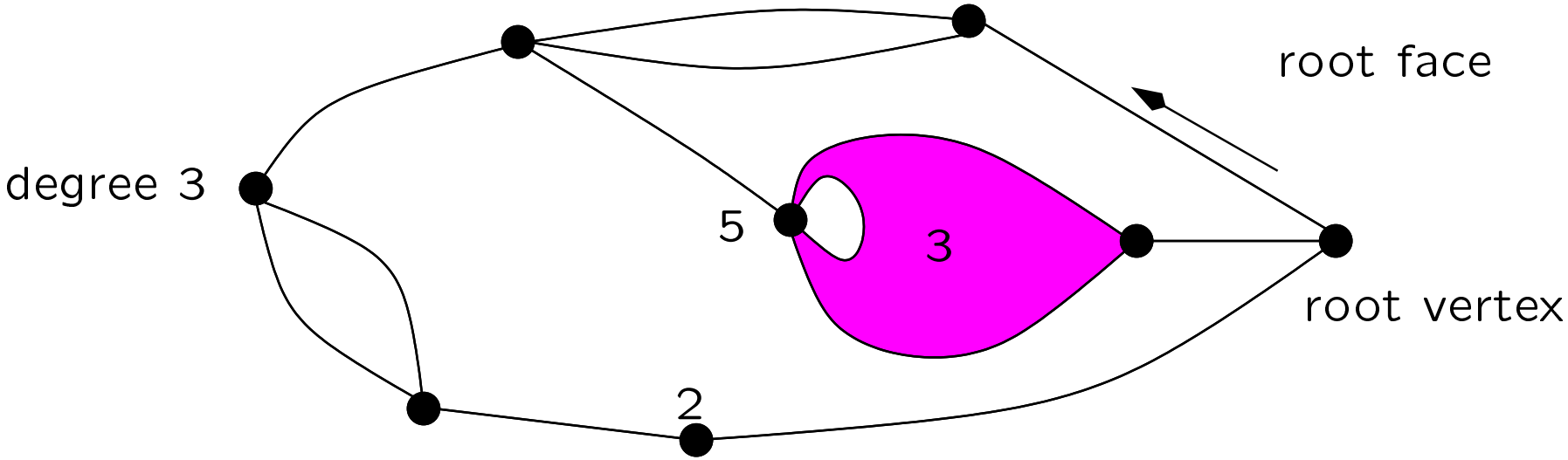


Planar maps



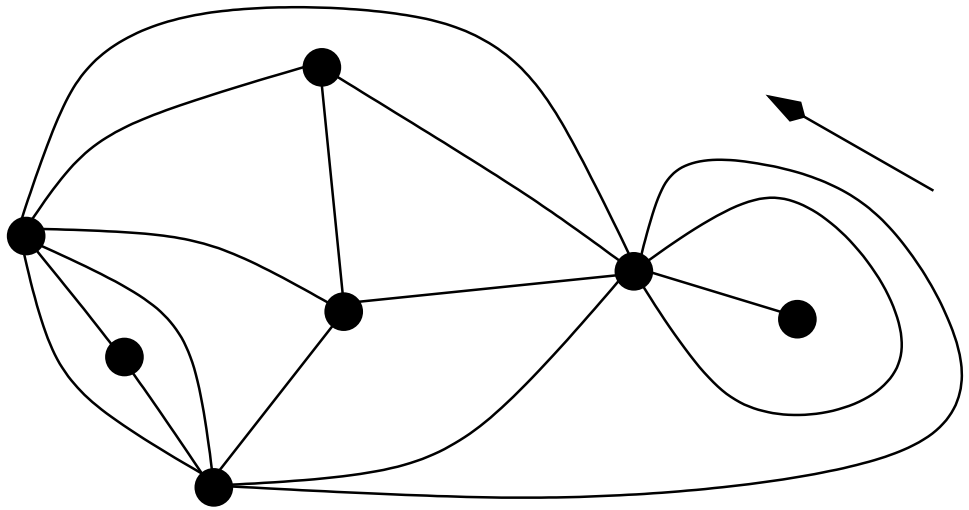
- vertices
- edges
- and faces

Rooted planar maps



- vertices
- edges
- and faces

Triangulations

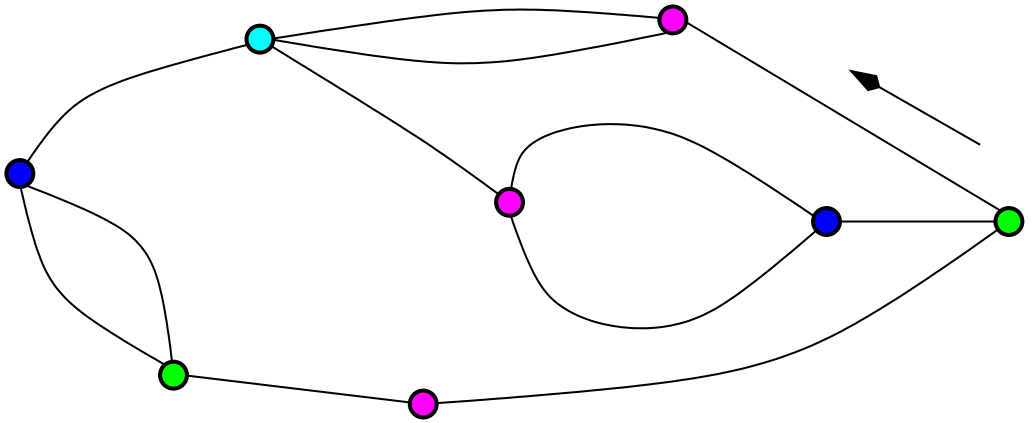


Every face has degree 3.

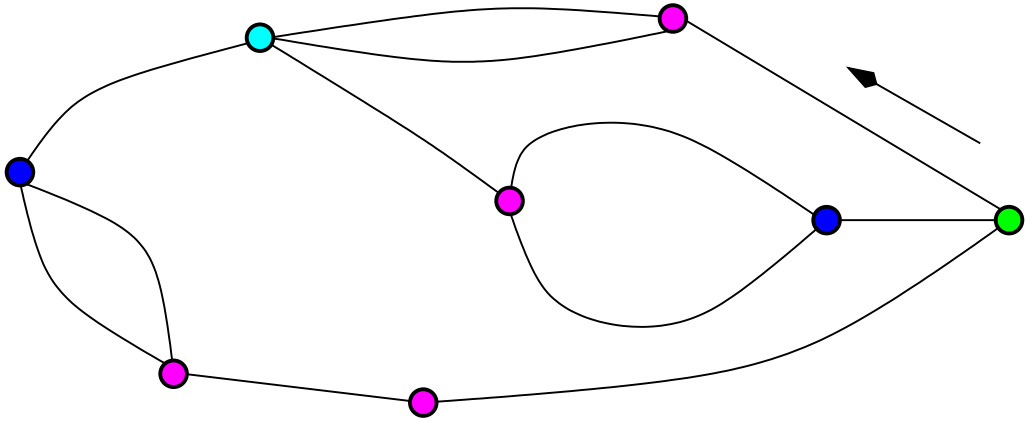
Loops and multiple edges are allowed.

Colourings

Proper



Non-proper (general)



Monochromatic edge

Map enumeration

Let $\mathcal{M}$ be a set of planar maps, and let $m(n)$ be the number of maps of $\mathcal{M}$ of size (= edge number) n.

- Objective: determine the sequence $m(n)$

$$m(n) = \cdots \qquad \text{or} \qquad M(t) := \sum_{n \geq 0} m(n) t^n = \sum_{M \in \mathcal{M}} t^{\mathrm{e}(M)} = \cdots$$

where $M(t)$ is the generating function of $\mathcal{M}$.

- Multi-parameter enumeration:

$$m(n;k) = \cdots \qquad \text{or} \qquad M(t;x) := \sum_{n \geq 0} m(n;k) t^n x^k = \cdots$$

Examples

- The generating function of planar maps, counted by edges, is

$$M(t) = \sum_{M} t^{\mathrm{e}(M)} = \sum_{n \geq 0} \frac{2 \cdot 3^n}{n(n+1)} \binom{2n}{n} t^n = \frac{(1-12t)^{3/2} - 1 + 18\,t}{54t^2}$$

[Tutte 60's]

Examples

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[Tutte 60's]

- The generating function of planar triangulations, counted by vertices, is

$$T(t) = \sum_T t^{\mathrm{v}(T)} = \sum_{n \geq 2} \frac{2 \cdot 4^{n-2}\,(3n-6)!!}{n!(n-2)!!}\, t^n$$

with $k!! = k(k-2)(k-4)\cdots$, and satisfies the cubic equation

$$64\,T(t)^3 + (1-96\,t)\,T(t)^2 - t\,(1-30\,t)\,T(t) + t^3\,(1-27\,t) = 0$$

[Mullin, Nemeth, Schellenberg 70]

Maps equipped with an additional structure

In combinatorics, but mostly in statistical physics

How many maps equipped with...

- a spanning tree?
  [Mullin 67]
- a spanning forest?
  [Bouttier et al., Sportiello et al.]
- a self-avoiding walk?
  [Duplantier-Kostov 88]
- a proper q-colouring?
  [Tutte 74, Bouttier et al. 02]

What is the expected partition function of...

- the Ising model?
  [Boulatov, Kazakov, MBM, Schaeffer, Bouttier et al.]
- the hard-particle model?
  [MBM, Schaeffer, Jehanne, Bouttier et al. 02, 07]
- the Potts model?
  [Eynard-Bonnet 99, Baxter 01, MBM-Bernardi 09, Guionnet et al. 10

The Potts model on planar maps

• Count all q-colourings of some family $\mathcal{M}$ of planar maps, keeping track of the number $m(M)$ of monochromatic edges:

$$M(q,\nu,t) := \sum_{M \ q-\text{coloured}} t^{\mathrm{e}(M)}\, \nu^{m(M)}$$

The Potts generating function of maps.

• In other words,

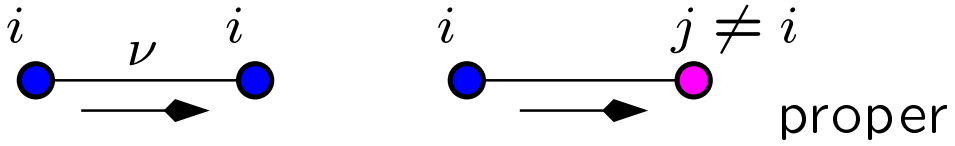
$$M(q,\nu,t) = \sum_{M} \mathrm{Z}_M(q,\nu) t^{\mathrm{e}(M)}$$

where

$$\mathrm{Z}_M(q,\nu) = \sum_{c:V(M)\to\{1,2,\ldots,q\}} \nu^{m(c)}$$

is the Potts partition function of M.

Example: When M has one edge and two vertices, $\mathrm{Z}_M(q,\nu) = q\nu + q(q-1)$



The Potts model on planar maps

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The Potts generating function of maps.

• In particular,

$$M(q,0,t) := \sum_{M \ q-\text{prop. coloured}} t^{\mathrm{e}(M)} = \sum_{M} \chi_M(q) t^{\mathrm{e}(M)}$$

counts properly coloured maps.

The Potts model on planar maps

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The Potts generating function of maps.

• Equivalently, find

$$\sum_{M\in\mathcal{M}} T_M(x,y)\ t^{\mathrm{e}(M)} = \cdots$$

where $T_M(x,y)$ is the Tutte polynomial of M. Connection:

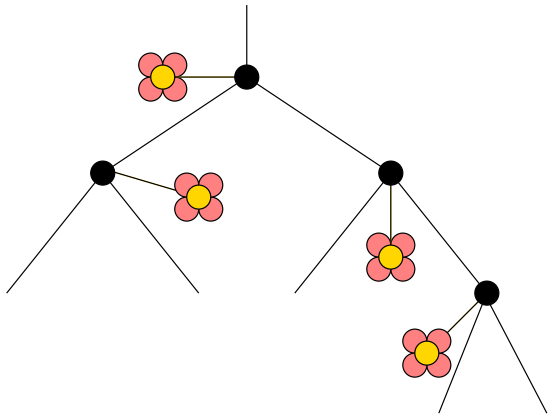
$$(x-1)(y-1)^{\mathrm{v}(M)} T_M(x,y) = \sum_{q-\text{colourings of } M} \nu^{m(M)}$$

with $q=(x-1)(y-1)$ and $\nu = y-1$.

Three approaches for counting maps

- [Tutte 60's]: recursive description of maps

$\Rightarrow$ functional equations for the GF



- [Schaeffer 97]: bijections with trees

- [Brézin-Itzykson-Parisi-Zuber 78]: matrix integrals

$$\lim_{N\to\infty} \frac{1}{N^2} \log \int dH e^{N\mathrm{tr}\left(-\frac{H^2}{2}+t\frac{H^3}{3}\right)}.$$

H: hermitian $N \times N$ matrix

I.1. Uncoloured enumeration: the recursive approach

Planar maps (tend to) have algebraic generating functions

$M(t_1, \ldots, t_k)$ map GF: there exists a polynomial Q such that

$$Q(t_1, \ldots, t_k; M(t_1, \ldots, t_k)) = 0.$$

Example:

$$M(t) = \frac{(1-12t)^{3/2} - 1 + 18\,t}{54t^2} \quad \Rightarrow \quad \left(54t^2 M(t) + 1 - 18t\right)^2 = (1-12t)^3$$

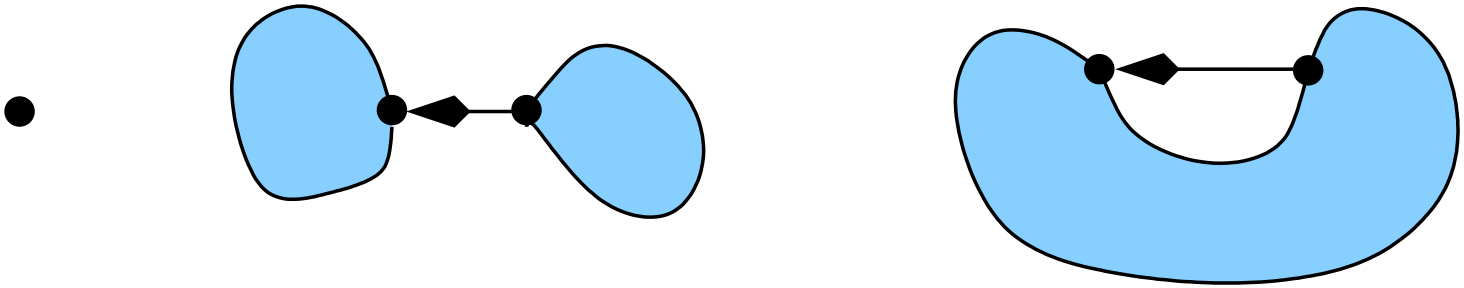
Arquès Bauer Bédard Bender Bernardi Bessis Bodirsky Bousquet-Mélou
Boulatov Bouttier Brézin Brown Canfield Chauve Cori Di Francesco
Duplantier Eynard Fusy Gao Goupil Goulden Guitter t'Hooft Itzykson Jackson
Jacquard Kazakov Kostov Krikun Labelle Lehman Leroux Liskovets Liu Machì
Mehta Mullin Parisi Poulalhon Richmond Robinson
Schaeffer Schellenberg Strehl **Tutte** Vainshtein Vauquelin Visentin Walsh
Wanless Wormald Zinn-Justin Zuber Zvonkine...

Recursive description of planar maps: deleting the root-edge

Let

$$F(t;x) \equiv F(x) = \sum_M t^{\mathsf{e}(M)} x^{\mathsf{df}(M)} = \sum_{d \geq 0} F_d(t) x^d$$

where $\mathsf{e}(M)$ is the number of edges and $\mathsf{df}(M)$ the degree of the outer face.



Recursive description of planar maps: deleting the root-edge

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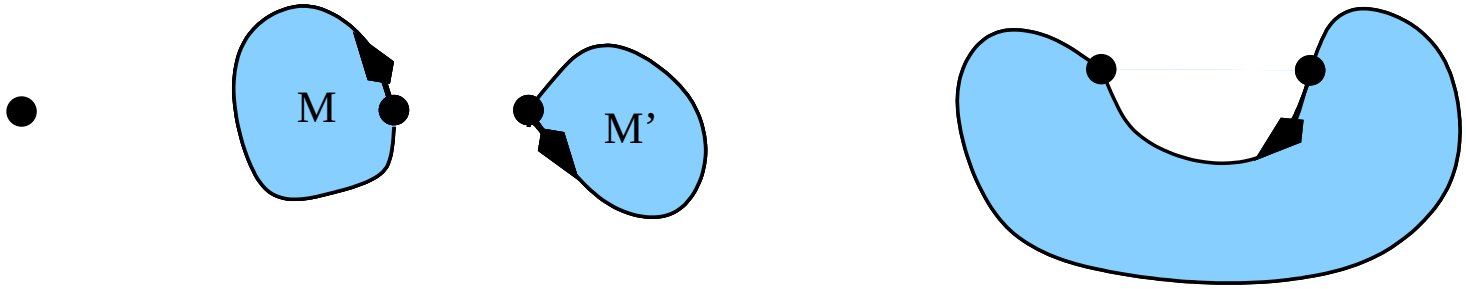

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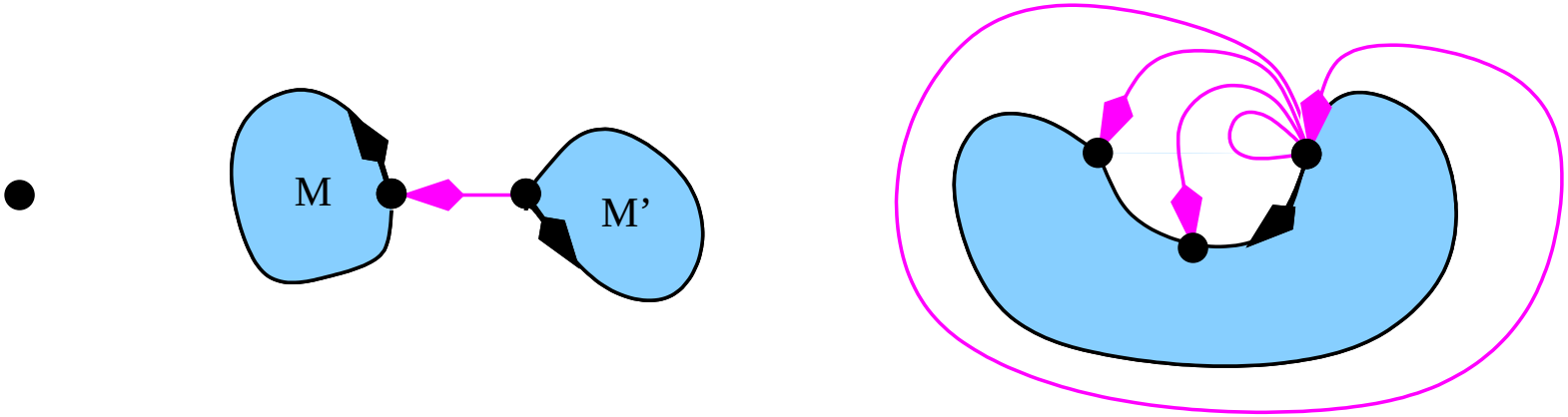

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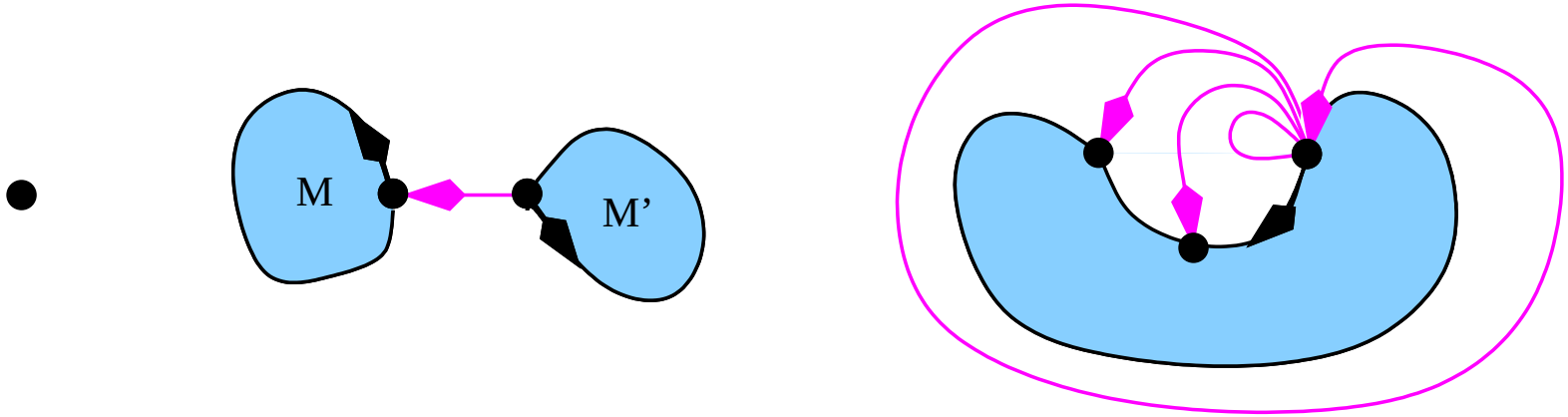


$$\begin{aligned} F(x) &= 1 + tx^2F(x)^2 + t\sum_{d\geq 0} F_d(t)\left(x^{d+1}+x^d+\cdots+x\right) \\ &= 1 + tx^2F(x)^2 + tx\,\frac{xF(x)-F(1)}{x-1} \end{aligned}$$

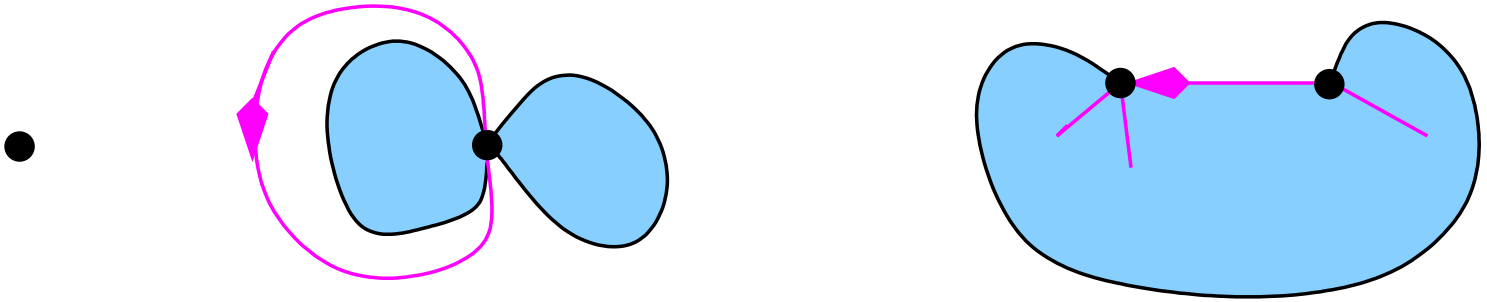
[Tutte 68] A quadratic equation with one catalytic variable, x

Recursive description of planar maps: contracting the root-edge

Let

$$F(t;y) \equiv F(y) = \sum_M t^{\mathrm{e}(M)} y^{\mathrm{dv}(M)} = \sum_{d\geq 0} F_d(t) y^d$$

where $\mathrm{e}(M)$ is the number of edges and $\mathrm{dv}(M)$ the degree of the root vertex.



$$F(y) = 1 + ty^2F(y)^2 + t\sum_{d\geq 0} F_d(t)\left(y^{d+1}+y^d+\cdots+y\right)$$

$$= 1 + ty^2F(y)^2 + ty\,\frac{yF(y)-F(1)}{y-1}$$

The same equation... (duality)

Polynomial equations with one catalytic variable

Examples:

$$F(x) = 1 + tx^2F(x)^2 + tx\,\frac{xF(x) - F(1)}{x-1}$$

$$F(x) = 1 + txF(x)^3 + tx(2F(x) + F(1))\,\frac{F(x) - F(1)}{x-1} + tx\frac{F(x) - F(1) - (x-1)F'(1)}{(x-1)^2}$$

Solution:

- Guess and check [Tutte 60's]
- Brown's quadratic method [Brown 65]
- A generalization to all polynomial equations with one catalytic variable [mbm-Jehanne 06]

Polynomial equations with one catalytic variable

- General framework: A polynomial equation:

$$P(F(x), F_1, \ldots, F_m, t, x) = 0 \tag{1}$$

where $F(x) \equiv F(t; x)$ is a series in t with polynomial coefficients in x, and $F_i \equiv F_i(t)$ does not depend on x.

- Results

1. The solution of every proper equation of this type is algebraic: There exists Q such that $Q(t, x, F(t; x)) = 0$.
2. A practical (but heavy) strategy allows to solve specific examples (that is, to derive from (1) an algebraic equation for $F(x)$, or $F_1, \ldots, F_m$).

[MBM-Jehanne 06]

Example

$$F(t;x) \equiv F(x) = 1 + tx^2F(x)^2 + tx\,\frac{xF(x) - F(1)}{x-1}$$

$$\Downarrow$$

$$F(t;1) = \frac{(1-12t)^{3/2} - 1 + 18\,t}{54t^2}$$

with two lines of Maple.

Equivalently,

$$F(t;1) = A(t) - tA(t)^3 \quad \text{where} \quad A(t) = 1 + 3tA(t)^2$$

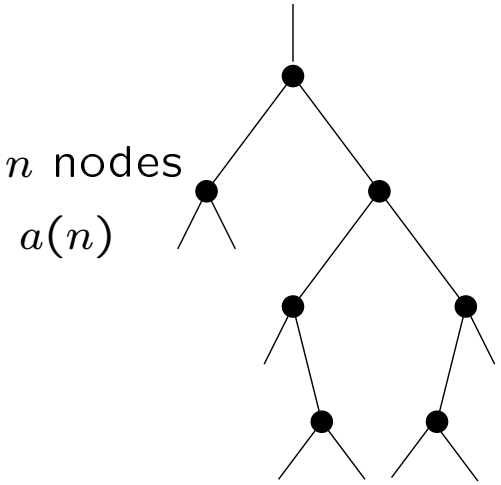
$\Rightarrow$ Many map families have an algebraic generating function

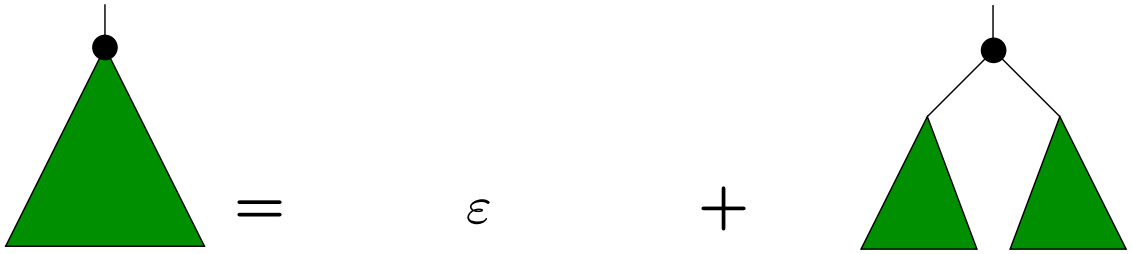
I.2. Uncoloured enumeration: bijections

The algebraicity of planar maps generating functions can be explained via bijections with plane trees

Algebraic series count plane trees

Algebraic series usually arise from tree enumeration:



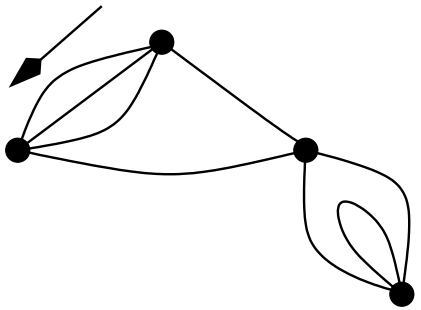


$$A(t) = 1 + tA(t)^2$$

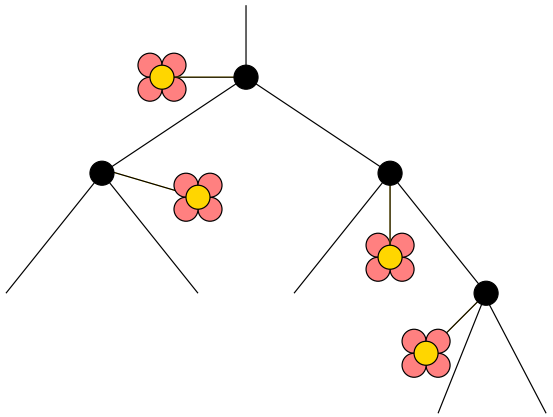
Are there correspondences between maps and trees?

Four-valent maps: a proof from The Book?

• Bijection between 4-valent planar maps (n vertices) and balanced blossoming trees (n nodes) [Schaeffer 97]



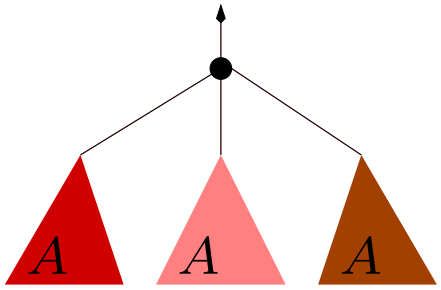
A planar map (GF M)



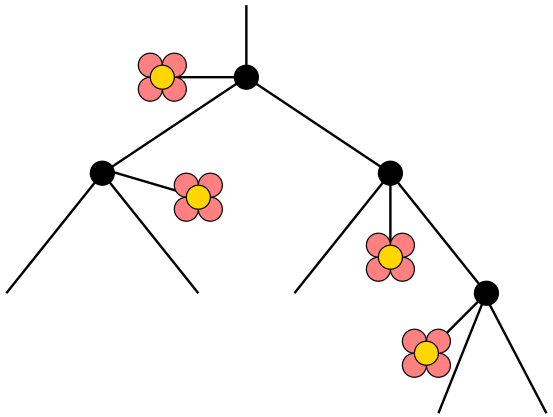
A blossoming tree (GF $A = 1 + 3tA^2$)

• An unbalanced blossoming tree is, in essence, a 3-tuple of blossoming subtrees [Bouttier et al. 02]:

$$\Rightarrow M = A - tA^3.$$



Blossoming trees and 4-valent maps



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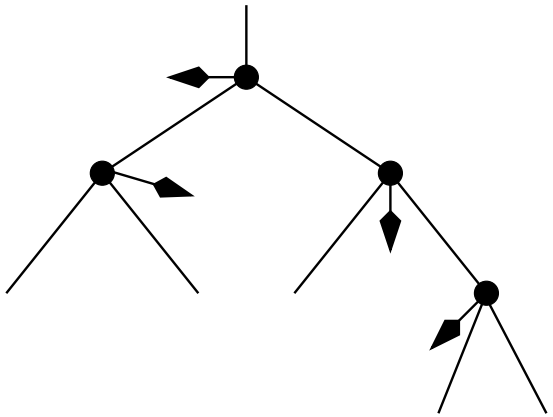

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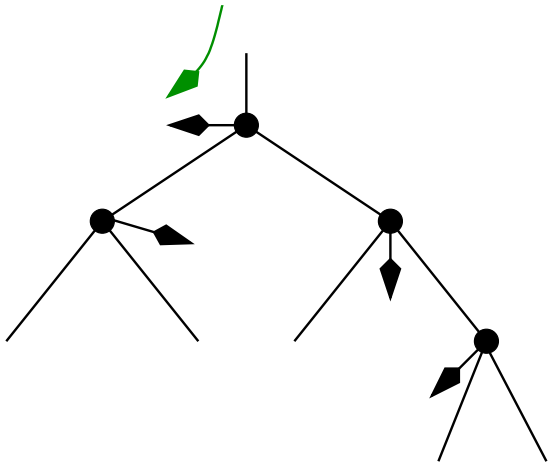

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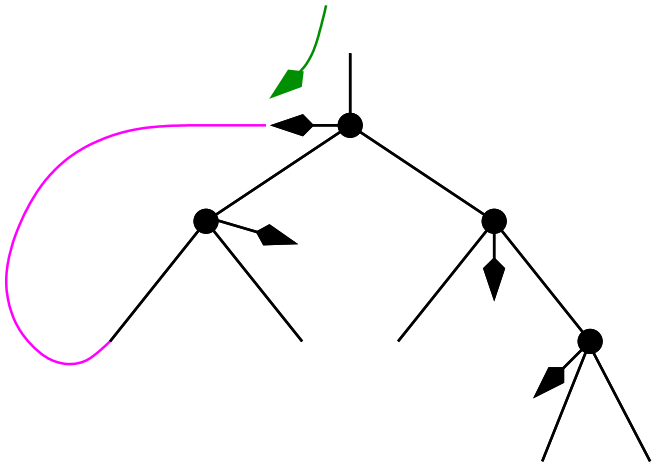

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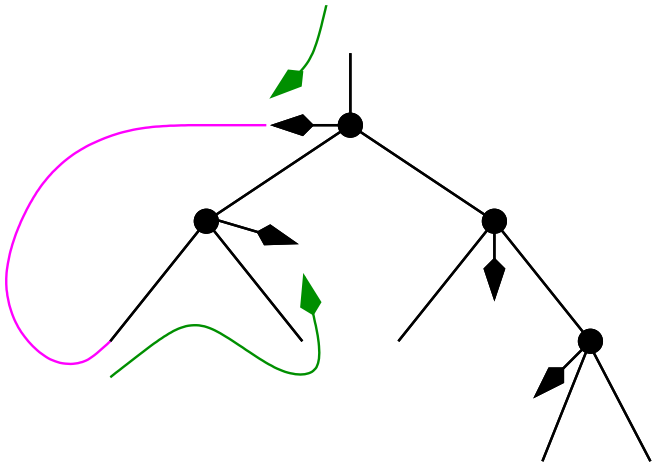

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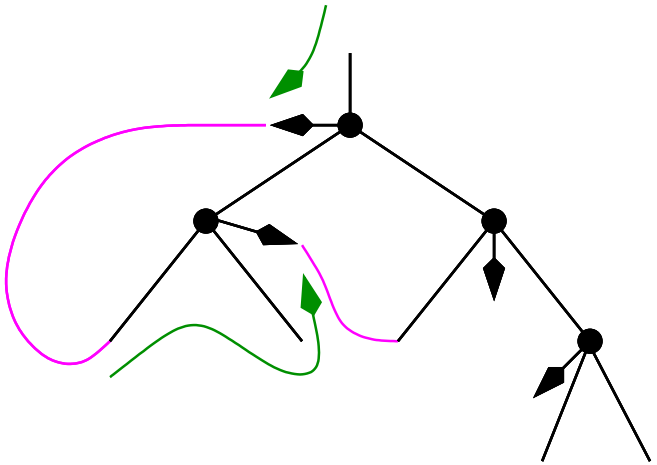

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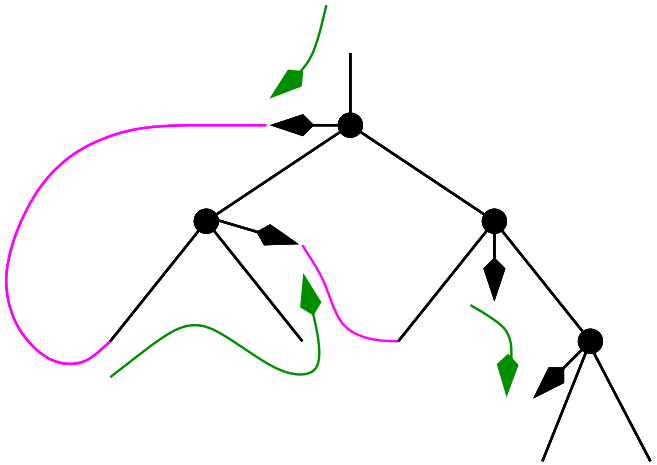

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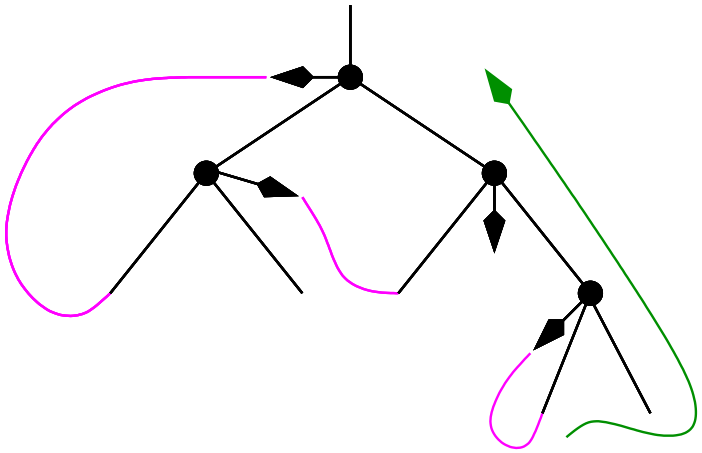

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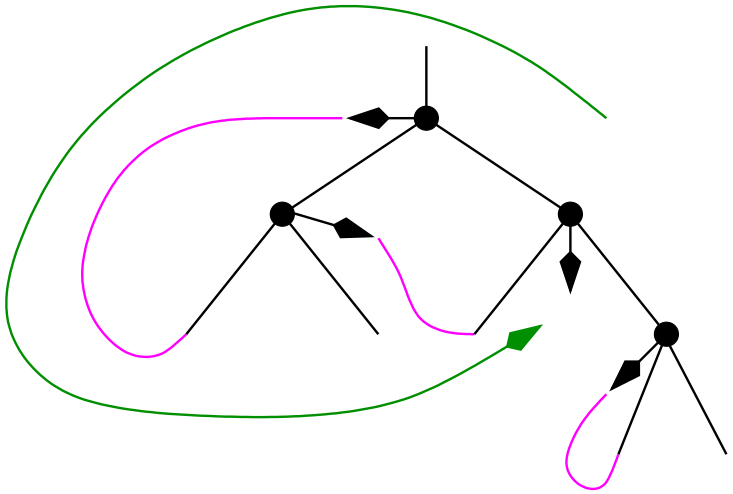

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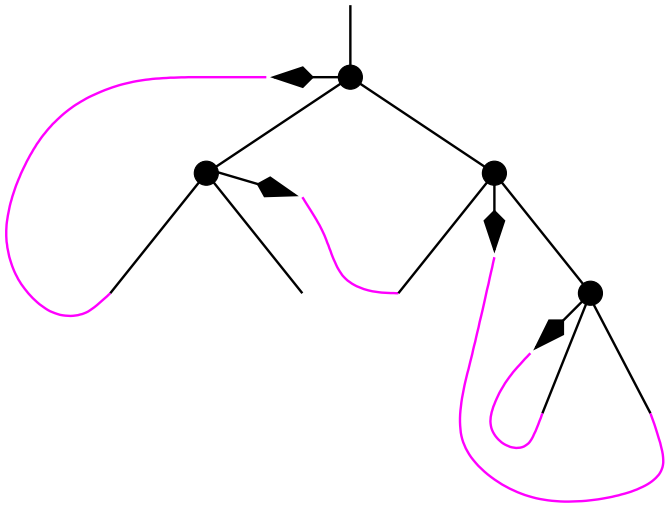

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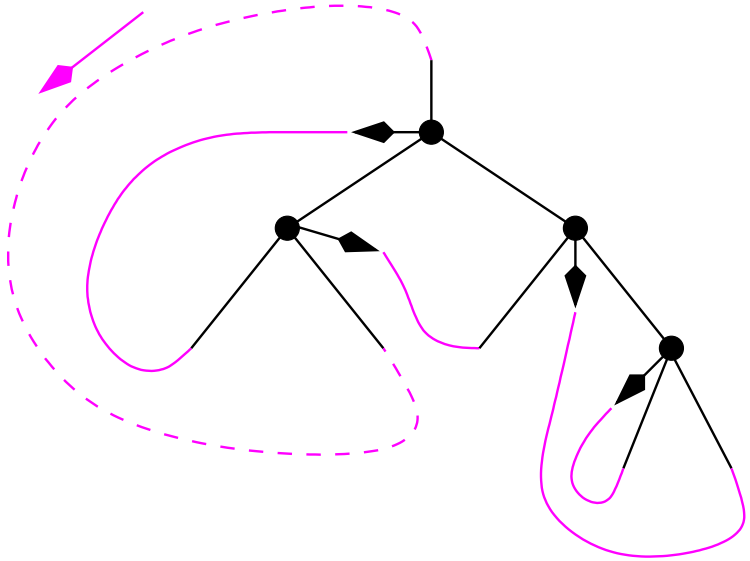

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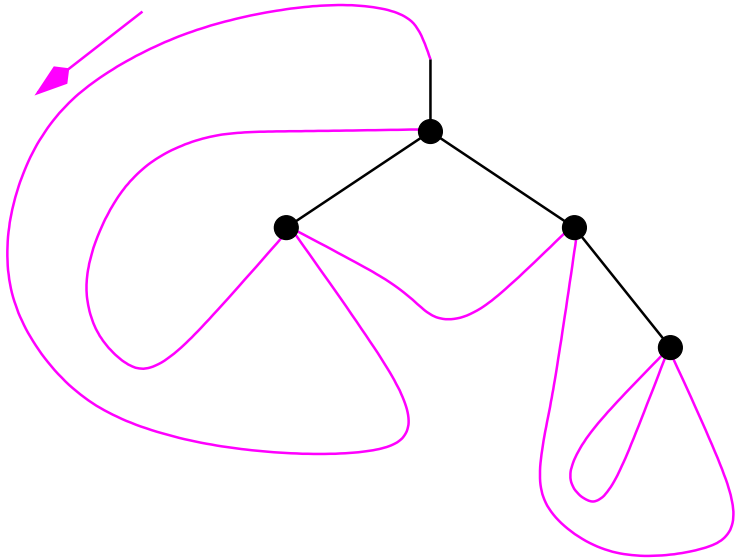
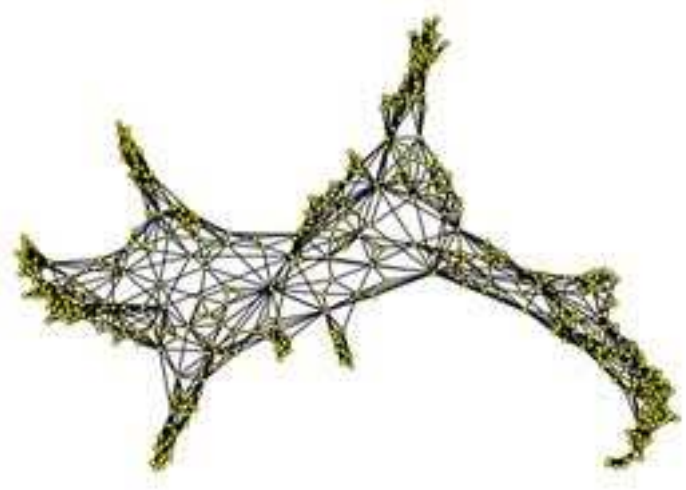


Bijections between maps and trees...

• Improve our understanding of the nature of planar maps

• Lead to new results, e.g. on the geometry of random planar maps. For instance, the average diameter of a random map of size n scales like $n^{1/4}$ [Chassaing-Schaeffer 02]



• These bijections are the starting point of many recent results in probability theory on the asymptotic properties of large random maps [Le Gall, Miermont, Marckert, Paulin...]

II.1. Coloured enumeration: the recursive approach

(joint work with Olivier Bernardi)

$$M(q,\nu,t) := \sum_{M \ q-\text{coloured}} t^{\mathrm{e}(M)}\, \nu^{m(M)}$$

- Other approaches: [Eynard-Bonnet 99], [Guionnet et al. 10]

Forget algebraicity!

Theorem [Tutte 73]: For planar triangulations,

$$\sum_T \chi'_T(1)\, t^{\mathsf{v}(T)} = \sum_n (-1)^n b(n) t^{n+2}$$

where

$$b(n) = \frac{2\,(3n)!}{n!(n+1)!(n+2)!} \sim 27^n n^{-4},$$

and this asymptotic behaviour prevents the series $B(t) := \sum b_n t^n$ from being algebraic.

However, it satisfies a linear differential equation.

Catalytic variables

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WHY IS THAT SO?

- The recursive description of the Potts partition function

$$Z_G(q,\nu) = Z_{G\setminus e}(q,\nu) + (\nu-1)\, Z_{G/e}(q,\nu)$$

calls for a recursive description of maps by contraction and deletion of edges.

- This is possible if one keeps track of the degree of the outer face, and the degree of the root-vertex.

Equations with two catalytic variables

- Let

$$M(x,y) \equiv M(q,\nu,t;x,y) = \frac{1}{q}\sum_M Z_M(q,\nu) t^{\mathsf{e}(M)} x^{\mathsf{dv}(M)} y^{\mathsf{df}(M)},$$

where $\mathsf{dv}(M)$ (resp. $\mathsf{df}(M)$) is the degree of the root-vertex (resp. root-face).

- The Potts generating function of planar maps satisfies:

$$\begin{aligned} M(x,y) = 1 &+ xyt\left((\nu-1)(y-1)+qy\right) M(x,y)M(1,y) \\ &+ xyzt(x\nu-1)M(x,y)M(x,1) \\ &+ xyt(\nu-1)\frac{xM(x,y)-M(1,y)}{x-1} + xyzt\frac{yM(x,y)-M(x,1)}{y-1}. \end{aligned}$$

[Tutte 68]

This equation has been sleeping for 40 years

In the footsteps of W. Tutte

- For the GF $T(q,t;x,y) \equiv T(x,y)$ of properly q-coloured triangulations:

$$T(x,y) = xy^2q(q-1) + \frac{xt}{yq}T(1,y)T(x,y) + xt\frac{T(x,y) - y^2T_2(x)}{y} - x^2yt\frac{T(x,y) - T(1,y)}{x-1}$$

where $T_2(x)$ is the coefficient of y^2 in $T(x,y)$.

[Tutte 73] Chromatic sums for rooted planar triangulations: the cases $\lambda = 1$ and $\lambda = 2$
[Tutte 73] Chromatic sums for rooted planar triangulations, II : the case $\lambda = \tau + 1$
[Tutte 73] Chromatic sums for rooted planar triangulations, III : the case $\lambda = 3$
[Tutte 73] Chromatic sums for rooted planar triangulations, IV : the case $\lambda = \infty$
[Tutte 74] Chromatic sums for rooted planar triangulations, V : special equations
[Tutte 78] On a pair of functional equations of combinatorial interest
[Tutte 82] Chromatic solutions
[Tutte 82] Chromatic solutions II
[Tutte 84] Map-colourings and differential equations

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[Tutte 95]: Chromatic sums revisited

In the footsteps of W. Tutte

- For the GF $T(q,t;x,y) \equiv T(x,y)$ of properly q-coloured triangulations:

$$T(x,y) = xy^2q(q-1) + \frac{xt}{yq}T(1,y)T(x,y) + xt\frac{T(x,y) - y^2T_2(x)}{y} - x^2yt\frac{T(x,y) - T(1,y)}{x-1}$$

where $T_2(x)$ is the coefficient of y^2 in $T(x,y)$.

Theorem [Tutte]

- For $q = 2+2\cos\frac{2\pi}{m}$, $q \neq 4$, the series $T(1,y) \equiv T(t;1,y)$ satisfies a polynomial equation with one catalytic variable y.

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• For the GF $T(q,t;x,y) \equiv T(x,y)$ of properly q-coloured triangulations:

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Theorem [Tutte]

• For $q = 2+2\cos\frac{2\pi}{m}$, $q \neq 4$, the series $T(1,y) \equiv T(t;1,y)$ satisfies a polynomial equation with one catalytic variable y.

• When q is generic, the generating function of properly q-coloured planar triangulations is differentially algebraic:

$$2q^2(1-q)t + (qt + 10H - 6tH')H'' + q(4-q)(20H - 18tH' + 9t^2H'') = 0$$

with $H(t) = t^2T_2(q,\sqrt{t};1)/q$.

Adapt this to other equations!

[Tutte 73] Chromatic sums for rooted planar triangulations: the cases $\lambda = 1$ and $\lambda = 2$
[Tutte 73] Chromatic sums for rooted planar triangulations, II : the case $\lambda = \tau + 1$
[Tutte 73] Chromatic sums for rooted planar triangulations, III : the case $\lambda = 3$
[Tutte 73] Chromatic sums for rooted planar triangulations, IV : the case $\lambda = \infty$
[Tutte 74] Chromatic sums for rooted planar triangulations, V : special equations
[Tutte 78] On a pair of functional equations of combinatorial interest
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[Tutte 82] Chromatic solutions II
[Tutte 84] Map-colourings and differential equations

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[Tutte 95]: Chromatic sums revisited

Our results

- Let $M(q,\nu,t;x,y)$ be the Potts generating function of planar maps:

$$M(x,y) \equiv M(q,\nu,t;x,y) = \frac{1}{q}\sum_{M} Z_M(q,\nu) t^{\mathrm{e}(M)} x^{\mathrm{dv}(M)} y^{\mathrm{df}(M)},$$

where $\mathrm{dv}(M)$ (resp. $\mathrm{df}(M)$) is the degree of the root-vertex (resp. root-face).

Theorem

- For $q = 2 + 2\cos\frac{j\pi}{m}$, $q \neq 0, 4$, the series $M(q,\nu,t;1,y) \equiv M(1,y)$ satisfies a polynomial equation with one catalytic variable y, and the complete Potts generating function $M(q,\nu,t;x,y)$ is algebraic.

- When q is generic, $M(q,\nu,t;1,1)$ is differentially algebraic:

(an explicit system of differential equations)

[mbm-Bernardi 09] Counting colored planar maps: algebraicity results. Arxiv:0909:1695

[mbm-Bernardi 11] Counting colored planar maps: differential equations

Example: The Ising model on planar maps ($q = 2$)

Let A be the series in t, with polynomial coefficients in ν, defined by

$$A = t\,\frac{\left(1 + 3\,\nu\,A - 3\,\nu\,A^2 - \nu^2 A^3\right)^2}{1 - 2\,A + 2\,\nu^2 A^3 - \nu^2 A^4}.$$

Then the Ising generating function of planar maps is

$$M(2, \nu, t; 1, 1) = \frac{1 + 3\,\nu\,A - 3\,\nu\,A^2 - \nu^2 A^3}{\left(1 - 2\,A + 2\,\nu^2 A^3 - \nu^2 A^4\right)^2}\; P(\nu, A)$$

where

$$P(\nu, A) = \nu^3 A^6 + 2\,\nu^2(1-\nu)A^5 + \nu\,(1 - 6\,\nu)A^4 \\ - \nu\,(1 - 5\,\nu)A^3 + (1 + 2\,\nu)A^2 - (3 + \nu)A + 1.$$

$\rightsquigarrow$ Asymptotics: Phase transition at $\nu_c = \frac{3+\sqrt{5}}{2}$, critical exponents...

Example: properly 3-coloured planar maps ($q = 3, \nu = 0$)

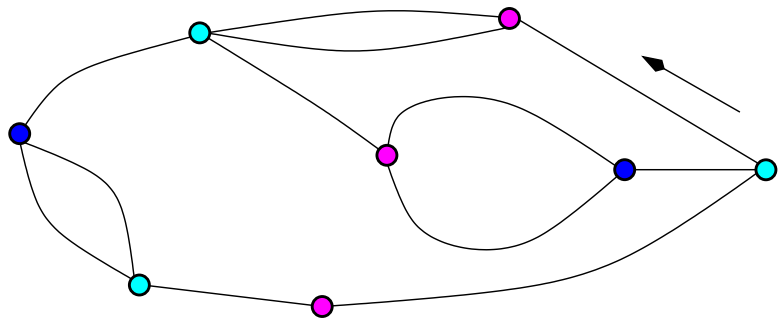
Let A be the quartic series in t defined by

$$A = t\,\frac{(1+2A)^3}{(1-2A^3)}.$$

Then the generating function of properly 3-coloured planar maps is

$$M(3,0,t;1,1) = \frac{(1+2A)(1-2A^2-4A^3-4A^4)}{(1-2A^3)^2}$$

$\rightsquigarrow$ Asymptotics: A random loopless planar map with n edges has approximately $(1.42\ldots)^n$ proper 3-colourings



Our results: when q is generic

- Let $M(q,\nu,t;x,y)$ be the Potts generating function of planar maps:

$$M(x,y) \equiv M(q,\nu,t;x,y) = \frac{1}{q}\sum_{M} \mathsf{Z}_M(q,\nu) t^{\mathsf{e}(M)} x^{\mathsf{dv}(M)} y^{\mathsf{df}(M)},$$

where $\mathsf{dv}(M)$ (resp. $\mathsf{df}(M)$) is the degree of the root-vertex (resp. root-face).

Theorem

- For $q = 2 + 2\cos\frac{j\pi}{m}$, $q \neq 4$, the series $M(q,\nu,t;1,y) \equiv M(1,y)$ satisfies a polynomial equation with one catalytic variable y, and the complete Potts generating function $M(q,\nu,t;x,y)$ is algebraic.

- When q is generic, $M(q,\nu,t;1,1)$ is differentially algebraic:

(an explicit system of differential equations)

An explicit system of differential equations

Let $D(t,v) = q\nu + (\nu-1)^2 - q(\nu+1)v + \Big(q + t(\nu-1)(q-4)(q+\nu-1)\Big)v^2$.

• There exists a unique 8-tuple $(P_1(t),\dots,P_4(t),Q_1(t),Q_2(t),R_1(t),R_2(t))$ of series in t with polynomial coefficients in q and ν such that

$$\frac{1}{v^2R}\,\frac{\partial}{\partial v}\left(\frac{v^4R^2}{PD^2}\right) = \frac{1}{Q}\,\frac{\partial}{\partial t}\left(\frac{Q^2}{PD^2}\right),$$

where

$$\begin{aligned} P(t,v) &= P_4(t)v^4 + P_3(t)v^3 + P_2(t)v^2 + P_1(t)v + 1,\\ Q(t,v) &= Q_2(t)v^2 + Q_1(t)v + 1,\\ R(t,v) &= R_2(t)v^2 + R_1(t)v + q + \nu - 3, \end{aligned}$$

with the initial conditions (at $t=0$):

$$P(0,v) = (1-v)^2 \quad \text{and} \quad Q(0,v) = 1-v.$$

An explicit system of differential equations (cont'd)

• The Potts generating function of planar maps, $M(1,1) \equiv M(q,\nu,t;1,1)$, satisfies

$$12\,t^2\left(q\nu+(\nu-1)^2\right)M(q,\nu,t;1,1) =$$
$$8\,t(q+\nu-3)Q_1(t)-Q_1(t)^2+P_2(t)-2\,Q_2(t)-4\,t\,(2-3\,\nu-q)-12\,t^2\,(q+\nu-3)^2\,.$$

Questions

1. Use the structure of

$$\frac{1}{v^2R}\frac{\partial}{\partial v}\left(\frac{v^4R^2}{PD^2}\right)=\frac{1}{Q}\frac{\partial}{\partial t}\left(\frac{Q^2}{PD^2}\right),$$

to obtain a single differential equation (or an expression?) for $M(q,\nu,t;1,1)$.

2. Relate this to elliptic functions, and to the papers of [Bonnet & Eynard 99], and [Guionnet, Jones, Shlyakhtenko & Zinn-Justin 10]

An analogous system for triangulations

Let $D(t,v) = q\nu^2 + (\nu-1)\,(4(\nu-1)+q)\,v + \Big(q\nu(\nu-1)(q-4)t + (\nu-1)^2\Big)\,v^2$.

• There exists a unique 7-tuple $(P_1(t),\ldots,P_3(t),Q_1(t),Q_2(t),R_0(t),R_1(t))$ of series in t with polynomial coefficients in q and ν such that

$$\frac{1}{v^2R}\,\frac{\partial}{\partial v}\left(\frac{v^5R^2}{PD^2}\right) = \frac{1}{Q}\,\frac{\partial}{\partial t}\left(\frac{Q^2}{PD^2}\right),$$

where

$$\begin{aligned} P(t,v) &= P_3(t)v^3 + P_2(t)v^2 + P_1(t)v + 1, \\ Q(t,v) &= Q_2(t)v^2 + Q_1(t)v + 2\nu, \\ R(t,v) &= R_1(t)v + R_0(t), \end{aligned}$$

with the initial conditions (at $t=0$):

$$P(0,v) = 1 + v/4 \quad \text{and} \quad Q(0,v) = 2\nu + v.$$

• Expression of the Potts GF of triangulations in terms of the P_i and Q_i

... and for properly q-coloured triangulations ($\nu = 0$)

Let $D(v) = v + 4 - q$.

• There exists a unique 4-tuple (P_1, P_2, P_3, Q_1) of zeries in t with polynomial coefficients in q such that

$$-\frac{4t}{v}\,\frac{\partial}{\partial v}\left(\frac{v^3}{P}\right) = \frac{1}{Q}\,\frac{\partial}{\partial t}\left(\frac{Q^2}{PD}\right).$$

where

$$\begin{aligned} P(t,v) &= P_3(t)v^3 + P_2(t) + P_1(t)v + 1, \\ Q(t,v) &= Q_1(t)v + 1, \end{aligned}$$

with the initial conditions (at $t = 0$):

$$P(0,v) = 1 + v/4 \quad \text{and} \quad Q(0,v) = 1.$$

• From the system, one can derive Tutte's differential equation,

$$2q^2(1-q)t + (qt + 10H - 6tH')H'' + q(4-q)(20H - 18tH' + 9t^2H'') = 0$$

with $H(t) = t^2 T_2(q, \sqrt{t}; 1)/q$.

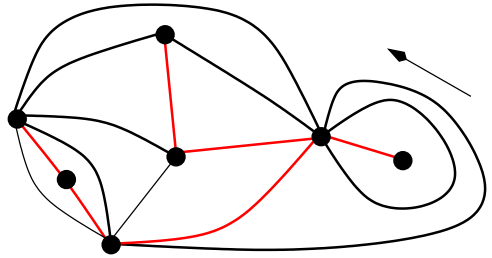
II.2. Coloured enumeration: bijections?

Some bijections exist in special cases... but most remain to be found

Some existing bijections

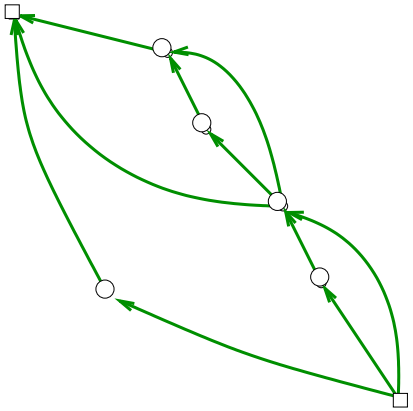
- Maps equipped with a spanning tree ($\mathsf{T}_M(1,1)$)

[Mullin 67], [Bernardi 07]



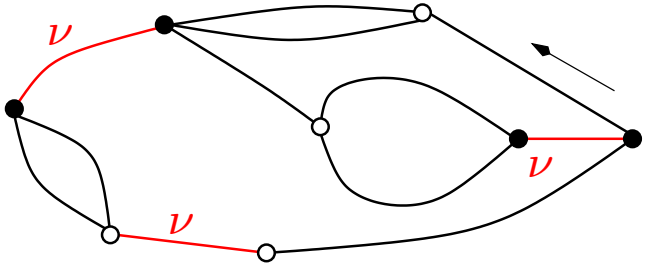
- Maps equipped with a bipolar orientation ($(-1)^{\mathsf{v}(M)}\chi'_M(1)$)

[Felsner-Fusy-Noy-Orden 08],
[Fusy-Poulalhon-Schaeffer 08],
[Bonichon-mbm-Fusy 08]

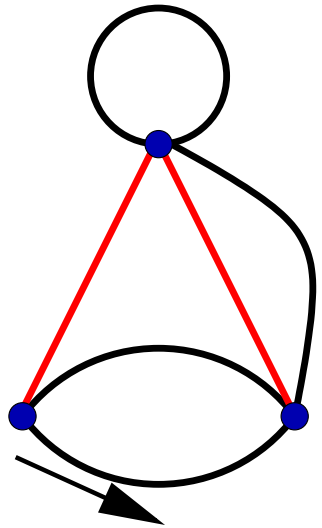


- The Ising model on planar maps (case $q = 2$)

[MBM-Schaeffer 02], [Bouttier et al.]

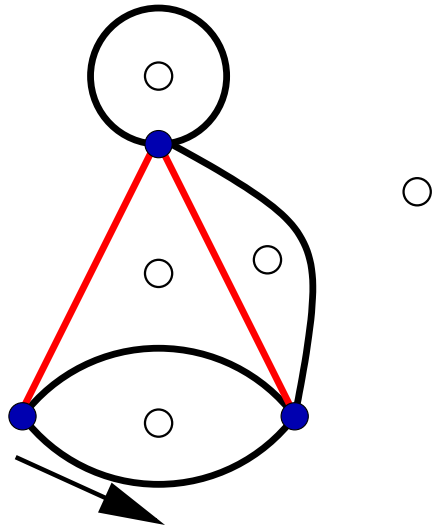


Bijective counting of maps equipped with a spanning tree



n edges, $k+1$ vertices ($\Rightarrow k$ edges in the tree)

Bijective counting of maps equipped with a spanning tree



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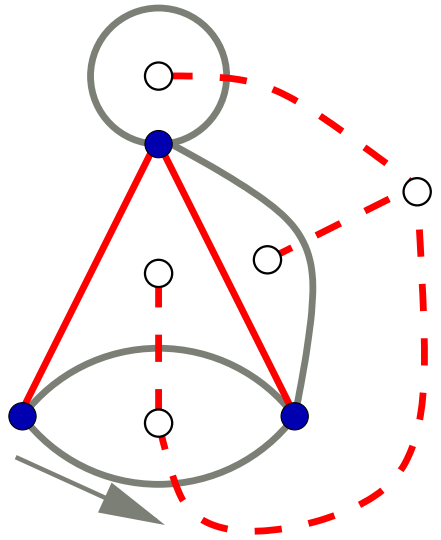

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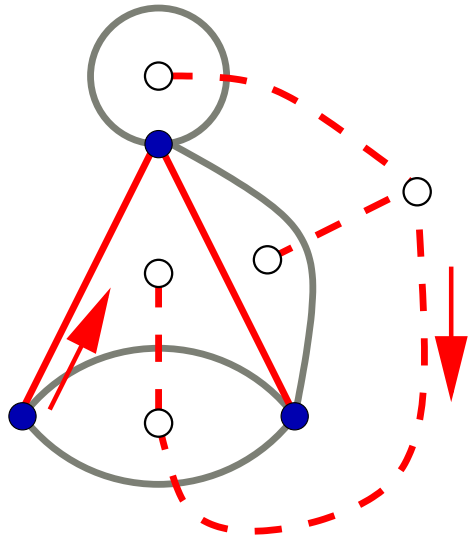

n edges, $k+1$ vertices ($\Rightarrow k$ edges in the tree)

A shuffle of two plane trees

$$\binom{2n}{2k} C_k C_{n-k}$$

with $C_k = \binom{2k}{k}/(k+1)$ counts rooted trees with k edges.

Many questions are left...

A. More combinatorics

- Understand algebraic series, e.g., for 3-coloured planar maps:

$$M(3,0,t;1,1) = \frac{(1+2A)(1-2A^2-4A^3-4A^4)}{(1-2A^3)^2} \quad \text{with} \quad A = t\frac{(1+2A)^3}{(1-2A^3)}$$

- Understand differential equations, e.g., for properly q-coloured triangulations:

$$2q^2(1-q)t + (qt+10H-6tH')H'' + q(4-q)(20H-18tH'+9t^2H'') = 0$$

B. Equations with two catalytic variables

- Elimination in the systems of differential equations
- Connections with elliptic functions
- Connections with [Eynard-Bonnet 99] and [Guionnet et al. 10]

C. Asymptotics

- Asymptotic number of properly q-coloured maps?

(done for triangulations $q \in (28/11, 4) \cup [5, \infty)$ [Odlyzko-Richmond 83])

- More generally, phase transitions and critical exponents of the Potts model