

Tutte's invariants in enumerative combinatorics

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with

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The Tutte polynomial is often called a **graph invariant**.

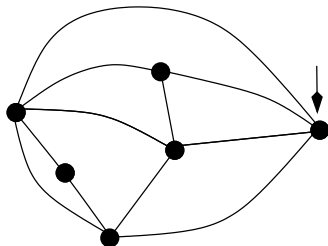
But this talk does **not** deal with the Tutte polynomial (or not much).

- The number of triangulations with $n + 3$ vertices is

$$a(n + 3) = \frac{2}{n(n + 1)} \binom{4n + 1}{n - 1}$$

- The **generating function** of triangulations, counted by vertices, is

$$A(t) := \sum_n a(n)t^n = \sum_{T \text{ triangulation}} t^{v(T)}$$



$$v(T) = 6$$

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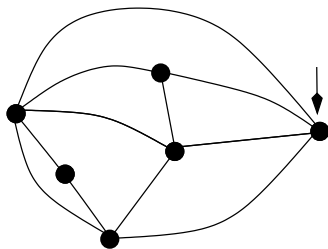
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- The **generating function** of triangulations, counted by vertices, is

$$A(t) := \sum_n a(n)t^n = \sum_{T \text{ triangulation}} t^{v(T)}$$

It is an **algebraic** series:

$$16t^{10} - t^9 - t^6(20t - 1)A(t) + t^4(8t + 3)A(t)^2 + 3t^2A(t)^3 + A(t)^4 = 0.$$



$$v(T) = 6$$

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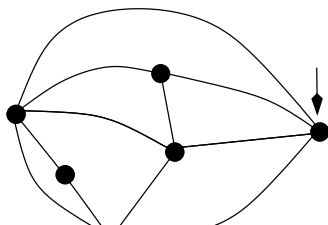
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- The **generating function** of triangulations, counted by vertices, is

$$A(t) := \sum_n a(n)t^n = \sum_{\tau \text{ triangulation}} t^{v(\tau)}$$

- Refine the enumeration by recording the degree of the root:

$$A(x, t) := \sum_{\tau \text{ triangulation}} x^{d(\tau)} t^{v(\tau)}$$



$$d(T) = 4$$

$$v(T) = 6$$

[Tutte, 1973] "Chromatic sums for rooted planar triangulations: the cases $\lambda = 1$ and $\lambda = 2$ "

- Let $T(x, y; t) \equiv T(x, y)$ be the unique formal power series in t , with polynomial coefficients in q , x and y , satisfying

$$T(x, y) = xq(q - 1) + \frac{xyt}{q} T(x, y) T(1, y) \\ + xt \frac{T(x, y) - T(x, 0)}{y} - x^2 yt \frac{T(x, y) - T(1, y)}{x - 1}$$

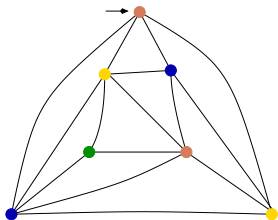
A quadratic equation with two **catalytic** variables x and y [Zeilberger 00]

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- Then $T(1, 0; t)$ counts properly q -coloured triangulations by the number of triangles.

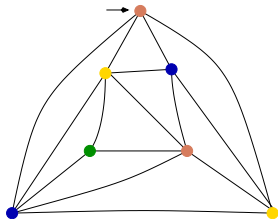


[Tutte, 1984] “Map-colourings and differential equations”

- The number $c(n)$ of q -coloured triangulations with $2n$ triangles satisfies:

$$q(n+1)(n+2)c(n) = q(q-4)(3n-1)(3n-2)c(n-1) + 2 \sum_{i=1}^n i(i+1)(3n-3i+1)c(i-1)c(n-i),$$

with $c(0) = q(q-1)$.



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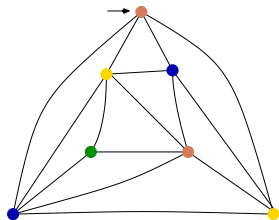
with $c(0) = q(q-1)$.

- The associated generating function

$$C(t) = \sum_n c(n)t^{n+2}$$

is **differentially algebraic**, and satisfies

$$2q^2(1-q)t + (qt + 10C - 6tC')C'' + q(4-q)(20C - 18tC' + 9t^2C'') = 0$$



What happened inbetween?

[Tutte 73] Chromatic sums for rooted planar triangulations: the cases $\lambda = 1$ and $\lambda = 2$

[Tutte 73] Chromatic sums for rooted planar triangulations, II : the case $\lambda = \tau + 1$

[Tutte 73] Chromatic sums for rooted planar triangulations, III : the case $\lambda = 3$

[Tutte 73] Chromatic sums for rooted planar triangulations, IV : the case $\lambda = \infty$

[Tutte 74] Chromatic sums for rooted planar triangulations, V : special equations

[Tutte 78] On a pair of functional equations of combinatorial interest

[Tutte 82] Chromatic solutions

[Tutte 82] Chromatic solutions II

[Tutte 84] Map-colourings and differential equations

[Tutte 95] Chromatic sums revisited

What happened inbetween?

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More equations of this type

- *Planar maps weighted by their... Tutte polynomial $T_M(\mu, \nu)$ [Tutte 68]*

$$M(x, y) = 1 + xyt(y\mu - 1)M(x, y)M(1, y) + xyt(x\nu - 1)M(x, y)M(x, 1) \\ + xyt \frac{xM(x, y) - M(1, y)}{x - 1} + xyt \frac{yM(x, y) - M(x, 1)}{y - 1}.$$

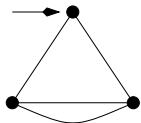
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- One has:

$$M(1, 1) \equiv M(1, 1; t, \mu, \nu) = \sum_{M \text{ planar map}} t^{e(M)} T_M(\mu, \nu)$$



$$T_M(\mu, \nu) = \mu^2 + \nu^2 + \mu\nu + \mu + \nu$$

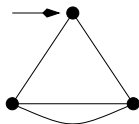
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$$T_M(\mu, \nu) = \mu^2 + \nu^2 + \mu\nu + \mu + \nu$$

- **The Potts model:** This is equivalent to counting maps coloured in q colours with a weight ν per monochromatic edge ($q = (\mu - 1)(\nu - 1)$).

More equations of this type

- *Planar maps weighted by their... Tutte polynomial $T_M(\mu, \nu)$ [Tutte 68]*
- *Walks confined to a quadrant with steps NE, W, S*

$$Q(x, y) = 1 + txyQ(x, y) + t \frac{Q(x, y) - Q(0, y)}{x} + t \frac{Q(x, y) - Q(x, 0)}{y}$$

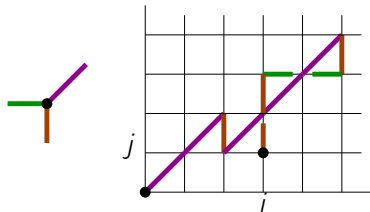
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$$Q(x, y) \equiv Q(x, y; t) = \sum_{w \text{ in the quadrant}} x^{i(w)} y^{j(w)} t^{\ell(w)}$$

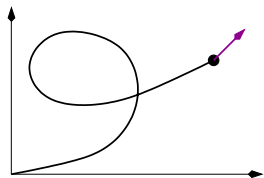
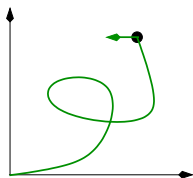
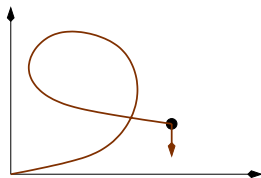


$$x^3 y^1 t^{11}$$

A simple functional equation



$$Q(x, y) = 1 + txyQ(x, y) + t \frac{Q(x, y) - Q(0, y)}{x} + t \frac{Q(x, y) - Q(x, 0)}{y}$$

 \oplus  \oplus 

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A simple functional equation



$$Q(x, y) = 1 + txyQ(x, y) + t \frac{Q(x, y) - Q(0, y)}{x} + t \frac{Q(x, y) - Q(x, 0)}{y}$$

or, with $\bar{x} = 1/x$ and $\bar{y} = 1/y$,

$$(1 - t(xy + \bar{x} + \bar{y}))Q(x, y) = 1 - t\bar{x}Q(0, y) - t\bar{y}Q(x, 0),$$

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or

$$(1 - t(xy + \bar{x} + \bar{y}))xyQ(x, y) = xy - tyQ(0, y) - txQ(x, 0)$$

- The polynomial $1 - t(xy + \bar{x} + \bar{y})$ is the **kernel** of this equation
- The equation is linear, with two **catalytic** variables x and y (tautological at $x = 0$ or $y = 0$)

More equations with two catalytic variables

- *Planar maps weighted by their... Tutte polynomial* $T_M(\mu, \nu)$ [Tutte 68]

$$M(x, y) = 1 + xyt(y\mu - 1)M(x, y)M(1, y) + xyt(x\nu - 1)M(x, y)M(x, 1) \\ + xyt \frac{xM(x, y) - M(1, y)}{x - 1} + xyt \frac{yM(x, y) - M(x, 1)}{y - 1}.$$

- *Walks confined to a quadrant*



$$(1 - t(\bar{x} + \bar{y} + xy))xyQ(x, y) = xy - tyQ(0, y) - txQ(x, 0)$$

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- *Walks confined to a quadrant [mbm-Mishna 10]*



$$(1 - t(\bar{x} + \bar{y} + xy))xyQ(x, y) = xy - tyQ(0, y) - txQ(x, 0)$$



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- And more! (e.g. permutations classes)

Equations with ONE catalytic variable abund in map enumeration

[Tutte 62]

$$\Psi(y) = 1 + \frac{t}{y} \left(\frac{\Psi(y)}{1 - y\Psi(y)} - \Psi(0) \right)$$

[Brown 65]

$$U(y) = (1 + y^2 U(y))^3 + 2tU(y)(1 + y^2 U(y)) - t^2 U(0)U(y) + t^2 \frac{U(y) - U(0)}{y^2}$$

[Tutte 68] Planar maps:

$$F(y) = 1 + ty^2 F(y)^2 + ty \frac{yF(y) - F(1)}{y - 1}$$

[Mullin, Nemeth, Schellenberg 70]

$$T(y) = y(1 + yT(y))^2 + t \frac{T(y) - T(0)}{y}$$

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$$T(y) = y(1 + yT(y))^2 + t \frac{T(y) - T(0)}{y}$$

→ Algebraic solutions! (Guess-and-check, Brown's quadratic method)

Equations with ONE catalytic variable

Theorem [mbm-Jehanne 06]

Let $P(t, y, S(y; t), A_1(t), \dots, A_k(t))$ be a proper polynomial equation in one catalytic variable y (it defines uniquely $S(y; t), A_1(t), \dots, A_k(t)$ as formal power series in t). Then each of these series is algebraic.

The proof is constructive, and generalizes Brown's quadratic method.

Example 1: planar maps

$$S(y; t) = 1 + ty^2S(y; t)^2 + ty \frac{yS(y; t) - A_1(t)}{y - 1}$$

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Example 1: planar maps

$$S(y; t) = 1 + ty^2S(y; t)^2 + ty \frac{yS(y; t) - A_1(t)}{y - 1}$$

implies $A_1(t) = S(1; t)$

Equations with ONE catalytic variable

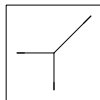
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The proof is constructive, and generalizes Brown's quadratic method.

Example 2: for $S(y; t) = Q(0, y; t)$ and $A_1(t) = Q(0, 0; t)$,

$$\frac{t}{y^2} - \frac{1}{y} - ty = t \left(tyS(y; t) + \frac{1}{y} \right)^2 - \left(tyS(y; t) + \frac{1}{y} \right) - 2t^2A_1(t).$$



Equations with ONE catalytic variable

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The proof is constructive, and generalizes Brown's quadratic method.

⇒ A special case of an Artin approximation theorem with “nested” conditions [Popescu 86], [Swan 98]

But what about TWO catalytic variables?

- Properly q -coloured triangulations [Tutte 73]

$$T(x, y) = xq(q-1) + \frac{xyt}{q} T(x, y) T(1, y) \\ + xt \frac{T(x, y) - T(x, 0)}{y} - x^2 yt \frac{T(x, y) - T(1, y)}{x-1}$$

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- Walks confined to a quadrant [mbm-Mishna 10]



$$(1 - t(\bar{x} + \bar{y} + xy))xyQ(x, y) = xy - tyQ(0, y) - txQ(x, 0)$$



$$(1 - t(y + \bar{x} + x\bar{y}))xyQ(x, y) = xy - tyQ(0, y) - tx^2Q(x, 0)$$



$$(1 - t(x + \bar{x} + \bar{y} + xy))xyQ(x, y) = xy - tyQ(0, y) - txQ(x, 0)$$

A hierarchy of formal power series

- Rational series

$$A(t) = \frac{P(t)}{Q(t)}$$

- Algebraic series

$$P(t, A(t)) = 0$$

- Differentially finite series (D-finite)

$$\sum_{i=0}^d P_i(t) A^{(i)}(t) = 0$$

- D-algebraic series

$$P(t, A(t), A'(t), \dots, A^{(d)}(t)) = 0$$



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Multi-variate series: one DE per variable



A hierarchy of formal power series



A hierarchy of formal power series

triangulations



A hierarchy of formal power series

triangulations

Rat.

Alg.

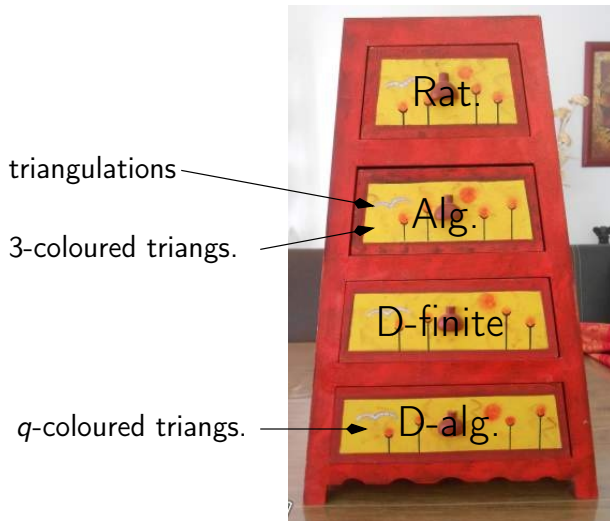
D-finite

q -coloured triangs.

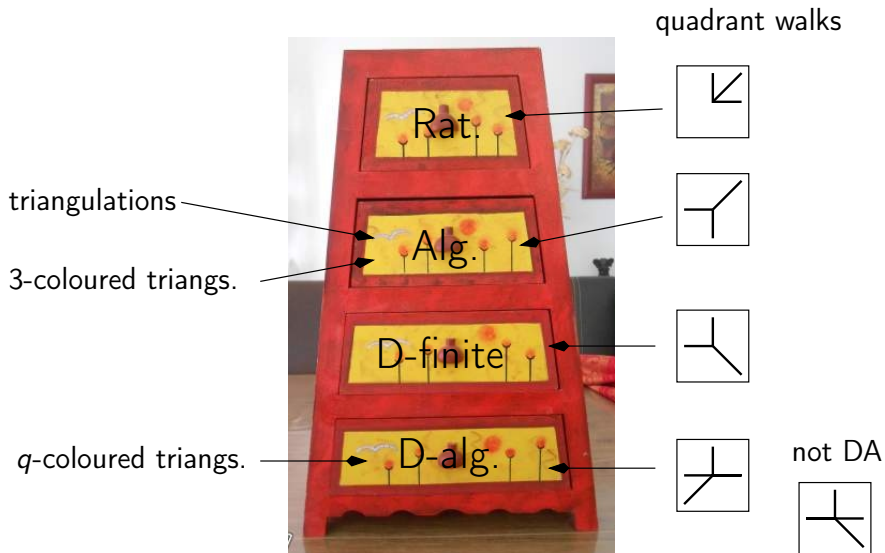
D-alg.



A hierarchy of formal power series

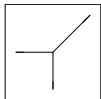


A hierarchy of formal power series



II. Tutte's invariants are, primarily, a tool for proving the algebraicity of solutions of functional equations

$$(1 - t(\bar{x} + \bar{y} + xy))xyQ(x, y) = xy - tyQ(0, y) - txQ(x, 0)$$





- The equation (with $\bar{x} = 1/x$ and $\bar{y} = 1/y$):

$$\begin{aligned} (1 - t(\bar{x} + \bar{y} + xy))xyQ(x, y) &= xy - txQ(x, 0) - tyQ(0, y) \\ &= xy - R(x) - S(y) \end{aligned}$$

with $Q(x, y) \equiv Q(x, y; t)$.

Theorem [Kreweras 65]

- The number of walks of length $3n + 2j$ ending at $(0, j)$ is

$$q(0, j; 3n + 2j) = \frac{4^n(2j + 1)}{(n + j + 1)(2n + 2j + 1)} \binom{2j}{j} \binom{3n + 2j}{n}$$

- The associated generating function $Q(0, y; t)$ is algebraic [Gessel 86].
- So is $Q(x, y; t)$



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- Cancel the kernel:** If we take $x = t + ut^2$, both roots of the kernel,

$$Y_{0,1} = \frac{x - t \pm \sqrt{(x - t)^2 - 4t^2x^3}}{2tx^2}$$

are (Laurent) series in t with rational coefficients in u , and can be legally substituted for y in $Q(x, y)$.



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- Cancel the kernel:** If we take $x = t + ut^2$, both roots of the kernel,

$$Y_{0,1} = \frac{x - t \pm \sqrt{(x - t)^2 - 4t^2x^3}}{2tx^2}$$

are (Laurent) series in t with rational coefficients in u , and can be legally substituted for y in $Q(x, y)$. This gives four equations:

| Kernel | RHS |
|--------------------------------|------------------------|
| $xY_0 = t(Y_0 + x + x^2Y_0^2)$ | $xY_0 = R(x) + S(Y_0)$ |
| $xY_1 = t(Y_1 + x + x^2Y_1^2)$ | $xY_1 = R(x) + S(Y_1)$ |



- Four equations relating x , $R(x)$, Y_0 , Y_1 , $S(Y_0)$, $S(Y_1)$ and t :

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- Eliminate x from the kernel equations:

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- Eliminate $R(x)$ from the RHS equations, and then x :

$$S(Y_0) + \frac{1}{Y_0} = S(Y_1) + \frac{1}{Y_1}$$

The two following functions are **invariants**:

$$I(y) = \frac{t}{y^2} - \frac{1}{y} - ty \quad \text{and} \quad J(y) = S(y) + \frac{1}{y}$$

The invariant lemma



We have

$$I(Y_0) = I(Y_1) \quad \text{and} \quad J(Y_0) = J(Y_1)$$

with

$$I(y) = \frac{t}{y^2} - \frac{1}{y} - ty \quad \text{and} \quad J(y) = tyQ(0, y) + \frac{1}{y}.$$

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There are few invariants: $I(y)$ must be a polynomial in $J(y)$ whose coefficients are series in t .

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$$I(y) = \frac{t}{y^2} - \frac{1}{y} - ty = t \left(tyQ(0, y) + \frac{1}{y} \right)^2 - \left(tyQ(0, y) + \frac{1}{y} \right) + c$$

Expanding at $y = 0$ gives the value of c .

The invariant lemma



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


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Polynomial equation with one catalytic variable $\Rightarrow Q(0, y; t)$ is algebraic

What invariants are good for

- start with an equation with two catalytic variables x and y (degree 1 in the main series $Q(x, y)$)
- cancel the kernel $\Rightarrow Y_0, Y_1$
- find a pair of invariants in y
- relate them algebraically (the invariant lemma)
- obtain an equation with one catalytic variable only \Rightarrow algebraicity

Other equations: are there invariants?

| |  |  |  |  |
|-------------|---|---|---|--|
| from kernel | × | × | — | — |
| from RHS | × | — | × | — |

Other equations: are there invariants?

Thm. [Bernardi, mbm, Raschel 15(a)]

Exactly 4 quadrant models with small steps have two invariants \Rightarrow uniform **algebraic** solution via the solution of an equation with one catalytic variable



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Gessel's model

- conjecture for the number $q(0, 0; n)$ of excursions [Gessel \simeq 00]
- proof of this conjecture [Kauers, Koutschan & Zeilberger 08]
- $Q(x, y; t)$ are algebraic! [Bostan & Kauers 09a]
- new proof via complex analysis [Bostan, Kurkova & Raschel 13(a)]
- an elementary and constructive proof [mbm 15(a)]

Other equations: are there invariants?

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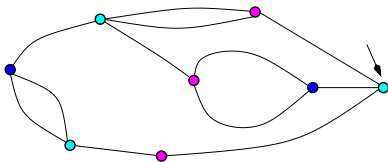
Exactly 4 quadrant models with small steps have two invariants \Rightarrow uniform algebraic solution via the solution of an equation with one catalytic variable



Thm. [Tutte 74], [Bernardi-mbm 11]

For q -coloured triangulations (or Potts on planar maps, or on triangulations), there is one invariant for any q , and a second one if $q = 4 \cos^2 \frac{k\pi}{m}$, with $q \neq 0, 4$. \Rightarrow Algebraicity

Example: Properly 3-coloured planar maps



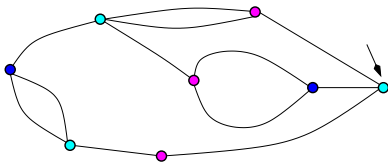
- Two catalytic variables [Tutte 68]

$$M(x, y) = 1 + xyt (1 + 2y) M(x, y)M(1, y) - xytM(x, y)M(x, 1) \\ - xyt \frac{xM(x, y) - M(1, y)}{x - 1} + xyt \frac{yM(x, y) - M(x, 1)}{y - 1}$$

- One catalytic variable [Bernardi-mbm 11]

$$P(M(1, y), M_0, M_1, M_2, t, y) = 0$$

Example: Properly 3-coloured planar maps



$$P(M(1, y), M_0, M_1, M_2, t, y) = 0$$

$$\begin{aligned}
 &= 36 y^6 t^3 (2y+1)(y-1)^3 M(y)^4 + 2 t^2 y^4 (y-1)^2 (42 t y^3 + 12 y^2 t - 26 y^3 - 39 y^2 + 39 y + 26) M(y)^3 \\
 &\quad + (-36 y^6 t^3 (y-1)^2 M_0 + (y-1) y^2 t (32 t^2 y^5 + 4 y^4 t^2 + 2 t y^5 - 120 t y^4 + 8 y^5 + 78 t y^3 + 38 y^4 \\
 &\quad + 40 y^2 t - 25 y^3 - 71 y^2 + 25 y + 25)) M(y)^2 + (-36 y^5 t^3 (y-1)^2 M_0^2 - 6 t^2 (y-1) y^4 (6 y^2 t - 2 y t \\
 &\quad - 9 y^2 + 5 y + 4) M_0 - 12 M_1 t^3 y^7 + 24 M_1 t^3 y^6 + 4 y^7 t^3 - 12 y^5 t^3 M_1 + 10 t^2 y^7 - 42 t^2 y^6 - 26 t y^7 \\
 &\quad + 28 t^2 y^5 + 52 t y^6 + 4 y^4 t^2 + 32 t y^5 - 4 y^6 - 94 t y^4 - 2 y^5 + 14 t y^3 + 16 y^4 + 22 y^2 t - 16 y^2 + 2 y + 4) M \\
 &\quad - 36 y^4 t^3 (y-1)^2 M_0^3 - 2 t^2 (y-1) y^3 (22 y^2 t - 16 y t - 33 y^2 + 27 y + 6) M_0^2 - 2 y^2 t (18 M_1 t^2 y^4 \\
 &\quad - 36 M_1 t^2 y^3 + 6 y^4 t^2 + 18 M_1 t^2 y^2 - 6 y^3 t^2 - 4 t y^4 + 2 y^2 t^2 - 7 t y^3 + 16 y^4 + 13 y^2 t - 23 y^3 - 2 y t \\
 &\quad + 5 y + 2) M_0 - (y-1) (12 y^5 t^3 M_1 + 2 M_2 t^3 y^5 - 8 y^4 t^3 M_1 - 22 M_1 t^2 y^5 - 2 y^4 t^3 M_2 \\
 &\quad + 18 M_1 t^2 y^4 + 4 M_1 t^2 y^3 - 11 t y^5 + 21 t y^4 - 4 y^5 - 9 t y^3 - 6 y^4 - y^2 t + 10 y^3 + 10 y^2 - 6 y - 4).
 \end{aligned}$$

- *Properly 3-coloured planar maps*

Let A be the quartic series in t defined by $A(0) = 0$ and

$$A = t \frac{(1 + 2A)^3}{1 - 2A^3}.$$

Then the generating function of **properly** 3-coloured planar maps is

$$\sum_{M \text{ prop. 3-coloured}} t^{e(M)} = \frac{(1 + 2A)(1 - 2A^2 - 4A^3 - 4A^4)}{(1 - 2A^3)^2}$$

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- *The Ising model on planar maps ($q = 2$)*

Let A be the series in t , with polynomial coefficients in ν , defined by

$$A = t \frac{(1 + 3\nu A - 3\nu A^2 - \nu^2 A^3)^2}{1 - 2A + 2\nu^2 A^3 - \nu^2 A^4}.$$

Then the Ising generating function of planar maps is rational in ν and A :

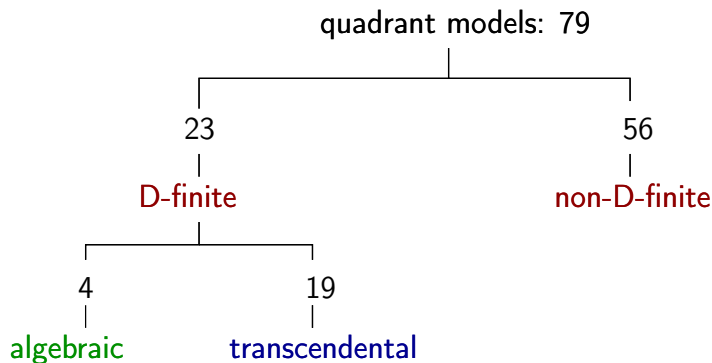
$$\sum_{M \text{ 2-coloured}} t^{e(M)} \nu^{\text{monochr}(M)} = \text{Rat}(\nu, A)$$

III. An analytic version of Tutte's invariants

Applications to differential algebraicity of quadrant walks

[Bernardi, mbm, Raschel]

Classification of quadrant walks with small steps



Theorem

The series $Q(x, y; t)$ is D-finite for 23 models, and algebraic for 4 of them.

[mbm-Mishna 10], [Bostan-Kauers 10]

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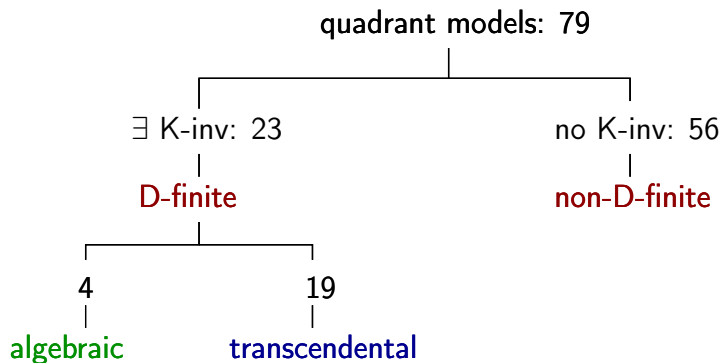
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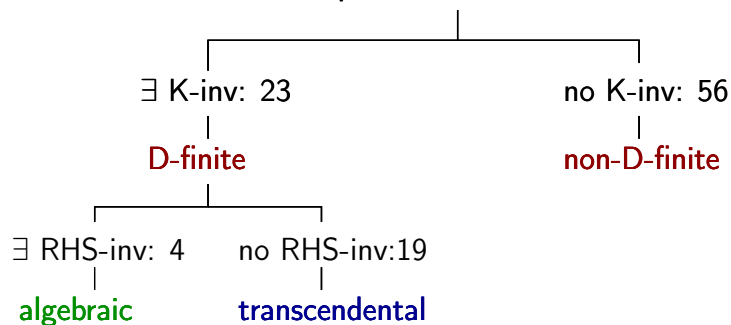
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Classification of quadrant walks with small steps

quadrant models: 79



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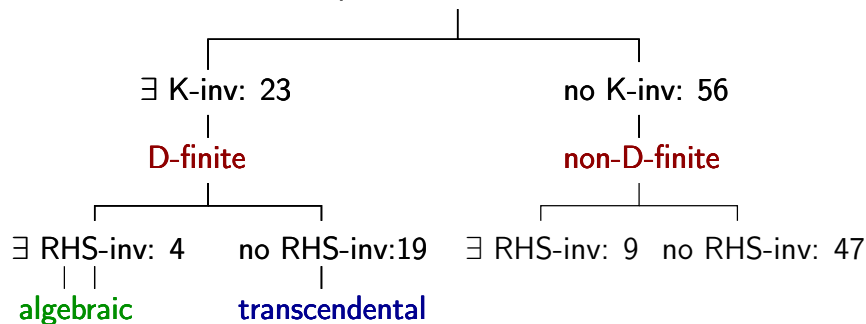
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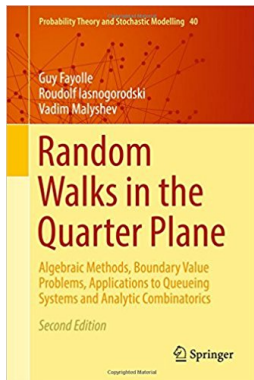
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A weaker (and analytic) notion of invariants

- Still require that $I(Y_0) = I(Y_1)$, where $Y_0(x)$, $Y_1(x)$ are the roots of the kernel
... but only for **some** values of x (and t).
- meromorphicity condition in a domain

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[Fayolle, Iasnogorodski, Malyshev 99], [Kurkova, Raschel 12]

Can we find weak invariants?

Theorem [Fayolle et al. 99, Raschel 12]

For each non-singular model, there exists an (explicit) weak invariant of the form

$$I(y; t) = \wp(\mathcal{Z}(y; t), \omega_1(t), \omega_3(t))$$

where

- \wp is Weierstrass elliptic function
- its periods ω_1 and ω_3 are elliptic integrals
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Proposition [Bernardi-mbm-Raschel]

$I(y; t)$ is D-algebraic in y and t .

The invariant lemma



For appropriate values of x , we have:

$$I(Y_0) = I(Y_1) \quad \text{and} \quad J(Y_0) = J(Y_1)$$

where $I(y)$ is the weak invariant and $J(y) = t(1 + y)Q(0, y) + 1/y$.

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The invariant lemma [Litvinchuk 00]

There are few invariants: $J(y)$ must be a rational function in $I(y)$. The value of this rational function is found by looking at the poles and zeroes of $J(y)$.

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$$J(y) = t(1+y)Q(0,y) + \frac{1}{y} = \frac{I'(0)}{I(y) - I(0)} - \frac{I'(0)}{I(-1) - I(0)} - 1$$

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Expression of $Q(0,y;t)$ in terms of the (explicit) weak invariant $I(y)$

The invariant lemma



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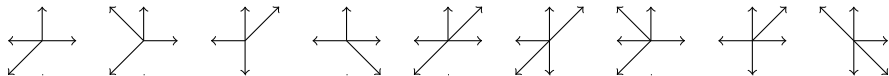
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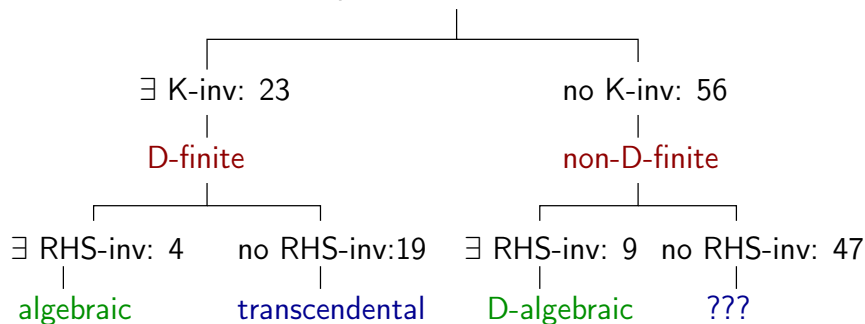
Corollary [Bernardi-mbm-Raschel 15(a)]

For the 9 models having no K-invariant but a RHS-invariant, the series $Q(x, y; t)$ is D-algebraic.



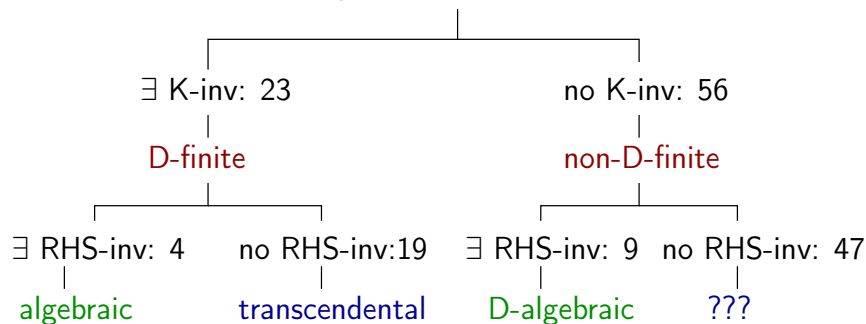
Completed classification of quadrant walks with small steps

quadrant models: 79



Completed classification of quadrant walks with small steps

quadrant models: 79



Not DA in y for 42 models

[Dreyfus, Hardouin, Roques, Singer 17(a)]

Conclusion: Invariants are back!

- Used once by Tutte to count q -coloured triangulations
- Apply to more equations (Potts model on maps, quadrant walks...)
- Extension to prove D-algebraicity for quadrant walks
- Prove more algebraicity or D-algebraicity results with them? (e.g. conjectures on lattice walks confined to/avoiding a quadrant)