

Counting walks in the quarter plane

Mireille Bousquet-Mélou

CNRS, LaBRI, Université Bordeaux 1

Marni Mishna

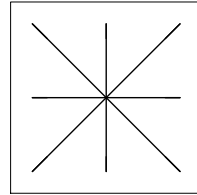
Simon Fraser University, Vancouver, Canada

<http://www.labri.fr/~bousquet>

arXiv:0810.4387

The question

Let $\mathcal{S} \subset \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$ be a set of “small” steps in the plane.



Consider walks with steps in \mathcal{S} , that start from $(0, 0)$ and are confined to the positive quadrant $\{(i, j) : i \geq 0, j \geq 0\}$. Let $q(i, j; n)$ be the number of such walks that have length n and end at position (i, j) . Let

$$Q(x, y; t) = \sum_{i, j, n \geq 0} q(i, j; n) x^i y^j t^n.$$

Can we determine $Q(x, y; t)$? $Q(0, 0; t)$? $Q(1, 1; t)$?

What is the **nature** of these series?

A hierarchy of formal power series

- The formal power series $A(t)$ is **rational** if it can be written

$$A(t) = \frac{P(t)}{Q(t)}$$

where $P(t)$ and $Q(t)$ are polynomials in t .

- The formal power series $A(t)$ is **algebraic** (over $\mathbb{Q}(t)$) if it satisfies a (non-trivial) polynomial equation:

$$P(t, A(t)) = 0.$$

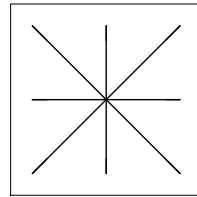
- The formal power series $A(t)$ is **D-finite** if it satisfies a (non-trivial) linear differential equation:

$$P_k(t)A^{(k)}(t) + \cdots + P_0(t)A(t) = 0.$$

+ extension to several variables + closure properties

The question

Let $\mathcal{S} \subset \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$ be a set of “small” steps in the plane.



Consider walks with steps in \mathcal{S} , that start from $(0, 0)$ and are confined to the positive quadrant $\{(i, j) : i \geq 0, j \geq 0\}$. Let $q(i, j; n)$ be the number of such walks that have length n and end at position (i, j) . Let

$$Q(x, y; t) = \sum_{i, j, n \geq 0} q(i, j; n) x^i y^j t^n.$$

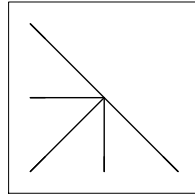
Can we determine $Q(x, y; t)$? $Q(0, 0; t)$? $Q(1, 1; t)$?

What is the **nature** of these series?

⇒ After all... there are only 2^8 such problems!

Some of these problems are trivial

If \mathcal{S} contains no step with both coordinates non-negative,

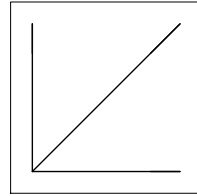


then

$$Q(x, y; t) = 1.$$

Some of these problems are very easy and rational

If \mathcal{S} contains **only** steps with both coordinates non-negative,



then

$$Q(x, y; t) = \frac{1}{1 - \sum_{(i,j) \in \mathcal{S}} x^i y^j t}$$

Some of these problems are routine, and algebraic

If the quarter plane condition for \mathcal{S} can be described as a half-plane condition, the walks have a simple context-free decomposition, and an algebraic generating function.

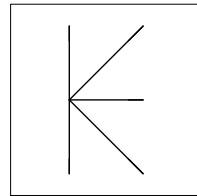
This happens when...

Some of these problems are routine, and algebraic

If the quarter plane condition for \mathcal{S} can be described as a half-plane condition, the walks have a simple context-free decomposition, and an algebraic generating function.

This happens when:

- \mathcal{S} contains no x -negative step (generalized Dyck paths)



Example: for the above set,

$$Q(x, 0) = 1 + txQ(x, 0) + t^2(x + 1)^2Q(x, 0)^2,$$

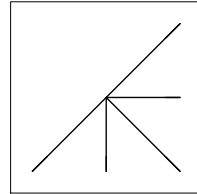
$$Q(x, y) = \frac{Q(x, 0)}{1 - ty(x + 1)Q(x, 0)}$$

Some of these problems are routine, and algebraic

If the quarter plane condition for \mathcal{S} can be described as a half-plane condition, the walks have a simple context-free decomposition, and an algebraic generating function.

This happens when:

- \mathcal{S} contains only sub-diagonal steps: then any walk lying in the upper half-plane automatically lies in the right half-plane (\simeq Dyck paths)

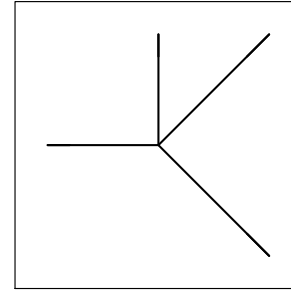
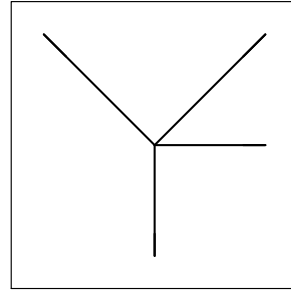


Example: for the above set,

$$Q(x, 0) = 1 + txQ(x, 0) + t^2x(1 + x + \bar{x})Q(x, 0)^2,$$

$$Q(x, y) = \frac{Q(x, 0)}{1 - txyQ(x, 0)} \quad \text{with } \bar{x} = 1/x.$$

Finally, some problems are equivalent by symmetry



(Symmetry across the first diagonal)

Good news

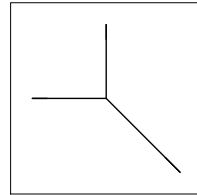
⇒ There are only 79 distinct problems to solve

I. A step-by-step decomposition

A functional equation

Example: $\mathcal{S} = \{N, W, SE\}$

$$Q(x, y; t) \equiv Q(x, y) = 1 + t(y + \bar{x} + x\bar{y})Q(x, y) - t\bar{x}Q(0, y) - tx\bar{y}Q(x, 0),$$



A functional equation

Example: $\mathcal{S} = \{N, W, SE\}$

$$Q(x, y; t) \equiv Q(x, y) = 1 + t(y + \bar{x} + x\bar{y})Q(x, y) - t\bar{x}Q(0, y) - tx\bar{y}Q(x, 0),$$

or

$$(1 - t(y + \bar{x} + x\bar{y}))Q(x, y) = 1 - t\bar{x}Q(0, y) - tx\bar{y}Q(x, 0),$$

A functional equation

Example: $\mathcal{S} = \{N, W, SE\}$

$$Q(x, y; t) \equiv Q(x, y) = 1 + t(y + \bar{x} + x\bar{y})Q(x, y) - t\bar{x}Q(0, y) - tx\bar{y}Q(x, 0),$$

or

$$(1 - t(y + \bar{x} + x\bar{y}))Q(x, y) = 1 - t\bar{x}Q(0, y) - tx\bar{y}Q(x, 0),$$

or

$$((1 - t(y + \bar{x} + x\bar{y}))xy)Q(x, y) = xy - tyQ(0, y) - txQ(x, 0).$$

A functional equation

Example: $\mathcal{S} = \{N, W, SE\}$

$$Q(x, y; t) \equiv Q(x, y) = 1 + t(y + \bar{x} + x\bar{y})Q(x, y) - t\bar{x}Q(0, y) - tx\bar{y}Q(x, 0),$$

or

$$(1 - t(y + \bar{x} + x\bar{y}))Q(x, y) = 1 - t\bar{x}Q(0, y) - tx\bar{y}Q(x, 0),$$

or

$$((1 - t(y + \bar{x} + x\bar{y}))xy)Q(x, y) = xy - tyQ(0, y) - txQ(x, 0).$$

The polynomial $1 - t(y + \bar{x} + x\bar{y})$ is the **kernel** of this equation

The equation is linear, with two **catalytic** variables x and y

A functional equation

General case:

$$K(x, y)xyQ(x, y) = xy - txA_{-1}(x)Q(x, 0) - tyB_{-1}(y)Q(0, y) + t\epsilon Q(0, 0)$$

where

$$K(x, y) = 1 - t \sum_{(i,j) \in \mathcal{S}} x^i y^j$$

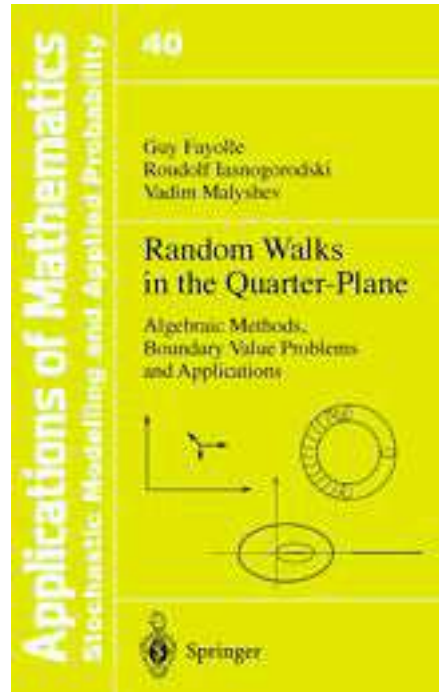
is the **kernel** of the equation,

$$A_{-1}(x) = \sum_{(i,-1) \in \mathcal{S}} x^i, \quad B_{-1}(y) = \sum_{(-1,j) \in \mathcal{S}} y^j$$

and ϵ is 1 if $(-1, -1) \in \mathcal{S}$ and 0 otherwise.

II. The group of the walk

[Fayolle et al. 99]



What is this group?

The kernel reads

$$K(x, y) = 1 - tS(x, y)$$

where

$$S(x, y) = \sum_{(i,j) \in \mathcal{S}} x^i y^j = \sum_{i=-1}^1 x^i B_i(y) = \sum_{j=-1}^1 y^j A_j(x)$$

Observation: $S(x, y)$ (and thus $K(x, y)$) is left unchanged by the rational transformations

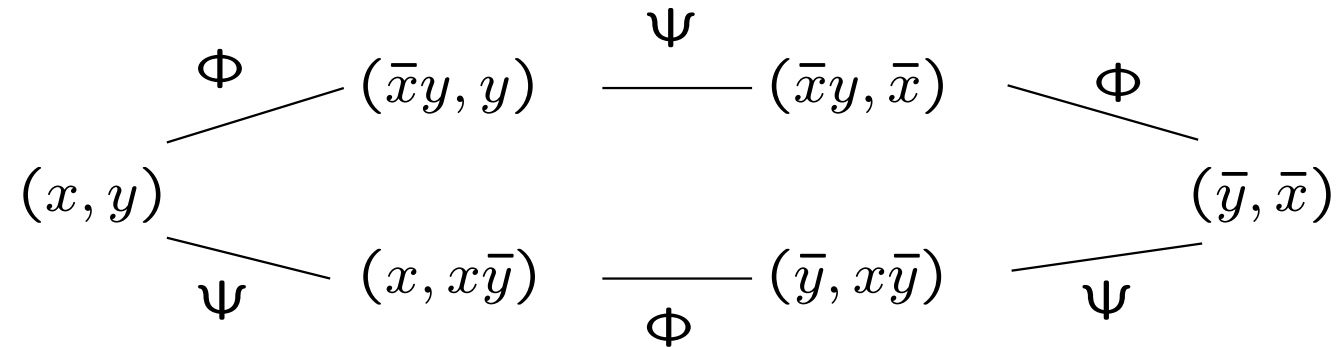
$$\Phi : (x, y) \mapsto \left(\bar{x} \frac{B_{-1}(y)}{B_1(y)}, y \right) \quad \text{and} \quad \Psi : (x, y) \mapsto \left(x, \bar{y} \frac{A_{-1}(x)}{A_1(x)} \right).$$

Lemma

- Φ and Ψ are involutions
- They generate a (dihedral) group denoted $G \equiv G(S)$

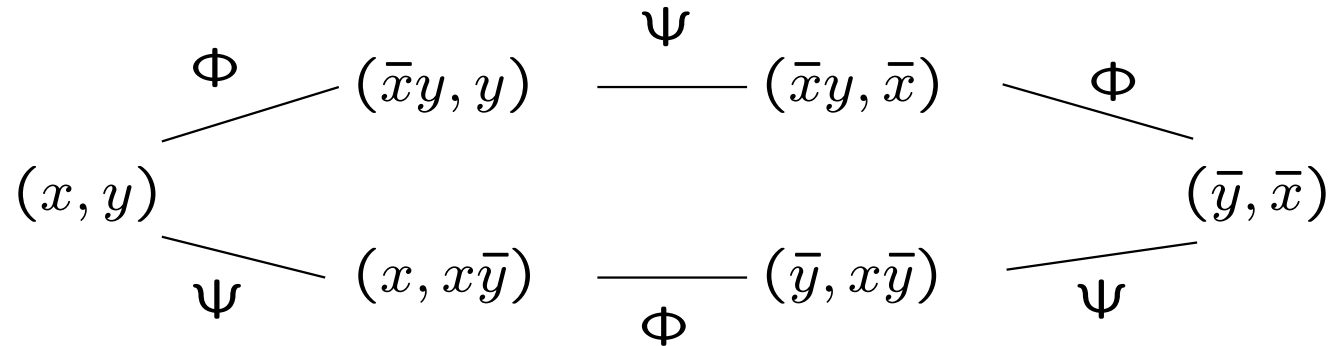
Two examples

- If $\mathcal{S} = \{N, W, SE\}$, then $\Phi : (x, y) \mapsto (\bar{x}y, y)$ and $\Psi : (x, y) \mapsto (x, x\bar{y})$ generate a group of order 6:



Two examples

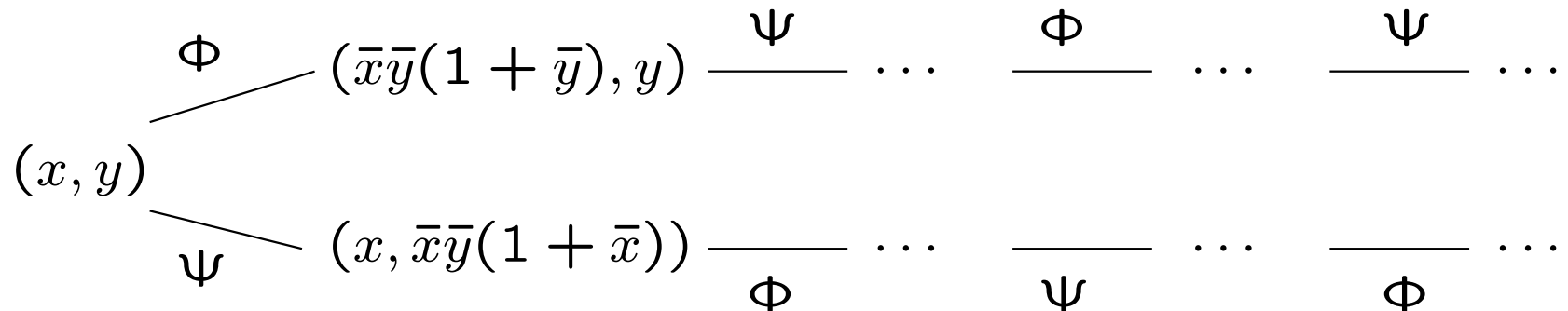
- If $\mathcal{S} = \{N, W, SE\}$, then $\Phi : (x, y) \mapsto (\bar{x}y, y)$ and $\Psi : (x, y) \mapsto (x, x\bar{y})$ generate a group of order 6:



- If $\mathcal{S} = \{S, W, SW, NE\}$:

$$\Phi : (x, y) \mapsto (\bar{x}\bar{y}(1 + \bar{y}), y) \quad \text{and} \quad \Psi : (x, y) \mapsto (x, \bar{x}\bar{y}(1 + \bar{x}))$$

seem to generate an infinite group



Classification

Proposition. Among the 79 quarter plane models, exactly 23 have a finite group:

- 16 have a vertical symmetry and a group of order 4,
- 5 have a group of order 6,
- 2 have a group of order 8.

Classification

Proposition. Among the 79 quarter plane models, exactly 23 have a finite group.

Theorem. For all models with a finite group, $Q(x, y; t)$ is D-finite.

Classification

Proposition. Among the 79 quarter plane models, exactly 23 have a finite group.

Theorem. For all models with a finite group, $Q(x, y; t)$ is D-finite.

([mbm-Mishna] for 22 models, [Bostan-Kauers 09] for the 23rd)

Classification

Proposition. Among the 79 quarter plane models, exactly 23 have a finite group.

Theorem. For all models with a finite group, $Q(x, y; t)$ is D-finite.

([mbm-Mishna] for 22 models,
[Bostan-Kauers 09] for the 23rd)

Conjecture. For all models with an infinite group, $Q(x, y; t)$ is non-D-finite

(proved in two cases [Mishna-Rechnitzer 07]).

III. Models with a finite group: The orbit sum

Example: $\mathcal{S} = \{N, W, SE\}$

- The equation reads

$$K(x, y) \quad xyQ(x, y) = xy - txQ(x, 0) - tyQ(0, y) \quad \text{with } K(x, y) = 1 - t(y + \bar{x} + x\bar{y}).$$

- The orbit of (x, y) under $G = G(\mathcal{S})$ is

$$(x, y) \xleftrightarrow{\Phi} (\bar{x}y, y) \xleftrightarrow{\Psi} (\bar{x}y, \bar{x}) \xleftrightarrow{\Phi} (\bar{y}, \bar{x}) \xleftrightarrow{\Psi} (\bar{y}, x\bar{y}) \xleftrightarrow{\Phi} (x, x\bar{y}) \xleftrightarrow{\Psi} (x, y).$$

- All transformations of G leave $K(x, y)$ invariant. Hence

$$\begin{aligned} K(x, y) \quad xyQ(x, y) &= xy - txQ(x, 0) - tyQ(0, y) \\ K(x, y) \quad \bar{x}y^2Q(\bar{x}y, y) &= \bar{x}y^2 - t\bar{x}yQ(\bar{x}y, 0) - tyQ(0, y) \end{aligned}$$

Example: $\mathcal{S} = \{N, W, SE\}$

- The equation reads

$$K(x, y) \quad xyQ(x, y) = xy - txQ(x, 0) - tyQ(0, y) \quad \text{with } K(x, y) = 1 - t(y + \bar{x} + x\bar{y}).$$

- The orbit of (x, y) under $G = G(\mathcal{S})$ is

$$(x, y) \xleftrightarrow{\Phi} (\bar{x}y, y) \xleftrightarrow{\Psi} (\bar{x}y, \bar{x}) \xleftrightarrow{\Phi} (\bar{y}, \bar{x}) \xleftrightarrow{\Psi} (\bar{y}, x\bar{y}) \xleftrightarrow{\Phi} (x, x\bar{y}) \xleftrightarrow{\Psi} (x, y).$$

- All transformations of G leave $K(x, y)$ invariant. Hence

$$\begin{aligned} K(x, y) \quad xyQ(x, y) &= xy - txQ(x, 0) - tyQ(0, y) \\ K(x, y) \quad \bar{x}y^2Q(\bar{x}y, y) &= \bar{x}y^2 - t\bar{x}yQ(\bar{x}y, 0) - tyQ(0, y) \\ K(x, y) \quad \bar{x}^2yQ(\bar{x}y, \bar{x}) &= \bar{x}^2y - t\bar{x}yQ(\bar{x}y, 0) - t\bar{x}Q(0, \bar{x}) \\ &\dots = \dots \\ K(x, y) \quad x^2\bar{y}Q(x, x\bar{y}) &= x^2\bar{y} - txQ(x, 0) - tx\bar{y}Q(0, x\bar{y}). \end{aligned}$$

Example: $\mathcal{S} = \{N, W, SE\}$

- All transformations of G leave $K(x, y)$ invariant. Hence

$$\begin{aligned}
 K(x, y) \quad xyQ(x, y) &= xy - txQ(x, 0) - tyQ(0, y) \\
 K(x, y) \quad \bar{x}y^2Q(\bar{x}y, y) &= \bar{x}y^2 - t\bar{x}yQ(\bar{x}y, 0) - tyQ(0, y) \\
 K(x, y) \quad \bar{x}^2yQ(\bar{x}y, \bar{x}) &= \bar{x}^2y - t\bar{x}yQ(\bar{x}y, 0) - t\bar{x}Q(0, \bar{x}) \\
 \dots &= \dots \\
 K(x, y) \quad x^2\bar{y}Q(x, x\bar{y}) &= x^2\bar{y} - txQ(x, 0) - tx\bar{y}Q(0, x\bar{y}).
 \end{aligned}$$

\Rightarrow Form the alternating sum of the equation over all elements of the orbit:

$$\begin{aligned}
 K(x, y) \left(xyQ(x, y) - \bar{x}y^2Q(\bar{x}y, y) + \bar{x}^2yQ(\bar{x}y, \bar{x}) \right. \\
 \left. - \bar{x}\bar{y}Q(\bar{y}, \bar{x}) + x\bar{y}^2Q(\bar{y}, x\bar{y}) - x^2\bar{y}Q(x, x\bar{y}) \right) = \\
 xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}.
 \end{aligned}$$

Sneak preview: Why is this interesting?

$$\begin{aligned} xyQ(x, y) - \bar{x}y^2Q(\bar{x}y, y) + \bar{x}^2yQ(\bar{x}y, \bar{x}) \\ - \bar{x}\bar{y}Q(\bar{y}, \bar{x}) + x\bar{y}^2Q(\bar{y}, x\bar{y}) - x^2\bar{y}Q(x, x\bar{y}) = \\ \frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})} \end{aligned}$$

- Both sides are power series in t , with coefficients in $\mathbb{Q}[x, \bar{x}, y, \bar{y}]$.
- Extract the part with positive powers of x and y :

$$xyQ(x, y) = [x^{>0}y^{>0}] \frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}$$

is a D-finite series.

[Lipshitz 88]

The orbit sum in general

For the 23 models with a finite group,

$$\sum_{g \in G(\mathcal{S})} \text{sign}(g)g(xyQ(x, y; t)) = \frac{1}{K(x, y; t)} \sum_{g \in G(\mathcal{S})} \text{sign}(g)g(xy),$$

where

$$g(A(x, y)) := A(g(x, y))$$

- The right-hand side is a rational series.

IV. Solutions via orbit-sums (and half-orbit sums)

1. When the orbit sum is non-zero

Theorem. Among the 23 models with a finite group, the orbit sum is non-zero in exactly 19 cases:

$$\sum_{g \in G(\mathcal{S})} \text{sign}(g) g(xyQ(x, y; t)) = \frac{1}{K(x, y; t)} \sum_{g \in G(\mathcal{S})} \text{sign}(g(\mathcal{S})) g(xy).$$

In those 19 cases, extracting the positive part in x and y gives

$$xyQ(x, y; t) = [x^{>0}y^{>0}] \frac{1}{K(x, y; t)} \sum_{g \in G(\mathcal{S})} \text{sign}(g) g(xy),$$

a D-finite series.

Remark. In many cases, one can perform an explicit coefficient extraction and obtain closed form expressions.

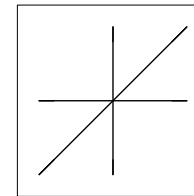
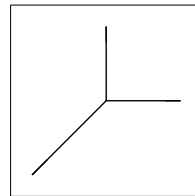
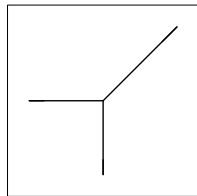
2. When the orbit sum is zero

In 4 cases, the orbit sum vanishes:

$$\sum_{g \in G(\mathcal{S})} \text{sign}(g) g(xyQ(x, y; t)) = \frac{1}{K(x, y; t)} \sum_{g \in G(\mathcal{S})} \text{sign}(g(\mathcal{S})) g(xy) = 0.$$

This happens:

- for Gessel's walks, $\mathcal{S} = \{E, W, NE, SW\}$
- for the Kreweras trilogy, $\mathcal{S} = \{W, S, NE\}$, $\{E, N, SW\}$, $\{W, S, E, N, NE, SW\}$.



In the latter case, the orbit sum reads

$$xyQ(x, y) - \bar{x}Q(\bar{x}\bar{y}, y) + \bar{y}Q(\bar{x}\bar{y}, x) = xyQ(y, x) - \bar{x}Q(y, \bar{x}\bar{y}) + \bar{y}Q(x, \bar{x}\bar{y}),$$

which is trivial because $Q(x, y) = Q(y, x)$.

2. When the orbit sum is zero

In 4 cases, the orbit sum vanishes:

$$\sum_{g \in G(\mathcal{S})} \text{sign}(g)g(xyQ(x, y; t)) = \frac{1}{K(x, y; t)} \sum_{g \in G(\mathcal{S})} \text{sign}(g(\mathcal{S}))g(xy) = 0.$$

This happens:

- for Gessel's walks, $\mathcal{S} = \{E, W, NE, SW\}$
- for the Kreweras trilogy, $\mathcal{S} = \{W, S, NE\}, \{E, N, SW\}, \{W, S, E, N, NE, SW\}$.

In all 4 cases, $Q(x, y; t)$ is algebraic.

([mbm-Mishna] for the Kreweras trilogy,
[Bostan-Kauers 09] for Gessel's model)

The Kreweras trilogy: solution

- Again a coefficient extraction
- Form the half-orbit sum:

$$xyQ(x, y) - \bar{x}Q(\bar{x}\bar{y}, y) + \bar{y}Q(\bar{x}\bar{y}, x) = \frac{xy - \bar{x} + \bar{y} - 2txA_{-1}(x)Q(x, 0) + t\epsilon Q(0, 0)}{K(x, y)}$$

with

$$K(x, y) = 1 - t \sum_{(i,j) \in \mathcal{S}} x^i y^j, \quad A_j(x) = \sum_{(i,j) \in \mathcal{S}} x^i$$

and ϵ is 1 if $(-1, -1) \in \mathcal{S}$ and 0 otherwise.

The Kreweras trilogy: solution

- Again a coefficient extraction

- Form the half-orbit sum:

$$xyQ(x, y) - \bar{x}Q(\bar{x}\bar{y}, y) + \bar{y}Q(\bar{x}\bar{y}, x) = \frac{xy - \bar{x} + \bar{y} - 2txA_{-1}(x)Q(x, 0) + t\epsilon Q(0, 0)}{K(x, y)}$$

- $1/K(x, y)$ is a rational function in t , x and y . Extract the coefficient of y^0 :

$$-\bar{x}Q_d(\bar{x}) = -\frac{x}{tA_1(x)} + \frac{1}{\sqrt{\Delta(x)}} \left(\frac{x(1 - tA_0(x))}{tA_1(x)} - \bar{x} - 2txA_{-1}(x)Q(x, 0) + t\epsilon Q(0, 0) \right)$$

where

$$\Delta(x) = (1 - tA_0(x))^2 - 4t^2A_{-1}(x)A_1(x),$$

$$Q_d(x) \equiv Q_d(x; t) = \sum_{n, i \geq 0} t^n x^i q(i, i; n).$$

- Write $\Delta(x)$ as

$$\Delta(x) = \Delta_0 \prod_{i=1}^{\delta} (1 - \bar{x}X_i) \prod_{i=\delta+1}^{\delta+d} (1 - x/X_i) = \Delta_0 \Delta_+(x) \Delta_-(\bar{x}),$$

where X_1, \dots, X_{δ} (resp. $X_{\delta+1}, \dots, X_{\delta+d}$) are the roots of $\Delta(x)$ that are finite (resp. infinite) at $t = 0$.

- The equation now reads

$$\sqrt{\Delta_-(\bar{x})} \left(\frac{x}{t} - \bar{x}A_1(x)Q_d(\bar{x}) \right) = \frac{1}{\sqrt{\Delta_0 \Delta_+(x)}} \left(\frac{x(1 - tA_0(x))}{t} - \bar{x}A_1(x) - 2txA_{-1}(x)A_1(x)Q(x, 0) + t\epsilon A_1(x)Q(0, 0) \right).$$

- Each term in this equation is a Laurent series in t with coefficients in $\mathbb{Q}[x, \bar{x}]$. Moreover,

1. few positive powers of x occur in the **left-hand** side,
2. few negative powers in x occur in the **right-hand** side.

\Rightarrow Extract the positive and negative parts in x to obtain algebraic expressions for the series $Q_d(x)$ and $Q(x, 0)$.

Kreweras' model: $\mathcal{S} = \{W, S, NE\}$

Let $W \equiv W(t)$ be the power series in t defined by $W = t(2 + W^3)$. Then

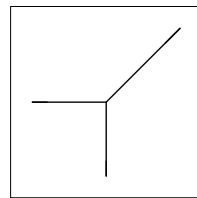
$$Q(x, 0; t) = \frac{1}{tx} \left(\frac{1}{2t} - \frac{1}{x} - \left(\frac{1}{W} - \frac{1}{x} \right) \sqrt{1 - xW^2} \right).$$

Consequently,

$$q(i, 0; 3m + 2i) = \frac{4^m (2i + 1)}{(m + i + 1)(2m + 2i + 1)} \binom{2i}{i} \binom{3m + 2i}{m}.$$

The generating function of walks ending on the diagonal is

$$Q_d(x; t) = \frac{W - \bar{x}}{t\sqrt{1 - xW(1 + W^3/4) + x^2W^2/4}} + \bar{x}/t.$$



Remark. $Q(0, y; t) = Q(y, 0; t)$ and $Q(x, y; t)$ can be recovered in terms of $Q(x, 0; t)$ and $Q(0, y; t)$ via the functional equation.

Reversing Kreweras' steps: $\mathcal{S} = \{E, N, SW\}$

Let $W \equiv W(t)$ be the power series in t defined by $W = t(2 + W^3)$. Then

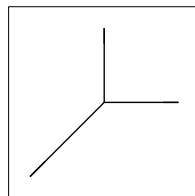
$$Q(x, 0; t) = \frac{W(4 - W^3)}{16t} - \frac{t - x^2 + tx^3}{2xt^2} - \frac{(2x^2 - xW^2 - W) \sqrt{1 - xW(W^3 + 4)/4 + x^2W^2/4}}{2txW}$$

The generating function of walks ending on the diagonal is

$$Q_d(x; t) = \frac{xW(x + W) - 2}{2tx^2 \sqrt{1 - xW^2}} + \frac{1}{tx^2}$$

Consequently,

$$q(i, i; 3m + 2i) = \frac{4^m (i + 1)^2}{(m + i + 1)(2m + 2i + 1)} \binom{2i + 1}{i} \binom{3m + 2i}{m}$$



Kreweras' steps and their reverse: $\mathcal{S} = \{W, S, E, N, NE, SW\}$

Let $Z \equiv Z(t)$ be defined by $Z(0) = 0$ and

$$Z = t \frac{1 - 2Z + 6Z^2 - 2Z^3 + Z^4}{(1 - Z)^2},$$

and denote

$$\Delta_+(x) = 1 - 2Z \frac{1 + Z^2}{(1 - Z)^2} x + Z^2 x^2.$$

Then

$$Q(x, 0; t) = \frac{\left(Z(1 - Z) + 2xZ - (1 - Z)x^2 \right) \sqrt{\Delta_+(x)}}{2txZ(1 - Z)(1 + x)^2} \\ \frac{Z(1 - Z)^2 + Z \left(Z^3 + 4Z^2 - 5Z + 2 \right) x - \left(1 - 2Z + 7Z^2 - 4Z^3 \right) x^2 + x^3 Z(1 - Z)^2}{2txZ(1 - Z)^2(1 + x)^2}$$

The generating function of walks ending on the diagonal is

$$Q_d(x; t) = \frac{1 - Z - 2xZ + x^2 Z(Z - 1)}{tx(1 + x)(Z - 1)\sqrt{\Delta_+(x)}} + \frac{1}{tx(1 + x)}$$

Kreweras' steps and their reverse: $\mathcal{S} = \{W, S, E, N, NE, SW\}$ Special cases

Let $Z \equiv Z(t)$ be defined by $Z(0) = 0$ and

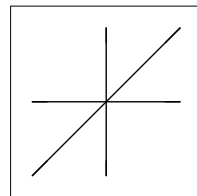
$$Z = t \frac{1 - 2Z + 6Z^2 - 2Z^3 + Z^4}{(1 - Z)^2},$$

- For walks ending at the origin:

$$Q(0, 0) = \frac{Z(1 - 2Z - Z^2)}{t(1 - Z)^2}$$

- All walks, counted by the length: $Q \equiv Q(1, 1)$ has degree 4:

$$Q(1 + tQ)(1 + 2tQ + 2t^2Q^2) = \frac{1}{1 - 6t}$$



V. Questions

Solve Gessel's model!

- The functional equation:

$$(1 - t(x + \bar{x} + xy + \bar{x}\bar{y}))xyQ(x, y) = 1 - tQ(x, 0) - t(1 + y)Q(0, y) + tQ(0, 0)$$

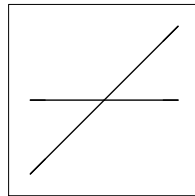
- The group $G(\mathcal{S})$ has order 8. The orbit of (x, y) reads:

$$\begin{aligned} (x, y) \xleftrightarrow{\Phi} (\bar{x}\bar{y}, y) \xleftrightarrow{\Psi} (\bar{x}\bar{y}, x^2y) \xleftrightarrow{\Phi} (\bar{x}, x^2y) \xleftrightarrow{\Psi} (\bar{x}, \bar{y}) \\ \xleftrightarrow{\Phi} (xy, \bar{y}) \xleftrightarrow{\Psi} (xy, \bar{x}^2\bar{y}) \xleftrightarrow{\Phi} (x, \bar{x}^2\bar{y}) \xleftrightarrow{\Psi} (x, y) \end{aligned}$$

- The orbit sum vanishes:

$$\begin{aligned} xyQ(x, y) - \bar{x}Q(\bar{x}\bar{y}, y) + xQ(\bar{x}\bar{y}, x^2y) - xyQ(\bar{x}, x^2y) \\ + \bar{x}\bar{y}Q(\bar{x}, \bar{y}) - xQ(xy, \bar{y}) + \bar{x}Q(xy, \bar{x}^2\bar{y}) - \bar{x}\bar{y}Q(x, \bar{x}^2\bar{y}) = 0 \end{aligned}$$

- Work with the half-orbit sum?

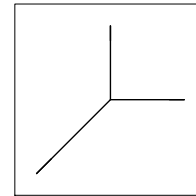
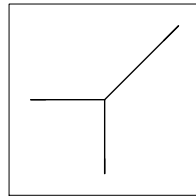


Explain closed form expressions

- Simple closed form expressions for Kreweras, reverse Kreweras

$$q_{\kappa}(i, 0; 3m + 2i) = \frac{4^m(2i + 1)}{(m + i + 1)(2m + 2i + 1)} \binom{2i}{i} \binom{3m + 2i}{m},$$

$$q_{\bar{\kappa}}(i, i; 3m + 2i) = \frac{4^m(i + 1)^2}{(m + i + 1)(2m + 2i + 1)} \binom{2i + 1}{i} \binom{3m + 2i}{m}.$$

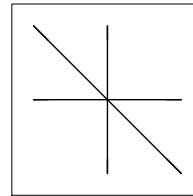
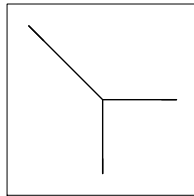


[Bernardi 07]: connection between $q_{\kappa}(0, 0; 3m)$ and planar triangulations.

Explain closed form expressions

- Motzkin numbers for $\mathcal{S}_1 = \{E, S, NW\}$ and $\mathcal{S}_2 = \{E, W, S, N, NW, SE\}$

$$q_1(-, -; n) = M_n, \quad q_2(-, -; n) = 2^n M_n.$$

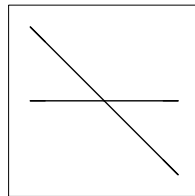


[Françon-Viennot 79]: bijection between $q_1(-, -; n)$ and Motzkin paths (via involutions)

Explain closed form expressions

- Many simple expressions for $\mathcal{S} = \{E, W, NW, SE\}$. For instance,

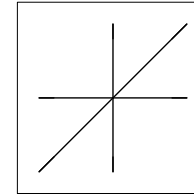
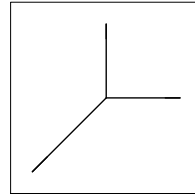
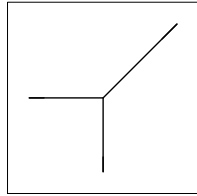
$$q(-, 0; 2m) = q(0, -; 2m) = \frac{(2m)!(2m+2)!}{m!(m+1)!^2(m+2)!}.$$



[Gouyou-Beauchamps 86]: combinatorial derivation of $q(-, 0; 2m)$

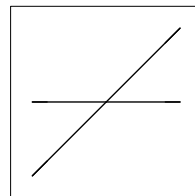
Explain algebraicity

- For the Kreweras' trilogy



[Bernardi 07] \simeq algebraicity for planar maps

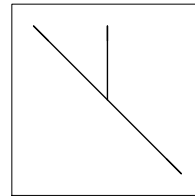
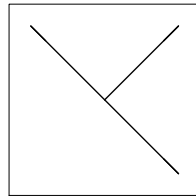
- For Gessel's walks...



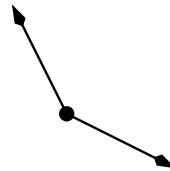
Prove non-D-finiteness

Conjecture. For all models with a finite group, $Q(x, y; t)$ is non-D-finite.

Proved in two cases [Mishna-Rechnitzer 07]: solution as an explicit infinite sum, and proof that the series has infinitely many singularities.



Also [mbm-Petkovšek 03] for knight walks ($\mathcal{S} = \{(-1, 2), (2, -1)\}$)



Construct differential equations

For the 19 models with a finite group and a non-zero orbit sum:

$$xyQ(x, y; t) = [x^{>0}y^{>0}] \frac{1}{K(x, y; t)} \sum_{g \in G} \text{sign}(g)g(xy),$$

a D-finite series.

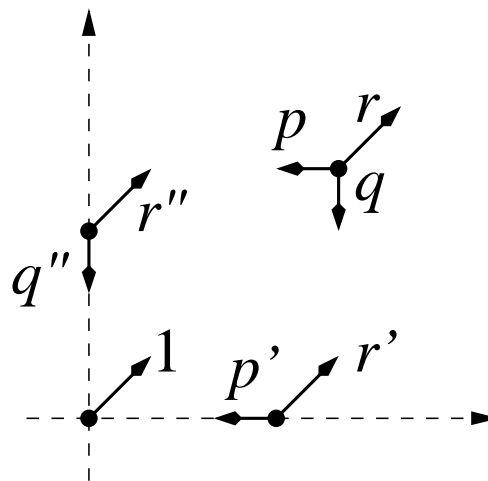
⇒ Work out (automatically?) a differential equation for $Q(0, 0; t)$ and/or $Q(1, 1; t)$

cf. [Bostan-Kauers 08]

Variations and extensions

- More parameters (e.g., contacts with the axes)
- Markov chains in the quarter plane [Fayolle et al. 99]: design new models with and algebraic stationary distribution?

$$\frac{p'}{r'} = \frac{p}{r}$$
$$\frac{q''}{r''} = \frac{q}{r}$$



[mbm 05]

- Higher dimension (cf. [Bostan-Kauers 08])