On self-avoiding walks

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Outline

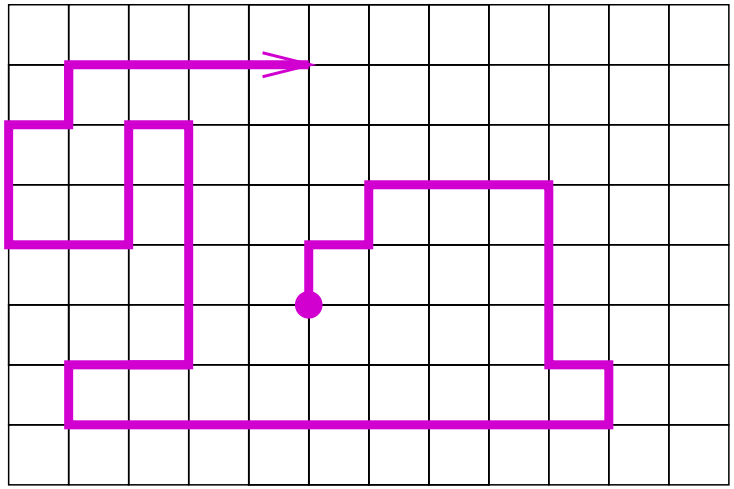
I. Self-avoiding walks (SAW): Generalities, predictions and results

II. Some exactly solvable models of SAW

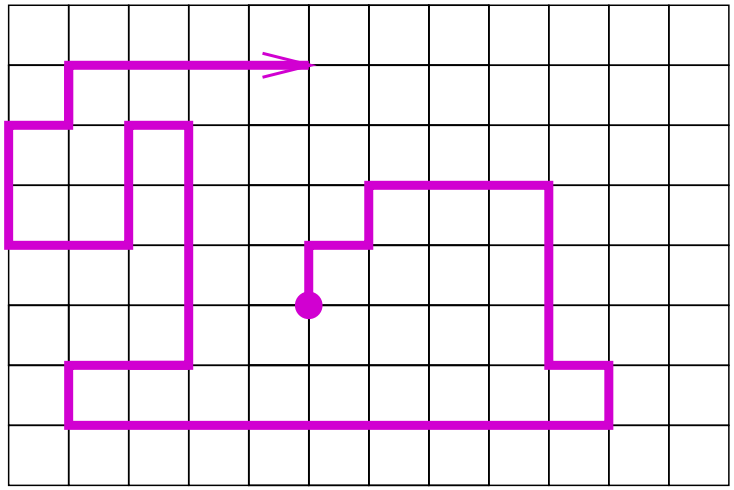
- II.0 A toy model: Partially directed walks
- II.1 Weakly directed walks
- II.2 Prudent walks
- II.3 Two related models

III. Au dessert : SAW on the honeycomb lattice,
d'après Duminil-Copin et Smirnov

I. Generalities



Self-avoiding walks (SAW)



What is $c(n)$, the number of n-step SAW?

$$
\begin{array}{lcll}
c(1) & = & 4 & \\
c(2) & = & c(1) \times 3 & = 12 \\
c(3) & = & c(2) \times 3 & = 36 \\
c(4) & = & c(3) \times 3 - 8 & = 100
\end{array}
$$

Not so easy! $c(n)$ is only known up to $n = 71$ [Jensen 04]

Problem: a highly non-markovian model

Some (old) conjectures/predictions

- The number of n-step SAW behaves asymptotically as follows:

$$c(n) \sim (\kappa)\, \mu^n\, n^\gamma$$

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- $\gamma = 11/32$ for all 2D lattices (square, triangular, honeycomb) [Nienhuis 82]

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where

- $\gamma = 11/32$ for all 2D lattices (square, triangular, honeycomb) [Nienhuis 82]

- $\mu = \sqrt{2+\sqrt{2}}$ on the honeycomb lattice [Nienhuis 82]
(proved this summer [Duminil-Copin & Smirnov])

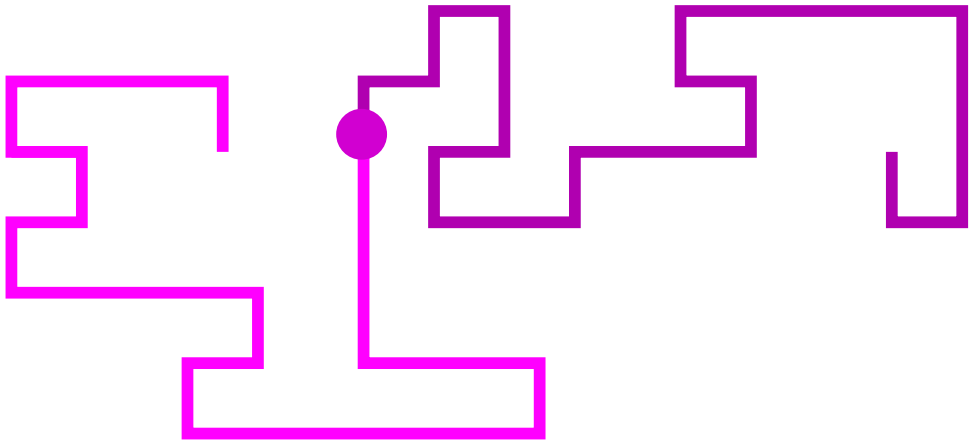
Some (old) conjectures/predictions

- The number of n-step SAW behaves asymptotically as follows:

$$c(n) \sim (\kappa)\, \mu^n\, n^\gamma$$

$\Rightarrow$ The probability that two n-step SAW starting from the same point do not intersect is

$$\frac{c(2n)}{c(n)^2} \sim n^{-\gamma}$$

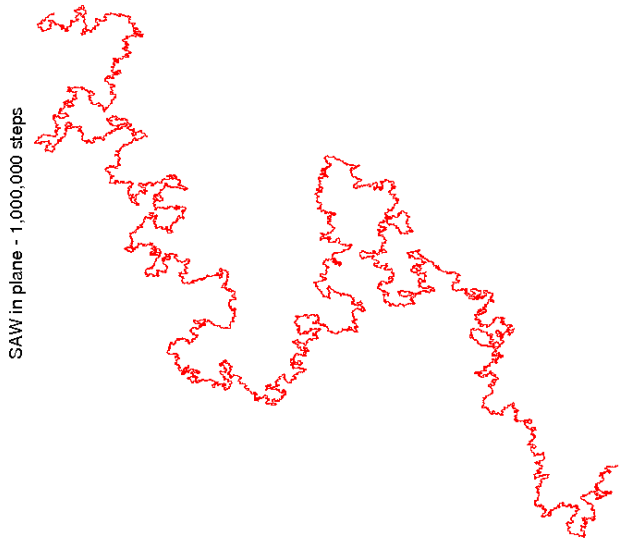


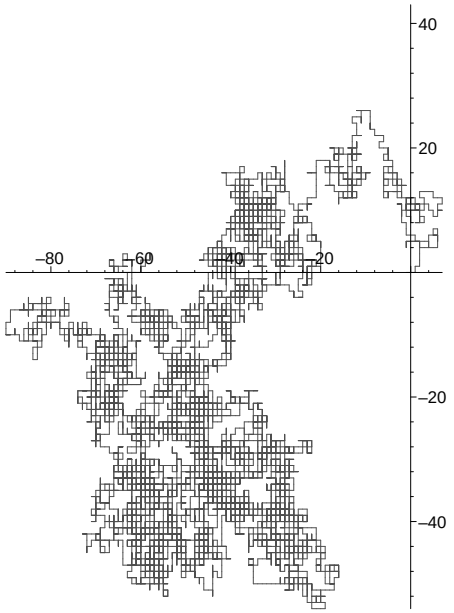
Some (old) conjectures/predictions

- The end-to-end distance is on average

$$\mathbb{E}(D_n) \sim n^{3/4}$$ (vs. $n^{1/2}$ for a simple random walk)

[Flory 49, Nienhuis 82]





Some (recent) conjectures/predictions

- Limit process: The scaling limit of SAW is $\mathrm{SLE}_{8/3}$.

(proved if the scaling limit of SAW exists and is conformally invariant [Lawler, Schramm, Werner 02])

This would imply

$$c(n) \sim \mu^n n^{11/32} \quad \text{and} \quad \mathbb{E}(D_n) \sim n^{3/4}$$

In 5 dimensions and above

- The critical exponents are those of the simple random walk:

$$c(n) \sim \mu^n n^0, \qquad \mathbb{E}(D_n) \sim n^{1/2}.$$

- The scaling limit exists and is the d-dimensional brownian motion

[Hara-Slade 92]

Proof: a mixture of combinatorics (the lace expansion) and analysis

II. Exactly solvable models

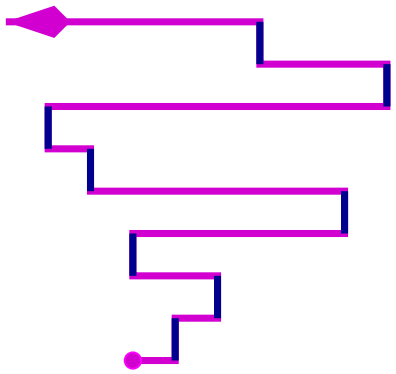
$\Rightarrow$ **Design simpler classes of SAW**, that should be natural, as general as possible... but still tractable

- solve better and better approximations of real SAW
- develop new techniques in exact enumeration

II.0. A toy model: Partially directed walks

Definition: A walk is partially directed if it avoids (at least) one of the 4 steps N, S, E, W.

Example: A NEW-walk is partially directed



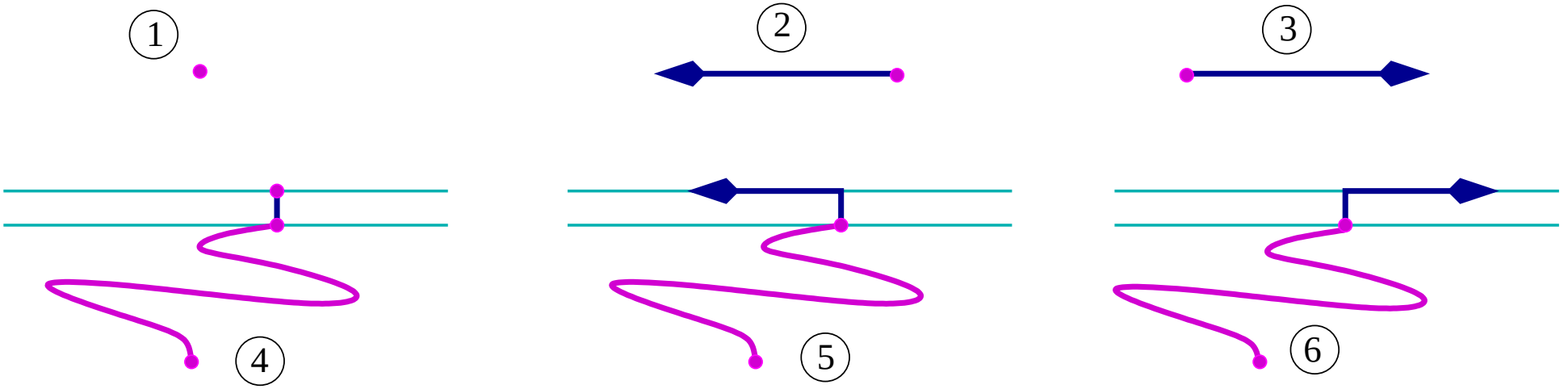
"Markovian with memory 1"

The self-avoidance condition is local.

Let $a(n)$ be the number of n-step NEW-walks.

A toy model: Partially directed walks

- Recursive description of NEW-walks:



$$\begin{aligned} a(0) &= 1 \\ a(n) &= 2 + a(n-1) + 2\sum_{k=0}^{n-2} a(k) \quad \text{for } n \geq 1 \end{aligned}$$

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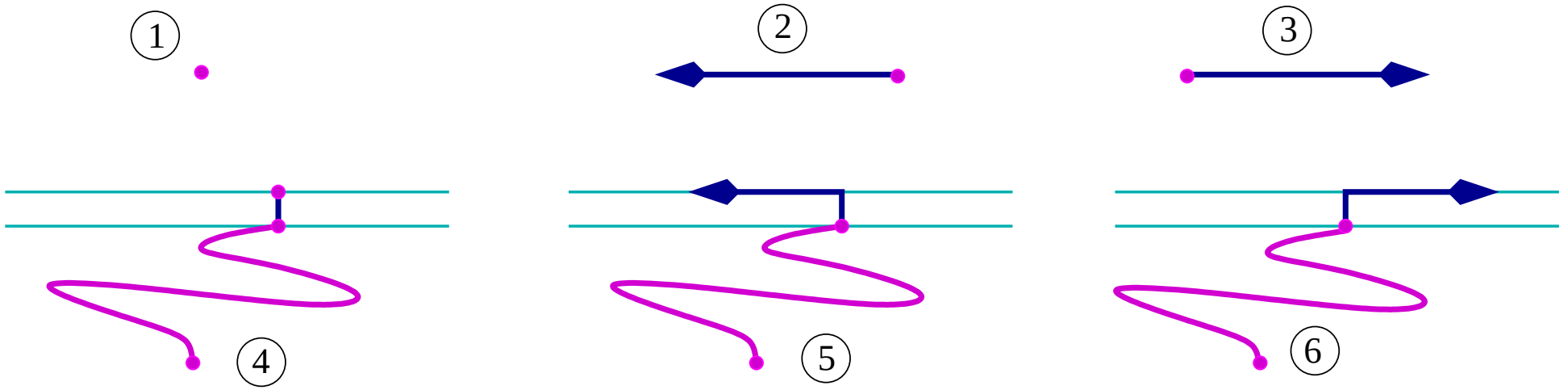

$$\begin{aligned} a(0) &= 1 \\ a(n) &= 2 + a(n-1) + 2\sum_{k=0}^{n-2} a(k) \quad \text{for } n \geq 1 \end{aligned}$$

- Generating function:

$$A(t) := \sum_{n\geq 0} a(n)t^n = 1 + 2\frac{t}{1-t} + tA(t) + 2A(t)\frac{t^2}{1-t}$$

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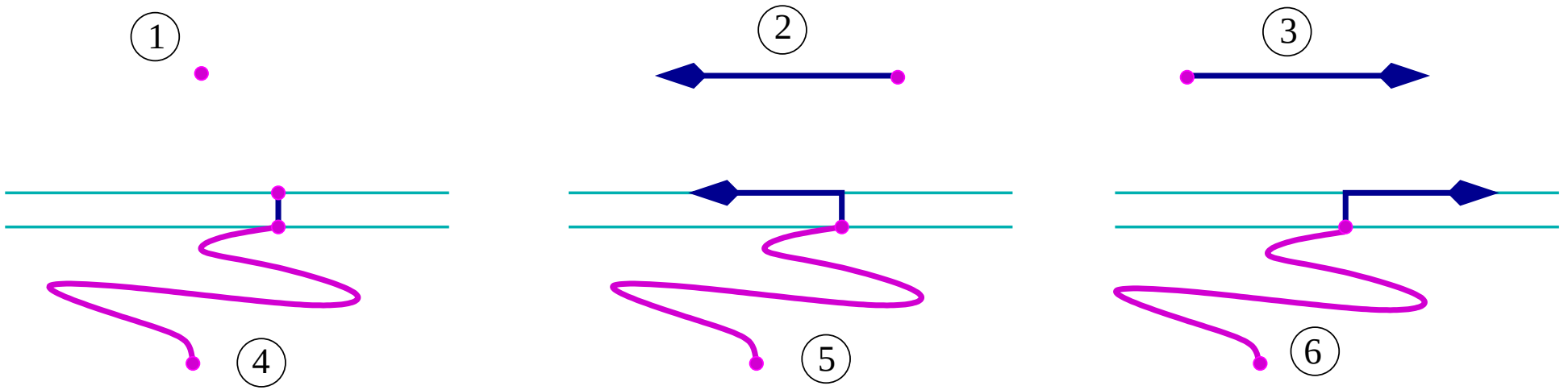

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$$A(t) = \frac{1+t}{1-2t-t^2} \quad \Rightarrow \quad a(n) \sim (1+\sqrt{2})^n \sim (2.41\ldots)^n$$

Generating functions

Let $\mathcal{A}$ be a set of discrete objects equipped with a size:

$$\begin{array}{rccc} size : \mathcal{A} & \to & \mathbb{N} \\ a & \mapsto & |a| \end{array}$$

Assume that for all n,

$$\mathcal{A}_n := \{a \in \mathcal{A} : |a| = n\} \text{ is finite.}$$

Let $a(n) = |\mathcal{A}_n|$.

The generating function of the objects of $\mathcal{A}$, counted by their size, is

$$\begin{aligned} A(t) &:= \sum_{n \geq 0} a(n) t^n \\ &= \sum_{a \in \mathcal{A}} t^{|a|}. \end{aligned}$$

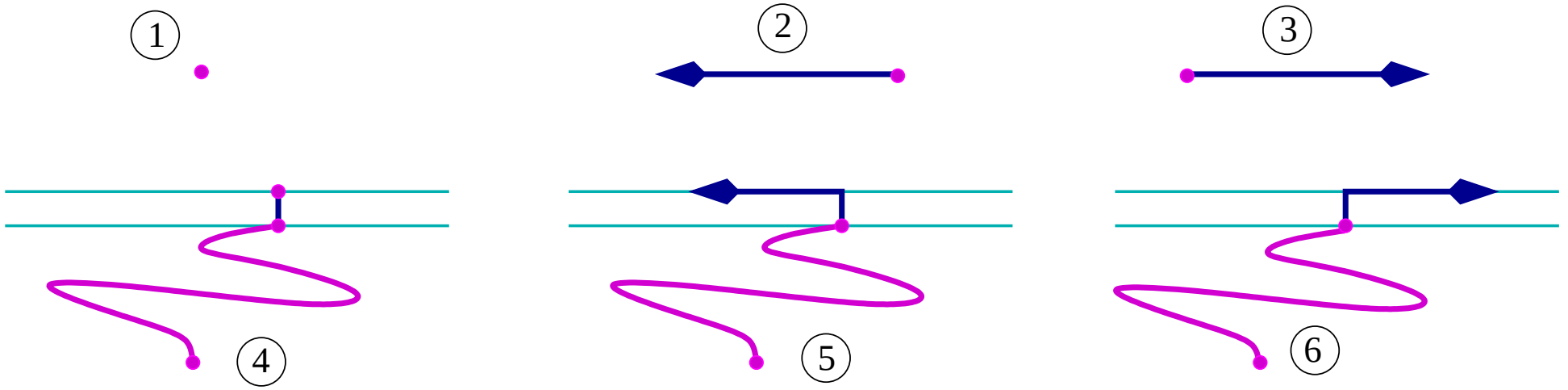
Notation: $[t^n]A(t) := a(n)$

Combinatorial constructions and operations on series: A dictionary

Construction	Numbers	Generating function
Union $\mathcal{A} = \mathcal{B} \cup \mathcal{C}$	$a(n) = b(n) + c(n)$	$A(t) = B(t) + C(t)$
Product $\mathcal{A} = \mathcal{B} \times \mathcal{C}$ $\lvert(\beta, \gamma)\rvert = \lvert\beta\rvert + \lvert\gamma\rvert$	$a(n) = b(0)c(n) + \cdots + b(n)c(0)$	$A(t) = B(t) \cdot C(t)$
Sequence $\mathcal{A} = \mathfrak{S}(\mathcal{B})$ $\mathcal{A} = \{\epsilon\} \cup \mathcal{B} \cup \mathcal{B} \times \mathcal{B} \cup \cdots$		$A(t) = \frac{1}{1-B(t)}$

Partially directed walks revisited

$$A(t) = 1 + 2\frac{t}{1-t} + tA(t) + 2A(t)\frac{t^2}{1-t}$$



A hierarchy of formal power series

- The formal power series $A(t)$ is rational if it can be written

$$A(t) = \frac{P(t)}{Q(t)}$$

where $P(t)$ and $Q(t)$ are polynomials in t.

- The formal power series $A(t)$ is algebraic (over $\mathbb{Q}(t)$) if it satisfies a (non-trivial) polynomial equation:

$$P(t, A(t)) = 0.$$

- The formal power series $A(t)$ is D-finite if it satisfies a (non-trivial) linear differential equation with polynomial coefficients:

$$P_0(t)A^{(k)}(t) + P_1(t)A^{(k-1)}(t) + \cdots + P_k(t)A(t) = 0.$$

$$\text{Rat} \subset \text{Alg} \subset \text{D-finite}$$

Some charms of this hierarchy...

- Closure properties ($+, \times$, derivatives, composition...)

- The series are "easy" to handle (partial fraction decomposition, Puiseux expansions, elimination, resultants, Gröbner bases...)

- A differential (resp. algebraic) equation can be guessed from the first coefficients (GFUN)

- The coefficients can be computed in a linear number of operations.

- (Almost) automatic asymptotics of the coefficients.

- May give insight on the (hidden) structure of the objects – or prove that they are inherently complex

What to do with a generating function?

- Extract the nth coefficient $a(n)$ (when nice...)

- The asymptotic behaviour of $a(n)$ can often be derived from the behaviour of $A(t)$ (seen as a function of a complex variable) in the neighborhood of its dominant singularities.

Example: $\limsup a(n)^{1/n} = \mu \iff A(t)$ has radius $1/\mu$

Transfer theorems: under certain hypotheses, if $A(t)$ has a unique dominant singularity at $1/\mu$,

$$A(t) \sim \frac{1}{(1-\mu t)^{1+\alpha}} \Longrightarrow a(n) \sim \frac{1}{\Gamma(\alpha+1)} \mu^n n^\alpha$$

Analytic combinatorics [Flajolet-Sedgewick 09]

Example: $A(t) = \frac{1+t}{1-2t-t^2}$ has a simple pole ($\alpha = 0$) at $t_c = \sqrt{2}-1$
$\Longrightarrow a(n) \sim \kappa(\sqrt{2}+1)^n n^0$

Multivariate generating functions

- Enumeration according to the size (main parameter) and another parameter:

$$A(t,x) = \sum_{a \in \mathcal{A}} t^{|a|} x^{p(a)}$$

- Then

$$[t^n] \left. \frac{\partial^i A}{\partial x^i}(t,x) \right|_{x=1} = \sum_{a:|a|=n} p(a)(p(a)-1)\cdots(p(a)-i+1)$$

$$\Rightarrow \quad \frac{[t^n]\frac{\partial^i A}{\partial x^i} A(t,1)}{[t^n]A(t,1)} = \mathbb{E}_n(p(p-1)\cdots(p-i+1))$$

is the ith factorial moment of $p(a)$, when a is taken uniformly among objects of size n.

- Asymptotic behaviour of the coefficients $\Rightarrow$ Asymptotic behaviour of the moments

Analytic combinatorics [Flajolet-Sedgewick 09]

Multivariate generating functions

- Enumeration according to the size (main parameter) and another parameter:

$$A(t,x) = \sum_{a \in \mathcal{A}} t^{|a|} x^{p(a)}$$

- Then

$$\frac{[t^n]A(t,x)}{[t^n]A(t,1)} = \frac{\sum_{a:|a|=n} x^{p(a)}}{\sum_{a:|a|=n} 1}$$

is the probability generating function of $p(a)$, when a is taken uniformly among objects of size n.

- Combined with:

– asymptotic behaviour of $[t^n]A(t,x)$ (for $x \in \mathbb{R}$)

– continuity theorems for (Laplace or Fourier) transforms,

this often yields a limit law for the additional parameter p.

Analytic combinatorics [Flajolet-Sedgewick 09]

Example: the number of North steps in a partially directed walk

- The bivariate generating function is

$$A(t,x) = \sum_{\omega} t^{|\omega|} x^{\mathsf{N}(\omega)} = \frac{1+t}{1-t-tx(1+t)}$$

- For x in a neighborhood of 1, there holds uniformly

$$\frac{[t^n]A(t,x)}{[t^n]A(t,1)} \sim \kappa(x) \left(\frac{1+x+\sqrt{1+6x+x^2}}{2(1+\sqrt{2})} \right)^n (1+O(1/n))$$

- By the quasi-power Theorem, the number N_n of North steps satisfies

$$\mathbb{E}(N_n) \sim \mathfrak{m}\, n, \qquad \mathbb{V}(N_n) \sim \mathfrak{s}^2\, n,$$

for some $\mathfrak{m}, \mathfrak{s}^2 > 0$, and the normalized random variable $\frac{N_n - \mathfrak{m}\, n}{\mathfrak{s}\sqrt{n}}$ converges in law to a standard normal distribution.

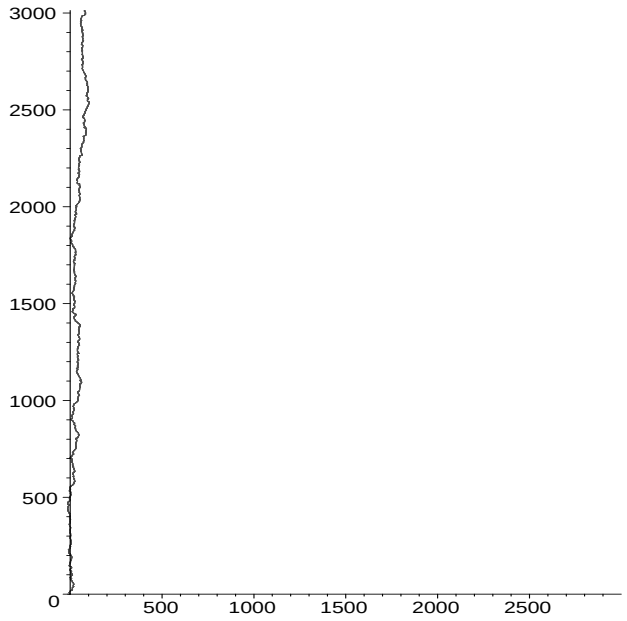
+ Similar result for the abscissa of the endpoint (with mean 0).

Random partially directed walks

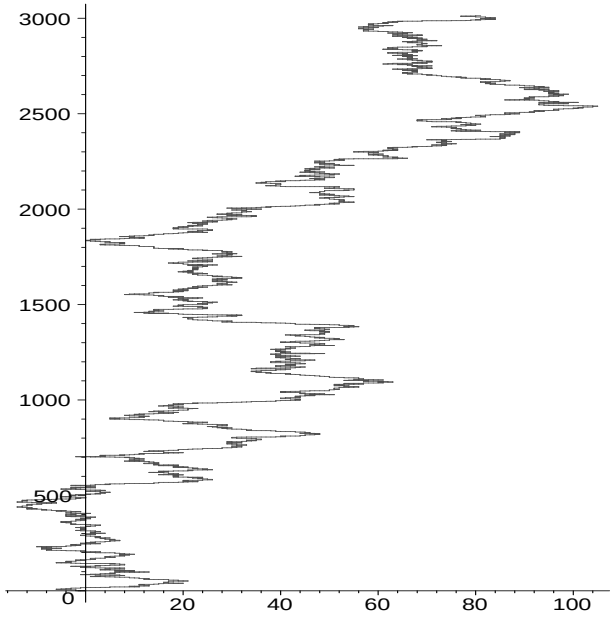
- Asymptotic properties: coordinates of the endpoint

$$\mathbb{E}(X_n) = 0, \quad \mathbb{E}(X_n^2) \sim n, \qquad \mathbb{E}(Y_n) \sim n$$

- Random NEW-walks:



Scaled by n (– and |)



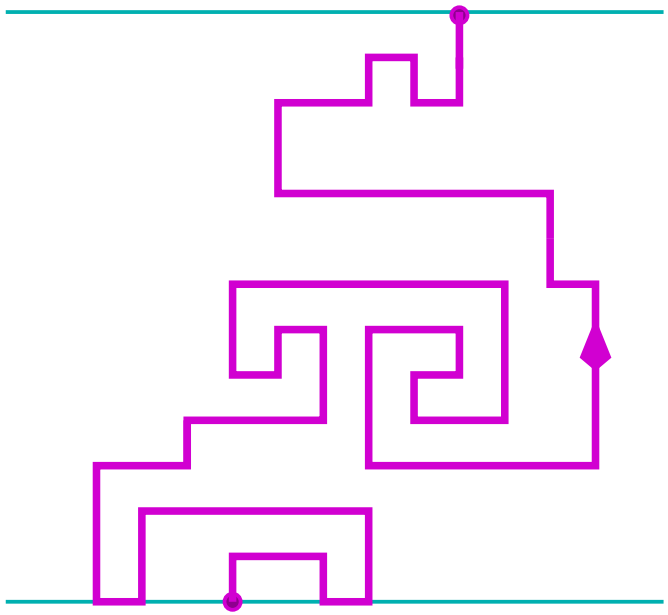
Scaled by $\sqrt{n}$ (–) and n (|)

II.1. Weakly directed walks

(joint work with Axel Bacher)

Bridges

• A walk with vertices $v_0, \ldots, v_i, \ldots, v_n$ is a bridge if the ordinates of its vertices satisfy $y_0 \leq y_i < y_n$ for $1 \leq i \leq n$.



• There are many bridges:

$$b(n) \sim \mu_{bridge}^{n} n^{\gamma'}$$

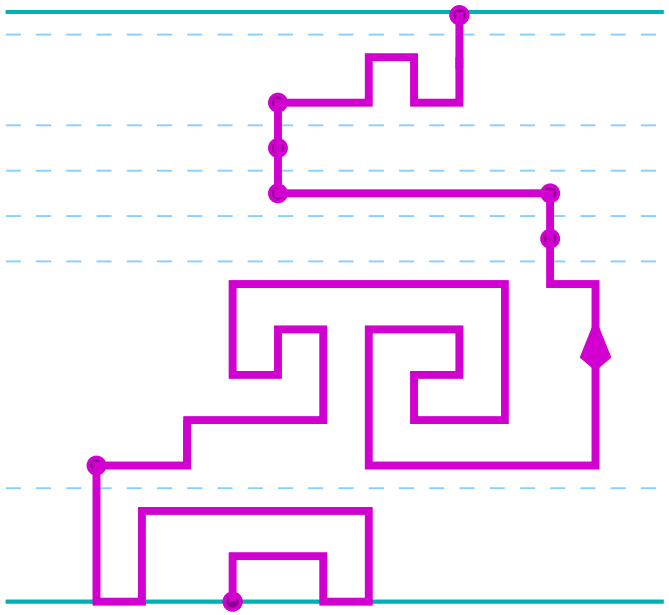
where

$$\mu_{bridge} = \mu_{SAW}$$

Irreducible bridges

Def. A bridge is irreducible if it is not the concatenation of two bridges.

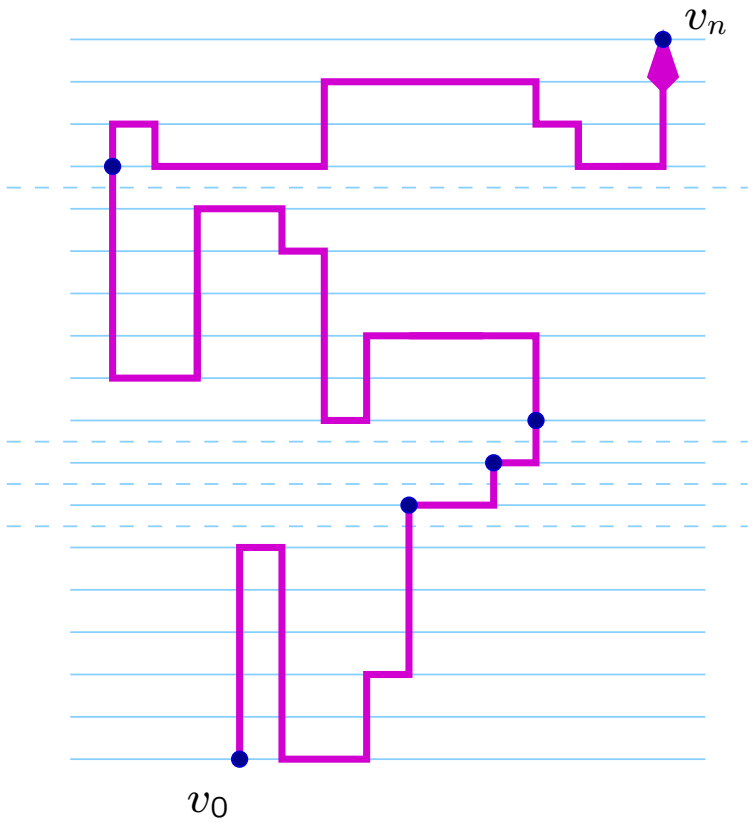
Observation: A bridge is a sequence of irreducible bridges



Weakly directed bridges

Definition: a bridge is weakly directed if each of its irreducible bridges avoids at least one of the steps N, S, E, W.

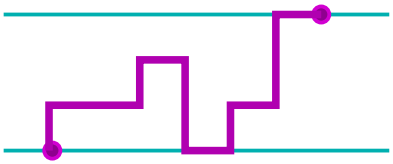
This means that each irreducible bridge is a NES- or a NWS-walk.



$\Rightarrow$ Count NES- (irreducible) bridges

Enumeration of NES-bridges

Proposition



- The generating function of NES-bridges of height $k+1$ is

$$B^{(k+1)}(t) = \sum_n b_n^{(k+1)} t^n = \frac{t^{k+1}}{G_k(t)},$$

where $G_{-1} = 1$, $G_0 = 1 - t$, and for $k \geq 0$,

$$G_{k+1} = (1 - t + t^2 + t^3)G_k - t^2 G_{k-1}.$$

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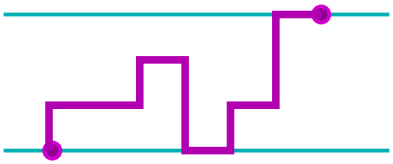

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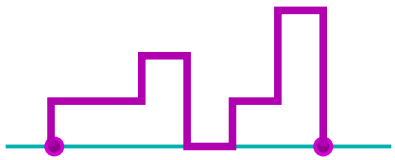
- The generating function of NES-excursions of height at most k is

$$E^{(k)}(t) = \frac{1}{t}\left(\frac{G_{k-1}}{G_k} - 1\right).$$

Excursion: $y_0 = 0 = y_n$ and $y_i \geq 0$ for $1 \leq i \leq n$.



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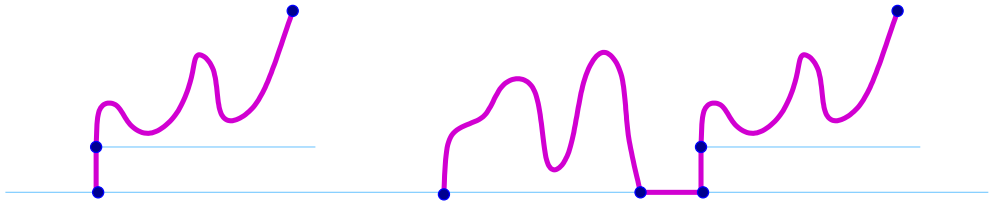
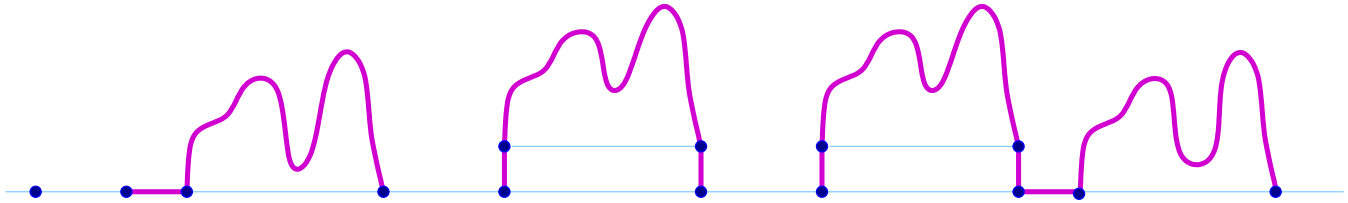


Last return to height 0



First return to height 0

- Bridges of height $k+1$:

$$B^{(k+1)} = tB^{(k)} + E^{(k)}t^2B^{(k)}$$

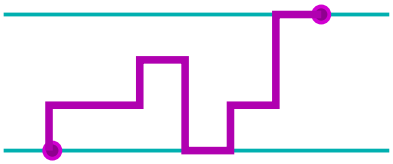
- Excursions of height at most k

$$E^{(k)} = 1 + tE^{(k)} + t^2\left(E^{(k-1)} - 1\right) + t^3\left(E^{(k-1)} - 1\right)E^{(k)}$$

- Initial conditions: $E^{(-1)} = 1$, $B^{(1)} = t/(1-t)$.

Enumeration of NES-bridges

Proposition



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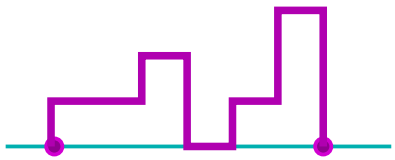

Enumeration of weakly directed bridges

- GF of NES-bridges:

$$B(t) = \sum_{k \geq 0} \frac{t^{k+1}}{G_k}$$

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- GF of irreducible NES-bridges:

$$B(t) = \frac{I(t)}{1 - I(t)} \Rightarrow I(t) = \frac{B(t)}{1 + B(t)}$$

Enumeration of weakly directed bridges

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- GF of irreducible NES-bridges:

$$B(t) = \frac{I(t)}{1 - I(t)} \Rightarrow I(t) = \frac{B(t)}{1 + B(t)}$$

- GF of weakly directed bridges (sequences of irreducible NES- or NWS-bridges):

$$W(t) = \frac{1}{1 - (2I(t) - t)} = \frac{1}{1 - \left(\frac{2B(t)}{1+B(t)} - t\right)}$$

with $G_{-1} = 1$, $G_0 = 1 - t$, and for $k \geq 0$,

$$G_{k+1} = (1 - t + t^2 + t^3)G_k - t^2 G_{k-1}.$$

[Bacher-mbm 10]

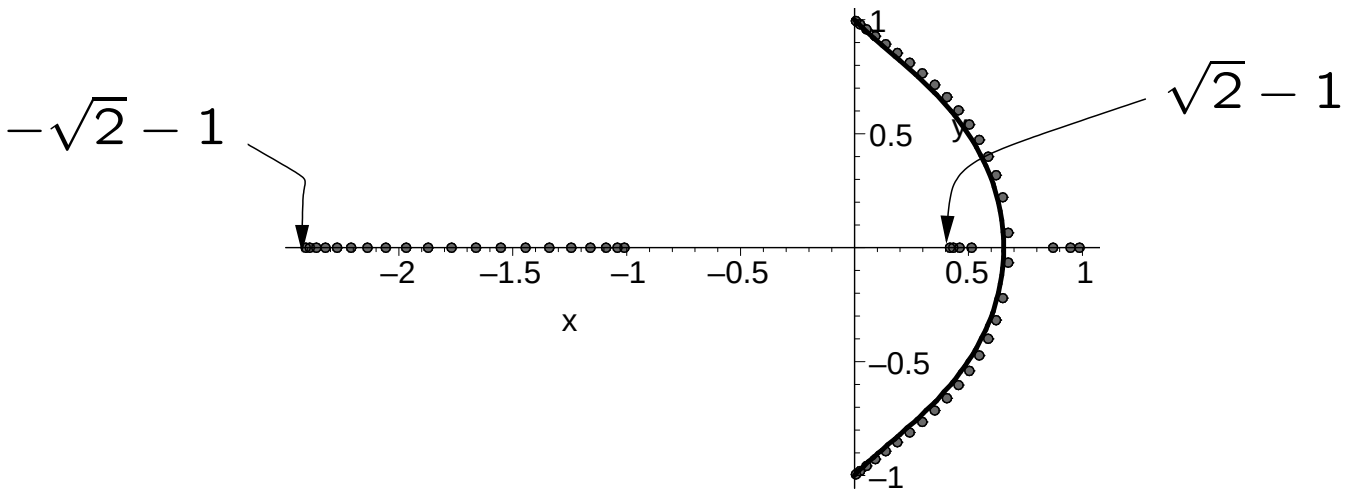
Asymptotic results and nature of the generating functions

$$B(t) = \sum_{k\geq 0} \frac{t^{k+1}}{G_k}, \qquad W(t) = \frac{1}{1 - \left(\frac{2B(t)}{1+B(t)} - t\right)}$$

with $G_{-1} = 1$, $G_0 = 1 - t$, and for $k \geq 0$,

$$G_{k+1} = (1 - t + t^2 + t^3)G_k - t^2 G_{k-1}.$$

The zeroes of G_k (here, $k = 20$):



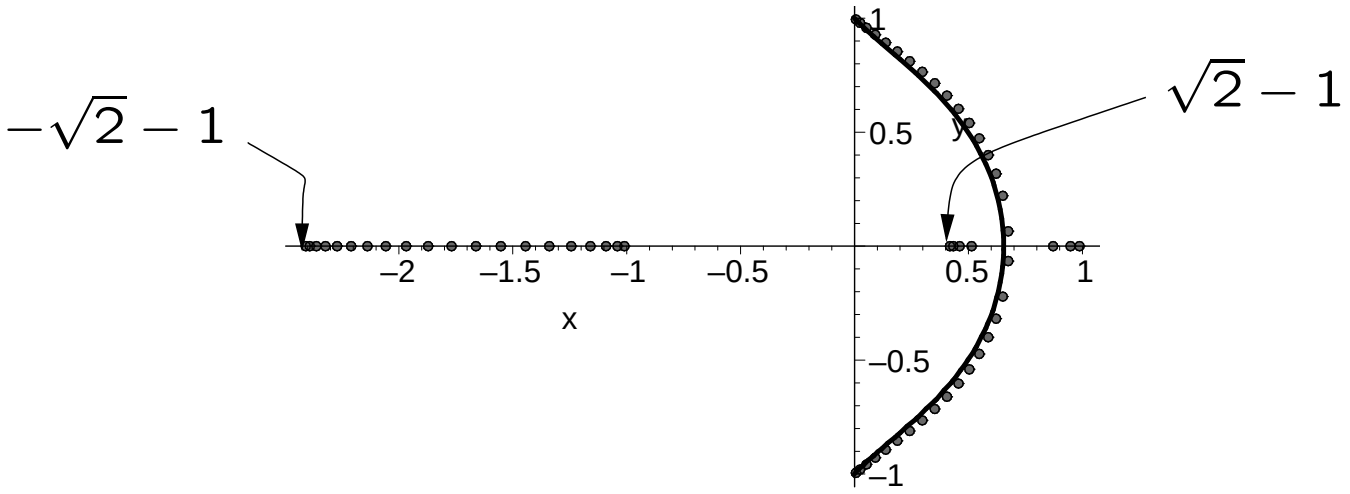
Asymptotic results and nature of the generating functions

$$B(t) = \sum_{k \geq 0} \frac{t^{k+1}}{G_k}, \qquad W(t) = \frac{1}{1 - \left(\frac{2B(t)}{1+B(t)} - t\right)}$$

• The series $B(t)$ and $W(t)$ are meromorphic in $\mathbb{C} \setminus \mathcal{E}$, where $\mathcal{E}$ consists of the two real intervals $[-\sqrt{2}-1, -1]$ and $[\sqrt{2}-1, 1]$, and of the curve

$$\mathcal{E}_0 = \left\{ x + iy : x \geq 0,\ y^2 = \frac{1 - x^2 - 2\,x^3}{1 + 2\,x} \right\}.$$

This curve is a natural boundary of B and W. These series thus have infinitely many singularities.



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• The series $B(t)$ has radius $\sqrt{2}-1$, while $W(t)$ has a simple pole ρ of smaller modulus (for which $1 = \frac{2B(\rho)}{1+B(\rho)} - \rho$).

• The number $w(n)$ of weakly directed bridges of length n satisfies

$$w(n) \sim \mu^n,$$

with $\mu \simeq 2.54$ (the current record).

The number of irreducible bridges

• The generating function of xeakly directed bridges, counted by the length and the number of irreducible bridges, is

$$W(t,x) = \frac{1}{1 - x\left(\frac{2B(t)}{1+B(t)} - t\right)}$$

• Let N_n denote the number N_n of irreducible bridges in a random weakly directed bridge of length n. Then

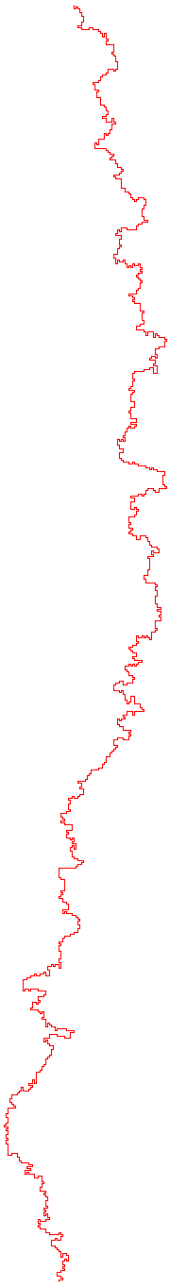
$$\mathbb{E}(N_n) \sim \mathfrak{m}\, n, \qquad \mathbb{V}(N_n) \sim \mathfrak{s}^2\, n,$$

where

$$\mathfrak{m} \simeq 0.318 \quad \text{and} \quad \mathfrak{s}^2 \simeq 0.7,$$

and the random variable $\frac{N_n - \mathfrak{m}\, n}{\mathfrak{s}\sqrt{n}}$ converges in law to a standard normal distribution. In particular, the average end-to-end distance, being bounded from below by $\mathbb{E}(N_n)$, grows linearly with n.

A random weakly directed bridge



II. 2. Prudent self-avoiding walks

Self-directed walks [Turban-Debierre 86]

Exterior walks **[Préa 97]**

Outwardly directed SAW [Santra-Seitz-Klein 01]

Prudent walks **[Duchi 05], [Dethridge, Guttmann, Jensen 07], [mbm 08]**

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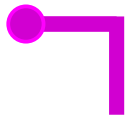

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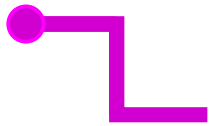

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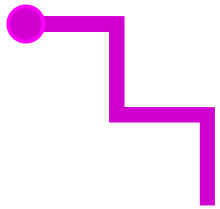

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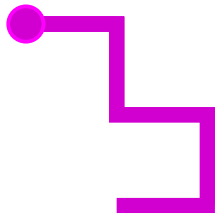

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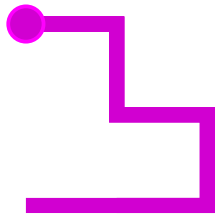

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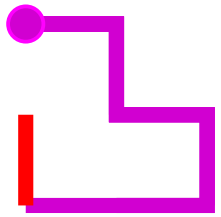

not prudent!

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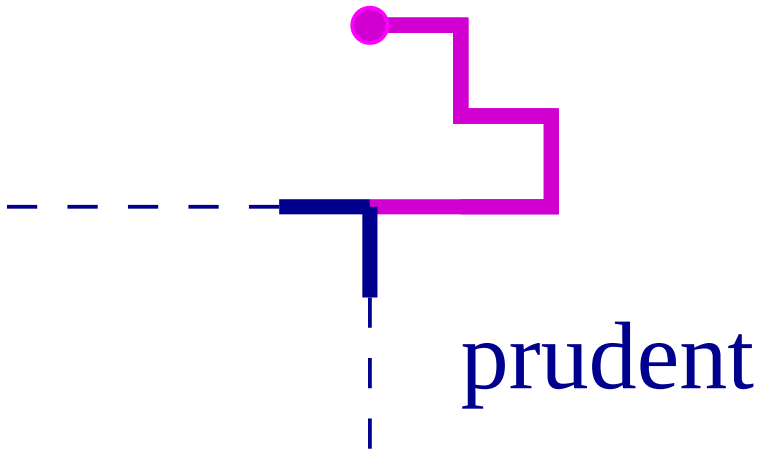

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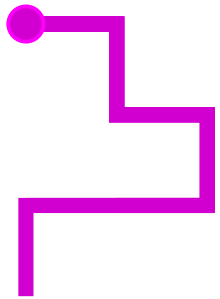

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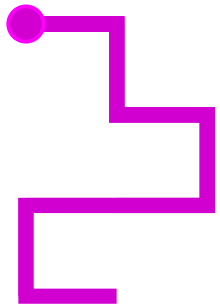

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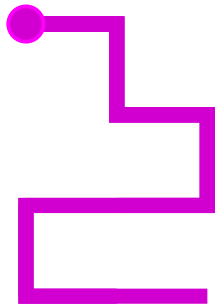

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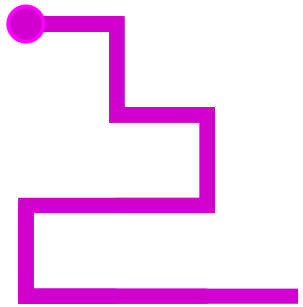

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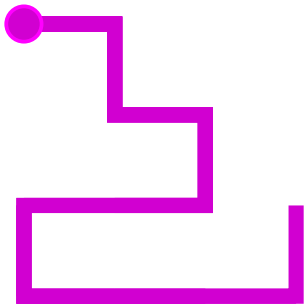

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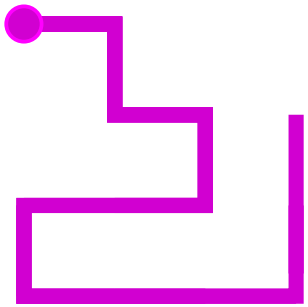

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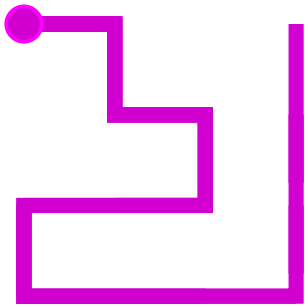

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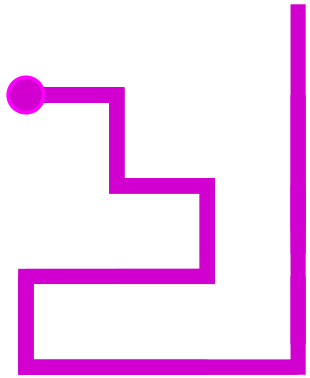

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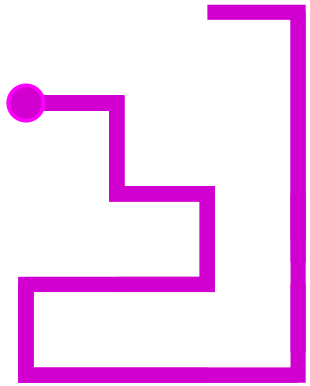

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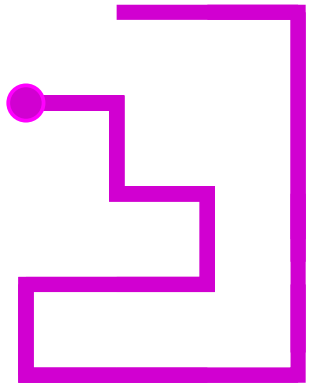

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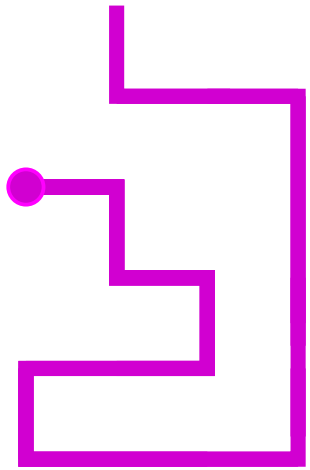

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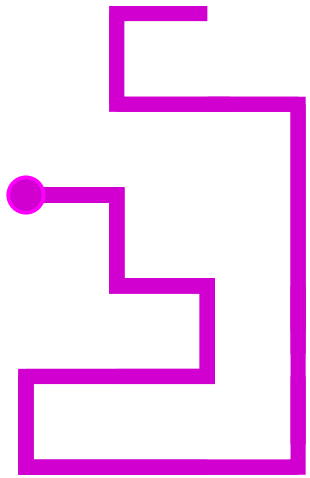

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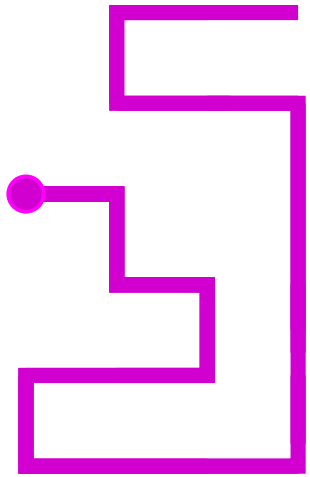

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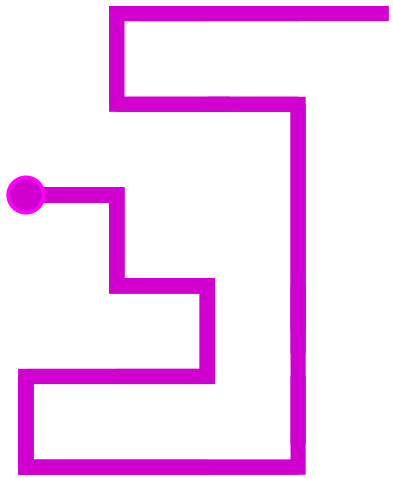

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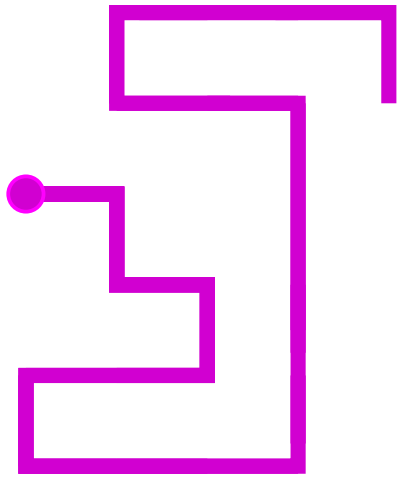

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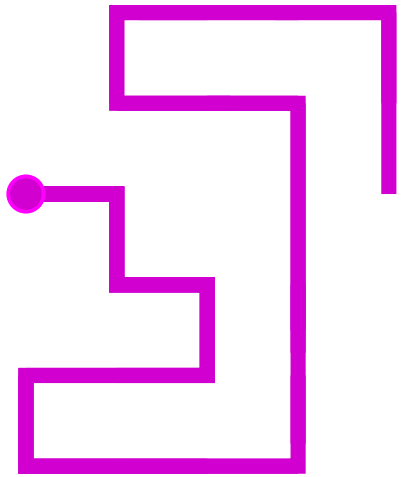

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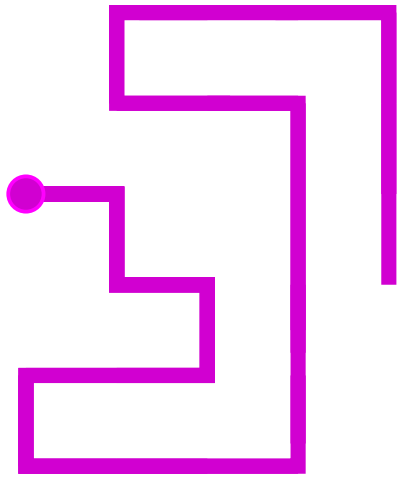

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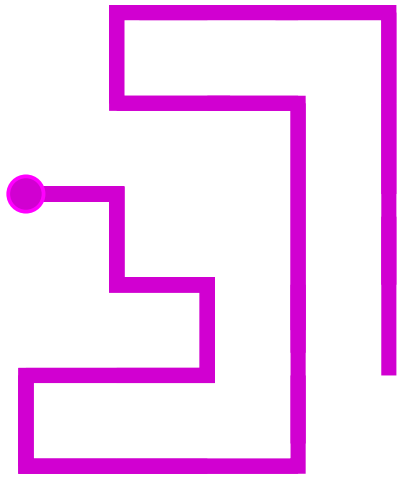

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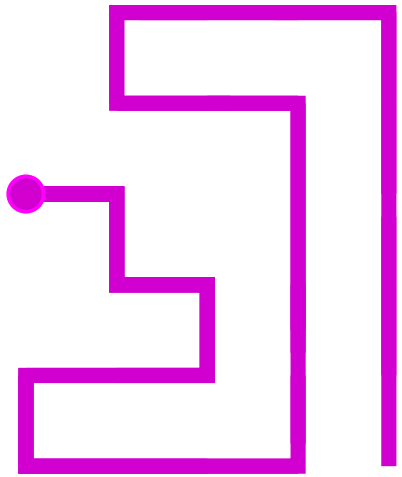

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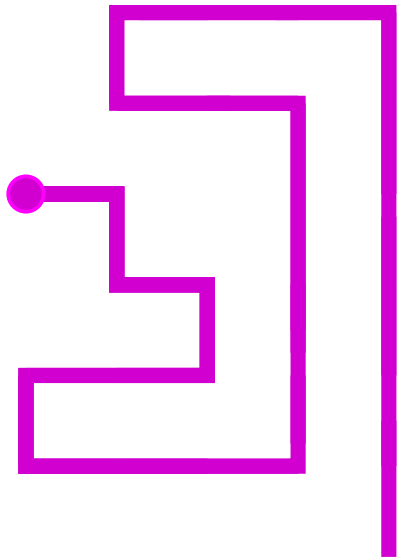

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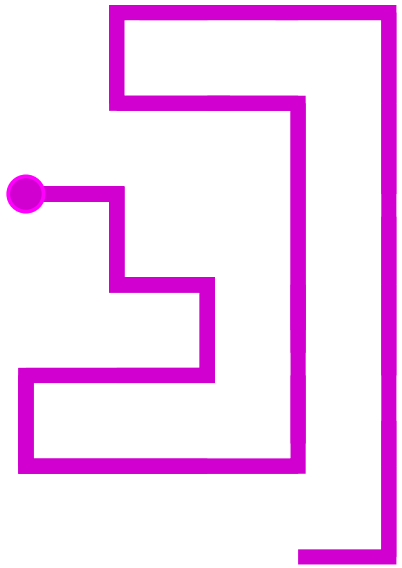

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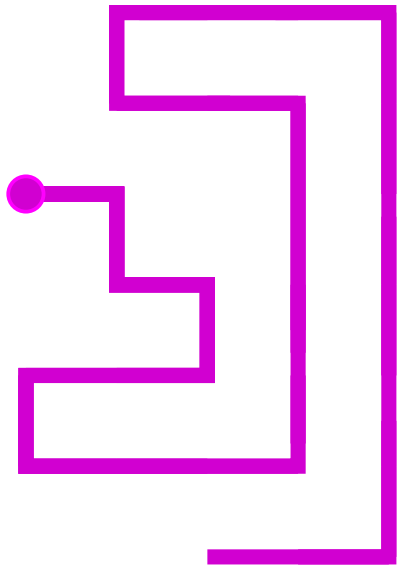

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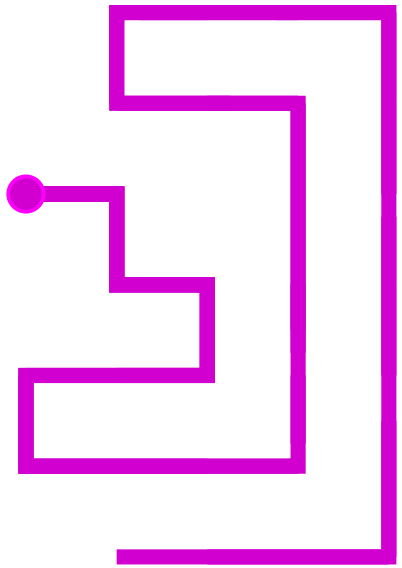

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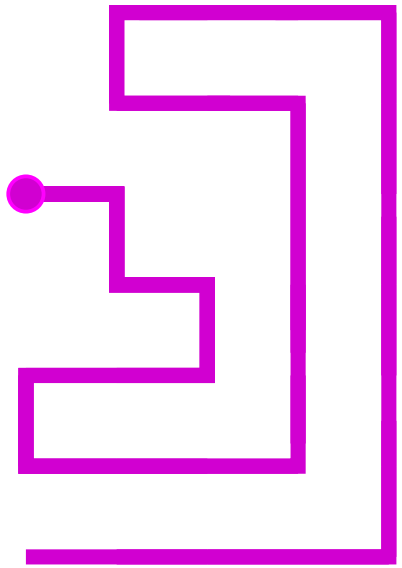

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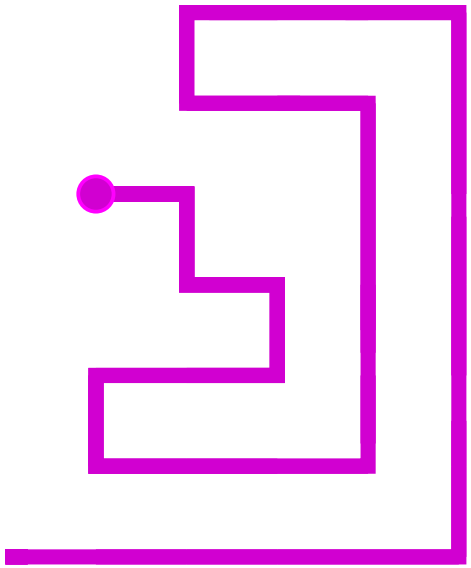

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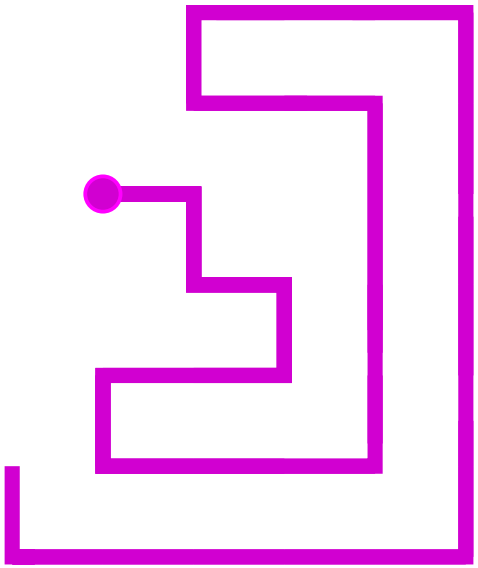

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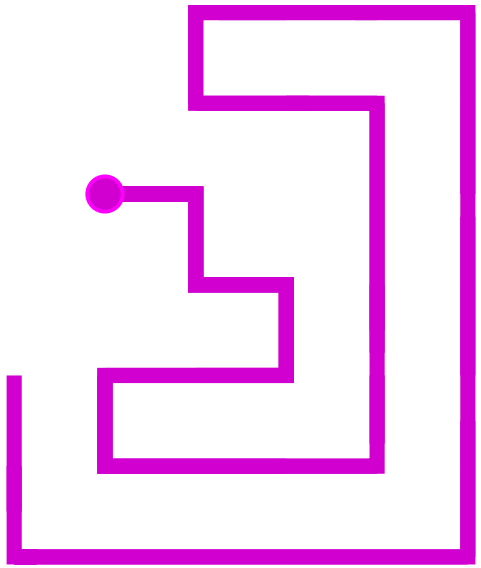

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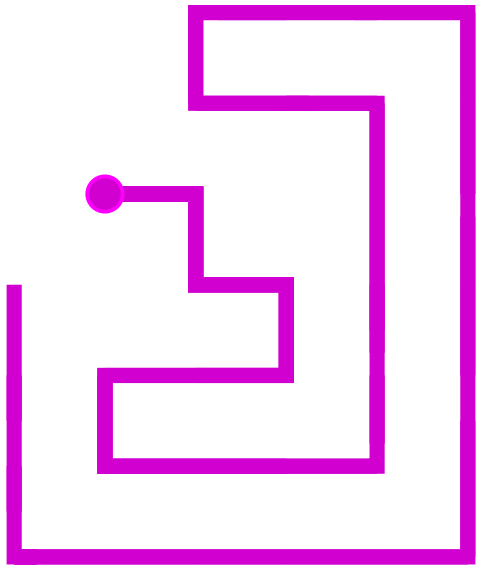

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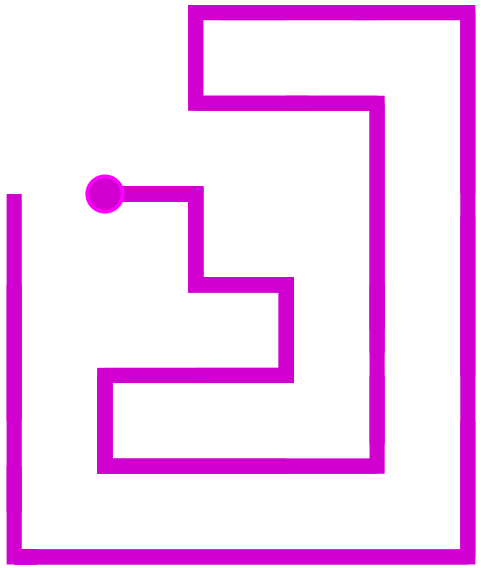

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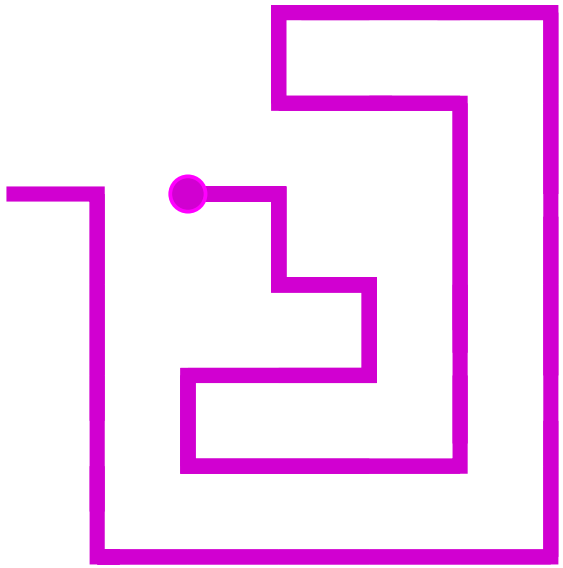

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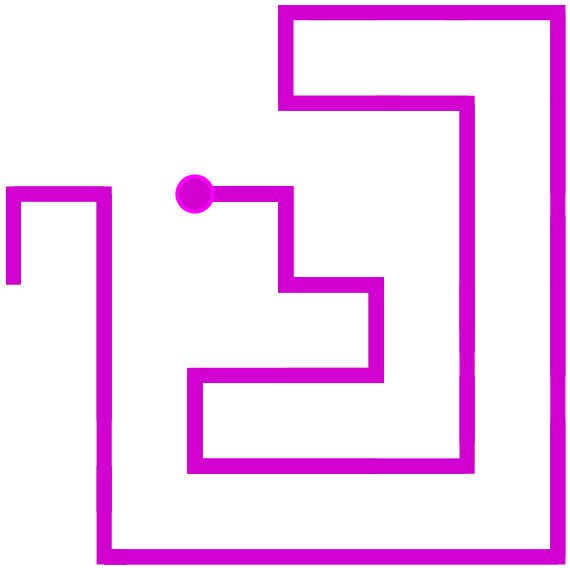

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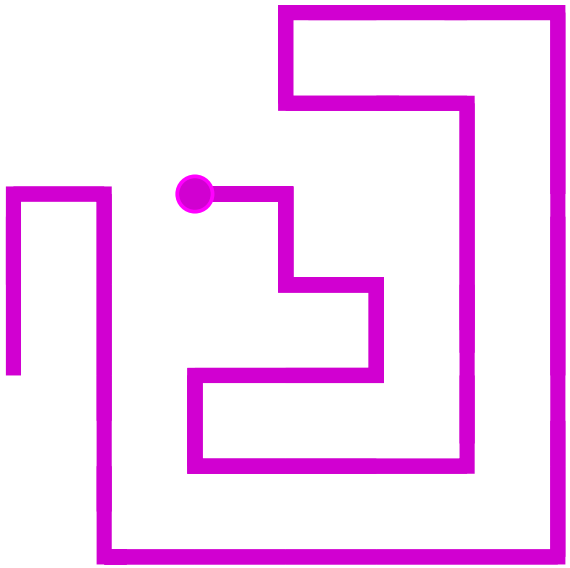

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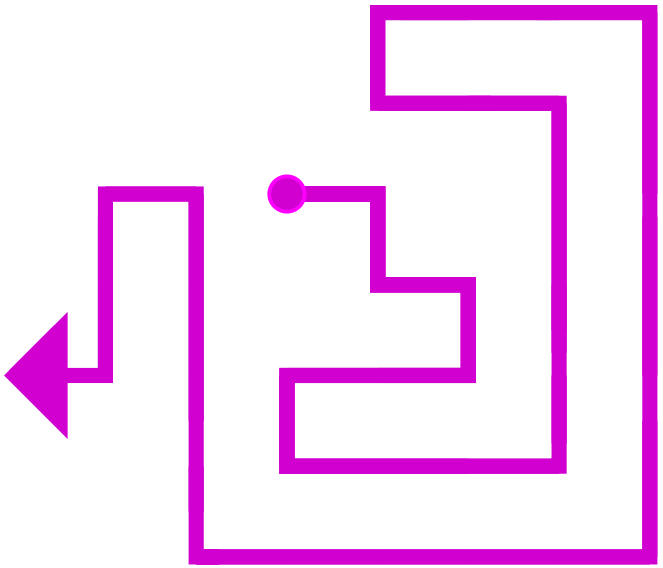
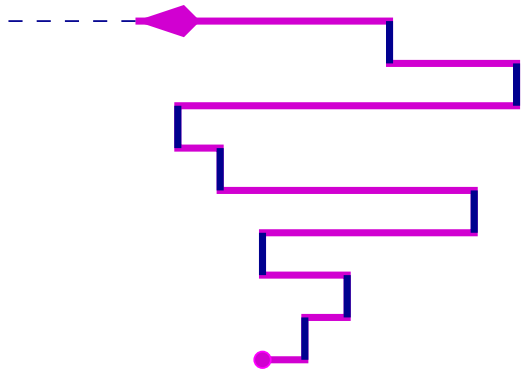
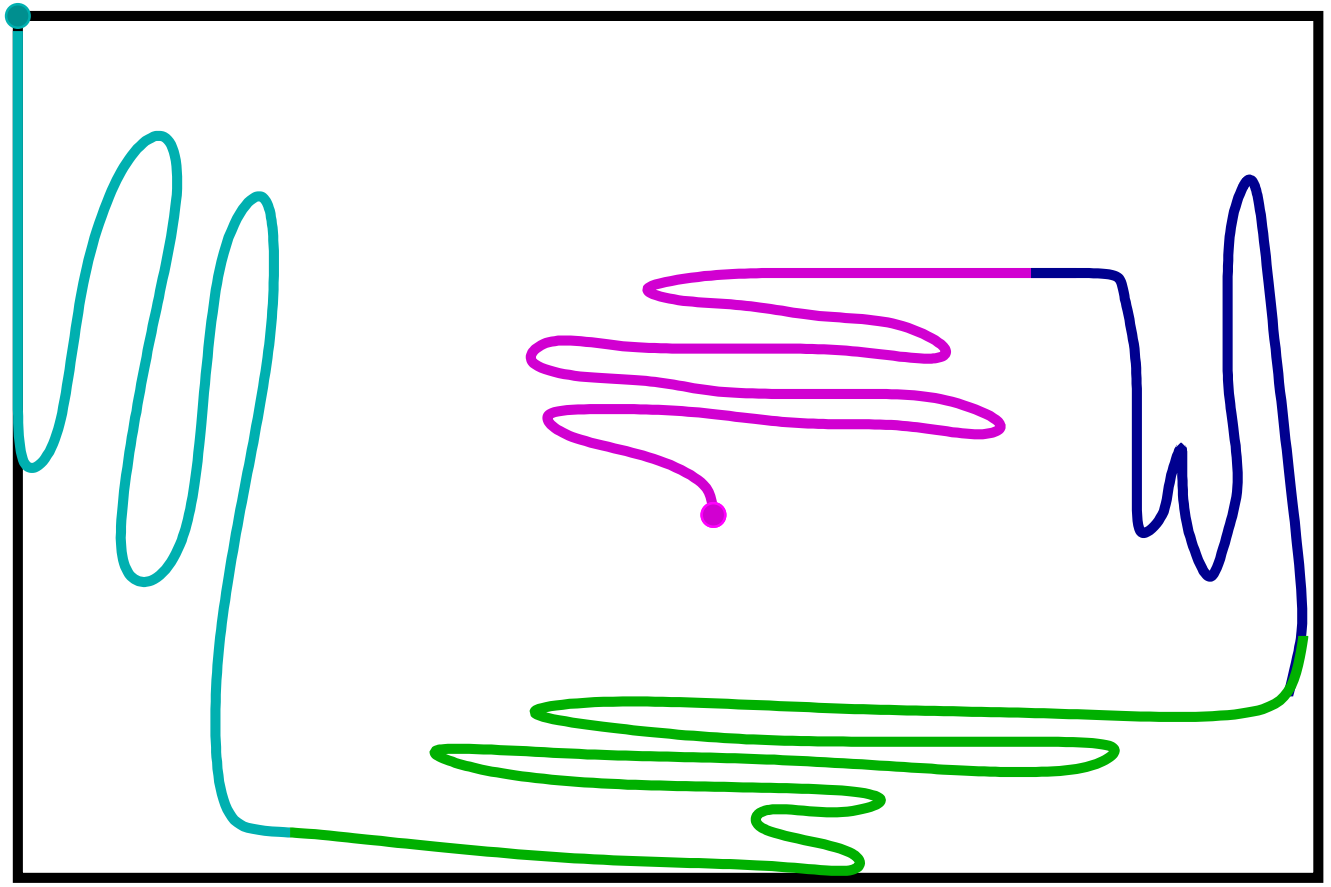


Remark: Partially directed walks are prudent



Some properties of prudent walks

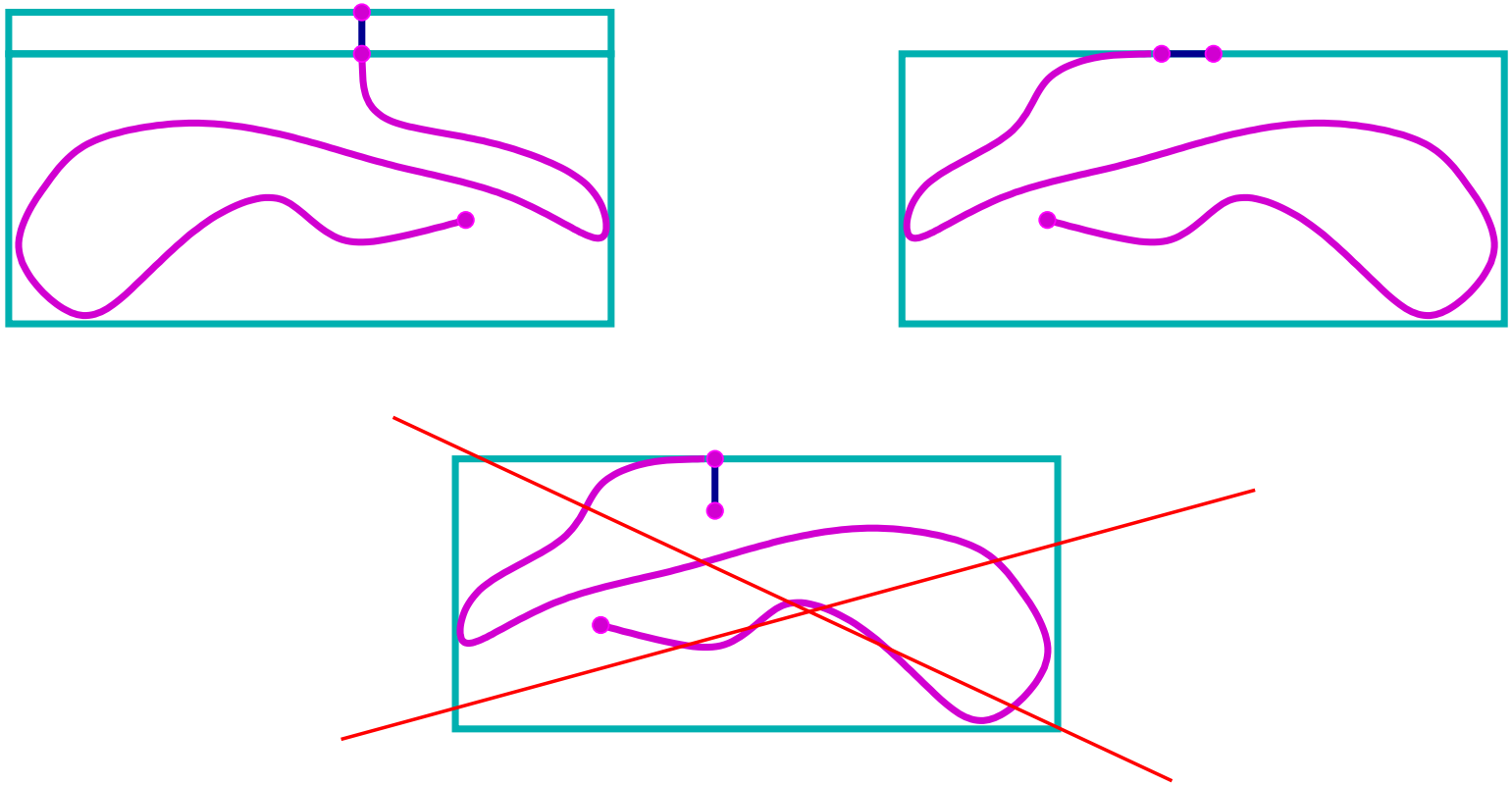
Some properties of prudent walks



The box of a prudent walk

Some properties of prudent walks

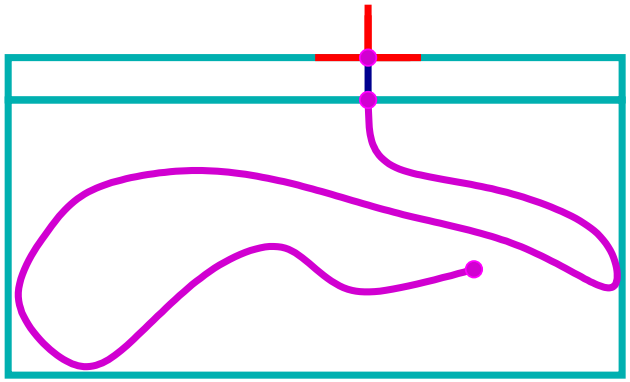
The endpoint of a prudent walk is always on the border of the box.

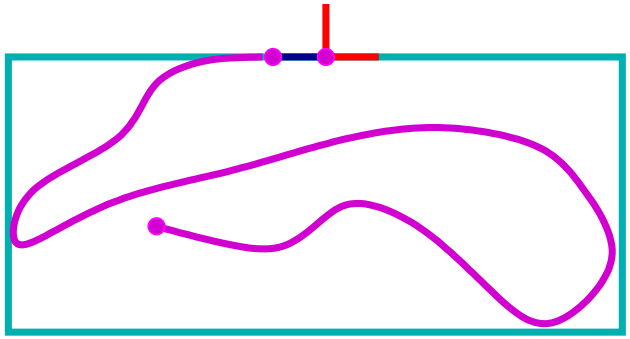


Each new step either inflates the box or walks (prudently) along the border.

Some properties of prudent walks

Each new step either inflates the box or walks (prudently) along the border.



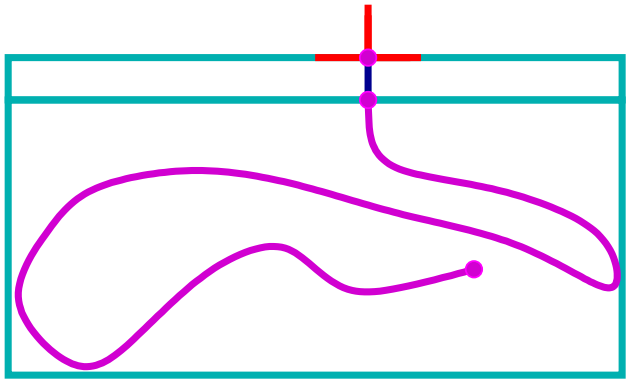


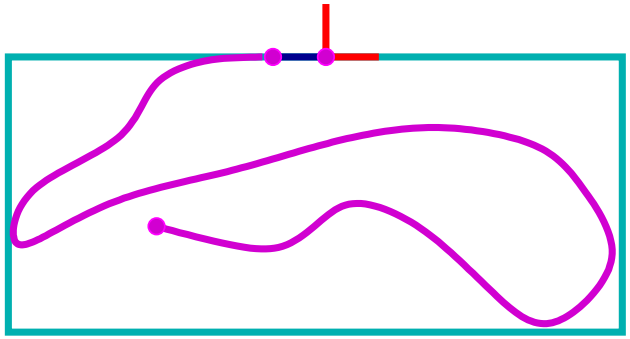
- After an inflating step, 3 possible extensions
- Otherwise, only 2.

$\Rightarrow$ Count prudent walks by looking for inflating steps

Some properties of prudent walks

Each new step either inflates the box or walks (prudently) along the border.





- After an inflating step, 3 possible extensions
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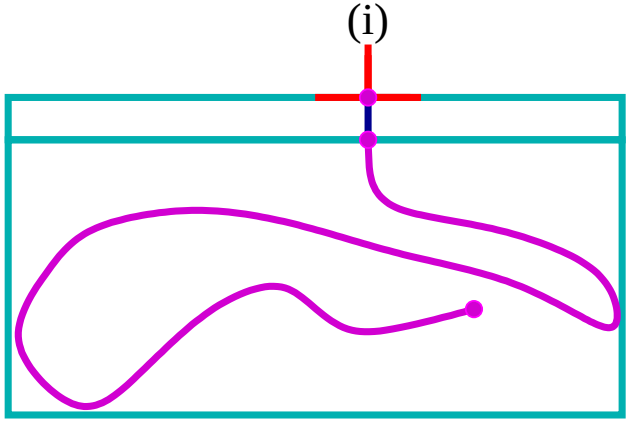
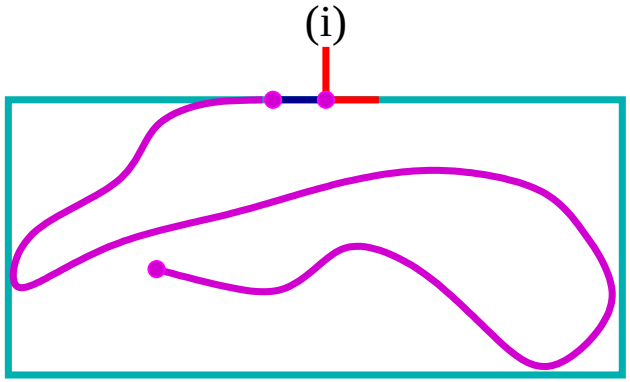
When do we create an inflating step?

Some properties of prudent walks

Each new step either inflates the box or walks (prudently) along the border.





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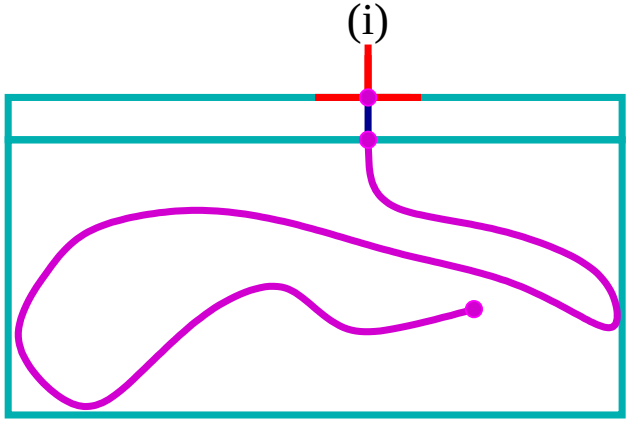

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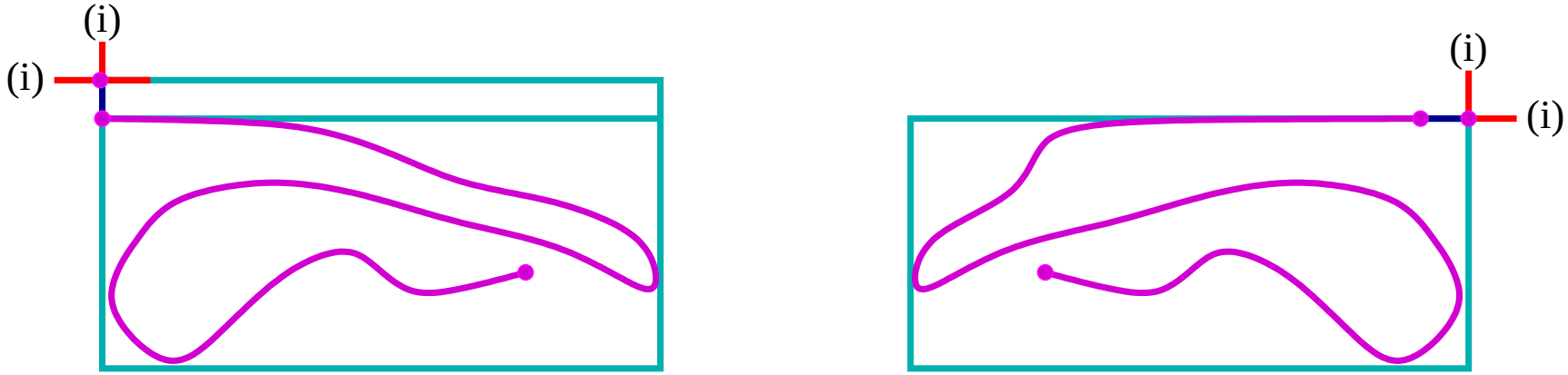

- After an inflating step, 3 possible extensions
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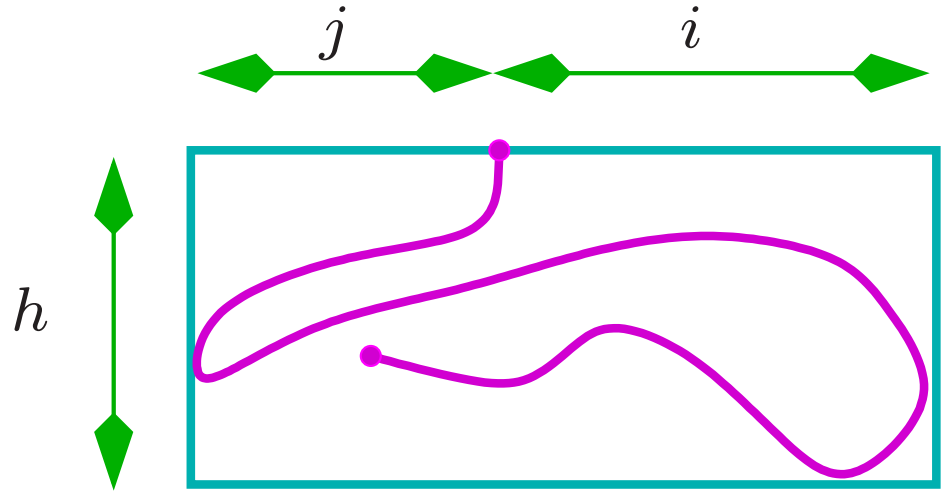
$\Rightarrow$ Count prudent walks by looking for inflating steps

When do we create an inflating step?

More parameters



If one knows:

- the direction of the last step,
- whether it is inflating or not,
- the distances i, j and h,

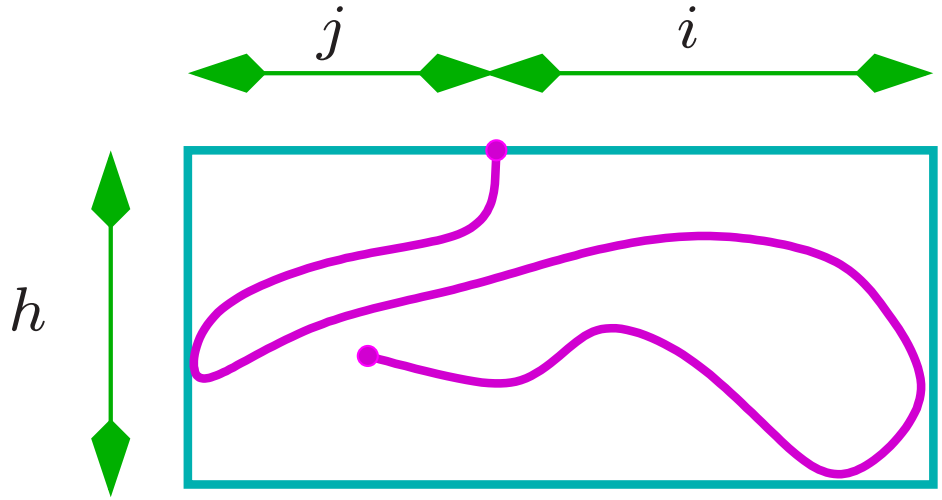
then one can decide which steps can be appended to the walk, and the new values of these parameters.

$\Rightarrow$ Count prudent walks by looking for inflating steps, keeping track of the distances i, j, h

Recursive construction of prudent walks

- Three more parameters

(*catalytic* parameters)



- Generating function of prudent walks ending on the top of their box:

$$\left(1-\frac{uvwt(1-t^2)}{(u-tv)(v-tu)}\right)T(t;u,v,w)=$$

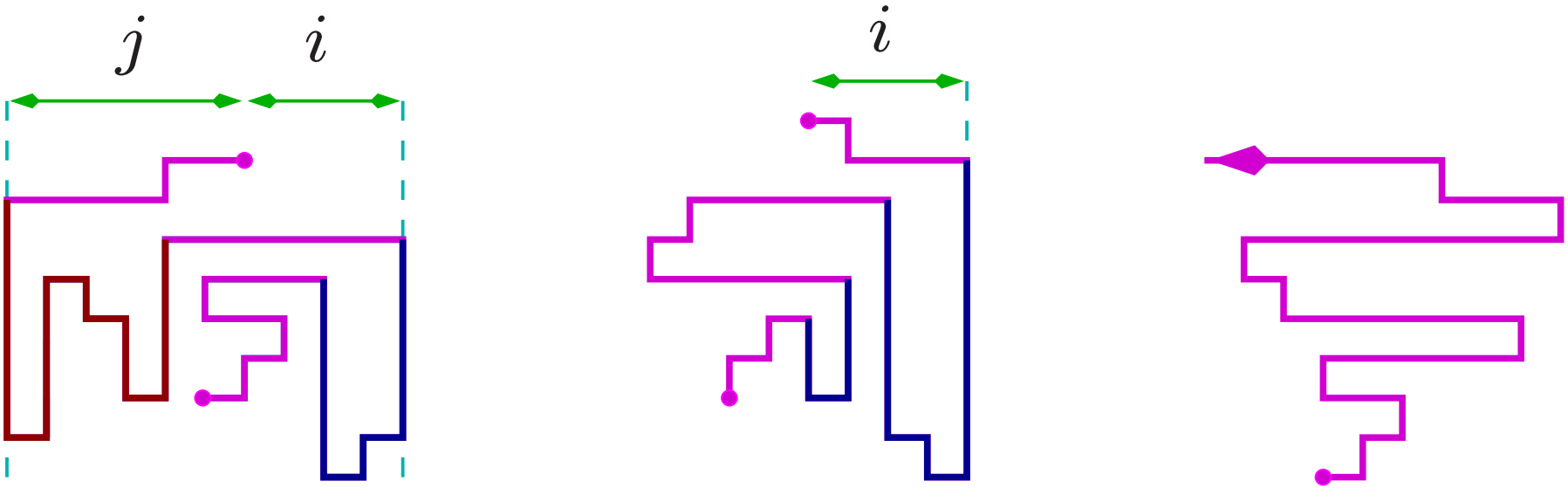
$$1+\mathcal{T}(t;w,u)+\mathcal{T}(t;w,v)-tv\frac{\mathcal{T}(t;v,w)}{u-tv}-tu\frac{\mathcal{T}(t;u,w)}{v-tu}$$

with $\mathcal{T}(t;u,v)=tvT(t;u,tu,v)$.

- Generating function of all prudent walks, counted by the length and the half-perimeter of the box:

$$P(t;u)=1+4T(t;u,u,u)-4T(t;0,u,u)$$

Simpler families of prudent walks [Préa 97]



3-sided 2-sided 1-sided

- The endpoint of a 3-sided walk lies always on the top, right or left side of the box
- The endpoint of a 2-sided walk lies always on the top or right side of the box
- The endpoint of a 1-sided walk lies always on the top side of the box (= partially directed!)

Functional equations for prudent walks: The more general the class, the more additional variables

(Walks ending on the top of the box)

- General prudent walks: three catalytic variables

$$\left(1-\frac{uvwt(1-t^2)}{(u-tv)(v-tu)}\right)T(t;u,v,w) = 1+\mathcal{T}(w,u)+\mathcal{T}(w,v)-tv\frac{\mathcal{T}(v,w)}{u-tv}-tu\frac{\mathcal{T}(u,w)}{v-tu}$$

with $\mathcal{T}(u,v) = tvT(t;u,tu,v)$.

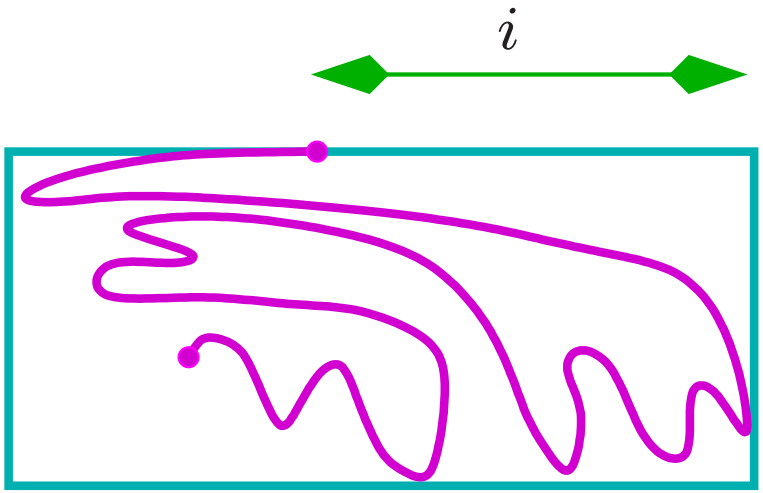
- Three-sided walks: two catalytic variables

$$\left(1-\frac{uvt(1-t^2)}{(u-tv)(v-tu)}\right)T(t;u,v) = 1+\cdots-\frac{t^2v}{u-tv}T(t;tv,v)-\frac{t^2u}{v-tu}T(t;u,tu)$$

- Two-sided walks: one catalytic variable

$$\left(1-\frac{tu(1-t^2)}{(1-tu)(u-t)}\right)T(t;u) = \frac{1}{1-tu}+t\,\frac{u-2t}{u-t}\,T(t;t)$$

Where do these equations come from? Two-sided walks



Generating function:

$$P(t;u) = \sum_{n;i} p(n;i)t^n u^i = \sum_i P_i(t)u^i \equiv P(u)$$

- For walks ending on the top of their box: series $T(t;u) \equiv T(u)$

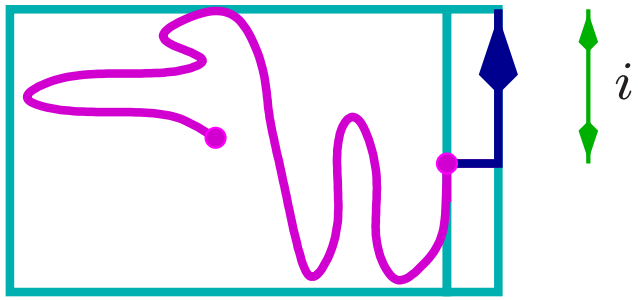
Two-sided walks

Take a walk ending on the top of its box: where is the last step that has moved the top or right side of the box?

- Either there was no such step: 

$$\frac{1}{1-tu}$$

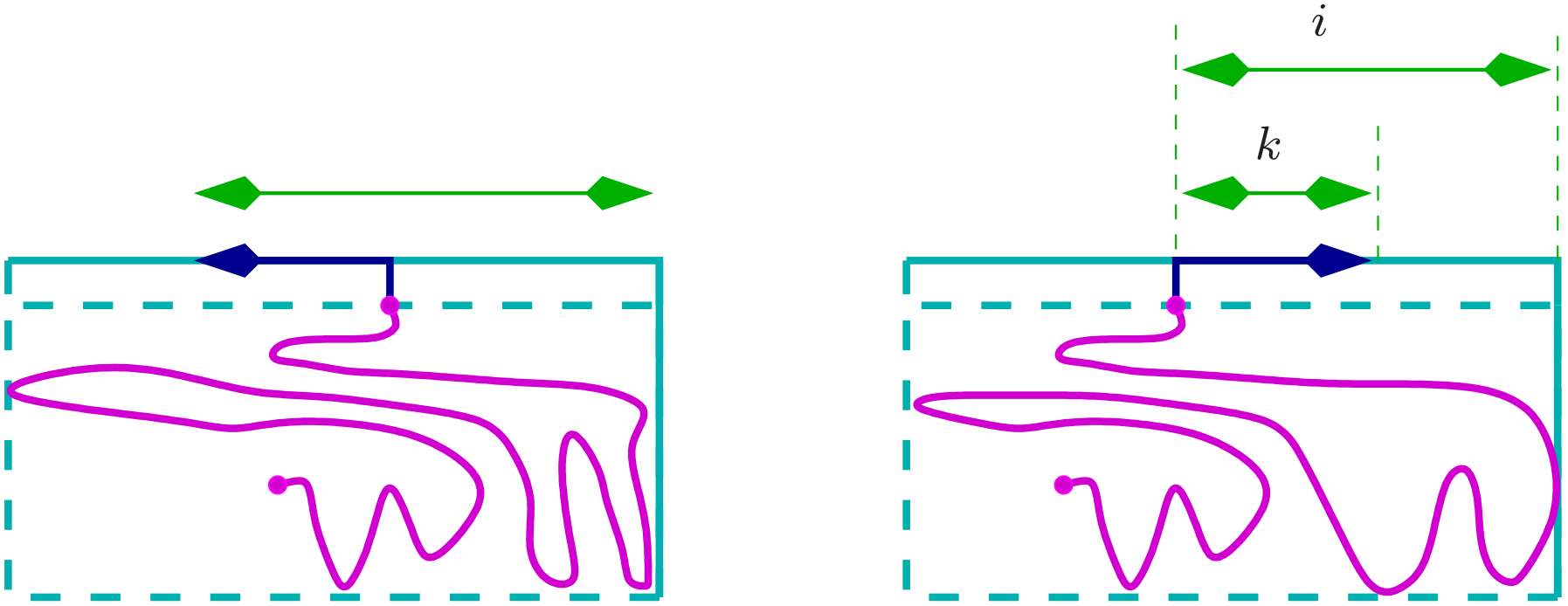
- ... or an East step moved the right side:



$$t \sum_i T_i(t) t^i = tT(t;t)$$

Two-sided walks

... or a North step moved the top side:



(at least one West step)

$$\frac{t^2u}{1-tu}T(t;u) + t\sum_{i\geq 0} T_i(t) \sum_{k=0}^{i} t^k u^{i-k} = \frac{t^2u}{1-tu}T(t;u) + \frac{t}{u-t}\left(uT(t;u) - tT(t;t)\right).$$

Two-sided walks

- The generating function$T(t;u) \equiv T(u)$ of two-sided walks ending on the top of their box satisfies:

$$\left(1 - \frac{tu(1-t^2)}{(1-tu)(u-t)}\right) T(u) = \frac{1}{1-tu} + t\,\frac{u-2t}{u-t} T(t)$$

- The length generating function $P(t)$ of all two-sided walks is

$$P(t) = 2T(t;1) - T(t;0)$$

(inclusion-exclusion)

Two-sided walks: the kernel method

$$\Big((1-tu)(u-t)-tu(1-t^2)\Big)\,T(t;u) = u - t + t(u-2t)(1-tu)T(t;t).$$

- If $u = U(t)$ cancels $(1-tu)(u-t)-tu(1-t^2)$, then

$$0 = U(t) - t + t(U(t)-2t)(1-tU(t))T(t;t),$$

that is,

$$T(t;t) = \frac{t - U(t)}{t(U(t)-2t)(1-tU(t))}$$

- We know such a series $U(t)$:

$$U(t) = \frac{1 - t + t^2 + t^3 - \sqrt{(1-t^4)(1-2t-t^2)}}{2t}$$

[Knuth 72], [mbm-Petkovšek 2000]

Two-sided walks

- The length generating function of 2-sided walks is

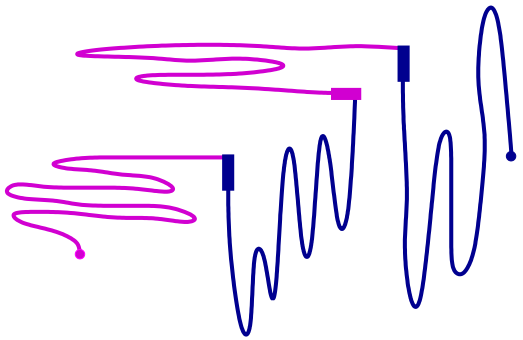
$$P(t) = \frac{1}{1-2t-2t^2+2t^3}\left(1+t-t^3+t(1-t)\sqrt{\frac{1-t^4}{1-2t-t^2}}\right)$$

- Dominant singularity: a simple pole for $1-2t-2t^2+2t^3=0$, that is, $t_c = 0.40303....$ Asymptotically,

$$p(n) \sim \kappa\,(2.48...)^n$$

Compare with 2.41... for partially directed walks.

- Another approach: factorization of walks [Duchi 05]



Functional equation for 3-sided walks

(Walks ending on the top of the box)

- Three-sided walks: two catalytic variables

$$\left(1-\frac{uvt(1-t^2)}{(u-tv)(v-tu)}\right)T(t;u,v) = 1+\cdots-\frac{t^2v}{u-tv}T(t;tv,v)-\frac{t^2u}{v-tu}T(t;u,tu)$$

Three-sided walks: two catalytic variables

- Functional equation for $T(t; u, v) \equiv T(u, v)$:

$$K(u, v)T(u, v) = A(u, v) + B(u, v)\,\Phi(u) + B(v, u)\,\Phi(v)$$

for polynomials $K(u, v), A(u, v), B(u, v)$, with $\Phi(u) = T(u, tu)$.

Three-sided walks: two catalytic variables

- Functional equation for $T(t; u, v) \equiv T(u, v)$:

$$K(u, v)T(u, v) = A(u, v) + B(u, v)\,\Phi(u) + B(v, u)\,\Phi(v)$$

for polynomials $K(u, v), A(u, v), B(u, v)$, with $\Phi(u) = T(u, tu)$.

- Cancellation of the kernel: $K(u, V(u)) = 0$ for a series $V(u) \equiv V(t; u)$

$$\Phi(u) = -\frac{A(u, V(u))}{B(u, V(u))} - \frac{B(V(u), u)}{B(u, V(u))}\,\Phi(V(u))$$

Three-sided walks: two catalytic variables

- Functional equation for $T(t; u, v) \equiv T(u, v)$:

$$K(u,v)T(u,v) = A(u,v) + B(u,v)\,\Phi(u) + B(v,u)\,\Phi(v)$$

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$$\Phi(u) = -\frac{A(u,V(u))}{B(u,V(u))} - \frac{B(V(u),u)}{B(u,V(u))}\,\Phi(V(u))$$

- If it is possible to iterate (...), denote $V^{(k)} = V(V(V(\cdots(u))))$ (k iterations):

$$\Phi(u) = \sum_{k\geq 0} (-1)^{k-1} \frac{B(V^{(1)},u)B(V^{(2)},V^{(1)})\cdots B(V^{(k)},V^{(k-1)})A(V^{(k)},V^{(k+1)})}{B(u,V^{(1)})B(V^{(1)},V^{(2)})\cdots B(V^{(k-1)},V^{(k)})B(V^{(k)},V^{(k+1)})}$$

Sum of algebraic series — iteration of algebraic functions

Three-sided prudent walks

- Let

$$U(w) = \frac{1 - tw + t^2 + t^3 w - \sqrt{(1-t^2)(1+t-tw+t^2w)(1-t-tw-t^2w)}}{2t},$$

and

$$q = U(1) = \frac{1 - t + t^2 + t^3 - \sqrt{(1-t^4)(1-2t-t^2)}}{2t}.$$

- The length generating function of three-sided prudent walks is:

$$P(t;1) = \frac{1}{1-2t-t^2}\left(\frac{1+3t+tq(1-3t-2t^2)}{1-tq} + 2t^2q\ T(t;1,t)\right)$$

where

$$T(t;1,t) = \sum_{k\geq 0}(-1)^k \frac{\prod_{i=0}^{k-1}\left(\frac{t}{1-tq} - U(q^{i+1})\right)}{\prod_{i=0}^{k}\left(\frac{tq}{q-t} - U(q^i)\right)}\left(1 + \frac{U(q^k)-t}{t(1-tU(q^k))} + \frac{U(q^{k+1})-t}{t(1-tU(q^{k+1}))}\right)$$

Three-sided prudent walks

The generating function of three-sided prudent walks is:

$$P(t;1) = \frac{1}{1-2t-t^2}\left(\frac{1+3t+tq(1-3t-2t^2)}{1-tq} + 2t^2q\ T(t;1,t)\right)$$

$$T(t;1,t) = \sum_{k\geq 0}(-1)^k \frac{\prod_{i=0}^{k-1}\left(\frac{t}{1-tq} - U(q^{i+1})\right)}{\prod_{i=0}^{k}\left(\frac{tq}{q-t} - U(q^i)\right)}\left(1 + \frac{U(q^k)-t}{t(1-tU(q^k))} + \frac{U(q^{k+1})-t}{t(1-tU(q^{k+1}))}\right)$$

• The series $P(t;1)$ has infinitely many poles, satisfying $\frac{tq}{q-t} = U(q^i)$ for some $i \geq 0$. Hence it is neither algebraic, nor even D-finite.

• Dominant singularity: (again) a simple pole for $1-2t-2t^2+2t^3 = 0$. Asymptotically,

$$p(n) \sim \kappa\,(2.48\ldots)^n$$

Two- and three-sided walks: asymptotic enumeration

- The numbers of 2-sided and 3-sided n-step prudent walks satisfy

$$p_2(n) \sim \kappa_2 \, \mu^n, \quad p_3(n) \sim \kappa_3 \, \mu^n$$

where $\mu \simeq 2.48...$ is such that

$$\mu^3 - 2\mu^2 - 2\mu + 2 = 0.$$

Compare with 2.41... for partially directed walks, 2.54... for weakly directed bridges, but 2.64... for general SAW.

- Conjecture: for general prudent walks

$$p_4(n) \sim \kappa_4 \, \mu^n$$

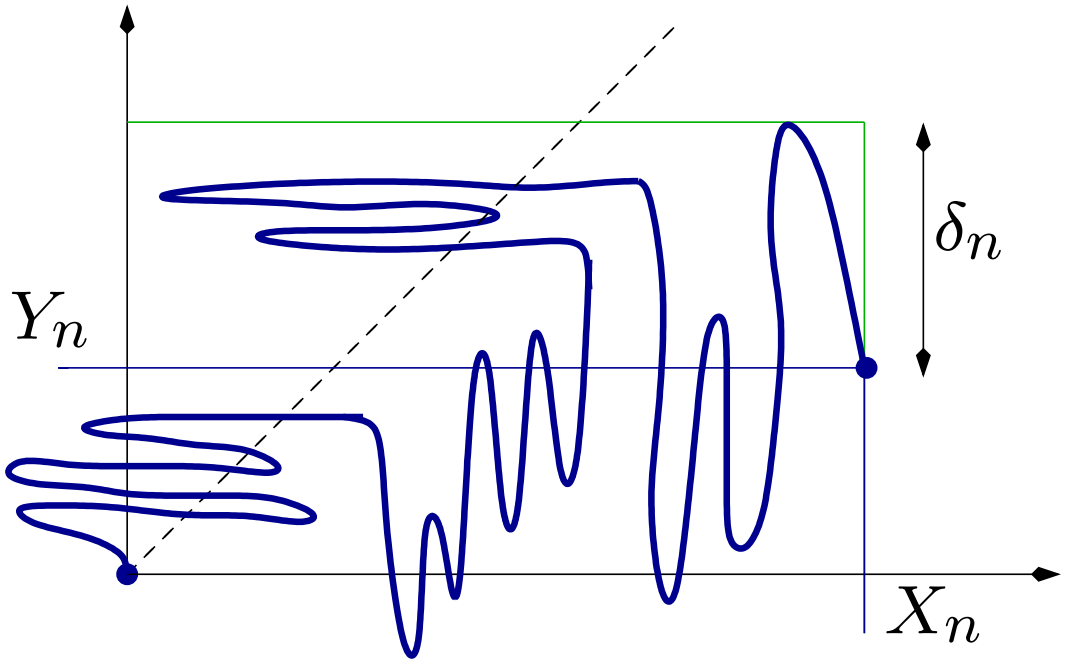
with the same value of μ as above [Dethridge, Guttmann, Jensen 07].

Two-sided walks: properties of large random walks (uniform distribution)

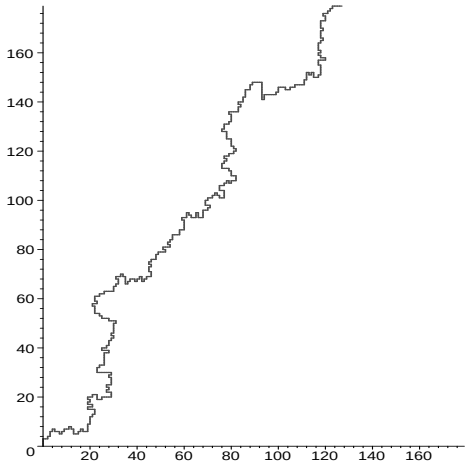
- The random variables X_n, Y_n and δ_n satisfy

$$\mathbb{E}(X_n) = \mathbb{E}(Y_n) \sim n \qquad \mathbb{E}((X_n - Y_n)^2) \sim n, \qquad \mathbb{E}(\delta_n) \sim 4.15\ldots$$

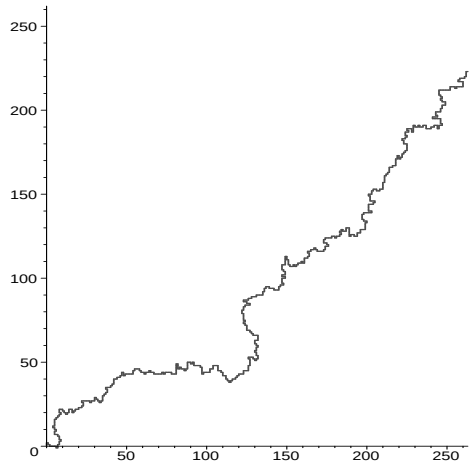
(+ gaussian limit law after normalization)



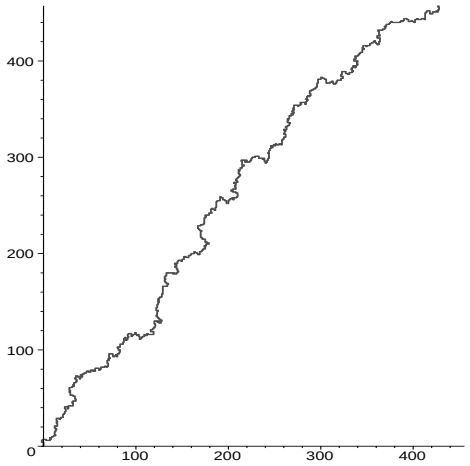
Two-sided walks: random generation (uniform distribution)



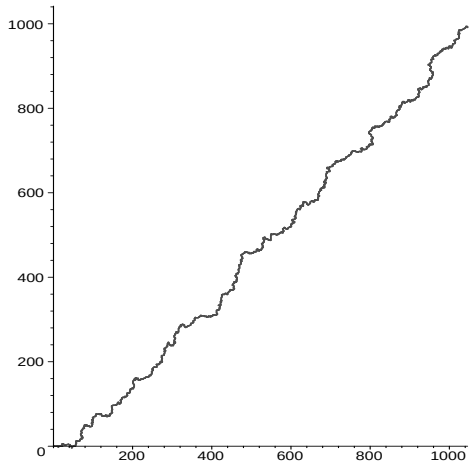
500 steps



780 steps



1354 steps



3148 steps

- Recursive step-by-step construction à la Wilf $\Rightarrow$ 500 steps (precomputation of $O(n^2)$ large numbers)

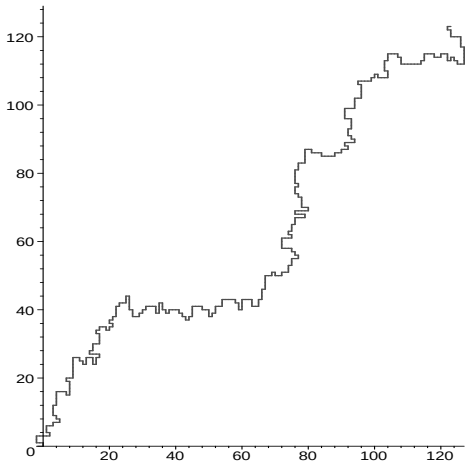
- Boltzmann sampling via a context-free grammar [Duchon-Flajolet-Louchard-Schaeffer 02]

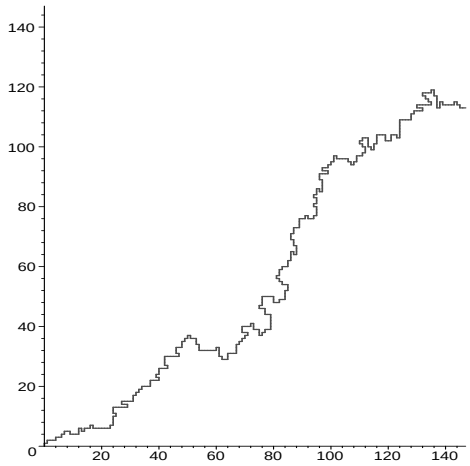
$$\mathbb{E}(X_n) = \mathbb{E}(Y_n) \sim n \qquad \mathbb{E}((X_n - Y_n)^2) \sim n, \qquad \mathbb{E}(\delta_n) \sim 4.15\ldots$$

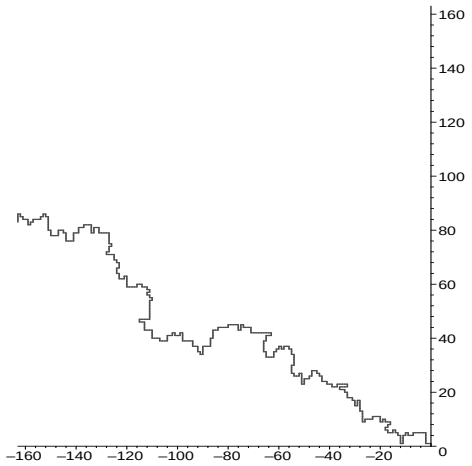
Three-sided prudent walks: random generation and asymptotic properties

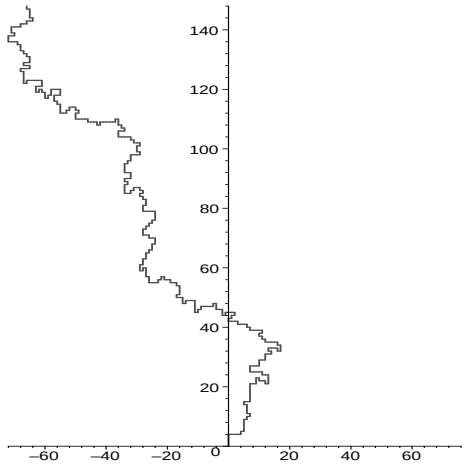
• Asymptotic properties: The average width of the box is $\sim \kappa n$ (variance $\sim n$, gausiian limit law after normalization)

• Random generation: Recursive method à la Wilf $\Rightarrow$ 400 steps (pre-computation of $O(n^3)$ numbers)









Four-sided (i.e. general) prudent walks

- An equation with 3 catalytic variables:

$$\left(1-\frac{uvwt(1-t^2)}{(u-tv)(v-tu)}\right)T(u,v,w)=1+\mathcal{T}(w,u)+\mathcal{T}(w,v)-tv\frac{\mathcal{T}(v,w)}{u-tv}-tu\frac{\mathcal{T}(u,w)}{v-tu}$$

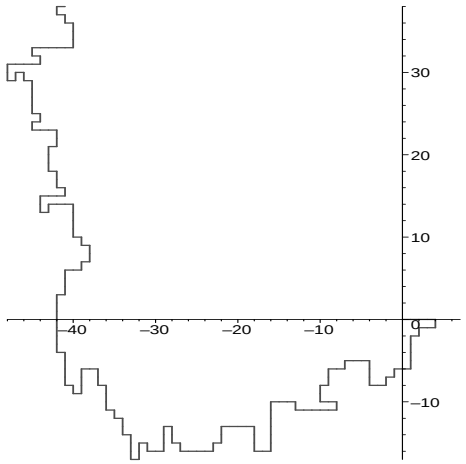
with $\mathcal{T}(u,v)=tvT(u,tu,v)$.

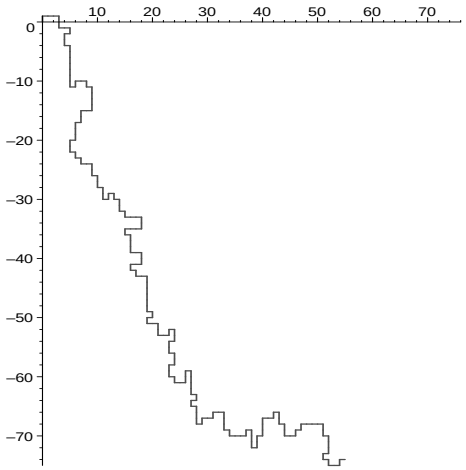
- Conjecture:

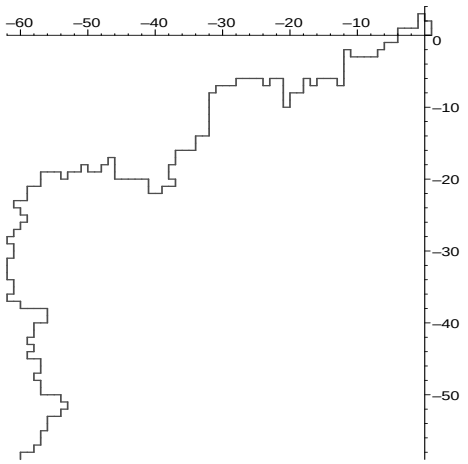
$$p_4(n)\sim\kappa_4\,\mu^n$$

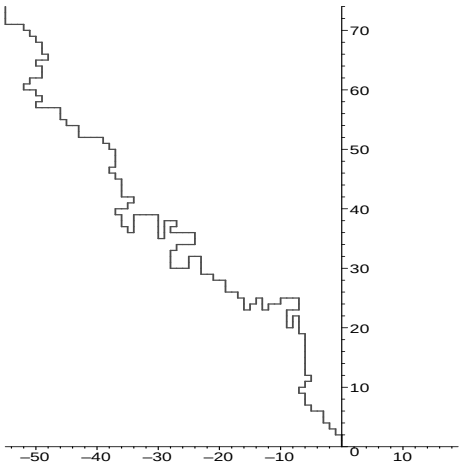
where $\mu\simeq 2.48$ satisfies $\mu^3-2\mu^2-2\mu+2=0$.

- Random prudent walks: recursive generation, 195 steps (sic! $O(n^4)$ numbers)





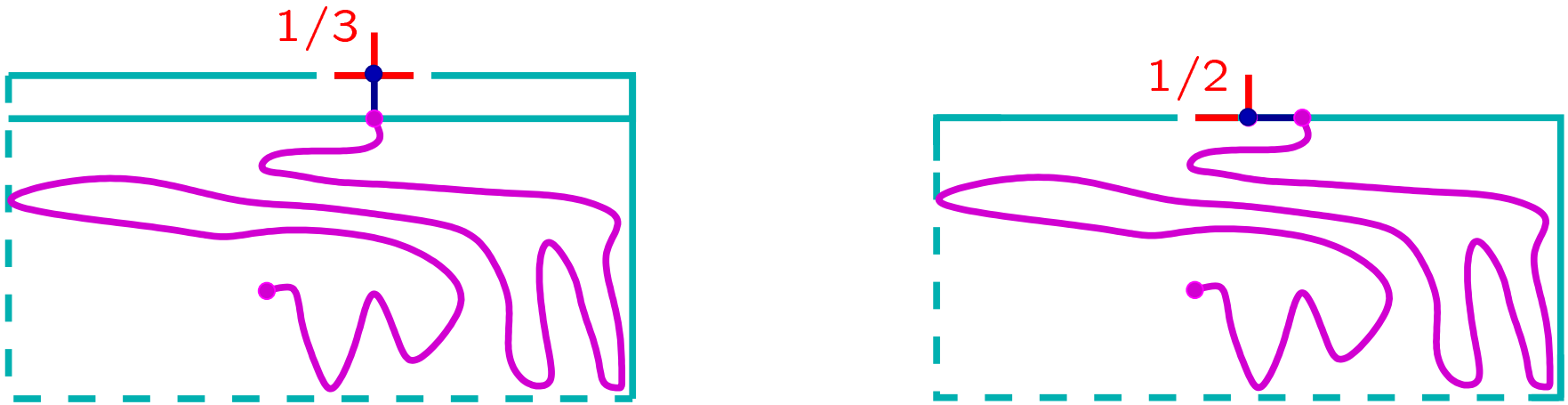




II.3. Another distribution: Kinetic prudent walks

At time n, the walk chooses one of the admissible steps with uniform probability.

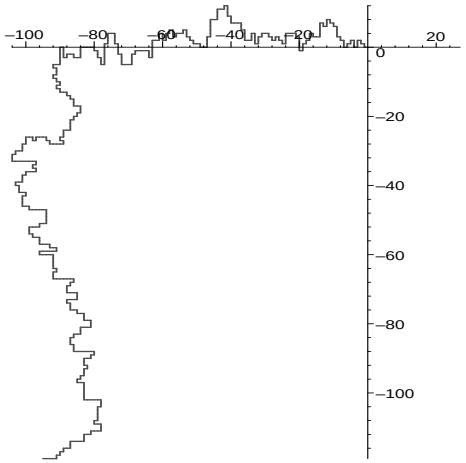
[An admissible step is one that gives a prudent walk]



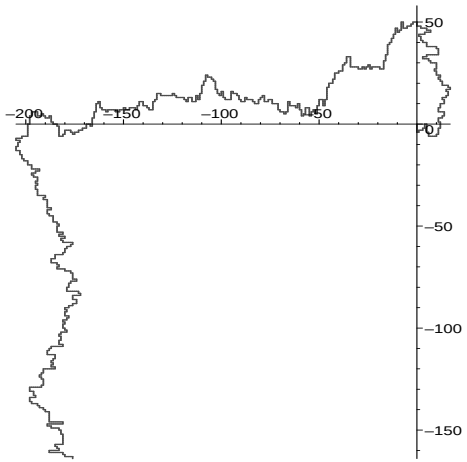
Remark: Walks of length n are no longer uniform

Another distribution: Kinetic prudent walks

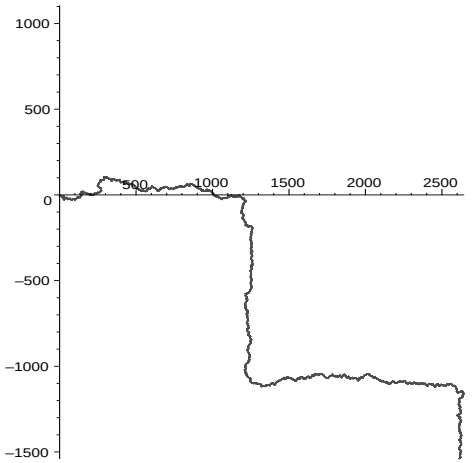
- Kinetic model: recursive generation with no precomputation



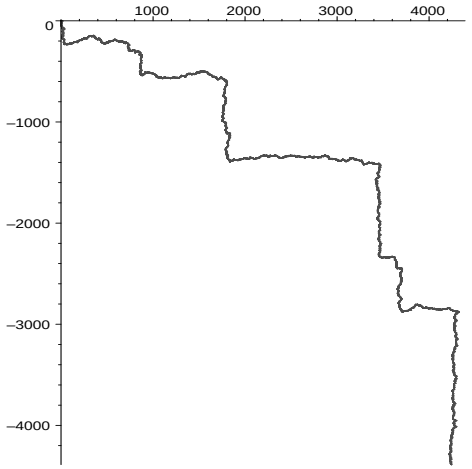
500 steps



1000 steps



10000 steps



20000 steps

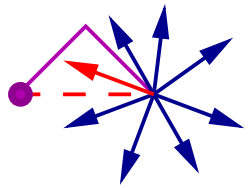
- Theorem: The walk chooses uniformly one quadrant, say the NE one, and then its scaling limit is given by

$$Z(u) = \int_0^{3u/7} \left(1_{W(s)\geq 0}\ e_1 + 1_{W(s)<0}\ e_2\right) ds$$

where e_1, e_2 form the canonical basis of $\mathbb{R}^2$ and $W(s)$ is a brownian motion. [Beffara, Friedli, Velenik 10]

A kinetic, continuous space version: The rancher's walk

At time n, the walk takes a uniform unit step in $\mathbb{R}^2$, conditioned so that the new step does not intersect the convex hull of the walk.



Theorem: the end-to-end distance is linear. More precisely, there exists a constant $a > 0$ such that

$$\liminf \frac{||\omega_n||}{n} \geq a.$$

[Angel, Benjamini, Virág 03], [Zerner 05]

Conjectures

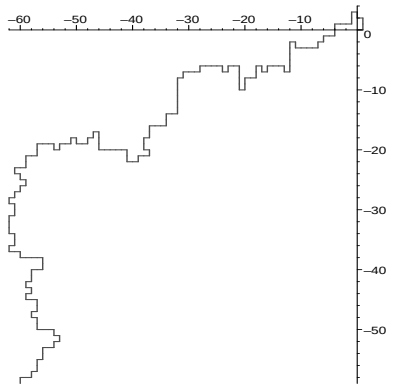
- Linear speed: There exists $a > 0$ such that $\frac{||\omega_n||}{n} \to a$ a.s.

- Angular convergence: $\frac{\omega_n}{||\omega_n||}$ converges a.s.

What's next? Exactly solvable models

• Exact enumeration: General prudent walks on the square lattice – Growth constant?

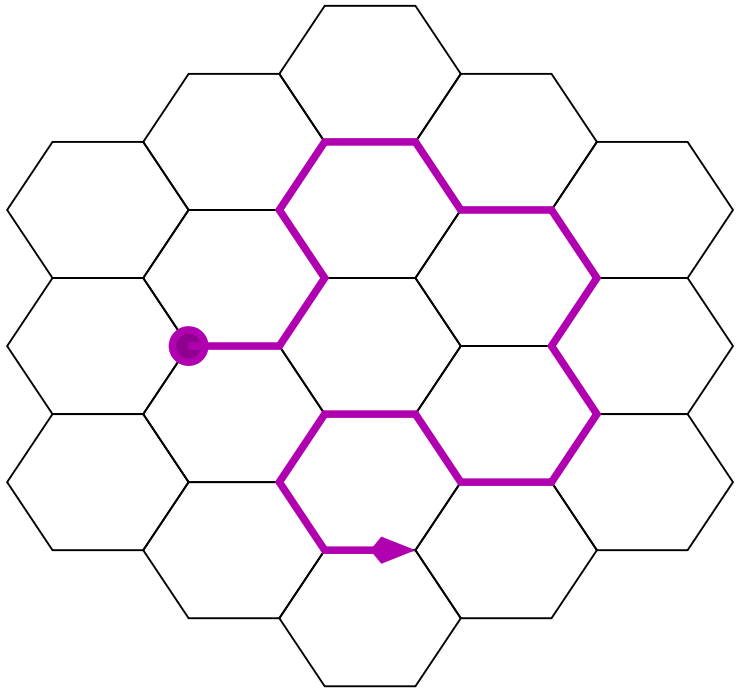
• Uniform random generation: better algorithms (maximal length 200 for general prudent walks...)



• A mixture of both models: walks formed of a sequence of prudent irreducible bridges?

III. Self-avoiding walks on the honeycomb lattice

d'après H. Duminil-Copin and S. Smirnov, ArXiv 1007.0575



Simple bounds on $c(n)$

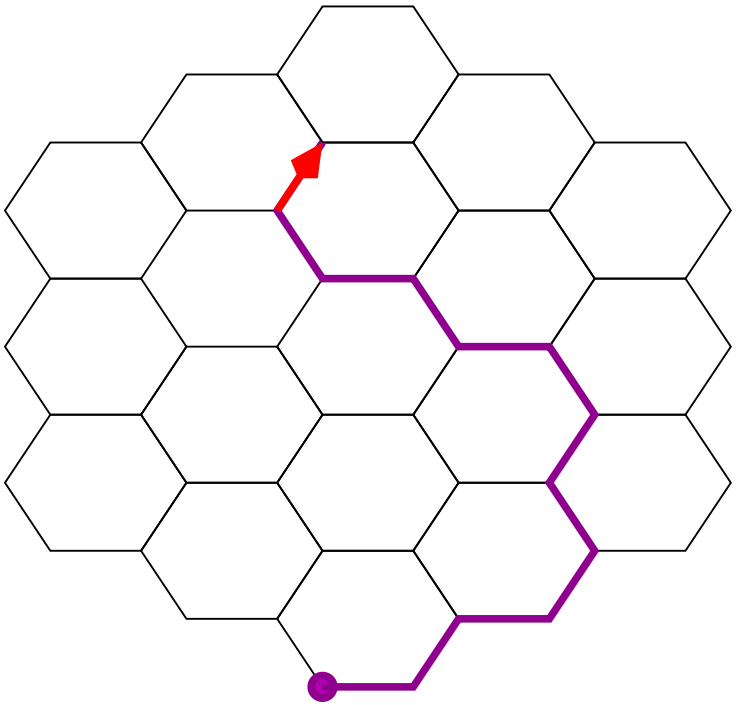
Let $c(n)$ be the number of n-step SAW. We have

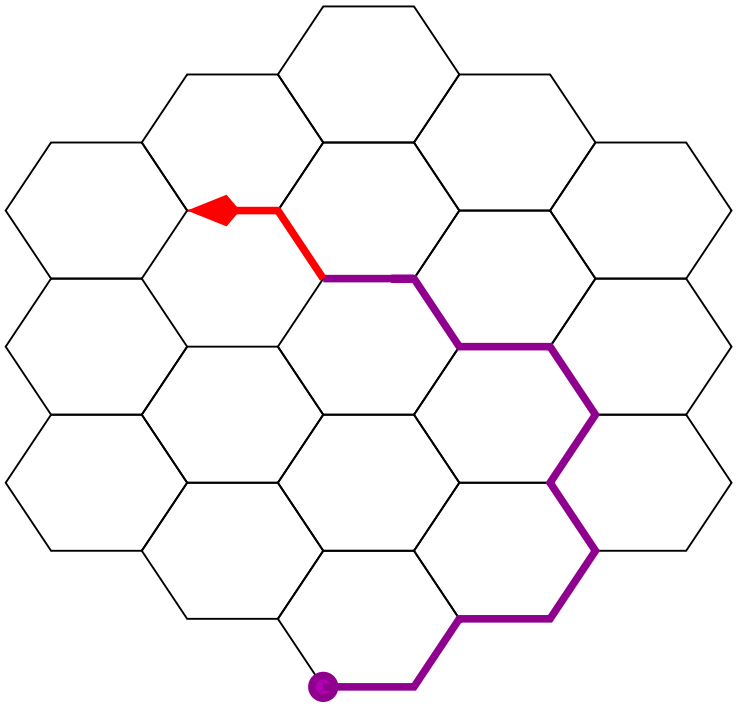
$$F_{n+1} \leq c(n) \leq 3 \cdot 2^{n-1}$$

with $F_0 = F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ (Fibonacci numbers).

- Upper bound: at most two possible extensions at each step

- Lower bound: SAW that never move in the South direction (partially directed).

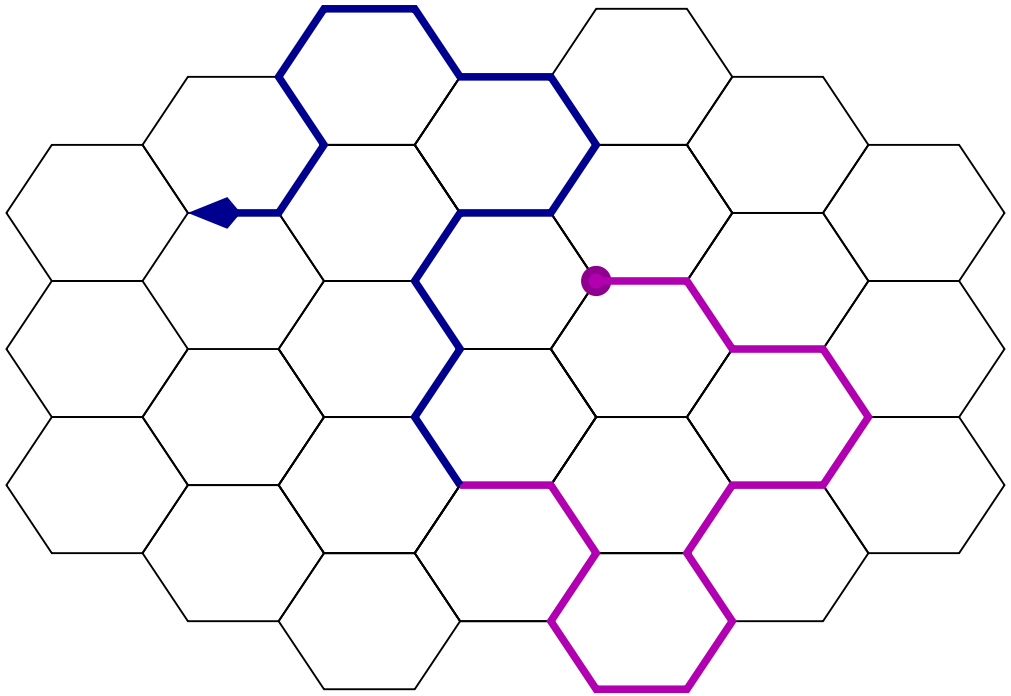




The connective constant

Clearly,

$$c(m+n) \leq c(m)c(n)$$



$\Rightarrow \lim c(n)^{1/n}$ exists and

$$\mu := \lim c(n)^{1/n} = \inf c(n)^{1/n}$$

The connective constant

$$\mu := \lim c(n)^{1/n} = \inf c(n)^{1/n}$$

Given the bounds

$$F_{n+1} \leq c(n) \leq 3 \cdot 2^{n-1},$$

there holds

$$\frac{1+\sqrt{5}}{2} \simeq 1.62 \leq \mu \leq 2$$

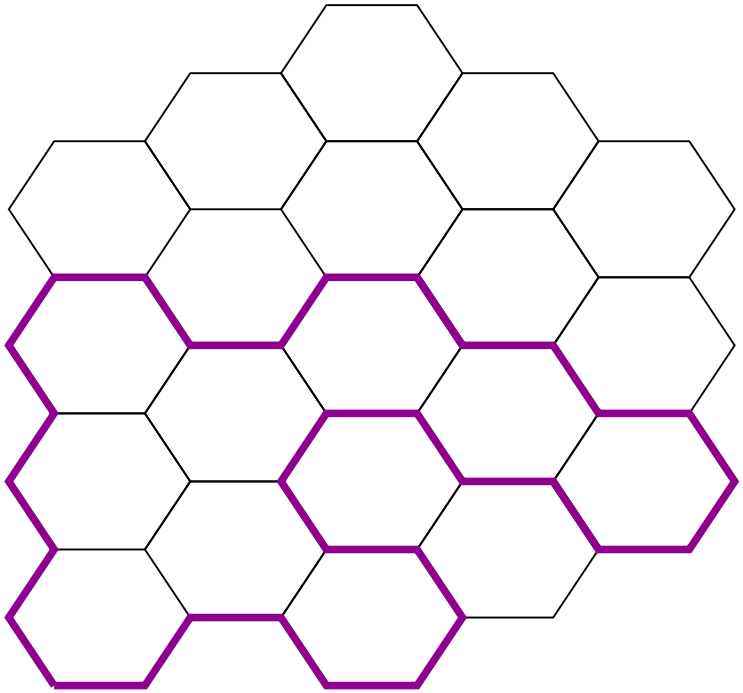
Theorem [Duminil-Copin & Smirnov 10]

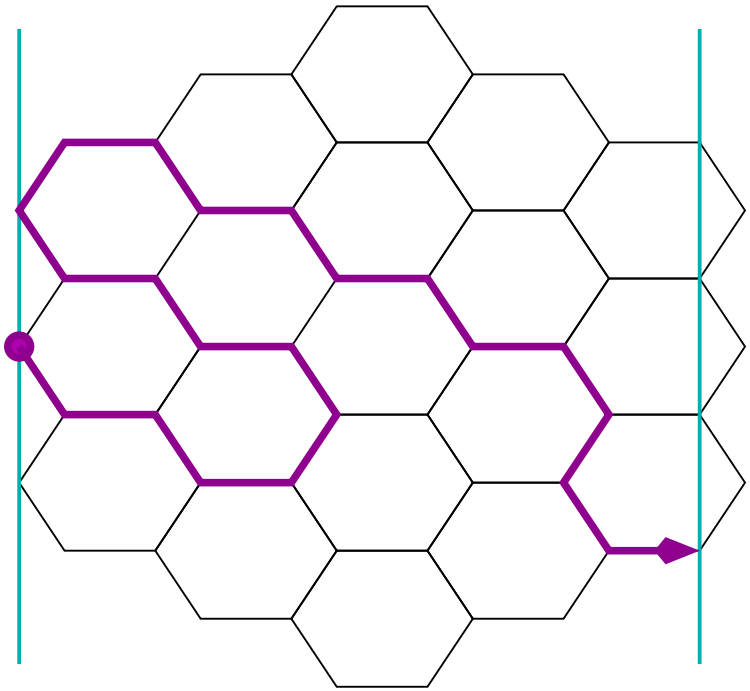
$$\mu = \sqrt{2+\sqrt{2}} \simeq 1.85$$

Conjectured by Nienhuis in 1982

Sub-families of SAW

Self-avoiding polygons and self-avoiding bridges





- Same growth constant:

$$p(n)^{1/n} \to \mu \qquad \text{and } b(n)^{1/n} \to \mu$$

[Hammersley 61]

Generating functions

For self-avoiding walks, polygons and bridges:

$$C(t) = \sum c(n)t^n, \qquad P(t) = \sum p(n)t^n, \qquad B(t) = \sum b(n)t^n$$

Theorem [Duminil-Copin & Smirnov 10]
The radius of convergence of these series is

$$t_c = \frac{1}{\sqrt{2+\sqrt{2}}}.$$

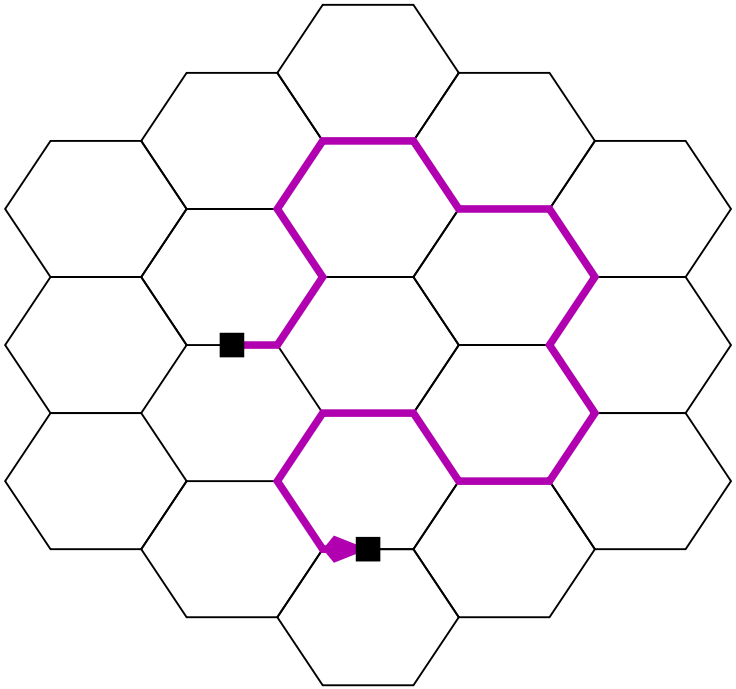
Moreover:

- $C(t)$ diverges at t_c,
- $P(t)$ converges at t_c.

Conjectures:

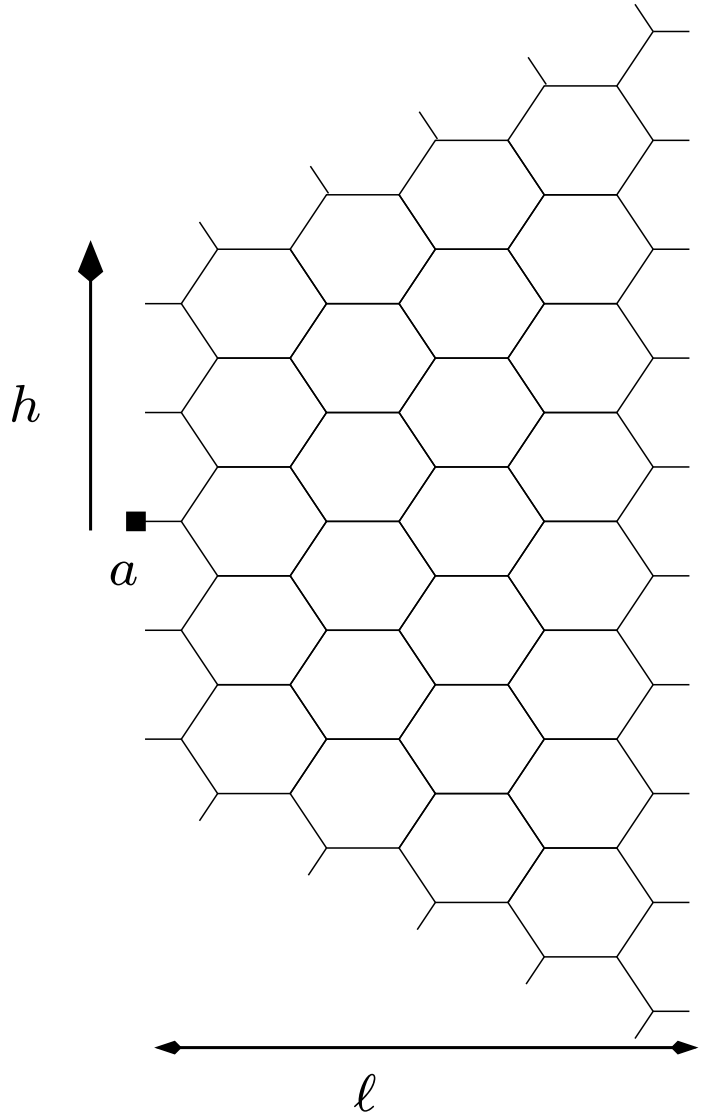
$$C(t) \sim \frac{1}{(1-t_c)^{43/32}}, \qquad P(t)_{sing} \sim (1-t_c)^{3/2}, \qquad B(t) \sim \frac{1}{(1-t_c)^{9/16}}$$

Walks starting and ending at mid-edges



A key identity

Consider the following domain $D_{h,\ell}$.



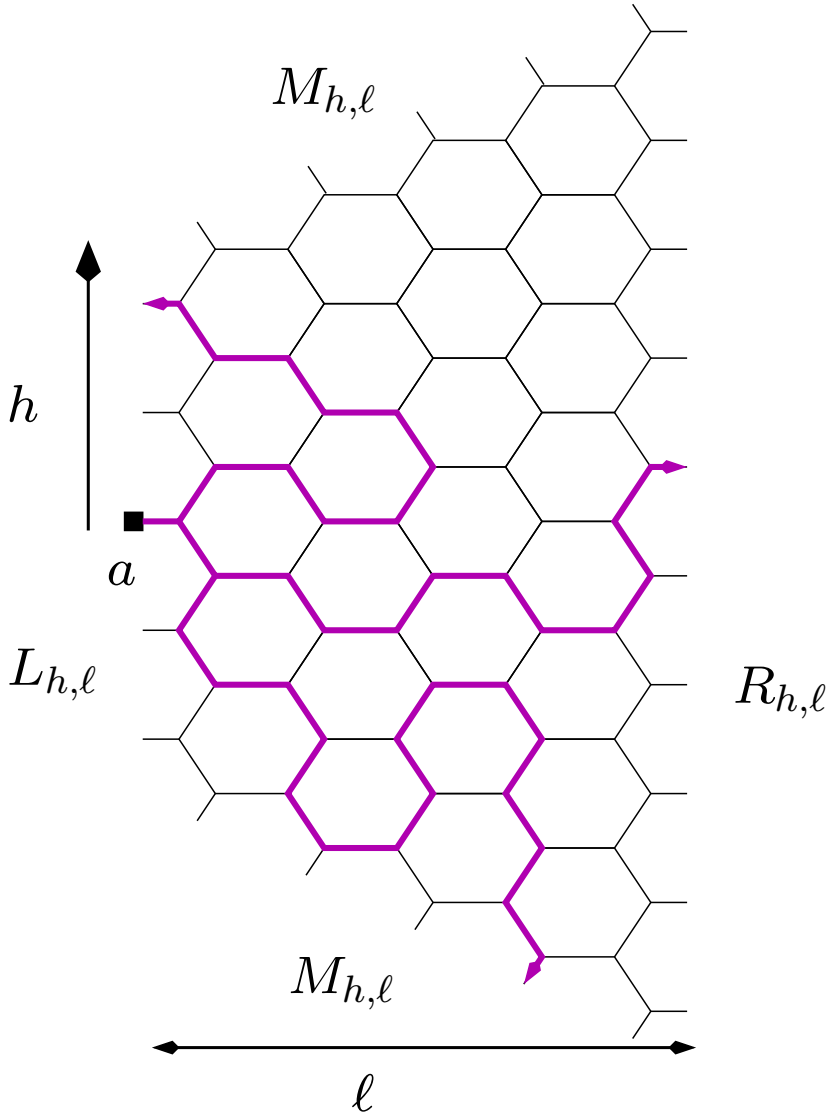
A key identity

Consider the following domain $D_{h,\ell}$.

Let $L_{h,\ell}(t)$ (resp. $R_{h,\ell}(t)$, $M_{h,\ell}(t)$) be the generating function of walks that start from a and end on the left (resp. right, middle) border of the domain $D_{h,\ell}$. These series are polynomials in t.

Then for all h and ℓ, at $t = t_c = 1/\sqrt{2+\sqrt{2}}$,

$$\frac{\sqrt{2-\sqrt{2}}}{2}\, L_{h,\ell}(t_c) + R_{h,\ell}(t_c) + \frac{1}{\sqrt{2}}\, M_{h,\ell}(t_c) = 1$$



An upper bound on μ

$$\frac{\sqrt{2-\sqrt{2}}}{2}\, L_{h,\ell}(t_c) + R_{h,\ell}(t_c) + \frac{1}{\sqrt{2}}\, M_{h,\ell}(t_c) = 1$$

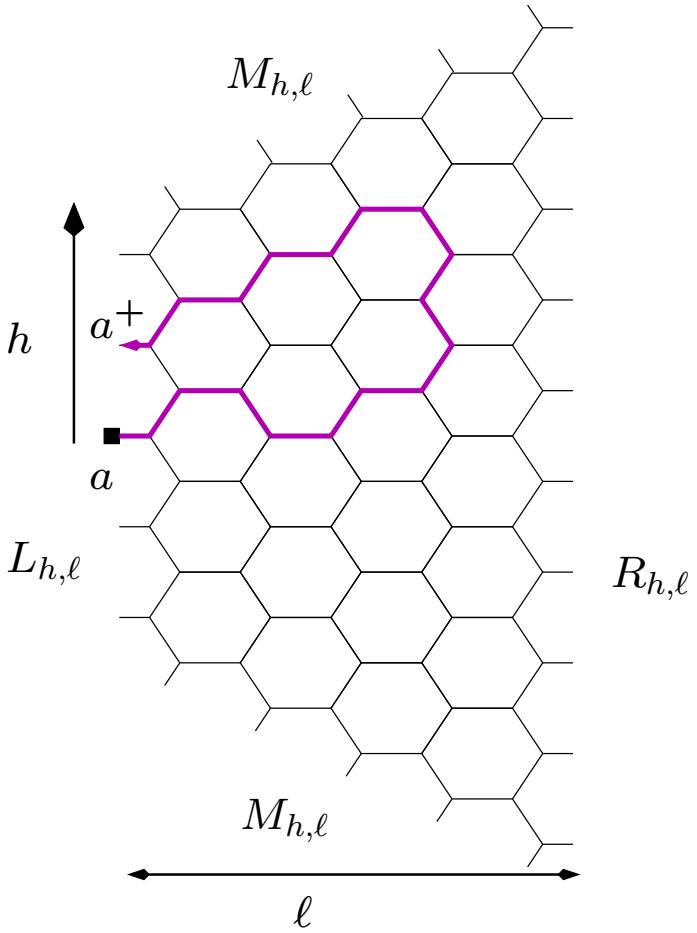
Consider walks in $D_{h,\ell}$ that go from a to a^+ ($\simeq$ self-avoiding polygons). Their generating function $\tilde{P}_{h,\ell}(t)$ satisfies

$$\tilde{P}_{h,\ell}(t_c) \leq L_{h,\ell}(t_c) \leq \frac{2}{\sqrt{2-\sqrt{2}}}$$

But

$$\tilde{P}_{h,\ell}(t_c) \to \tilde{P}(t_c) \geq t_c^2 P(t_c)$$

as $h, \ell \to \infty$. This implies that $P(t_c)$ is finite. Hence $\mu \leq 1/t_c = \sqrt{2+\sqrt{2}}$.



A lower bound on μ

$$\frac{\sqrt{2-\sqrt{2}}}{2}\, L_{h,\ell}(t_c) + R_{h,\ell}(t_c) + \frac{1}{\sqrt{2}}\, M_{h,\ell}(t_c) = 1$$

As $h \to \infty$, $L_{h,\ell}(t_c)$ and $R_{h,\ell}(t_c)$ increase to limit values $L_\ell(t_c)$ and $R_\ell(t_c)$. Hence $M_{h,\ell}(t_c)$ decreases to a limit value $M_\ell(t_c)$.

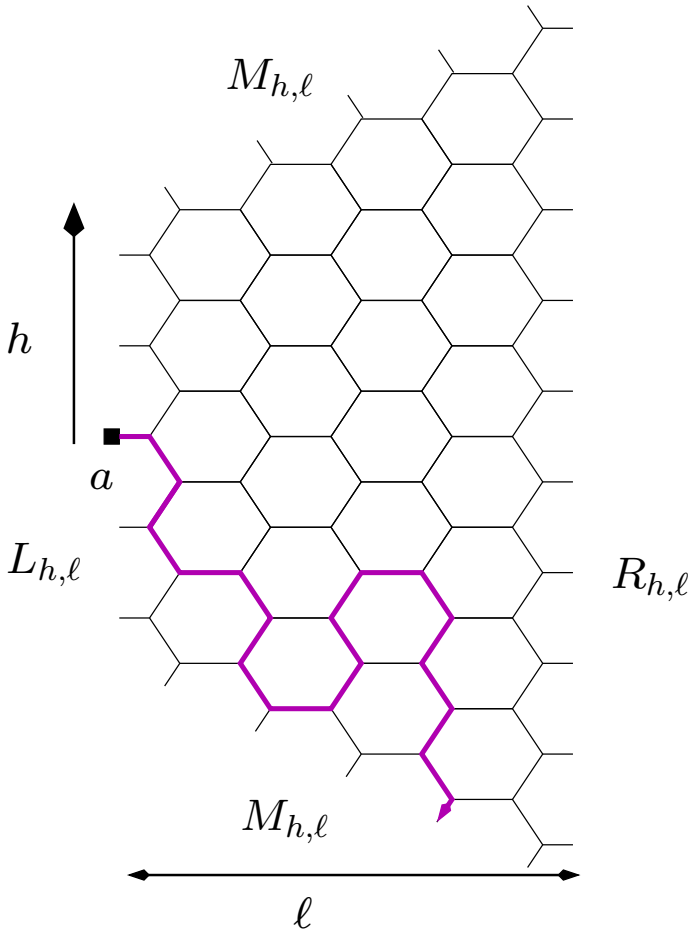
- If $M_\ell(t_c) > 0$ for some ℓ, the series

$$\sum_h M_{h,\ell}(t_c)$$

diverges. As it counts certain SAW, this implies that $\mu \geq 1/t_c$.

- If $M_\ell(t_c) = 0$ for all ℓ, another argument shows that $\sum_\ell R_\ell(t_c)$ (the generating function of bridges) diverges.

In both cases, $\mu \geq 1/t_c = \sqrt{2+\sqrt{2}}$.



A lower bound on μ (continued)

$$\alpha\ L_{h,\ell}(t_c) + R_{h,\ell}(t_c) + \frac{1}{\sqrt{2}}\ M_{h,\ell}(t_c) = 1$$

- If $M_\ell(t_c) = 0$ for all ℓ, then

$$\alpha\ L_\ell(t_c) + R_\ell(t_c) = 1$$

$$\Rightarrow\ \alpha\ \Big(L_{\ell+1}(t_c) - L_\ell(t_c)\Big) = R_\ell(t_c) - R_{\ell+1}(t_c)$$

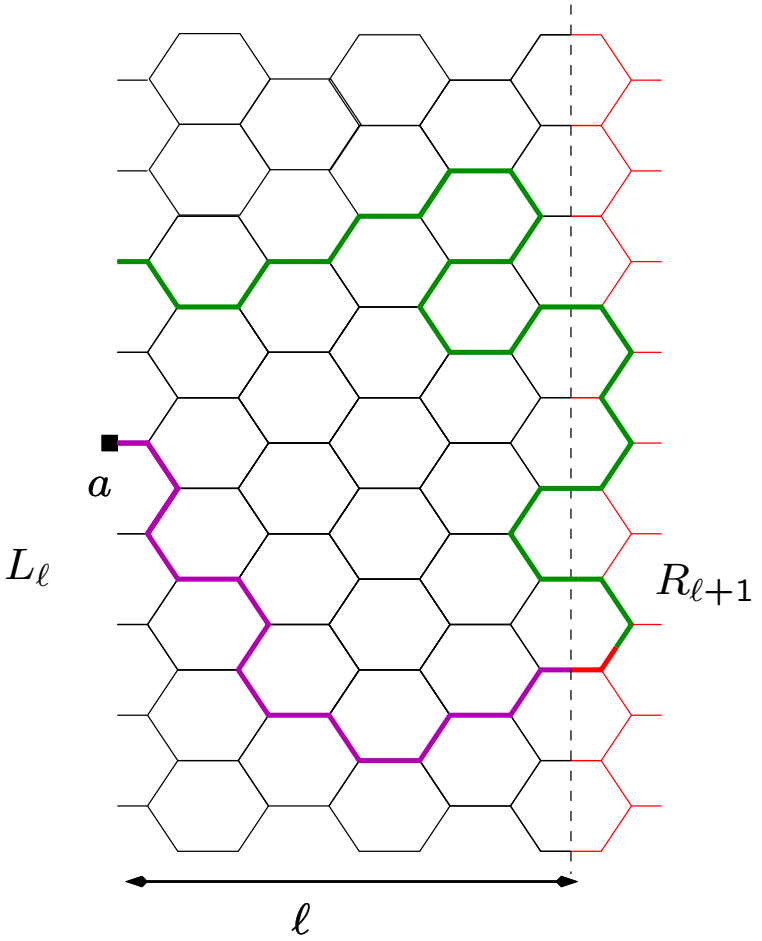
$$\Rightarrow\ 2\alpha t_c R_\ell R_{\ell+1} \geq R_\ell(t_c) - R_{\ell+1}(t_c)$$

$$\Rightarrow\ 2\alpha t_c \geq \frac{1}{R_{\ell+1}(t_c)} - \frac{1}{R_\ell(t_c)}$$

$$\Rightarrow\ 2\ell\alpha t_c + \frac{1}{R_1} \geq \frac{1}{R_\ell}$$

$$\Rightarrow R_\ell \geq \frac{1}{2\ell\alpha t_c + \beta},$$

so that $\sum_\ell R_\ell(t_c)$ (the generating function of bridges) diverges.



A key identity

$$\frac{\sqrt{2-\sqrt{2}}}{2}\, L_{h,\ell}(t_c) + R_{h,\ell}(t_c) + \frac{1}{\sqrt{2}}\, M_{h,\ell}(t_c) = 1$$

Where does it come from?

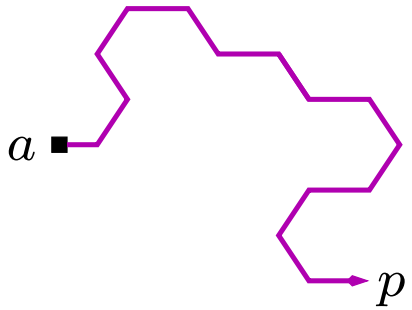
From a local identity that is re-summed over all vertices of the domain.

A local identity

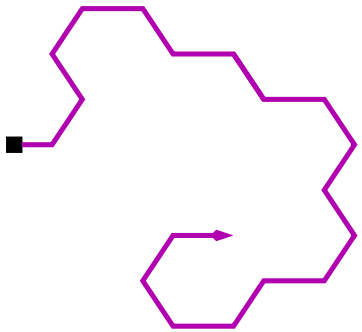
Let D be a finite, simply connected domain, and a a point on the boundary of D. For p a point in the domain, let

$$F(p) \equiv F(t, \alpha; p) = \sum_{\omega : a \rightsquigarrow p} t^{|\omega|} e^{i\alpha W(\omega)},$$

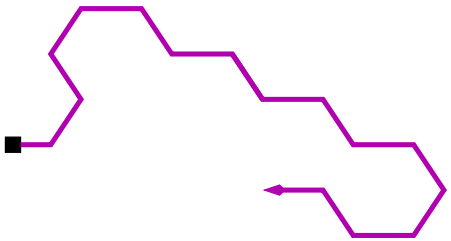
where $|\omega|$ is the length of ω, and $W(\omega)$ its winding number.



$W(\omega) = 0$



$W(\omega) = -2\pi$



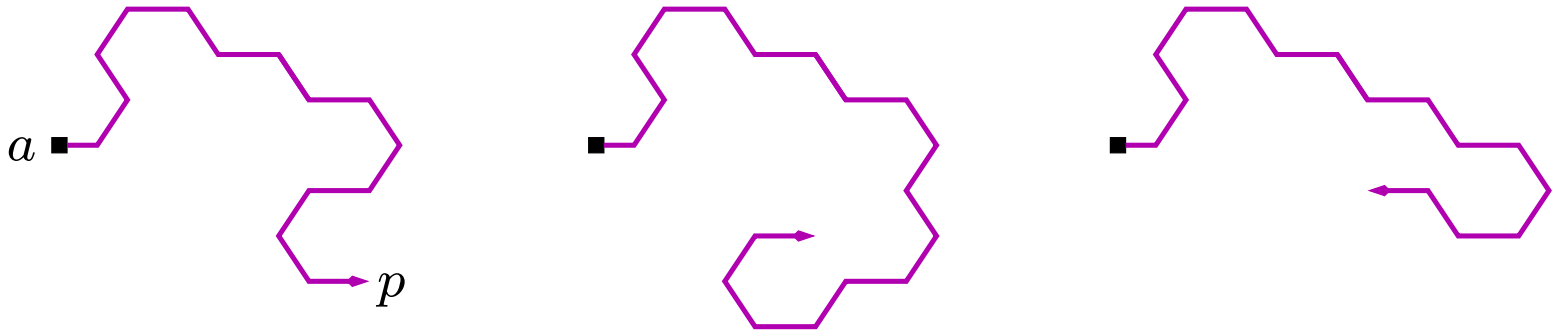
$W(\omega) = -\pi$

A local identity

Let D be a finite, simply connected domain, and a a point on the boundary of D. For p a point in the domain, let

$$F(p) \equiv F(t, \alpha; p) = \sum_{\omega : a \rightsquigarrow p} t^{|\omega|} e^{i\alpha W(\omega)},$$

where $|\omega|$ is the length of ω, and $W(\omega)$ its winding number.



$W(\omega) = 0$ $\qquad$ $W(\omega) = -2\pi$ $\qquad$ $W(\omega) = -\pi$

Rem. W is additive:

$$W(\omega) = \frac{\pi}{3}\,(\text{left turns} - \text{right turns})\,.$$

A local identity

Let

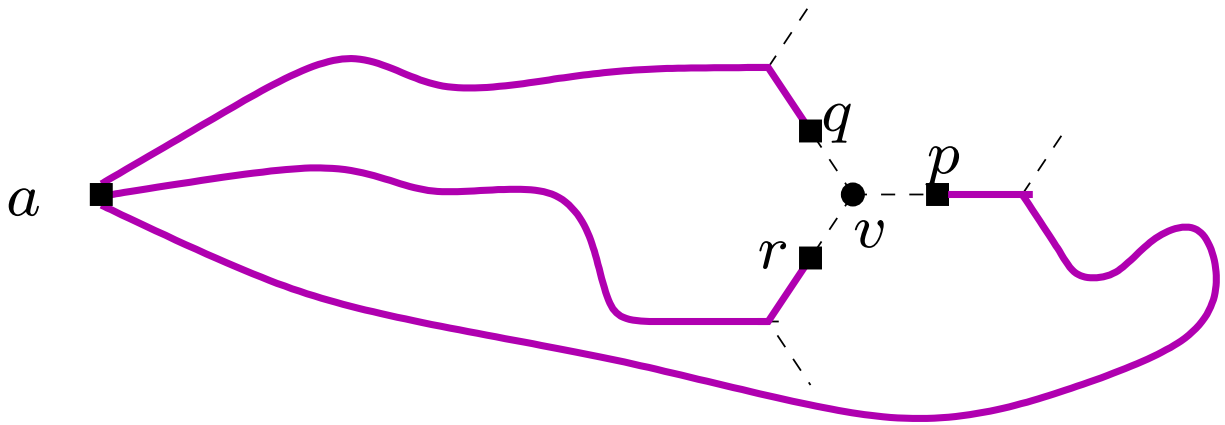
$$F(p) \equiv F(t, \alpha; p) = \sum_{\omega : a \rightsquigarrow p \text{ in } D} t^{|\omega|} e^{i\alpha W(\omega)},$$

If p, q and r are the 3 mid-edges around a vertex v of the honeycomb lattice, taken in counterclockwise order, then, for $t = t_c$ and $\alpha = -5/8$,

$$F(p) + jF(q) + j^2 F(r) = 0,$$

where $j = e^{2i\pi/3}$, or, more symmetrically,

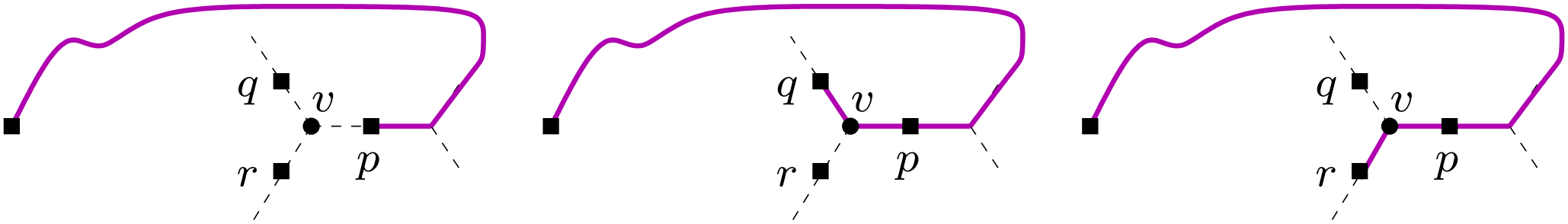
$$(p - v)F(p) + (q - v)F(q) + (r - v)F(r) = 0.$$



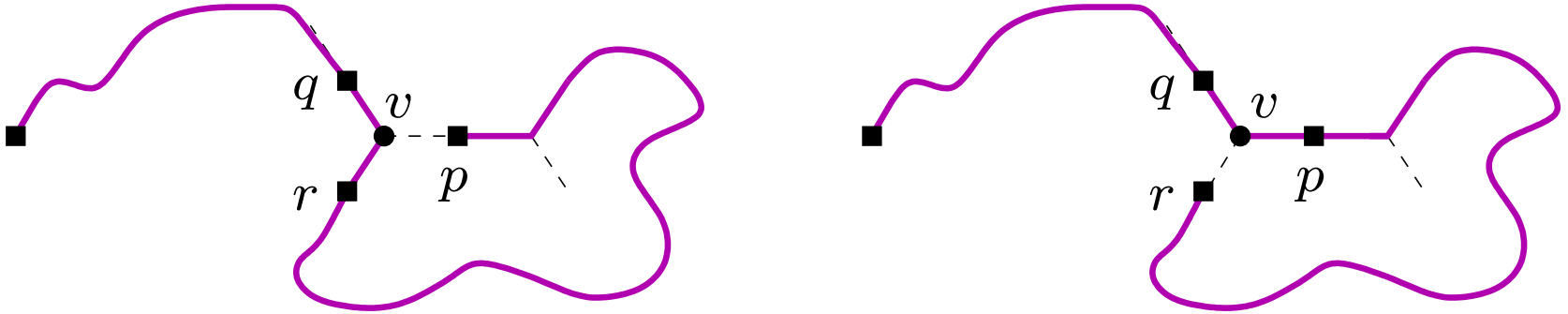
A local identity

Proof: Group walks three by three or two by two as follows

- Walks that avoid v + walks in which v is the last visited vertex:



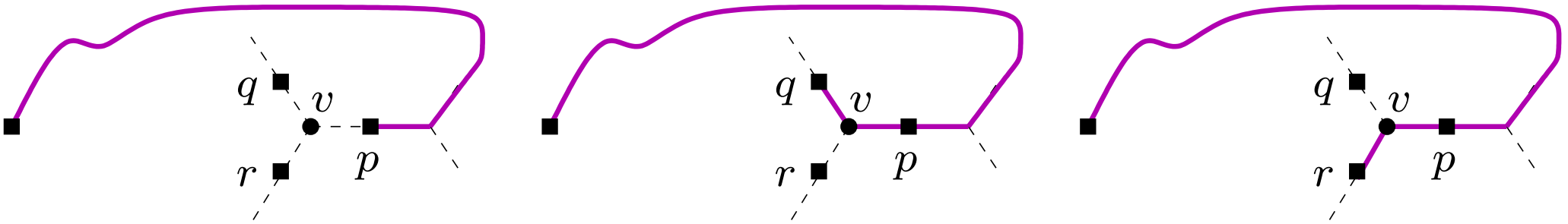
- Walks that visit v before their last vertex:



The contribution of all walks in a group is zero.

A local identity

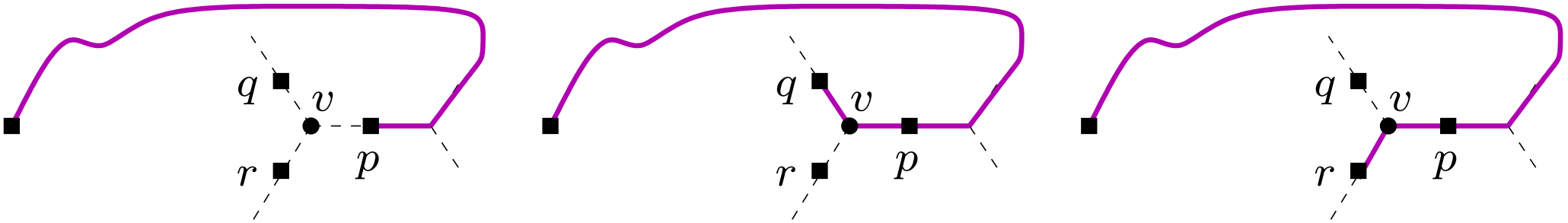
- Walks that avoid v + walks in which v is the last visited vertex:



$$1 + jt_c\ e^{-i\alpha\pi/3} + j^2 t_c\ e^{i\alpha\pi/3} = 0$$

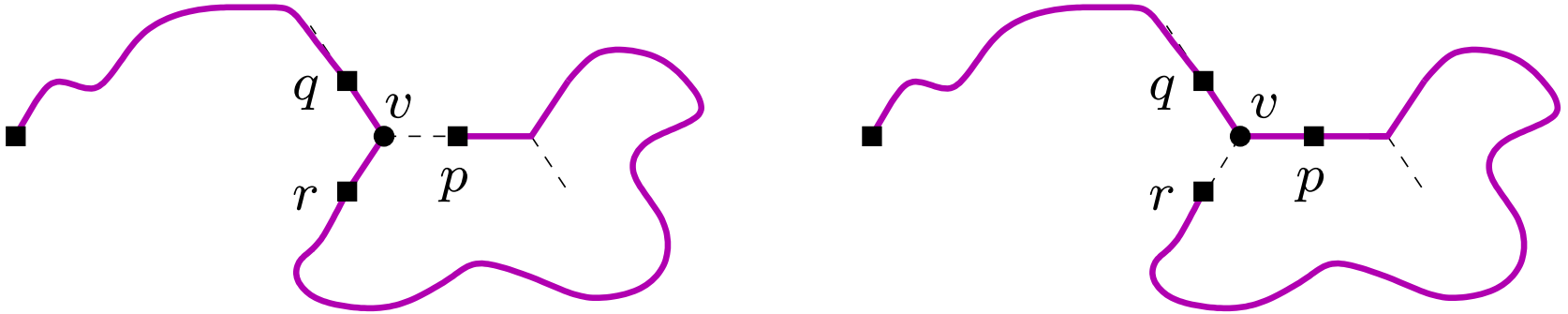
A local identity

- Walks that avoid v + walks in which v is the last visited vertex:



$$1 + jt_c\ e^{-i\alpha\pi/3} + j^2 t_c\ e^{i\alpha\pi/3} = 0$$

- Walks that visit v before their last vertex:



$$e^{-i\alpha\pi/3} e^{5i\alpha\pi/3} + j^2 e^{i\alpha\pi/3} e^{-5i\alpha\pi/3} = 2j\cos(\alpha\pi/3 - 2\pi/3) = 0$$

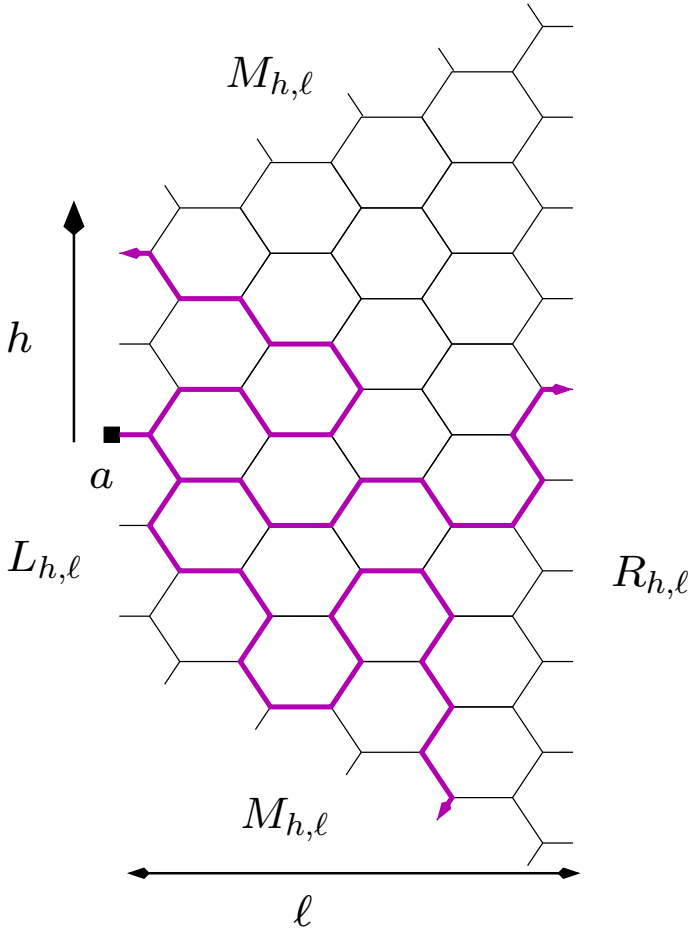
Proof of the key identity

Sum the local identity

$$(p-v)F(p)+(q-v)F(q)+(r-v)F(r)=0$$

over all vertices v of the domain $D_{h,\ell}$.

- The inner mid-edges do not contribute.
- The winding number of walks ending on the boundary is known.



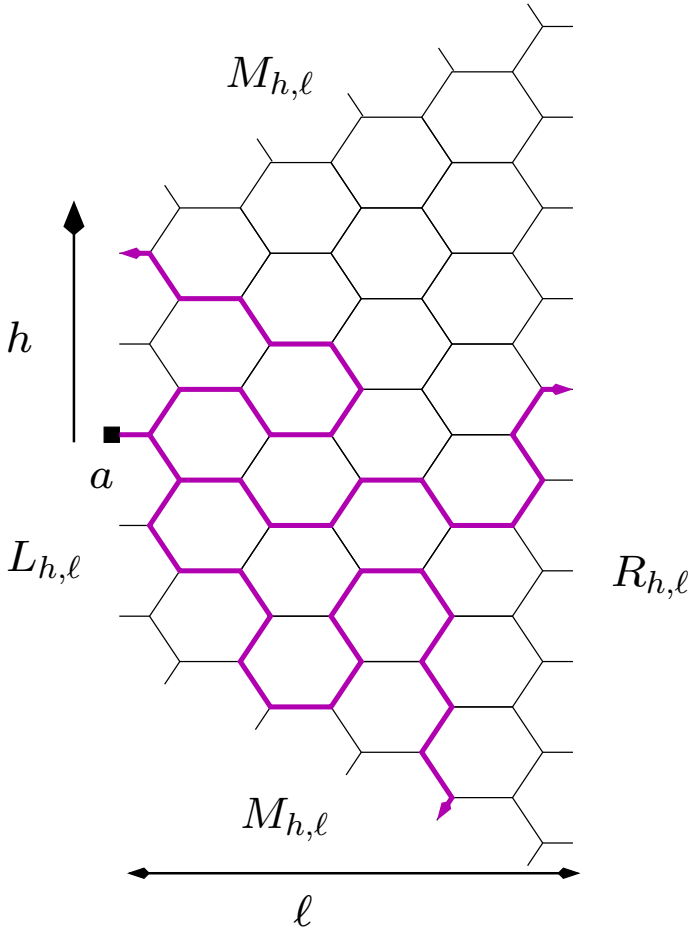
Proof of the key identity

Sum the local identity

$$(p-v)F(p)+(q-v)F(q)+(r-v)F(r)=0$$

over all vertices v of the domain $D_{h,\ell}$.

- The inner mid-edges do not contribute.
- The domain has an up-down symmetry.
- The winding number of walks ending on the boundary is known.



This gives:

$$\frac{\sqrt{2-\sqrt{2}}}{2}\,L_{h,\ell}(t_c)+R_{h,\ell}(t_c)+\frac{1}{\sqrt{2}}\,M_{h,\ell}(t_c)=1,$$

where $L_{h,\ell}(t)$ (resp. $R_{h,\ell}(t)$, $M_{h,\ell}(t)$) are the generating function of walks that start from a and end on the left (resp. right, middle) border of the domain $D_{h,\ell}$.