

Self-avoiding walks on the honeycomb lattice

d'après H. Duminil-Copin and S. Smirnov

ArXiv 1007.0575

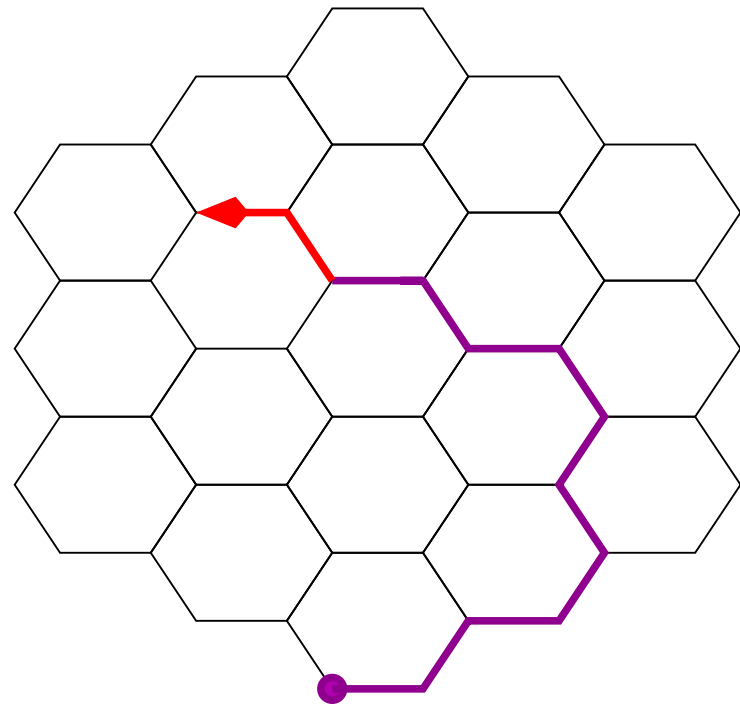
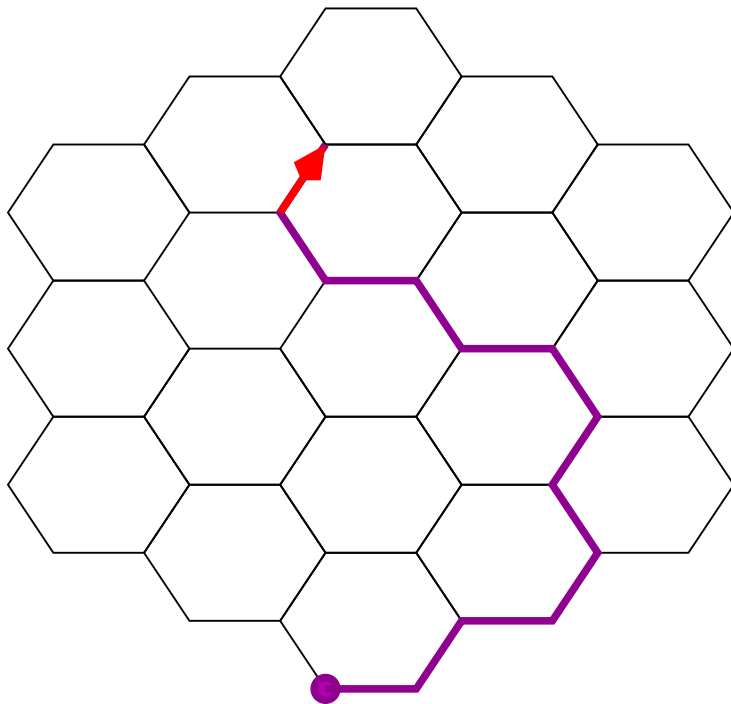
Simple bounds on $c(n)$

We have

$$F_{n+1} \leq c(n) \leq 3 \cdot 2^{n-1}$$

with $F_0 = F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ (Fibonacci numbers).

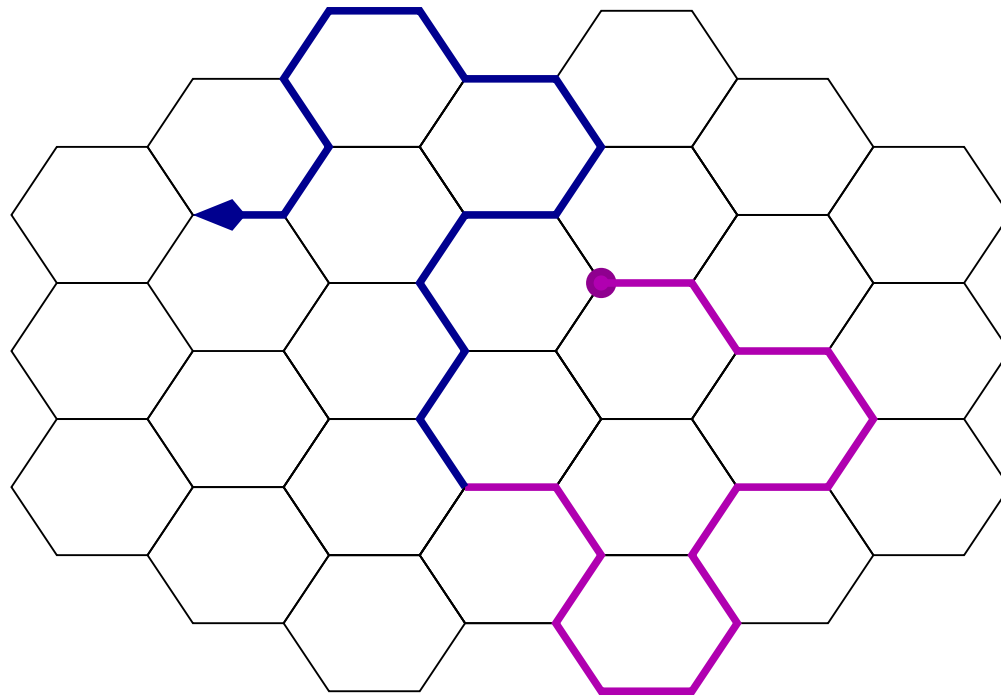
- Upper bound: at most two possible extensions at each step
- Lower bound: SAW that never move in the South direction (partially directed).



The connective constant

Clearly,

$$c(m+n) \leq c(m)c(n)$$



$\Rightarrow \lim c(n)^{1/n}$ exists and

$$\mu := \lim c(n)^{1/n} = \inf c(n)^{1/n}$$

The connective constant

$$\mu := \lim c(n)^{1/n} = \inf c(n)^{1/n}$$

Given the bounds

$$F_{n+1} \leq c(n) \leq 3 \cdot 2^{n-1},$$

there holds

$$\frac{1 + \sqrt{5}}{2} \simeq 1.62 \leq \mu \leq 2$$

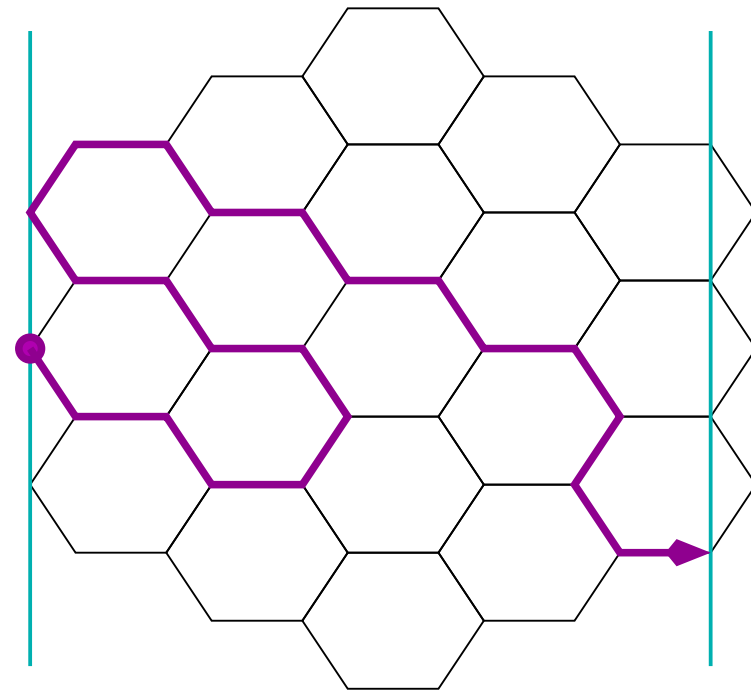
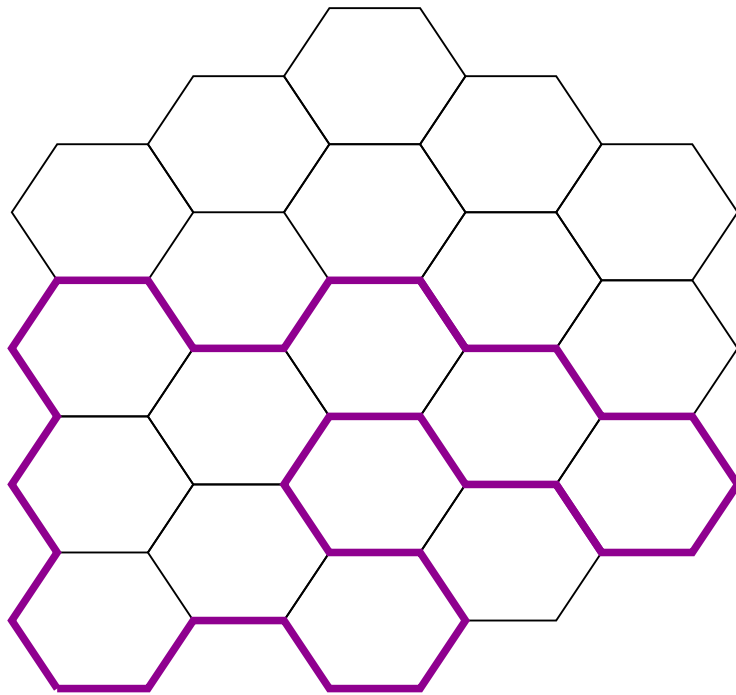
Theorem [Duminil-Copin & Smirnov 10]

$$\mu = \sqrt{2 + \sqrt{2}} \simeq 1.85$$

Conjectured by Nienhuis in 1982

Sub-families of SAW

Self-avoiding polygons and self-avoiding bridges



- Same growth constant:

$$p(n)^{1/n} \rightarrow \mu \quad \text{and} \quad b(n)^{1/n} \rightarrow \mu$$

[Hammersley 61]

Generating functions

For self-avoiding walks, polygons and bridges:

$$C(t) = \sum c(n)t^n, \quad P(t) = \sum p(n)t^n, \quad B(t) = \sum b(n)t^n$$

Theorem [Duminil-Copin & Smirnov 10]

The radius of convergence of these series is

$$t_c = \frac{1}{\sqrt{2 + \sqrt{2}}}.$$

Moreover:

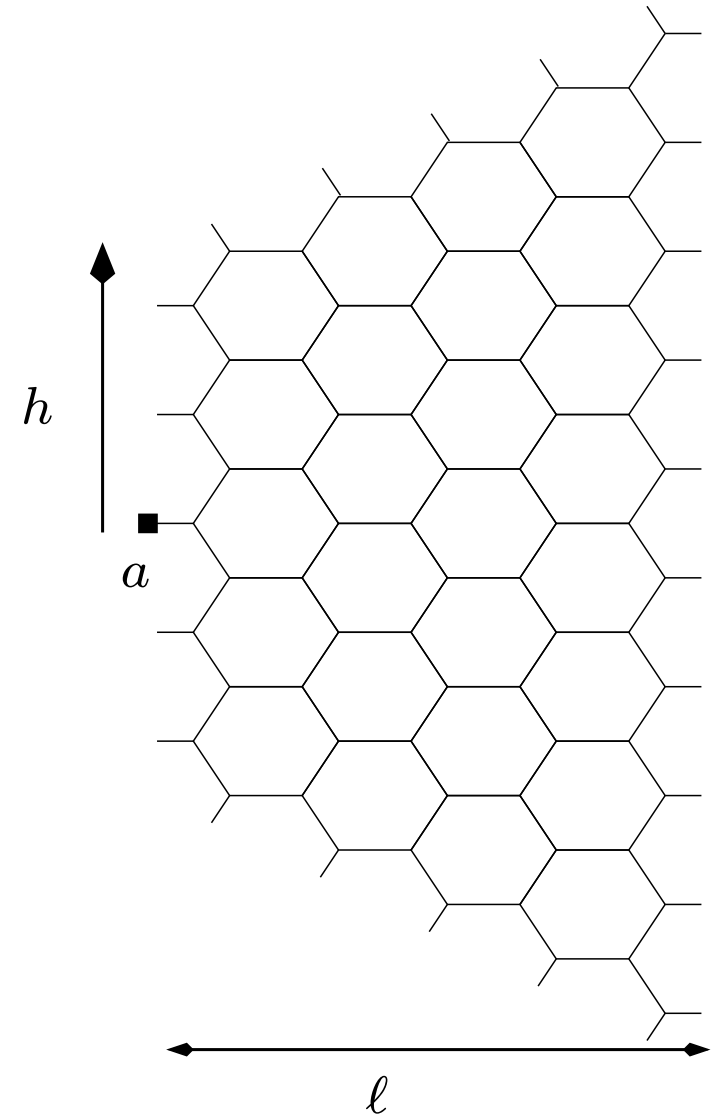
- $C(t)$ diverges at t_c ,
- $P(t)$ converges at t_c .

Conjectures:

$$C(t) \sim \frac{1}{(1 - t/t_c)^{43/32}}, \quad P(t)_{sing} \sim (1 - t/t_c)^{3/2}, \quad B(t) \sim \frac{1}{(1 - t/t_c)^{9/16}}$$

A key identity

Consider the following domain $D_{h,\ell}$.



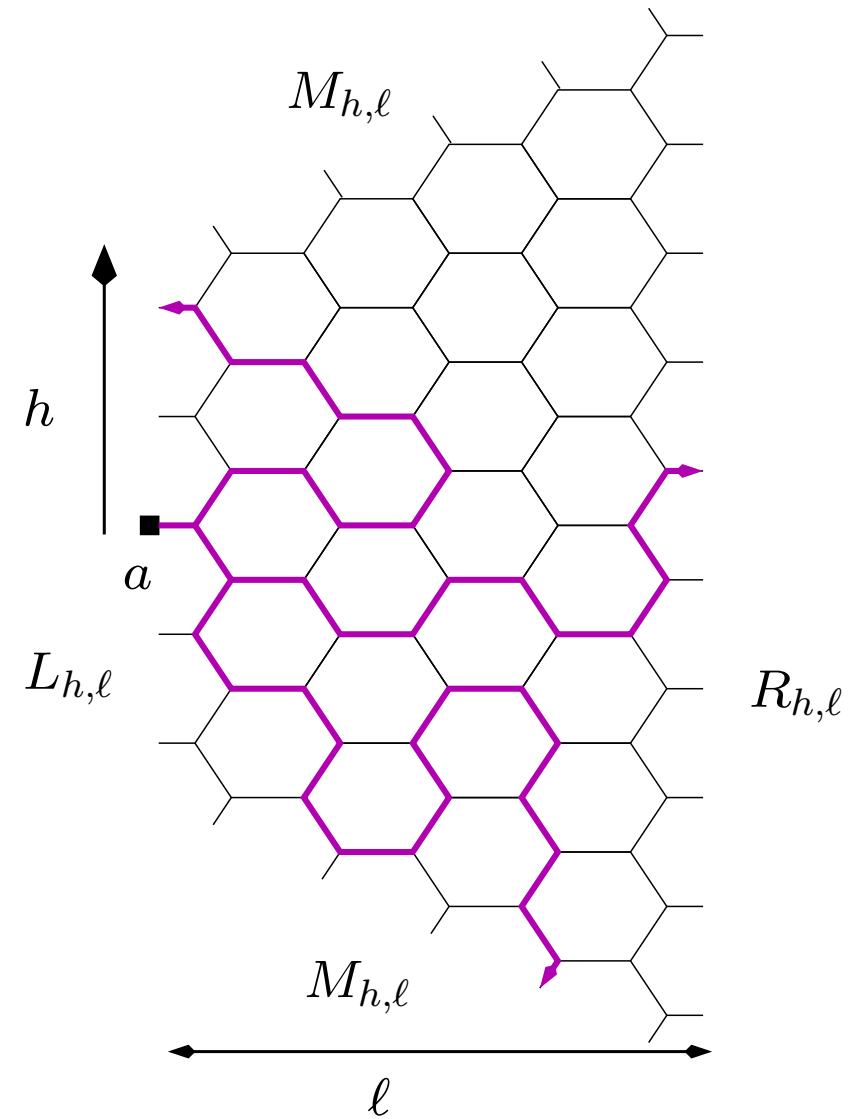
A key identity

Consider the following domain $D_{h,\ell}$.

Let $L_{h,\ell}(t)$ (resp. $R_{h,\ell}(t)$, $M_{h,\ell}(t)$) be the generating function of walks that start from a and end on the left (resp. right, middle) border of the domain $D_{h,\ell}$. These series are polynomials in t .

Then for all h and ℓ , at $t = t_c = 1/\sqrt{2 + \sqrt{2}}$,

$$\frac{\sqrt{2 - \sqrt{2}}}{2} L_{h,\ell}(t_c) + R_{h,\ell}(t_c) + \frac{1}{\sqrt{2}} M_{h,\ell}(t_c) = 1$$



An upper bound on μ

$$\frac{\sqrt{2 - \sqrt{2}}}{2} L_{h,\ell}(t_c) + R_{h,\ell}(t_c) + \frac{1}{\sqrt{2}} M_{h,\ell}(t_c) = 1$$

Consider walks in $D_{h,\ell}$ that go from a to a^+ (\simeq self-avoiding polygons). Their generating function $\tilde{P}_{h,\ell}(t)$ satisfies

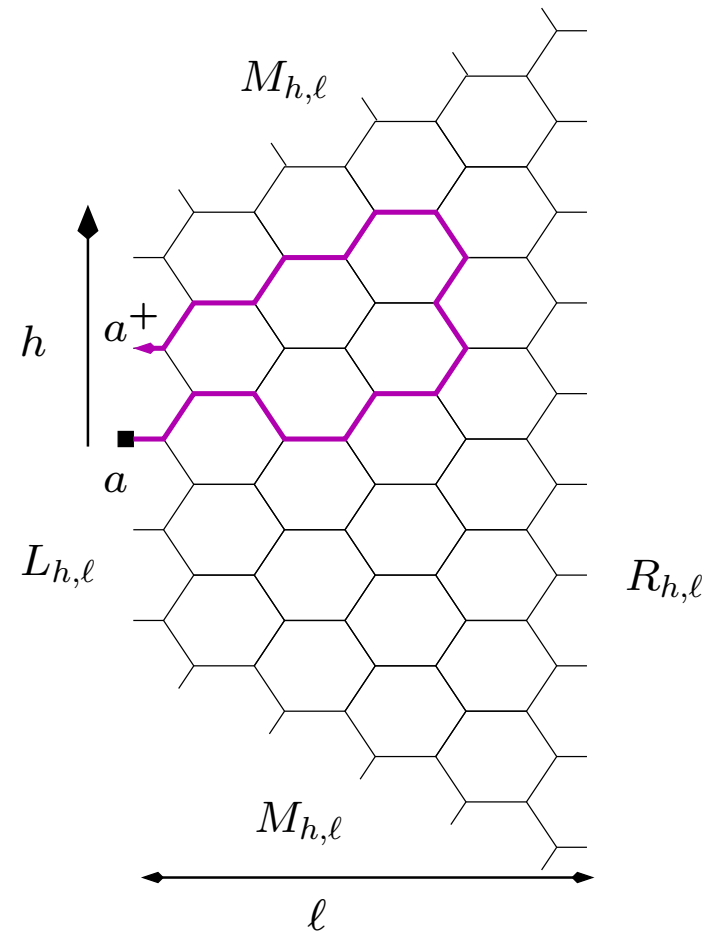
$$\tilde{P}_{h,\ell}(t_c) \leq L_{h,\ell}(t_c) \leq \frac{2}{\sqrt{2 - \sqrt{2}}}$$

But

$$\tilde{P}_{h,\ell}(t_c) \rightarrow \tilde{P}(t_c) \geq t_c^2 P(t_c)$$

as $h, \ell \rightarrow \infty$. This implies that $P(t_c)$ is finite.

Hence $\mu \leq 1/t_c = \sqrt{2 + \sqrt{2}}$.



A lower bound on μ

$$\frac{\sqrt{2 - \sqrt{2}}}{2} L_{h,\ell}(t_c) + R_{h,\ell}(t_c) + \frac{1}{\sqrt{2}} M_{h,\ell}(t_c) = 1$$

As $h \rightarrow \infty$, $L_{h,\ell}(t_c)$ and $R_{h,\ell}(t_c)$ increase to limit values $L_\ell(t_c)$ and $R_\ell(t_c)$. Hence $M_{h,\ell}(t_c)$ decreases to a limit value $M_\ell(t_c)$.

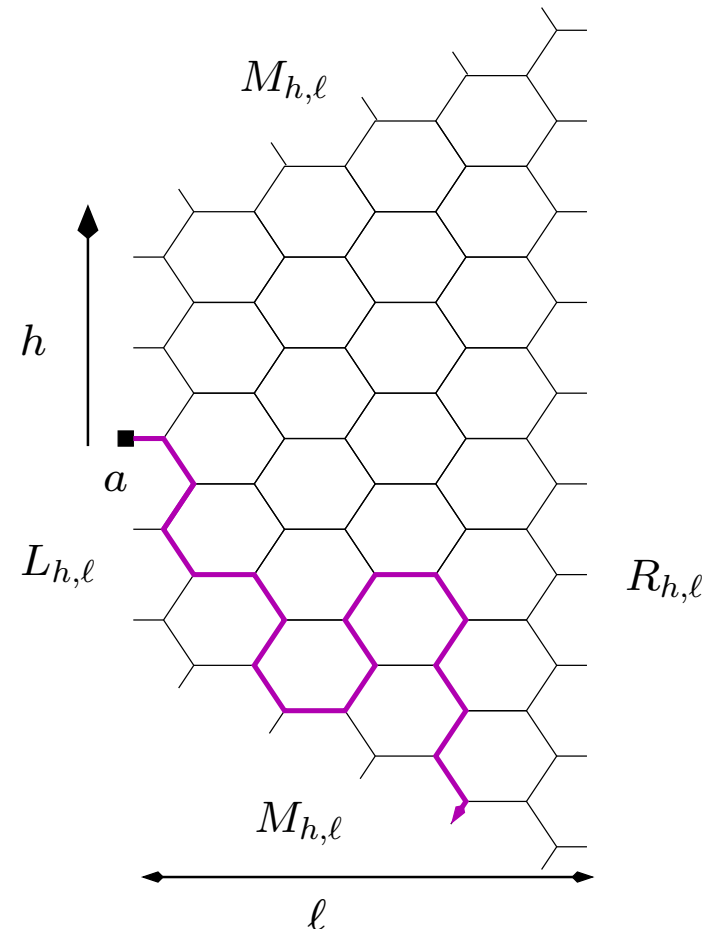
- If $M_\ell(t_c) > 0$ for some ℓ , the series

$$\sum_h M_{h,\ell}(t_c)$$

diverges. As it counts certain SAW, this implies that $\mu \geq 1/t_c$.

- If $M_\ell(t_c) = 0$ for all ℓ , another argument shows that $\sum_\ell R_\ell(t_c)$ (the generating function of bridges) diverges.

In both cases, $\mu \geq 1/t_c = \sqrt{2 + \sqrt{2}}$.



A lower bound on μ (continued)

$$\alpha L_{h,\ell}(t_c) + R_{h,\ell}(t_c) + \frac{1}{\sqrt{2}} M_{h,\ell}(t_c) = 1$$

- If $M_\ell(t_c) = 0$ for all ℓ , then

$$\alpha L_\ell(t_c) + R_\ell(t_c) = 1$$

$$\Rightarrow \alpha (L_{\ell+1}(t_c) - L_\ell(t_c)) = R_\ell(t_c) - R_{\ell+1}(t_c)$$

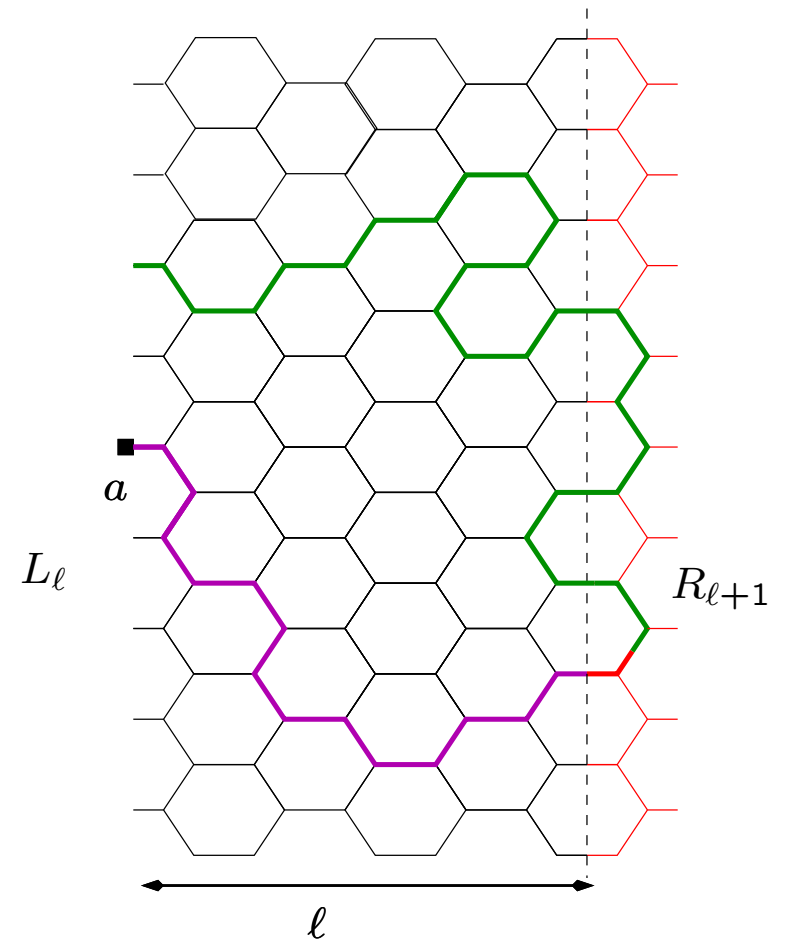
$$\Rightarrow 2\alpha t_c R_\ell R_{\ell+1} \geq R_\ell(t_c) - R_{\ell+1}(t_c)$$

$$\Rightarrow 2\alpha t_c \geq \frac{1}{R_{\ell+1}(t_c)} - \frac{1}{R_\ell(t_c)}$$

$$\Rightarrow 2\alpha t_c + \frac{1}{R_1} \geq \frac{1}{R_\ell}$$

$$\Rightarrow R_\ell \geq \frac{1}{2\alpha t_c + \beta},$$

so that $\sum_\ell R_\ell(t_c)$ (the generating function of bridges) diverges.



A key identity

$$\frac{\sqrt{2 - \sqrt{2}}}{2} L_{h,\ell}(t_c) + R_{h,\ell}(t_c) + \frac{1}{\sqrt{2}} M_{h,\ell}(t_c) = 1$$

Where does it come from?

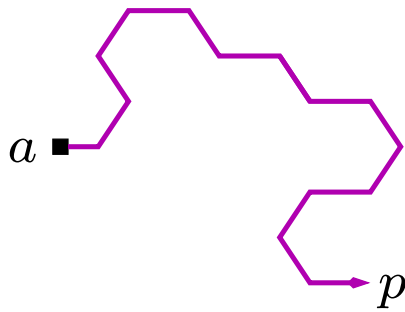
From a local identity that is re-summed over all vertices of the domain.

A local identity

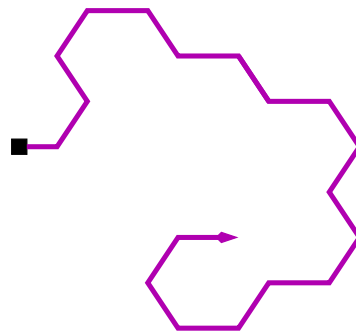
Let D be a finite, simply connected domain, and a a point on the boundary of D . For p a point in the domain, let

$$F(p) \equiv F(t, \alpha; p) = \sum_{\omega: a \rightsquigarrow p} t^{|\omega|} e^{i\alpha W(\omega)},$$

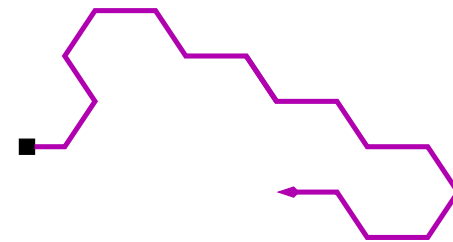
where $|\omega|$ is the length of ω , and $W(\omega)$ its winding number.



$$W(\omega) = 0$$



$$W(\omega) = -2\pi$$



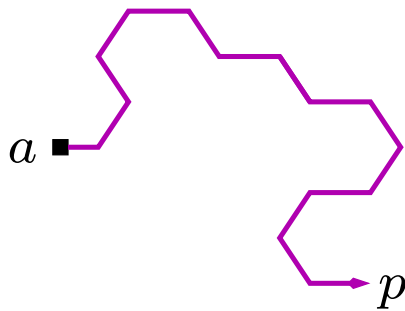
$$W(\omega) = -\pi$$

A local identity

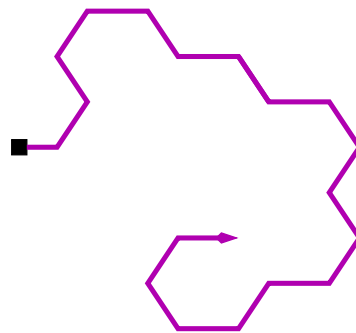
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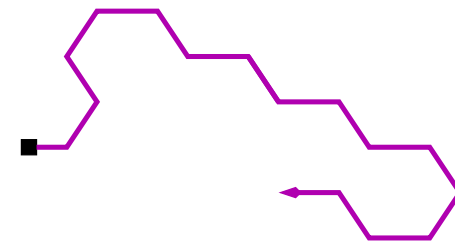
where $|\omega|$ is the length of ω , and $W(\omega)$ its **winding number**.



$$W(\omega) = 0$$



$$W(\omega) = -2\pi$$



$$W(\omega) = -\pi$$

Rem. W is additive:

$$W(\omega) = \frac{\pi}{3} (\text{left turns} - \text{right turns}).$$



A local identity

Let

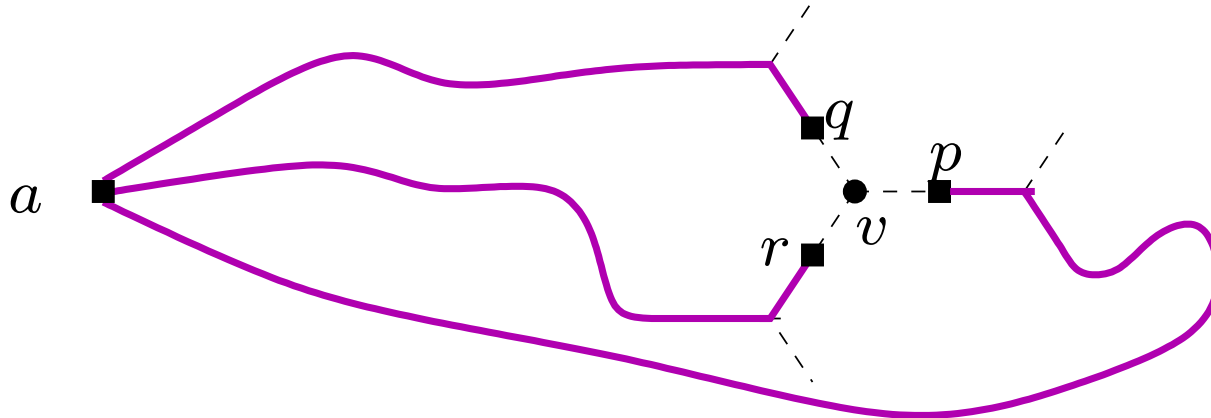
$$F(p) \equiv F(t, \alpha; p) = \sum_{\omega: a \rightsquigarrow p \text{ in } D} t^{|\omega|} e^{i\alpha W(\omega)},$$

If p , q and r are the 3 mid-edges around a vertex v of the honeycomb lattice, taken in counterclockwise order, then, for $t = t_c$ and $\alpha = -5/8$,

$$F(p) + jF(q) + j^2F(r) = 0,$$

where $j = e^{2i\pi/3}$, or, more symmetrically,

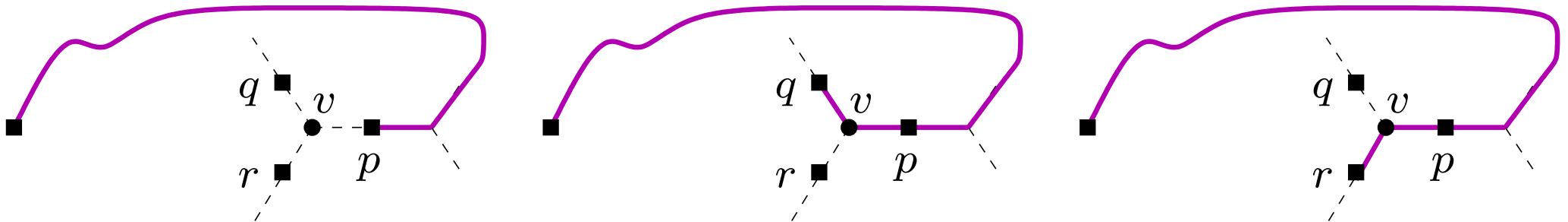
$$(p - v)F(p) + (q - v)F(q) + (r - v)F(r) = 0.$$



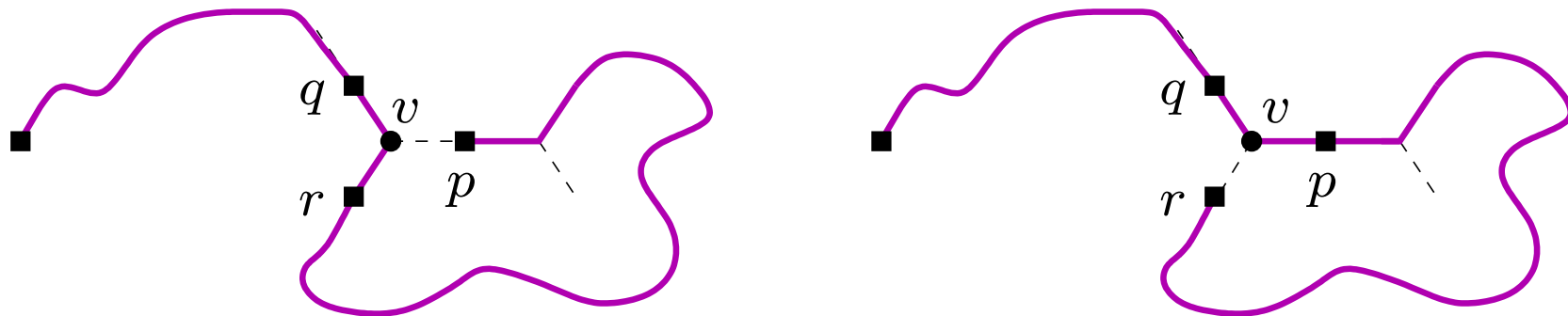
A local identity

Proof: Group walks three by three or two by two as follows

- Walks that avoid v + walks in which v is the last visited vertex:



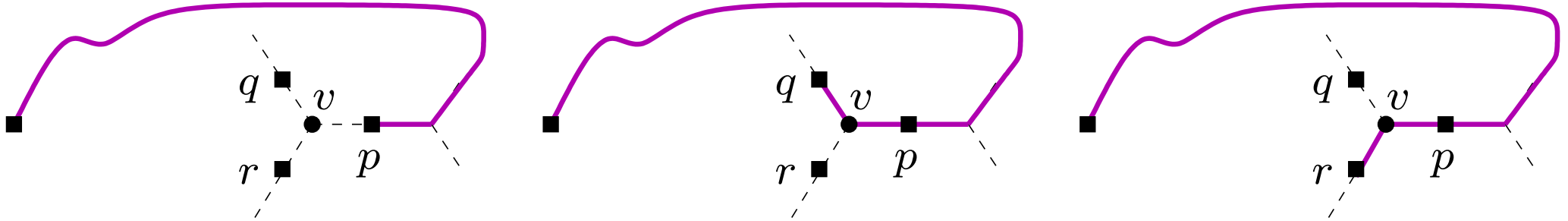
- Walks that visit v before their last vertex:



The contribution of all walks in a group is zero.

A local identity

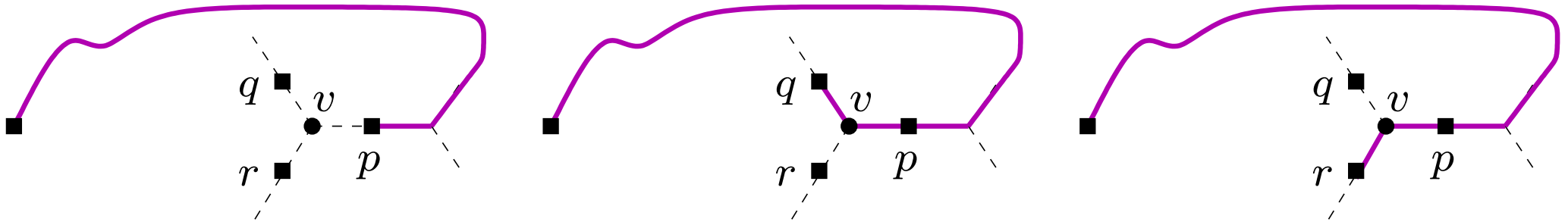
- Walks that avoid v + walks in which v is the last visited vertex:



$$1 + jt_c e^{-i\alpha\pi/3} + j^2 t_c e^{i\alpha\pi/3} = 0$$

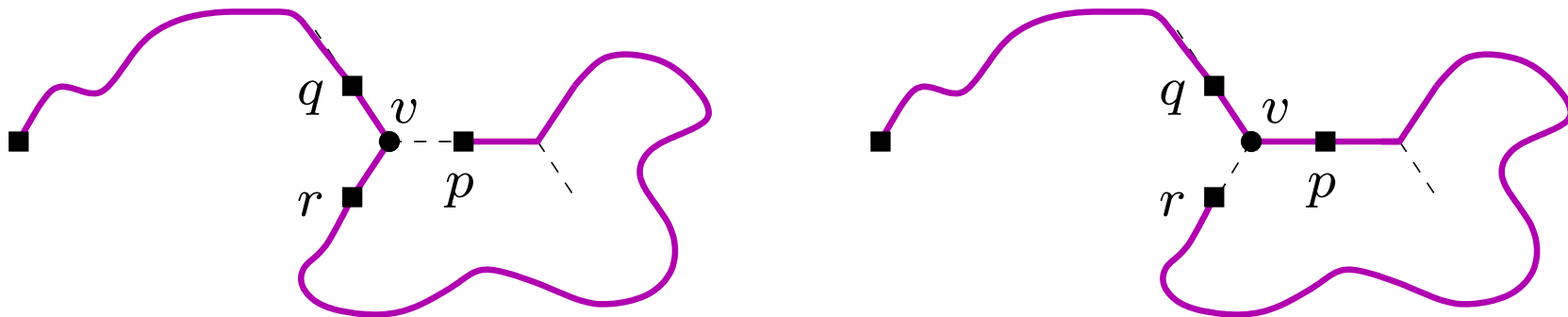
A local identity

- Walks that avoid v + walks in which v is the last visited vertex:



$$1 + jt_c e^{-i\alpha\pi/3} + j^2 t_c e^{i\alpha\pi/3} = 0$$

- Walks that visit v before their last vertex:



$$e^{4i\alpha\pi/3} + j^2 e^{-4i\alpha\pi/3} = 2/j \cos\left(\frac{\pi}{3} + 4\alpha\frac{\pi}{3}\right) = 0$$

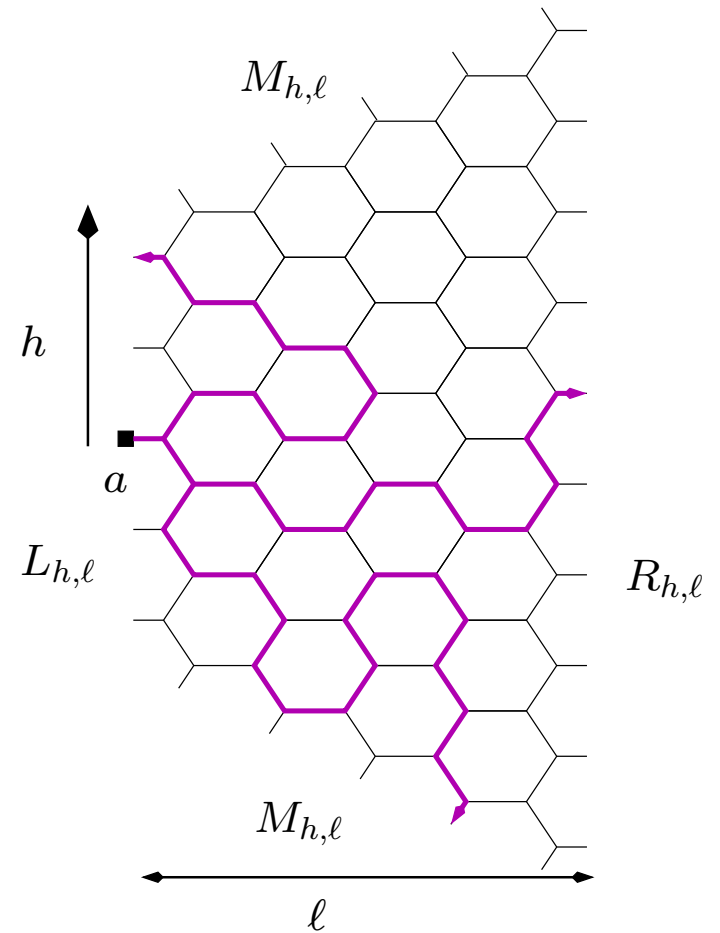
Proof of the key identity

Sum the local identity

$$(p - v)F(p) + (q - v)F(q) + (r - v)F(r) = 0$$

over all vertices v of the domain $D_{h,\ell}$.

- The inner mid-edges do not contribute.
- The winding number of walks ending on the boundary is known.



Proof of the key identity

Sum the local identity

$$(p - v)F(p) + (q - v)F(q) + (r - v)F(r) = 0$$

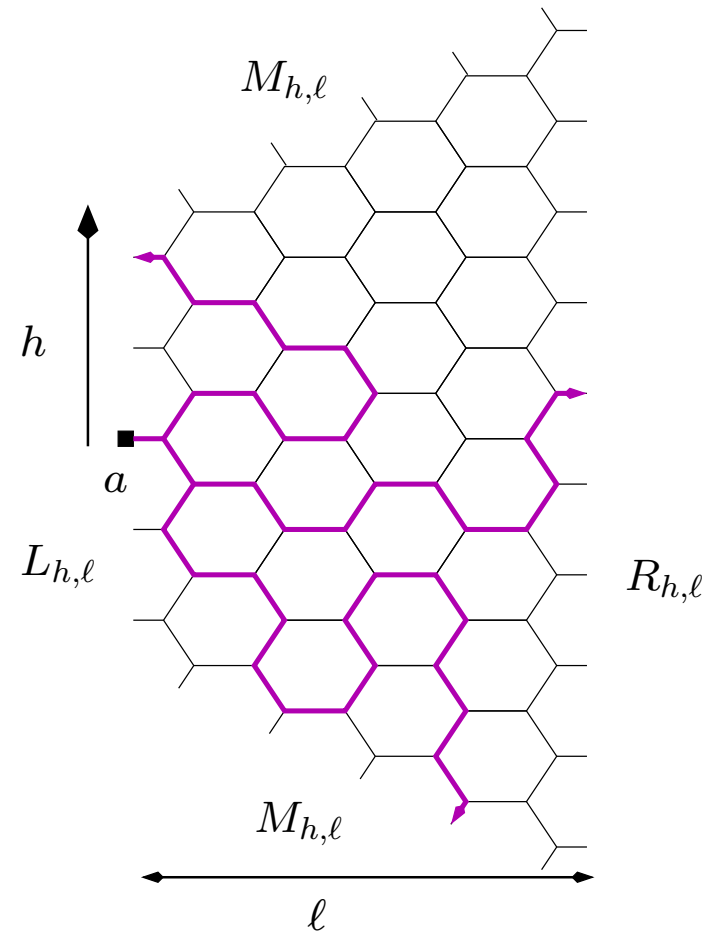
over all vertices v of the domain $D_{h,\ell}$.

- The inner mid-edges do not contribute.
- The domain has an up-down symmetry.
- The winding number of walks ending on the boundary is known.

This gives:

$$\frac{\sqrt{2 - \sqrt{2}}}{2} L_{h,\ell}(tc) + R_{h,\ell}(tc) + \frac{1}{\sqrt{2}} M_{h,\ell}(tc) = 1,$$

where $L_{h,\ell}(t)$ (resp. $R_{h,\ell}(t)$, $M_{h,\ell}(t)$) are the generating function of walks that start from a and end on the left (resp. right, middle) border of the domain $D_{h,\ell}$.



References

- ArXiv 1007.0575

- Some slides:

- Smirnov's lecture at ICM

- <http://www.unige.ch/~smirnov/slides/slides-hyderabad.pdf>

- A video of Duminil-Copin at Microsoft

- <http://research.microsoft.com/apps/video/dl.aspx?id=133046>