

# Walks with large steps in the quadrant

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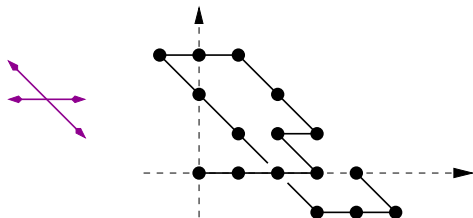
ArXiv: soon !

# Counting quadrant walks

Let  $\mathcal{S}$  be a finite subset of  $\mathbb{Z}^2$  (set of **steps**).

We look at walks starting at  $(0, 0)$  and formed of steps of  $\mathcal{S}$ .

**Example.**  $\mathcal{S} = \{10, \bar{1}0, 1\bar{1}, \bar{1}\bar{1}\}$



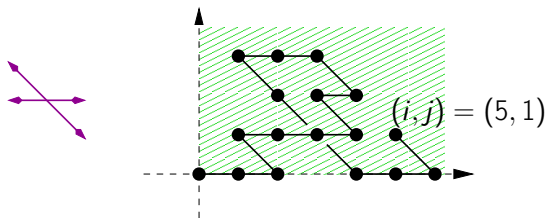
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- Functional equation:

$$\left(1 - t\left(x + \frac{1}{x} + \frac{y}{x} + \frac{x}{y}\right)\right) Q(x, y; t) = 1 - t \frac{1+y}{x} Q(0, y; t) - t \frac{x}{y} Q(x, 0; t)$$

What is the **nature** of this series?

# A hierarchy of formal power series

- Rational series

$$A(t) = \frac{P(t)}{Q(t)}$$

- Algebraic series

$$\text{Pol}(t, A(t)) = 0$$

- Differentially finite series (D-finite)

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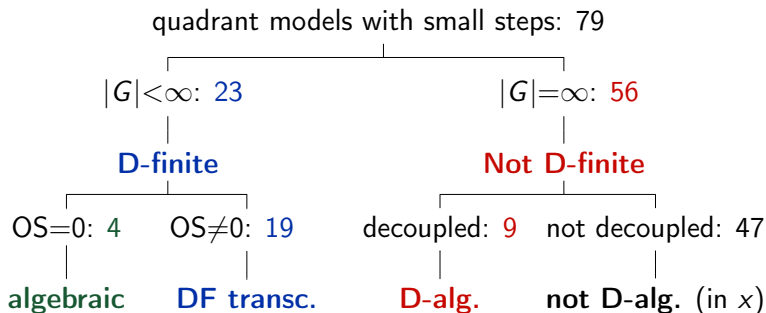
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Multi-variate series: one DE per variable



# Classification of quadrant walks with small steps



- $G$  the group of the model
- OS the orbit sum

[mbm-Mishna 10] [Bostan-Kauers 10] [Mishna-Rechnitzer 07]

[Melczer-Mishna 13] [Kurkova-Raschel 12] [Bostan-Raschel-Salvy 14]

[Bernardi-mbm-Raschel 17(a)] [Dreyfus-Hardouin-Roques-Singer 17(a), 18]

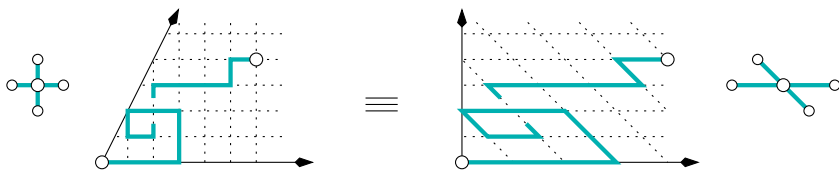


## Quadrant walks with arbitrary steps

- A mathematical challenge: the small step condition seems crucial in all approaches (apart from computer algebra)
- Large steps occur in “real life”: simple walk models,

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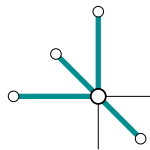
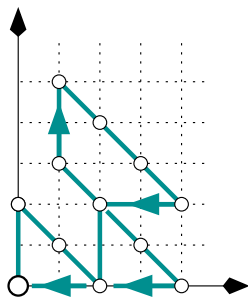
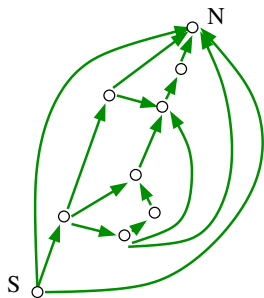
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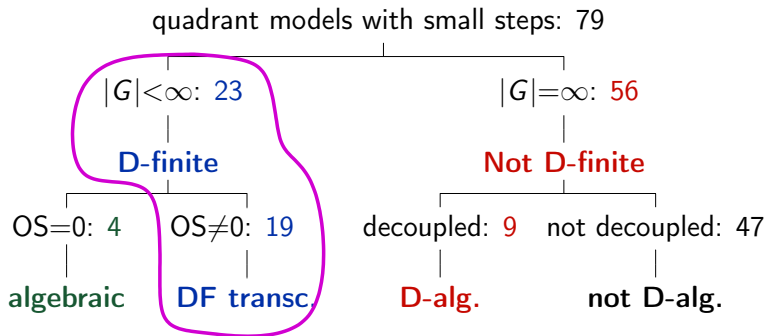
# Quadrant walks with arbitrary steps

- A mathematical challenge: the small step condition seems crucial in all approaches (apart from computer algebra)
- Large steps occur in “real life”: simple walk models, queuing theory, bipolar orientations ([Kenyon et al. 15(a)])



# A general approach for quadrant walks...

which solves *some* cases.



“The simple branch”  
[mbm-Mishna 10]

## A four step approach

1. Write a functional equation for the tri-variate series  $Q(x, y; t)$ .  
It involves bi-variate series  $Q(x, 0; t)$ ,  $Q(0, y; t)$ , ... (called **sections**)
2. Compute the “orbit” of  $(x, y)$
3. Combine the main equation and the orbit to find a functional equation **free from sections**
4. Extract from it  $Q(x, y; t)$

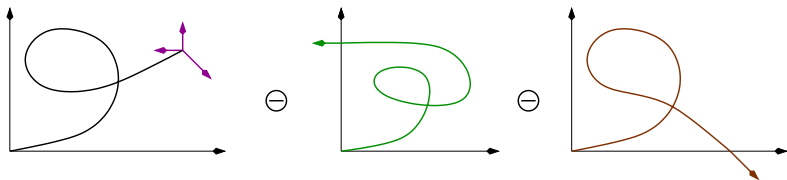
# Step 1: Write a step-by-step functional equation



**Example:**  $\mathcal{S} = \{01, \bar{1}0, 1\bar{1}\}$  (bipolar triangulations)

$$Q(x, y; t) \equiv Q(x, y) = 1 + t(y + \bar{x} + x\bar{y})Q(x, y) - t\bar{x}Q(0, y) - tx\bar{y}Q(x, 0)$$

with  $\bar{x} = 1/x$  and  $\bar{y} = 1/y$ .



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or

$$(1 - t(y + \bar{x} + x\bar{y}))Q(x, y) = 1 - t\bar{x}Q(0, y) - tx\bar{y}Q(x, 0)$$

- The polynomial  $1 - t(y + \bar{x} + x\bar{y})$  is the **kernel** of this equation



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- The polynomial  $1 - t(y + \bar{x} + x\bar{y})$  is the **kernel** of this equation
- The series  $Q(0, y)$  and  $Q(x, 0)$  are the **sections**.

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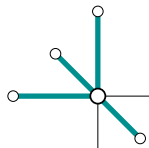
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**Example:** Bipolar quadrangulations

$$(1 - t(x\bar{y} + \bar{x}^2 + \bar{x}y + y^2))Q(x, y) = 1 - tx\bar{y}Q(x, 0) - t\bar{x}^2(Q_0(y) + xQ_1(y)) - t\bar{x}yQ_0(y),$$

where  $Q_i(y)$  counts quadrant walks ending at abscissa  $i$ .



$\Rightarrow$  More sections, kernel of higher degree

## Step 2: the group of the model (and its orbit)



- The step polynomial:

$$S(x, y) = \bar{x} + y + x\bar{y}$$

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$$\Phi : (x, y) \mapsto (\bar{x}y, y) \quad \text{and} \quad \Psi : (x, y) \mapsto (x, x\bar{y}).$$

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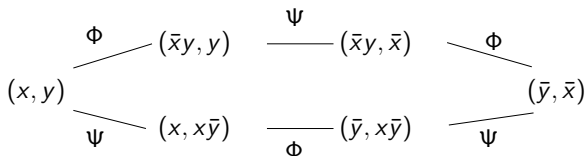
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- $\Phi$  and  $\Psi$  are involutions
- They generate a (dihedral) group



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  - they have one coordinate in common
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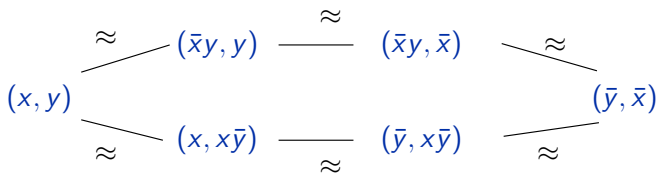
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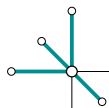
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  - they have one coordinate in common
  - they give the same value to the step polynomial  $S$
- Let  $\sim$  be the transitive closure of the adjacency relation. The orbit of  $(x, y)$  is its equivalence class.



## Step 2: the orbit of the model

**Example:** Bipolar quadrangulations

$$S(x, y) = \bar{x}^2 + \bar{x}y + y^2 + x\bar{y}$$



The equation (in  $x'$ )  $S(x, y) = S(x', y)$ , has 3 solutions, namely  $x$  and

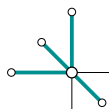
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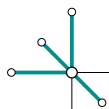
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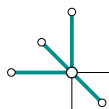
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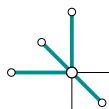
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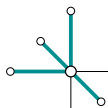
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- and so on.

## Step 2: the orbit of the model

**Example:** Bipolar quadrangulations

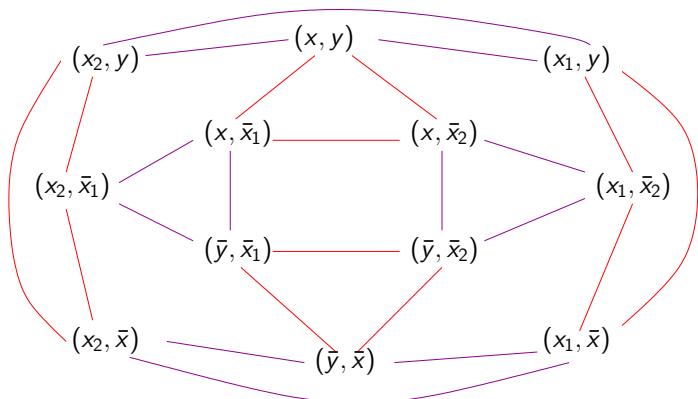
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• Orbit



### Step 3: Find a functional equation free from sections



- The equation reads (with  $K(x, y) = 1 - tS(x, y)$ ):

$$K(x, y)xyQ(x, y) = xy - tx^2Q(x, 0) - tyQ(0, y)$$

- The orbit of  $(x, y)$  is

$$(x, y) \approx (\bar{x}y, y) \approx (\bar{x}y, \bar{x}) \approx (\bar{y}, \bar{x}) \approx (\bar{y}, x\bar{y}) \approx (x, x\bar{y})$$

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- The value of  $S(x, y)$  (and  $K(x, y)$ ) is the same over the orbit. Hence

$$\begin{aligned} K(x, y) xyQ(x, y) &= xy - tx^2Q(x, 0) - tyQ(0, y) \\ K(x, y) \bar{x}y^2Q(\bar{x}y, y) &= \bar{x}y^2 - t\bar{x}^2y^2Q(\bar{x}y, 0) - tyQ(0, y) \\ K(x, y) \bar{x}^2yQ(\bar{x}y, \bar{x}) &= \bar{x}^2y - t\bar{x}^2y^2Q(\bar{x}y, 0) - t\bar{x}Q(0, \bar{x}) \\ \dots &= \dots \\ K(x, y) x^2\bar{y}Q(x, x\bar{y}) &= x^2\bar{y} - tx^2Q(x, 0) - tx\bar{y}Q(0, x\bar{y}). \end{aligned}$$

### Step 3: Find a functional equation free from sections



⇒ Form the alternating sum of the equation over all elements of the orbit:

$$\begin{aligned} K(x, y) & \left( xyQ(x, y) - \bar{x}y^2Q(\bar{x}y, y) + \bar{x}^2yQ(\bar{x}y, \bar{x}) \right. \\ & \left. - \bar{x}\bar{y}Q(\bar{y}, \bar{x}) + x\bar{y}^2Q(\bar{y}, x\bar{y}) - x^2\bar{y}Q(x, x\bar{y}) \right) = \\ & \qquad \qquad \qquad xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y} \\ & \qquad \qquad \qquad \text{(the orbit sum)}. \end{aligned}$$



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## Step 4: Extract $Q(x, y)$



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- Both sides are power series in  $t$ , with coefficients in  $\mathbb{Q}[x, \bar{x}, y, \bar{y}]$ .

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- Both sides are power series in  $t$ , with coefficients in  $\mathbb{Q}[x, \bar{x}, y, \bar{y}]$ .
- Extract the part with positive powers of  $x$  and  $y$ :

$$xyQ(x, y) = [x^{>0}y^{>0}] \frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}$$

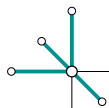
is a D-finite series.

[Lipshitz 88]

## Step 3: Find a functional equation free from sections

**Example:** Bipolar quadrangulations

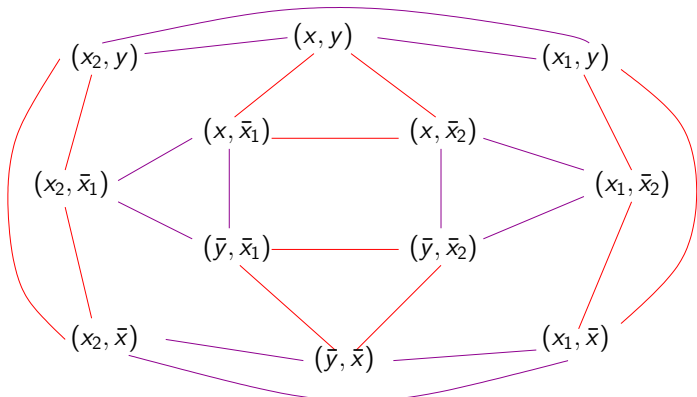
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The equation (in  $x'$ )  $S(x, y) = S(x', y)$ , has 3 solutions, namely  $x$  and

$$x_{1,2} = \frac{xy^2 + y \pm \sqrt{y(x^2y^3 + 4x^3 + 2xy^2 + y)}}{2x^2}.$$

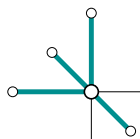
• Orbit



## Step 3: Find a functional equation free from sections

**Example:** Bipolar quadrangulations

$$S(x, y) = \bar{x}^2 + \bar{x}y + y^2 + x\bar{y}$$



Two section-free linear combinations (+ linear combinations):

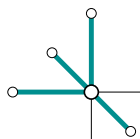
$$\begin{aligned} & xyQ(x, y) - \bar{x}_1xQ(x, \bar{x}_1) - \frac{x_1^2y(x-x_2)Q(x_1, y)}{x(x_1-x_2)} + \frac{x_1^2(x-x_2)Q(x_1, \bar{x})}{(x_1-x_2)x^2} \\ & + \frac{x_2^2y(x-x_1)Q(x_2, y)}{x(x_1-x_2)} - \frac{(x_1y-1)x_2^2(x-x_2)Q(x_2, \bar{x})}{x^2(x_1-x_2)(x_2y-1)} \\ & + \frac{(xy-1)x_2^2Q(x_2, \bar{x}_1)}{x_1x(x_2y-1)} + \frac{(x-x_2)Q(\bar{y}, \bar{x})}{y(x_2y-1)x^2} - \frac{(x-x_2)Q(\bar{y}, \bar{x}_1)}{yx_1x(x_2y-1)} \\ & = \frac{(y-\bar{x}_1)(xy-1)(\bar{y}-\bar{x}^2y-2\bar{x}^3)}{K(x, y)}, \end{aligned}$$

and the same equation with  $x_1$  and  $x_2$  exchanged.

## Step 4: Extract $Q(x, y)$

**Example:** Bipolar quadrangulations

$$S(x, y) = \bar{x}^2 + \bar{x}y + y^2 + x\bar{y}$$



- A section-free equation:

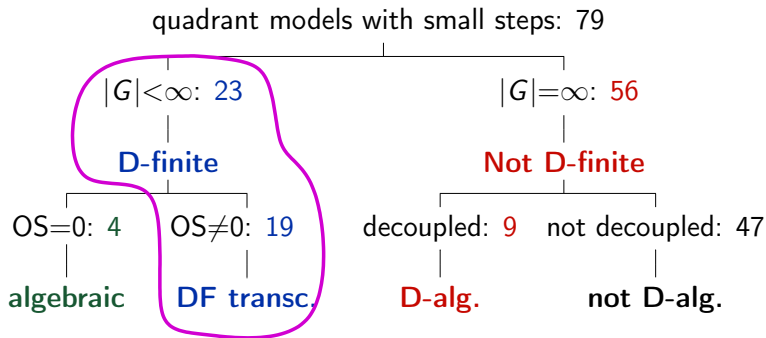
$$\begin{aligned} xyQ(x, y) - \bar{x}_1xQ(x, \bar{x}_1) - \frac{x_1^2y(x - x_2)Q(x_1, y)}{x(x_1 - x_2)} + \dots \\ = \frac{(y - \bar{x}_1)(xy - 1)(\bar{y} - \bar{x}^2y - 2\bar{x}^3)}{K(x, y)} \end{aligned}$$

- Then

$$xyQ(x, y) = [x^{>0}y^{>0}] \frac{(y - \bar{x}_1)(xy - 1)(\bar{y} - \bar{x}^2y - 2\bar{x}^3)}{K(x, y)},$$

provided the RHS is expanded first in  $t$ , then in  $\bar{y}$ , and finally in  $x$ .

# What can go wrong?



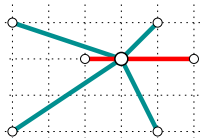
1. Functional equation: OK
2. The “orbit” may be infinite
3. There may be several section-free equations (or none?) [NEW]
4. The extraction may be tricky [NEW], or impossible

# Some cases that work

- Hadamard walks:

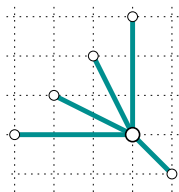
$$S(x, y) = U(x) + V(x)T(y)$$

(Small steps: 16 out of the 19 “simple” models)



- Bipolar maps [mbm, Fusy, Raschel 18]

$$\mathcal{S}_p = \{(-p, 0), (-p + 1, 1), \dots, (0, p), (1, -1)\}$$



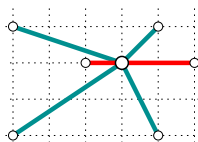
In all those cases, the orbit is finite and the series  $Q(x, y; t)$  D-finite, expressed as the non-negative part of an algebraic series.



# Hadamard walks in 2D

Assume

$$S(x, y) = U(x) + V(x)T(y)$$



Proposition [Bostan, mbm, Melczer 18]

The series  $Q(x, y)$  is D-finite, and reads

$$Q(x, y) = [x \geq y \geq] \frac{\prod_{i=1}^m (1 - \bar{x}x_i(y)) \prod_{j=1}^{m'} (1 - \bar{y}y_j)}{1 - tS(x, y)},$$

where

- the  $x_i(y)$  are the roots of  $S(x, y) = S(x', y)$  (solved for  $x'$ ), whose expansion in  $\bar{x}$  involves no positive power of  $x$ ,
- the  $y_j$  are the roots of  $S(x, y) = S(x, y')$ , or  $T(y) = T(y')$  (solved for  $y'$ ) whose expansion in  $\bar{y}$  involve no positive powers of  $y$ .

# Bipolar maps

## Proposition [mbm, Fusy, Raschel 18]

The generating function of bipolar maps with faces of degree  $p + 2$  is

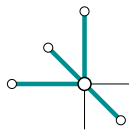
$$Q(x, y) = [x \geq y \geq] \frac{(y - \bar{x}_1)(1 - \bar{x}\bar{y}) S_x(x, y)}{1 - tS(x, y)},$$

where  $S(x, y)$  is the step polynomial:

$$S(x, y) = x\bar{y} + \bar{x}^p + \bar{x}^{p-1}y + \dots + \bar{x}y^{p-1} + y^p.$$

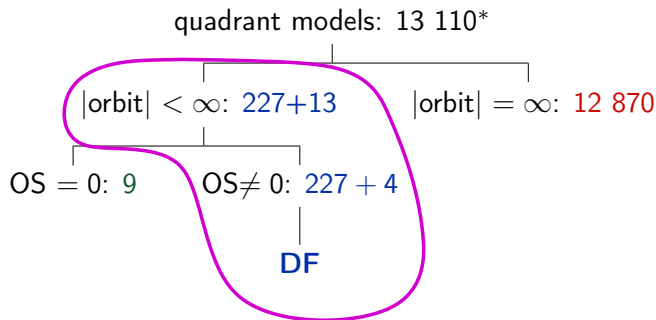
and  $x_1$  is the only root of  $S(x, y) = S(x', y)$  (solved for  $x'$ ) whose expansion in  $\bar{y}$  involves a positive power of  $y$ .

It is D-finite.



# Quadrant walks with steps in $\{-2, -1, 0, 1\}^2$

- In all cases, a unique section-free equation

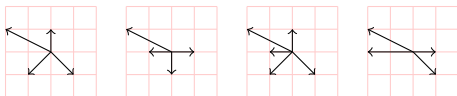


- 227 Hadamard models

(\*) Models with at least one occurrence of  $-2$

## Some interesting models

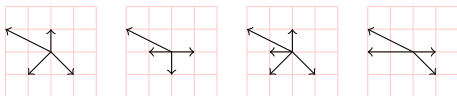
- Non-Hadamard, solvable via our approach (and D-finite):



$\Rightarrow Q(x, y; t)$  is the non-negative part (in  $x$  and  $y$ ) of a rational series

## Some interesting models

- Non-Hadamard, solvable via our approach (and D-finite):



$\Rightarrow Q(x, y; t)$  is the non-negative part (in  $x$  and  $y$ ) of a rational series

For the first model,

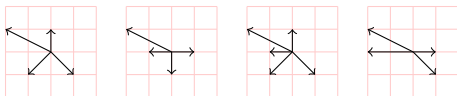
$$Q(x, y) = [x^{\geq 0} y^{\geq 0}] \frac{(x^3 - 2y^2 - x)(y^2 - x)(x^2y^2 - y^2 - 2x)}{x^5y^4(1 - t(y + x\bar{y} + \bar{x}\bar{y} + \bar{x}^2y))}.$$

The coefficients are nice: for  $n = 2i + j + 4m$ ,

$$q(i, j; n) = \frac{(i+1)(j+1)(i+j+2)n!(n+2)!}{m!(3m+2i+j+2)!(2m+i+1)!(2m+i+j+2)!}.$$

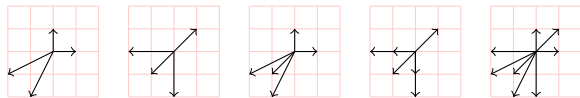
# Some interesting models

- Non-Hadamard, solvable via our approach (and D-finite):



⇒  $Q(x, y; t)$  is the non-negative part (in  $x$  and  $y$ ) of a rational series

- Non-Hadamard, orbit sum zero: let's guess! (what  $Q(0, 0; t)$  is)



DF

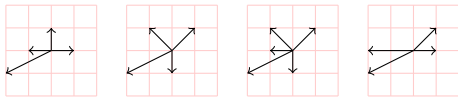
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à la Kreweras



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Alg

Alg

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à la Gessel

Thanks to Axel Bacher!

## Final comments

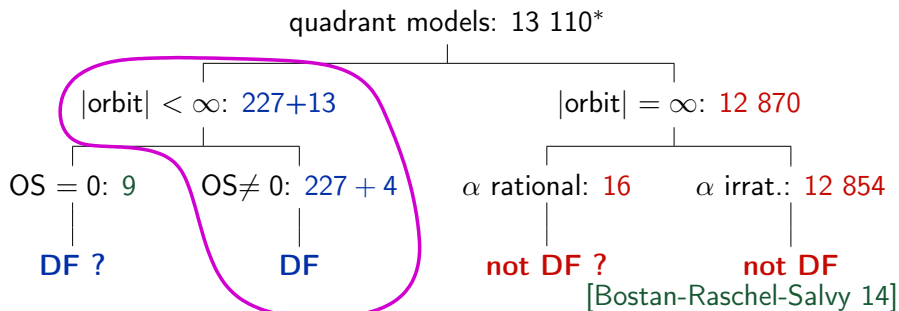
Still a lot to be done...

- Is there a unique section free equation when there are no large forward step?

# Final comments

Still a lot to be done...

- Is there a unique section free equation when there are no large forward step?
- Closer study for tricky examples (the 9 analogues of Kreweras' and Gessel's algebraic models)
- Nature of models where  $\alpha$  is rational but the orbit infinite





## Walks on a half-line ( $d = 1$ )

Let  $\mathcal{S} \subset \mathbb{Z}$  with  $\min \mathcal{S} = -m$ .

Proposition [Bostan, mbm, Melczer 18]

$$Q(x) = [x^{\geq 0}] \frac{\prod_{j=1}^m (1 - \bar{x}x_j)}{1 - tS(x)},$$

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Classical solution [Gessel 80, mbm-Petkovšek 00, Banderier-Flajolet 02...]

$$Q(x) = \frac{\prod_{j=1}^m (1 - \bar{x}X_j)}{1 - tS(x)}$$

where the  $X_j \equiv X_j(t)$  are the roots of  $1 - tS(x)$  whose expansion in  $t$  involves no negative power of  $t$ . The series  $Q(x)$  is algebraic.

These solutions are (of course) equivalent.