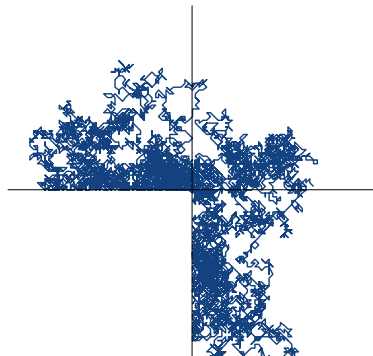
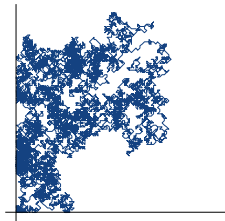


Counting lattice paths confined to cones

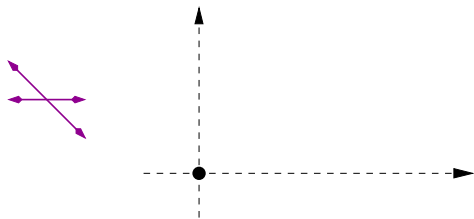
Mireille Bousquet-Mélou, CNRS, Bordeaux, France



A typical question

Let \mathcal{S} be a finite subset of \mathbb{Z}^d (set of **steps**) and $p_0 \in \mathbb{Z}^d$ (starting point).

Example. $\mathcal{S} = \{10, \bar{1}0, 1\bar{1}, \bar{1}\bar{1}\}$, $p_0 = (0, 0)$

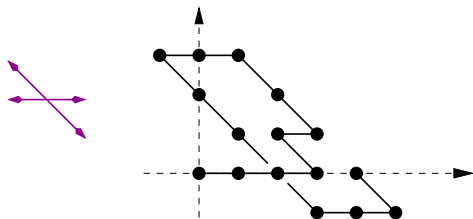


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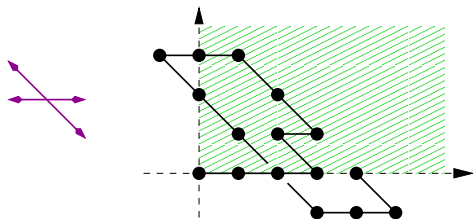
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Let C be a **cone** of \mathbb{R}^d .

Example. $\mathcal{S} = \{10, \bar{1}0, 1\bar{1}, \bar{1}1\}$, $p_0 = (0, 0)$ and $C = \mathbb{R}_+^2$.

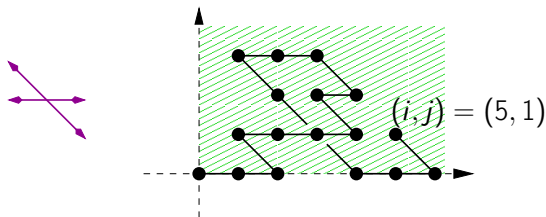


A typical question

Questions

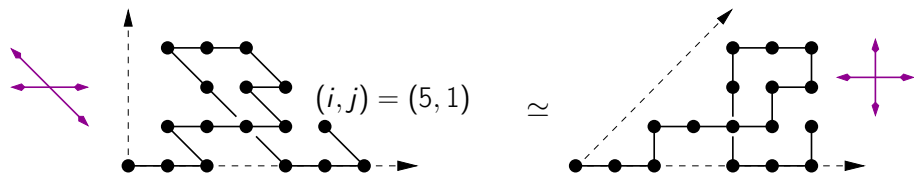
- What is the number $a(n)$ of n -step walks starting at p_0 and contained in C ?
- For $\mathbf{i} = (i_1, \dots, i_d) \in C$, what is the number $a(\mathbf{i}; n)$ of such walks that end at \mathbf{i} ?

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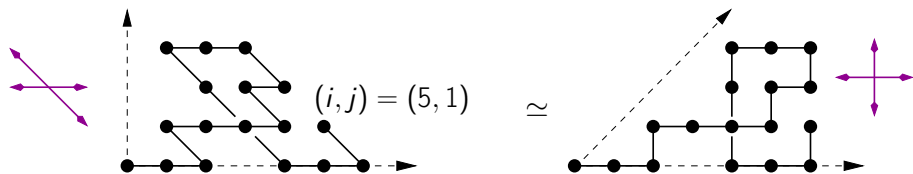
Example [Gouyou-Beauchamps 86], [mbm-Mishna 10]

Take $\mathcal{S} = \{10, \bar{1}0, 1\bar{1}, \bar{1}\bar{1}\}$, $p_0 = (0, 0)$ and $C = \mathbb{R}_+^2$



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Take $\mathcal{S} = \{10, \bar{1}0, 1\bar{1}, \bar{1}\bar{1}\}$, $p_0 = (0, 0)$ and $C = \mathbb{R}_+^2$



Nice numbers

If $n = 2m + \delta$, with $\delta \in \{0, 1\}$,

$$a(n) = \frac{n!(n+1)!}{m!(m+1)!(m+\delta)!(m+\delta+1)!}$$

Moreover, if $n = 2m + i$,

$$a(i, j; n) = \frac{(i+1)(j+1)(i+j+2)(i+2j+3)n!(n+2)!}{(m-j)!(m+1)!(m+i+2)!(m+i+j+3)!}$$

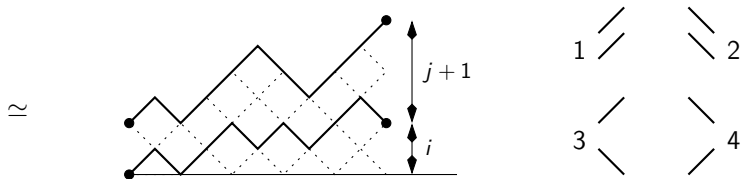
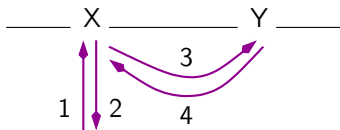
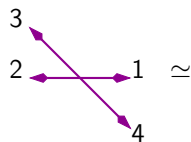
Why count walks in cones?

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- Many discrete objects can be encoded in that way:
 - ▶ in combinatorics, statistical physics...
 - ▶ in (discrete) probability theory: random walks, queuing theory...

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+ Young tableaux of height 4 [Gouyou-Beauchamps 89]

Why count walks in cones?

- Many discrete objects can be encoded in that way:
 - ▶ in combinatorics, statistical physics...
 - ▶ in (discrete) probability theory: random walks, queuing theory...
- To reach a better understanding of functional equations with **divided differences**

$$Q(x, y) = 1 + txyQ(x, y) + t \frac{Q(x, y) - Q(0, y)}{x} + t \frac{Q(x, y) - Q(x, 0)}{y}$$

Many contributions

Adan, Banderier, Bernardi, Bostan, Budd, Cori, Denisov, Duchon, Dulucq, Fayolle, Gessel, Fisher, Flajolet, Gouyou-Beauchamps, Guttmann, Guy, Janse van Rensburg, Johnson, Kauers, Koutschan, Krattenthaler, Kurkova, Kreweras, van Leeuwarden, MacMahon, Melczer, Mishna, Niederhausen, Petkovšek, Prellberg, Raschel, Rechnitzer, Sagan, Salvy, Viennot, Wachtel, Wilf, Yeats, Zeilberger...

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etc.

~~Specific question
Ad hoc solution~~



Systematic approach

A too ambitious question?

- Our original question: exact enumeration

$$a(n) = ? \quad a(\mathbf{i}; n) = ?$$

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- Generating functions:

$$\begin{aligned} A(t) &= \sum_{n \geq 0} a(n)t^n, & A(x_1, \dots, x_d; t) &= \sum_{\mathbf{i}, n} a(\mathbf{i}; n) \mathbf{x}^{\mathbf{i}} t^n \\ & & &= \sum_w \mathbf{x}^{i(w)} t^{|w|} \end{aligned}$$

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Remarks

- $A(1, \dots, 1; t) = A(t)$
- if $C \subset \mathbb{R}_+^d$, then $A(0, \dots, 0; t)$ counts walks ending at $(0, \dots, 0)$
- $A(0, x_2, \dots, x_d; t)$ counts walks ending on the hyperplane $i_1 = 0$

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Can one express these series? What is their *nature*?

A hierarchy of formal power series

- Rational series

$$A(t) = \frac{P(t)}{Q(t)}$$

- Algebraic series

$$P(t, A(t)) = 0$$

- Differentially finite series (D-finite)

$$\sum_{i=0}^d P_i(t) A^{(i)}(t) = 0$$

- D-algebraic series

$$P(t, A(t), A'(t), \dots, A^{(d)}(t)) = 0$$



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Multi-variate series: one DE per variable



A (very) basic cone: the full space

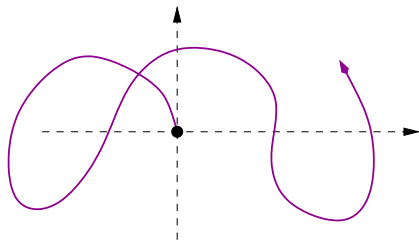
Rational series

If $\mathcal{S} \subset \mathbb{Z}^d$ is finite and $C = \mathbb{R}^d$, then $A(\mathbf{x}; t)$ is rational:

$$a(n) = |\mathcal{S}|^n \Leftrightarrow A(t) = \sum_{n \geq 0} a(n)t^n = \frac{1}{1 - |\mathcal{S}|t}$$

More generally:

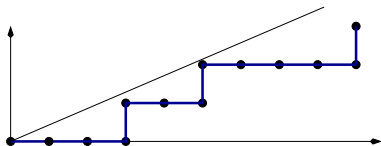
$$A(\mathbf{x}; t) = \frac{1}{1 - t \sum_{s \in \mathcal{S}} \mathbf{x}^s}.$$



Caveat: rational cones only!

The bounding hyperplanes are given by linear equations with integer/rational coefficients.

Example: the generating function of walks with N and E steps under a line of irrational slope is not known.

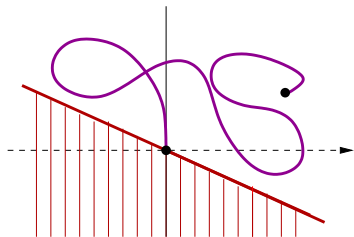


Also well-known: a (rational) half-space

Algebraic series

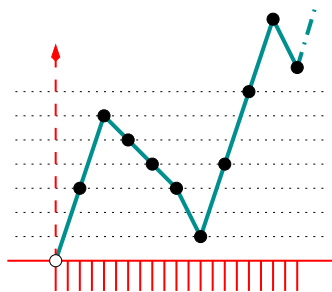
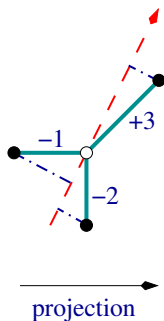
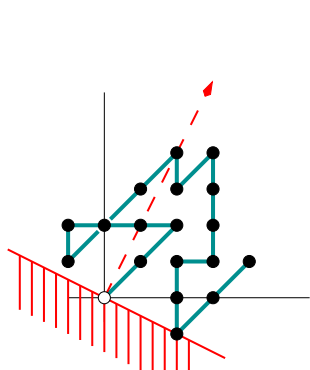
If $\mathcal{S} \subset \mathbb{Z}^d$ is finite and C is a rational half-space, then $A(\mathbf{x}; t)$ is algebraic, given by an explicit system of polynomial equations.

[Gessel 80]; [mbm-Petkovšek 00], [Duchon 00], [Banderier & Flajolet 02]...



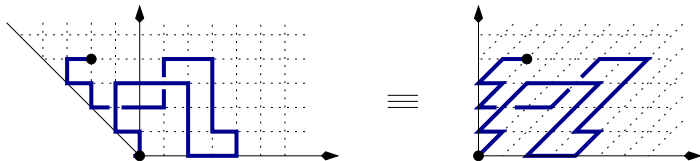
Also well-known: a (rational) half-space

By projection: Equivalent to walks in 1D confined to a half-line

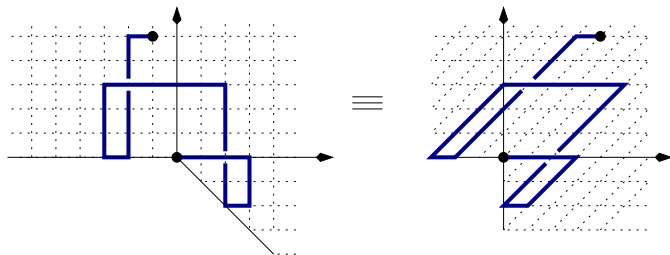


The “next” case: two bounding hyperplanes

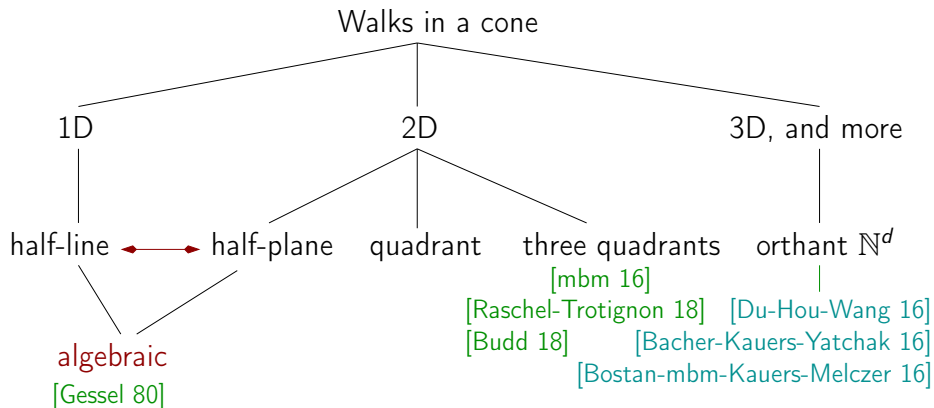
- Convex cone: walks in a quadrant



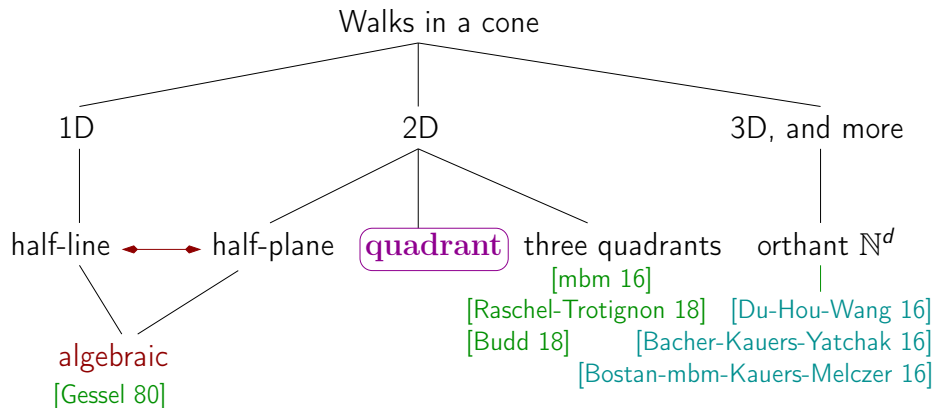
- Non-convex cone: walks avoiding a quadrant



Counting walks confined to cones



Counting walks confined to cones



A typical question

Let \mathcal{S} be a finite subset of \mathbb{Z}^2 (set of **steps**) and $p_0 \in \mathbb{Z}^d$ (starting point).

Questions

- What is the number $q(n)$ of n -step walks starting at p_0 and contained in the quadrant \mathbb{N}^2 ?
- For $(i, j) \in \mathbb{N}^2$, what is the number $q(i, j; n)$ of such walks that end at (i, j) ?

The associated generating function:

$$Q(x, y; t) = \sum_{i, j, n \geq 0} q(i, j; n) x^i y^j t^n = ?$$

II. Walks in a quadrant: asymptotic enumeration

1. Excursions (prescribed endpoint (i, j))
2. All quadrant walks

Expected:

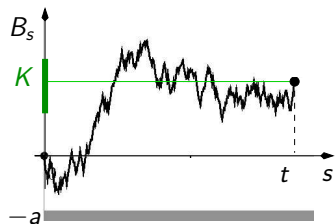
$$q(i, j; n) \sim \kappa \mu_e^n n^{\gamma_e} \qquad q(n) \sim \kappa \mu_w^n n^{\gamma_w}$$

possibly with periodicity conditions on n .

The 1D case: excursions on a half-line

- The brownian exponent: as $t \rightarrow \infty$,

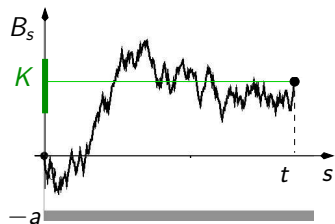
$$\mathbb{P}(B_s > -a \text{ for } s \in [0, t] \text{ and } B_t \in K) \sim \kappa t^{-3/2}$$



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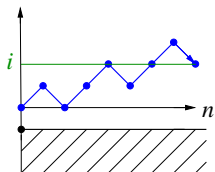
- Lattice walks: let \mathcal{S} be a (finite) step set, and let $S(x) = \sum_{s \in \mathcal{S}} x^s$. For walks on the non-negative half-line ending at position i ,

$$h(i; n) \sim \kappa \mu^n n^{-3/2}$$

where

$$\mu = \min_{x>0} S(x)$$

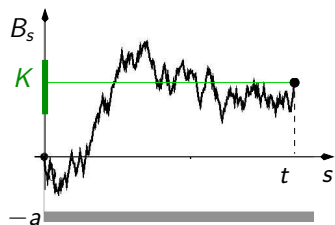
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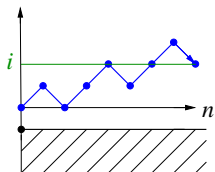
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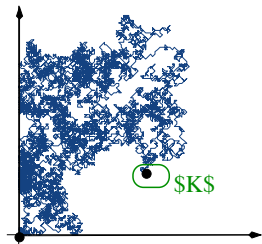
$$= S(x_c) \quad \text{with} \quad S'(x_c) = 0$$



Remark: $\mu \leq S(1)$, with equality iff $S'(1) = \sum_{s \in \mathcal{S}} s = 0 \Leftrightarrow$ **NO DRIFT**

The 2D case: excursions in a quadrant

- The (2D) brownian exponent: as $t \rightarrow \infty$,
$$\mathbb{P}(B_s > -\mathbf{a} \text{ for } s \in [0, t] \text{ and } B_t \in K) \sim \kappa t^{-3}$$






The 2D case: excursions in a quadrant

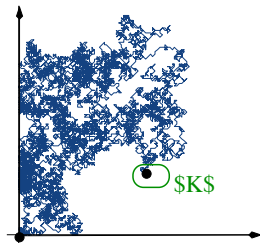
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- Lattice walks:

$$q(0, 0; n) \sim \kappa 4^n n^{-3}$$

				
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


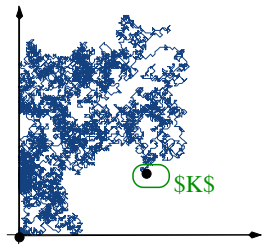
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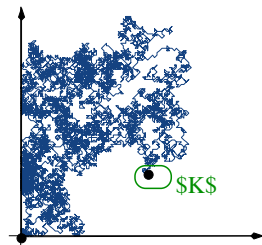
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The 2D case: excursions in a quadrant

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


What's happening?

The 2D case: excursions in a quadrant

- The (2D) brownian exponent: as $t \rightarrow \infty$,

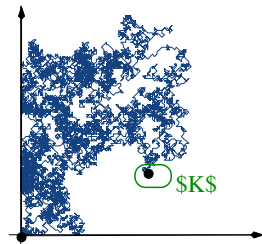
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What's happening?

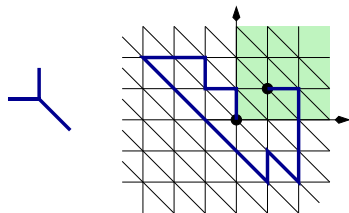
The x- and y-moves are **correlated**.



Excursions in a quadrant: an example



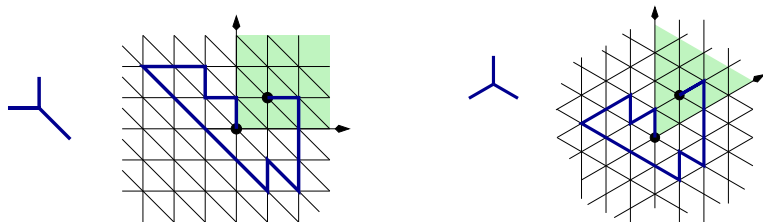
Take a uniform random walk with steps $(0, 1)$, $(-1, 0)$, $(1, -1)$. We have $\mathbb{E}(X) = \mathbb{E}(Y) = 0$, $\mathbb{E}(X^2) = \mathbb{E}(Y^2) = 2/3$ and $\mathbb{E}(XY) = -1/3$.



Excursions in a quadrant: an example



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Excursions in a quadrant: an example



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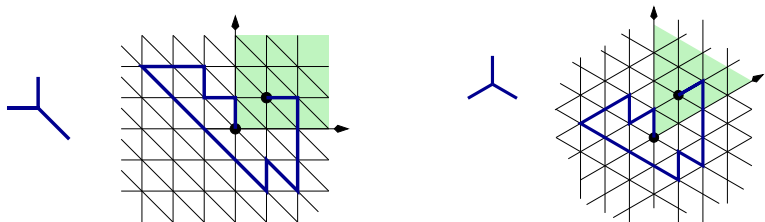
Define

$$Y' = \frac{1}{\sqrt{3}}(X + 2Y),$$

so that

$$\mathbb{E}(X) = \mathbb{E}(Y') = 0, \quad \mathbb{E}(X^2) = \mathbb{E}(Y'^2) = 2/3 \quad \text{and} \quad \mathbb{E}(XY') = 0.$$

But the confining quadrant has become a wedge of $\pi/3$!






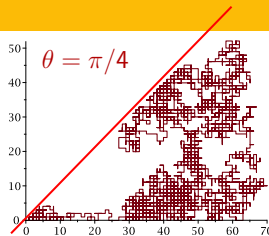
The 2D case: excursions in a wedge

- Brownian exponents: as $t \rightarrow \infty$,

$$\mathbb{P}(B_s \in W_\theta \text{ for } s \in [0, t] \text{ and } B_t \in K) \sim \kappa t^{-1-\pi/\theta}$$

- Lattice walks:

			
θ	$\pi/2$	$\pi/3$	$2\pi/3$
$q(0, 0; n) \sim$	$\kappa 4^n n^{-3}$	$\kappa 3^n n^{-4}$	$\kappa 3^n n^{-5/2}$



The 2D case: excursions in a wedge

- Brownian exponents: as $t \rightarrow \infty$,

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- Lattice walks: let $\mathcal{S} \subset \mathbb{Z}^2$ be a (finite) step set, and let $S(x, y) = \sum_{(k, \ell) \in \mathcal{S}} x^k y^\ell$. For walks in the quadrant \mathbb{N}^2 ending at (i, j) ,

$$q(i, j; n) \sim \kappa \mu^n n^{-1-\pi/\theta}$$

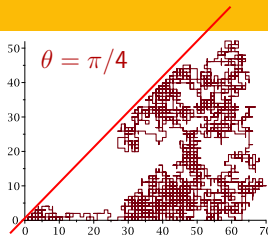
where

$$\begin{aligned} \mu &= \min_{x>0, y>0} S(x, y) \\ &= S(x_c, y_c) \quad \text{with} \quad S'_1(x_c, y_c) = S'_2(x_c, y_c) = 0 \end{aligned}$$

and

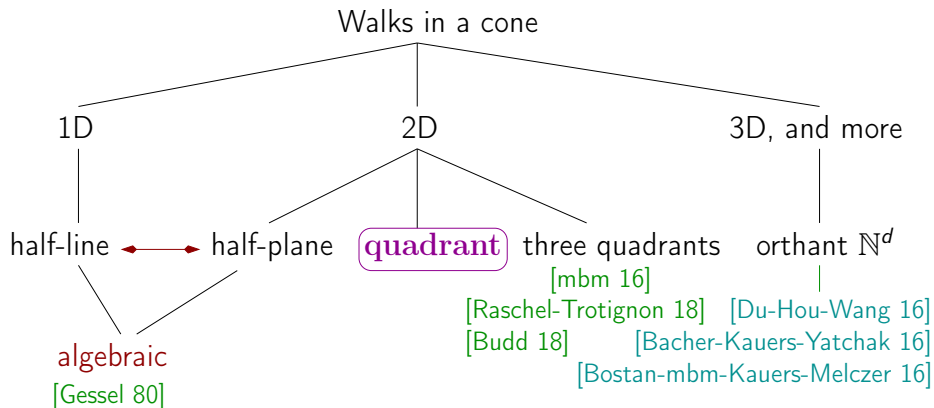
$$\theta = \arccos \left(- \frac{S''_{1,2}(x_c, y_c)}{\sqrt{S''_{1,1}(x_c, y_c) S''_{2,2}(x_c, y_c)}} \right)$$

[Denisov &
Wachtel 15]

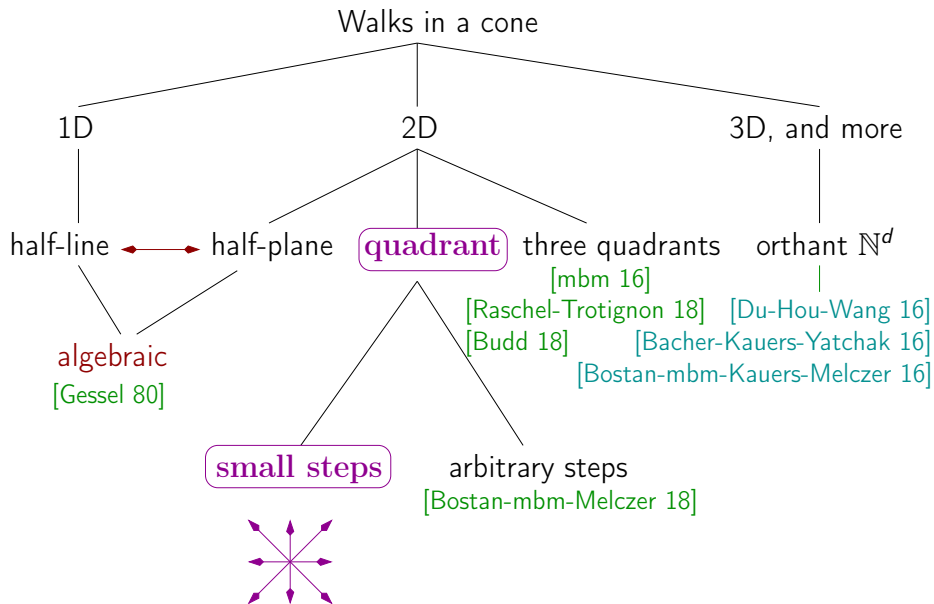


III. Walks in a quadrant: exact enumeration

Counting walks confined to cones



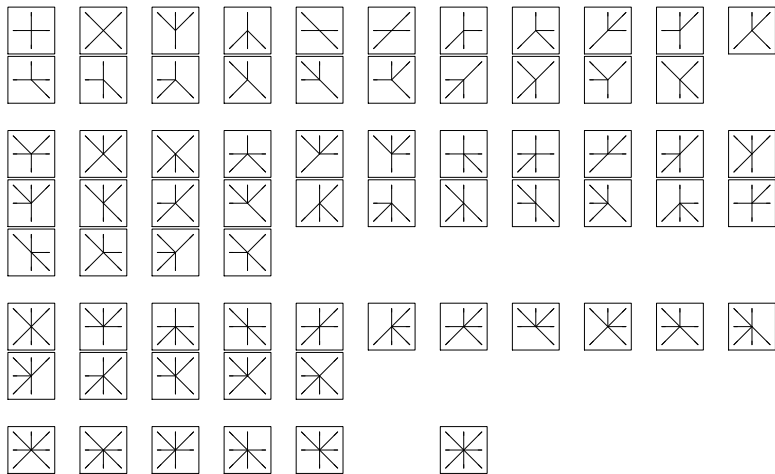
Counting walks confined to cones



The 79 interesting distinct quadrant models with small steps



Non-singular

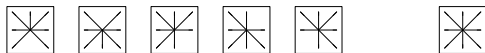
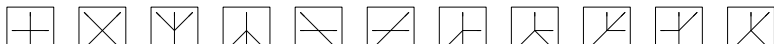


Singular

The 79 interesting distinct quadrant models with small steps



Non-singular



Singular

The starting point: a recurrence relation...

The numbers $q(i, j; n)$ satisfy

$$q(i, j; n) = \begin{cases} 0 & \text{if } i < 0 \text{ or } j < 0 \text{ or } n < 0, \\ \mathbb{1}_{i=j=0} & \text{if } n = 0, \\ \sum_{(i', j') \in \mathcal{S}} q(i - i', j - j'; n - 1) & \text{otherwise.} \end{cases}$$

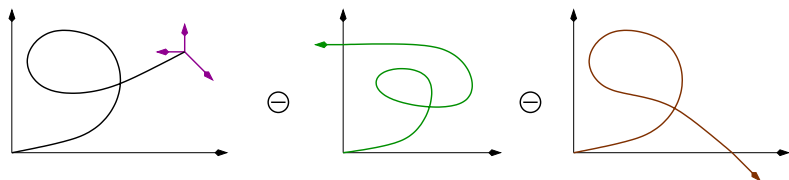
\Rightarrow an equation for

$$Q(x, y; t) = \sum_{i, j, n \geq 0} q(i, j; n) x^i y^j t^n$$



Example: $\mathcal{S} = \{01, \bar{1}0, 1\bar{1}\}$, with $\bar{x} := 1/x$ and $\bar{y} := 1/y$

$$Q(x, y; t) \equiv Q(x, y) = 1 + t(y + \bar{x} + x\bar{y})Q(x, y) - t\bar{x}Q(0, y) - tx\bar{y}Q(x, 0)$$



$$Q(x, y; t) = \sum_{i, j, n \geq 0} q(i, j; n) x^i y^j t^n$$

... and the corresponding functional equation



Example: $\mathcal{S} = \{01, \bar{1}0, 1\bar{1}\}$, with $\bar{x} := 1/x$ and $\bar{y} := 1/y$

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or

$$(1 - t(y + \bar{x} + x\bar{y}))Q(x, y) = 1 - t\bar{x}Q(0, y) - tx\bar{y}Q(x, 0),$$

... and the corresponding functional equation



Example: $\mathcal{S} = \{01, \bar{1}0, 1\bar{1}\}$, with $\bar{x} := 1/x$ and $\bar{y} := 1/y$

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or

$$(1 - t(y + \bar{x} + x\bar{y}))Q(x, y) = 1 - t\bar{x}Q(0, y) - tx\bar{y}Q(x, 0),$$

or

$$(1 - t(y + \bar{x} + x\bar{y}))xyQ(x, y) = xy - tyQ(0, y) - tx^2Q(x, 0)$$

- The polynomial $1 - t(y + \bar{x} + x\bar{y})$ is the **kernel** of this equation
- The equation is linear, with two **catalytic** variables x and y (tautological at $x = 0$ or $y = 0$)

Exact results: examples

- expressions for $q(n)$, for $q(i, j; n)$
- expressions for the generating functions

$$Q(t) := \sum_n q(n)t^n \quad \text{and} \quad Q(x, y; t) := \sum_{i, j, n} x^i y^j t^n q(i, j; n)$$

- **nature** of these generating functions

A hierarchy of formal power series

- Rational series

$$A(t) = \frac{P(t)}{Q(t)}$$

- Algebraic series

$$P(t, A(t)) = 0$$

- Differentially finite series (D-finite)

$$\sum_{i=0}^d P_i(t) A^{(i)}(t) = 0$$

- D-algebraic series

$$P(t, A(t), A'(t), \dots, A^{(d)}(t)) = 0$$



1. Expressions for numbers

- Square lattice walks

$$q(0, 0; 2n) = C_n C_{n+1} \quad \text{with} \quad C_n = \frac{1}{n+1} \binom{2n}{n}$$



- Kreweras' walks [Kreweras 65]

$$q(0, 0; 3n) = \frac{4^n}{(n+1)(2n+1)} \binom{3n}{n}$$



- Gessel's walks [Kauers-Zeilberger 09]

$$q(0, 0; 2n) = 16^n \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n}$$



where $(a)_n = a(a+1) \cdots (a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$ is the ascending factorial.

2. Expressions for series

- As positive parts of rational series in t (with $\bar{x} := 1/x$, $\bar{y} := 1/y$):

$$Q(x, y; t) = [x^{\geq 0} y^{\geq 0}] \frac{1 - \bar{x}^2 y + \bar{x}^3 - \bar{x}^2 \bar{y}^2 + \bar{y}^3 - x \bar{y}^2}{1 - t(y + \bar{x} + x \bar{y})}$$



[mbm-Mishna 09]

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- As positive parts of rational series in t (with $\bar{x} := 1/x$, $\bar{y} := 1/y$):

$$Q(x, y; t) = [x^{\geq 0} y^{\geq 0}] \frac{1 - \bar{x}^2 y + \bar{x}^3 - \bar{x}^2 \bar{y}^2 + \bar{y}^3 - x \bar{y}^2}{1 - t(y + \bar{x} + x \bar{y})}$$



[mbm-Mishna 09]

- As explicit integrals involving hypergeometric series

$$Q(0, 0; t) = \frac{2}{t^2} \int_0^t \int_0^u \frac{1}{(1 - 4v^2)^{3/2}} {}_2F_1\left(\frac{3}{4}, \frac{5}{4}; 2; 64 \frac{(v+1)v^3}{(1-4v^2)^2}\right) dv du$$



[Bostan-Chyzak-van Hoeij-Kauers-Pech 17]

2. Expressions for series (cont'd)

- In terms of Weierstrass' function and elliptic integrals

$$t(1+y)Q(0,y;t) + \frac{1}{y} = \frac{I'(0)}{I(y) - I(0)} - \frac{I'(0)}{I(-1) - I(0)} - 1$$

with

$$I(y) \equiv I(y;t) = \wp(\mathcal{R}(y;t), \omega_1(t), \omega_3(t))$$

where

- ▶ \wp is Weierstrass' elliptic function
- ▶ its periods ω_1 and ω_3 are explicit elliptic integrals
- ▶ its argument \mathcal{R} is also an explicit elliptic integral



[Bernardi-mbm-Raschel 17(a)]

2. Expressions for series (cont'd)

- Climax: an **integral** expression involving the same ingredients

$$\tilde{K}(0, y; t)Q(0, y; t) - \tilde{K}(0, 0; t)Q(0, 0; t) = yX_0(y; t) + \frac{1}{2i\pi} \int_{y_1(t)}^{y_2(t)} u [X_0(u; t) - X_1(u; t)] \left[\frac{\partial_u I(u; t)}{I(u; t) - I(y; t)} - \frac{\partial_u I(u; t)}{I(u; t) - I(0; t)} \right] du$$

where $\tilde{K}(x, y; t) = xy(1 - tS(x, y))$, X_0 , X_1 , y_1 and y_2 are explicit algebraic series and $I(y; t)$ is as given on the previous slide.

Valid for all (non-singular) small step models

[Raschel 12]

3. Nature of the series



Algebraic

[Kreweras 65, Gessel 86]

$$(1 - t(\bar{x} + \bar{y} + xy))_{xy}Q(x, y) = xy - tyQ(0, y) - txQ(x, 0)$$



D-finite, but transcendental

[Gessel 90]

$$(1 - t(y + \bar{x} + x\bar{y}))_{xy}Q(x, y) = xy - tyQ(0, y) - tx^2Q(x, 0)$$



Not D-finite, but D-algebraic

[Bernardi, mbm & Raschel 17]

$$(1 - t(x + \bar{x} + y + x\bar{y}))_{xy}Q(x, y) = xy - tyQ(0, y) - tx^2Q(x, 0)$$

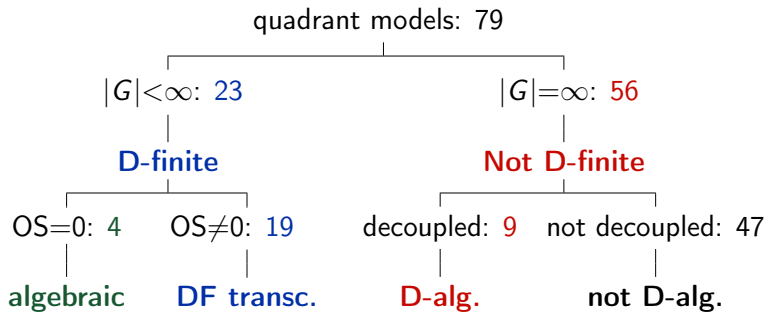


Not D-algebraic (in y)

[Dreyfus, Hardouin, Roques & Singer 17]

$$(1 - t(x\bar{y} + \bar{x} + \bar{y} + y))_{xy}Q(x, y) = xy - tyQ(0, y) - tx(1 + x)Q(x, 0)$$

Classification of quadrant walks: a variety of tools



Formal power series algebra

Complex analysis

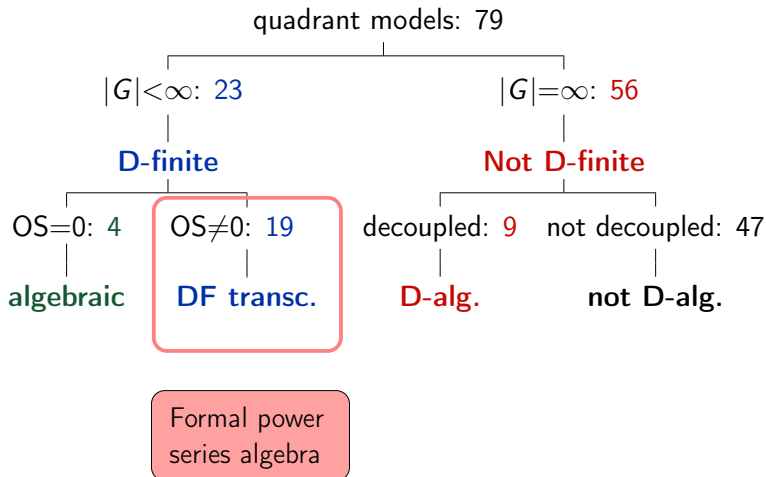
Differential Galois theory

Computer algebra

Random walks in probability

D-finite series
effective closure properties
arithmetic properties
asymptotics
G-functions

III.1. The group of the walk, and D-finite cases



The group of the model



Example. Take $\mathcal{S} = \{\bar{1}0, 01, 1\bar{1}\}$, with **step polynomial**

$$S(x, y) = \frac{1}{x} + y + \frac{x}{y} = \bar{x} + y + x\bar{y}$$



Example. Take $\mathcal{S} = \{\bar{1}0, 01, 1\bar{1}\}$, with **step polynomial**

$$S(x, y) = \frac{1}{x} + y + \frac{x}{y} = \bar{x} + y + x\bar{y}$$

Observation: $S(x, y)$ is left unchanged by the rational transformations

$$\Phi : (x, y) \mapsto (\bar{x}y, y) \quad \text{and} \quad \Psi : (x, y) \mapsto (x, x\bar{y}).$$

The group of the model



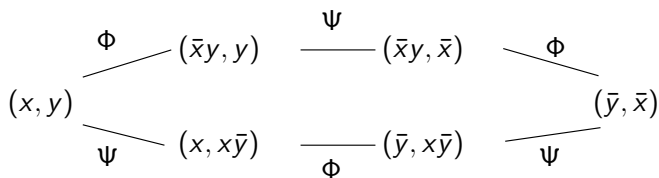
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They are involutions, and generate a finite dihedral group G :



The group of the model



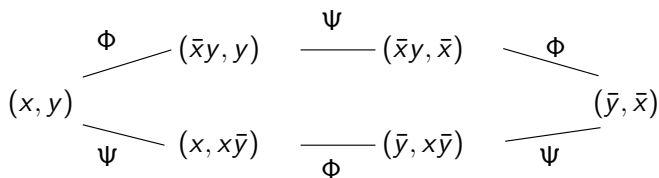
Example. Take $\mathcal{S} = \{\bar{1}0, 01, 1\bar{1}\}$, with **step polynomial**

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They are involutions, and generate a finite dihedral group G :



Remark. G can be defined for any quadrant model with small steps

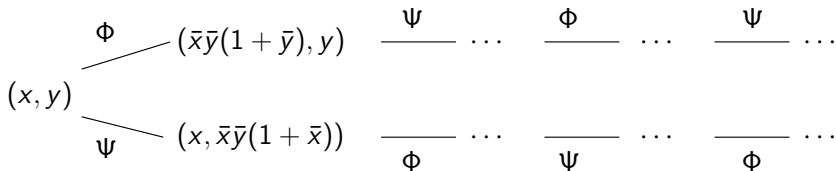
The group is not always finite



- If $S = \{0\bar{1}, \bar{1}\bar{1}, \bar{1}0, 11\}$, then $S(x, y) = \bar{x}(1 + \bar{y}) + \bar{y} + xy$ and

$$\Phi : (x, y) \mapsto (\bar{x}\bar{y}(1 + \bar{y}), y) \quad \text{and} \quad \Psi : (x, y) \mapsto (x, \bar{x}\bar{y}(1 + \bar{x}))$$

generate an infinite group:



The algebraic kernel method



- The equation reads (with $K(x, y) = 1 - t(y + \bar{x} + x\bar{y})$):

$$K(x, y)xyQ(x, y) = xy - tx^2Q(x, 0) - tyQ(0, y)$$

- The orbit of (x, y) under G is

$$(x, y) \xleftrightarrow{\Phi} (\bar{x}y, y) \xleftrightarrow{\Psi} (\bar{x}y, \bar{x}) \xleftrightarrow{\Phi} (\bar{y}, \bar{x}) \xleftrightarrow{\Psi} (\bar{y}, x\bar{y}) \xleftrightarrow{\Phi} (x, x\bar{y}) \xleftrightarrow{\Psi} (x, y).$$

The algebraic kernel method



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- The orbit of (x, y) under G is

$$(x, y) \xleftarrow{\Phi} (\bar{x}y, y) \xleftarrow{\Psi} (\bar{x}y, \bar{x}) \xleftarrow{\Phi} (\bar{y}, \bar{x}) \xleftarrow{\Psi} (\bar{y}, x\bar{y}) \xleftarrow{\Phi} (x, x\bar{y}) \xleftarrow{\Psi} (x, y).$$

- All transformations of G leave $K(x, y)$ invariant. Hence

$$K(x, y)xyQ(x, y) = xy - tx^2Q(x, 0) - tyQ(0, y)$$

$$K(x, y)\bar{x}y^2Q(\bar{x}y, y) = \bar{x}y^2 - t\bar{x}^2y^2Q(\bar{x}y, 0) - tyQ(0, y)$$

$$K(x, y)\bar{x}^2yQ(\bar{x}y, \bar{x}) = \bar{x}^2y - t\bar{x}^2y^2Q(\bar{x}y, 0) - t\bar{x}Q(0, \bar{x})$$

$$\dots = \dots$$

$$K(x, y)x^2\bar{y}Q(x, x\bar{y}) = x^2\bar{y} - tx^2Q(x, 0) - tx\bar{y}Q(0, x\bar{y}).$$



⇒ Form the alternating sum of the equation over all elements of the orbit:

$$\begin{aligned} K(x, y) & \left(xyQ(x, y) - \bar{x}y^2Q(\bar{x}y, y) + \bar{x}^2yQ(\bar{x}y, \bar{x}) \right. \\ & \left. - \bar{x}\bar{y}Q(\bar{y}, \bar{x}) + x\bar{y}^2Q(\bar{y}, x\bar{y}) - x^2\bar{y}Q(x, x\bar{y}) \right) = \\ & \quad xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y} \\ & \quad \text{(the orbit sum).} \end{aligned}$$

Why is this interesting?



$$\begin{aligned} &xyQ(x, y) - \bar{x}y^2Q(\bar{x}y, y) + \bar{x}^2yQ(\bar{x}y, \bar{x}) \\ &\quad - \bar{x}\bar{y}Q(\bar{y}, \bar{x}) + x\bar{y}^2Q(\bar{y}, x\bar{y}) - x^2\bar{y}Q(x, x\bar{y}) = \\ &\qquad\qquad\qquad \frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})} \end{aligned}$$

- Both sides are power series in t , with coefficients in $\mathbb{Q}[x, \bar{x}, y, \bar{y}]$.

Why is this interesting?



$$\begin{aligned} xyQ(x, y) - \bar{x}y^2Q(\bar{x}y, y) + \bar{x}^2yQ(\bar{x}y, \bar{x}) \\ - \bar{x}\bar{y}Q(\bar{y}, \bar{x}) + x\bar{y}^2Q(\bar{y}, x\bar{y}) - x^2\bar{y}Q(x, x\bar{y}) = \\ \frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})} \end{aligned}$$

- Both sides are power series in t , with coefficients in $\mathbb{Q}[x, \bar{x}, y, \bar{y}]$.
- Extract the part with positive powers of x and y :

$$xyQ(x, y) = [x^{>0}y^{>0}] \frac{xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + x\bar{y}^2 - x^2\bar{y}}{1 - t(y + \bar{x} + x\bar{y})}$$

is a D-finite series.

[Lipshitz 88]

The kernel method in general (finite groups)

- For all models with a finite group,

$$\sum_{g \in G} \text{sign}(g) g(xyQ(x, y; t)) = \frac{1}{K(x, y; t)} \sum_{g \in G} \text{sign}(g) g(xy) = \frac{OS}{K(x, y; t)},$$

where $g(Q(x, y)) := Q(g(x, y))$.

- The right-hand side is an explicit rational series.

[mbm-Mishna 10]

The kernel method in general (finite groups)

- For all models with a finite group,

$$\sum_{g \in G} \text{sign}(g) g(xyQ(x, y; t)) = \frac{1}{K(x, y; t)} \sum_{g \in G} \text{sign}(g) g(xy) = \frac{OS}{K(x, y; t)},$$

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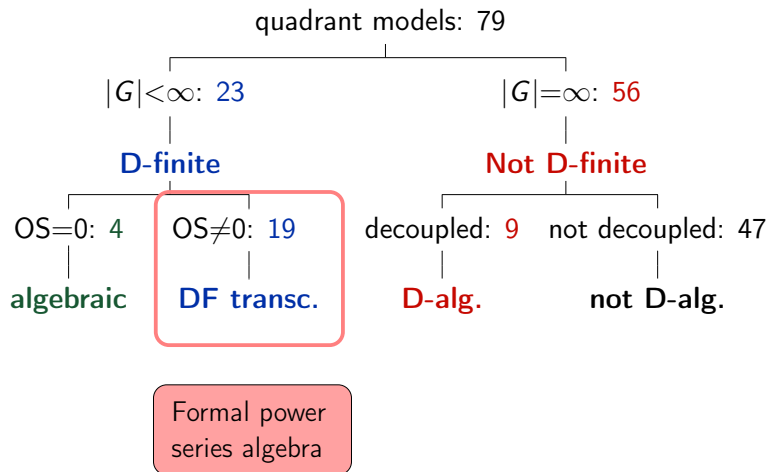
- The right-hand side is an explicit rational series.
- For the 19 models where the orbit sum is non-zero,

$$xyQ(x, y; t) = [x^{>0}y^{>0}] \frac{OS}{K(x, y; t)}$$

is a D-finite series.

[mbm-Mishna 10]

The group of the walk, and D-finite cases



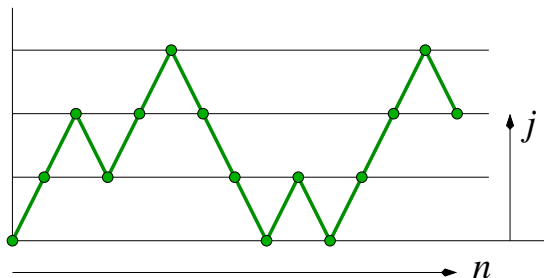
III.2. Computer algebra: a guess & check approach

In 1D: Walks on a half-line with steps $+1, -1$

- Generating function:

$$H(y; t) \equiv H(y) = \sum_{n \geq 0} \sum_{j \geq 0} h(j; n) y^j t^n$$

$h(j; n)$: number of n -step walks on the half-line ending at ordinate j



In 1D: Walks on a half-line with steps $+1, -1$

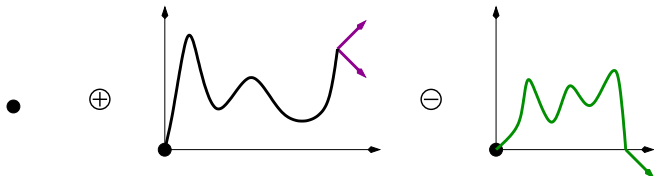
- Generating function:

$$H(y; t) \equiv H(y) = \sum_{n \geq 0} \sum_{j \geq 0} h(j; n) y^j t^n$$

- Step-by-step construction:

$$H(y) = 1 + t(y + \bar{y})H(y) - t\bar{y}H(0)$$

with $\bar{y} = 1/y$.



In 1D: Walks on a half-line with steps $+1, -1$

- Generating function:

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$$H(y) = 1 + t(y + \bar{y})H(y) - t\bar{y}H(0)$$

or

$$(1 - t(y + \bar{y}))H(y) = 1 - t\bar{y}H(0).$$

Guess & check for walks on a half-line

- The equation:

$$(1 - t(y + \bar{y}))H(y; t) = 1 - t\bar{y}H(0; t). \quad (1)$$

Guess & check for walks on a half-line

- The equation:

$$(1 - t(y + \bar{y}))H(y; t) = 1 - t\bar{y}H(0; t). \quad (1)$$

- A uniqueness result: there exists a unique series $H(y; t) \equiv H(y)$ in t , with coefficients in $\mathbb{Q}[y]$, satisfying the above equation.

Guess & check for walks on a half-line

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- Generate the first coefficients:

$$H(0) \equiv H(0; t) = 1 + t^2 + 2t^4 + 5t^6 + 14t^8 + 42t^{10} + O(t^{12}).$$

Guess & check for walks on a half-line

- The equation:

$$(1 - t(y + \bar{y}))H(y; t) = 1 - t\bar{y}H(0; t). \quad (1)$$

- A uniqueness result: there exists a unique series $H(y; t) \equiv H(y)$ in t , with coefficients in $\mathbb{Q}[y]$, satisfying the above equation.
- Generate the first coefficients:

$$H(0) \equiv H(0; t) = 1 + t^2 + 2t^4 + 5t^6 + 14t^8 + 42t^{10} + O(t^{12}).$$

- Guess that [Gfun]

$$H(0) = 1 + t^2H(0)^2 \quad \text{and} \quad H(y) = 1 + t(ty^2 + t - y)H(y)^2 + 2tyH(y).$$

Guess & check for walks on a half-line

- The equation:

$$(1 - t(y + \bar{y}))H(y; t) = 1 - t\bar{y}H(0; t). \quad (1)$$

- A uniqueness result: there exists a unique series $H(y; t) \equiv H(y)$ in t , with coefficients in $\mathbb{Q}[y]$, satisfying the above equation.
- Generate the first coefficients:

$$H(0) \equiv H(0; t) = 1 + t^2 + 2t^4 + 5t^6 + 14t^8 + 42t^{10} + O(t^{12}).$$

- Guess that [Gfun]

$$H(0) = 1 + t^2H(0)^2 \quad \text{and} \quad H(y) = 1 + t(ty^2 + t - y)H(y)^2 + 2tyH(y).$$

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- Check (using resultants, and the first coeffs.) that $\tilde{H}(y)$ satisfies (1).

Some guess & check quadrant results

- Algebraicity results for $Q(x, y; t)$
 - ▶ Kreweras' walks [Bostan, Kauers 10]
 - ▶ Gessel's walks [Bostan, Kauers 10]



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- D-finiteness results for $Q(x, y; t)$

- ▶ Gessel's walks (for $Q(0, 0; t)$ only) [Kauers, Zeilberger 09]



- ▶ For weighted D-finite models that resist the algebraic kernel method

[Bostan, mbm, Kauers, Melczer 16]



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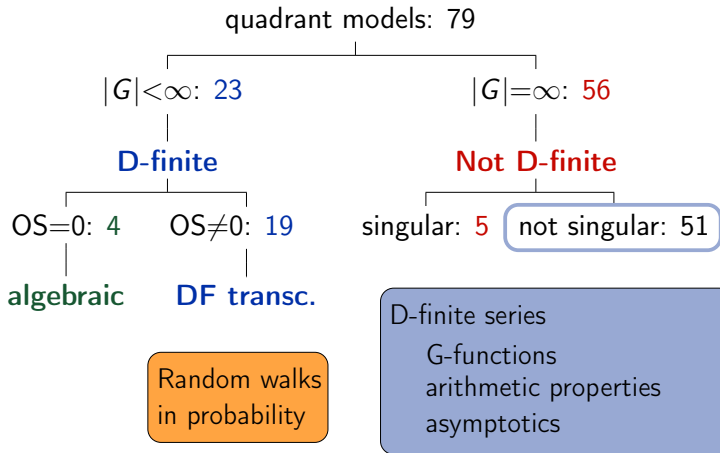
[Bostan, mbm, Kauers, Melczer 16]



For Gessel's walks, the polynomial annihilating $Q(x, y; t)$ has size about 30Gb.

Even with big computers, one needs to be clever!

III.3. Non-D-finiteness via asymptotics



Non-D-finiteness via asymptotics

- **The excursion exponent:** let $\mathcal{S} \subset \mathbb{Z}^2$ be a (finite) step set, and let $S(x, y) = \sum_{(k, \ell) \in \mathcal{S}} x^k y^\ell$. For walks in the quadrant \mathbb{N}^2 ending at $(0, 0)$:

$$q(0, 0; n) \sim \kappa \mu^n n^{-1-\pi/\theta}$$

where

$$\theta = \arccos(-c), \quad c = \frac{S''_{1,2}(x_c, y_c)}{\sqrt{S''_{1,1}(x_c, y_c) S''_{2,2}(x_c, y_c)}}$$

with $S'_1(x_c, y_c) = S'_2(x_c, y_c) = 0$.

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Asymptotics of D-finite generating functions (G-functions)
[André 89], [Chudnovsky² 85], [Katz 70]

If the exponent $-1 - \pi/\theta$ is irrational, then $Q(0, 0; t)$ cannot be D-finite.

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If c is not the root of a cyclotomic polynomial, then $Q(0, 0; t)$ cannot be D-finite.

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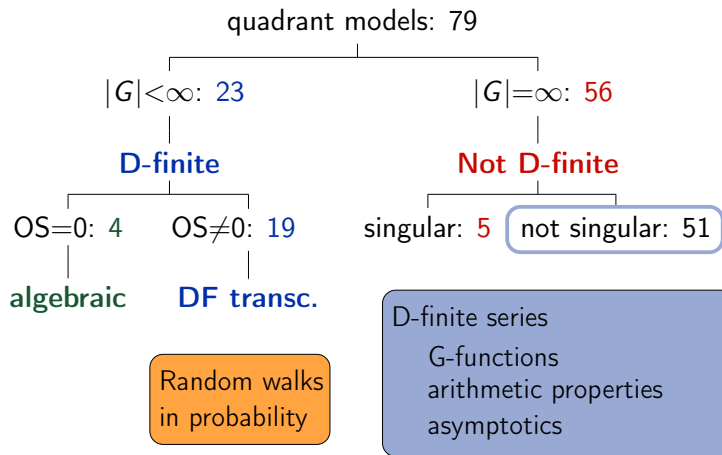
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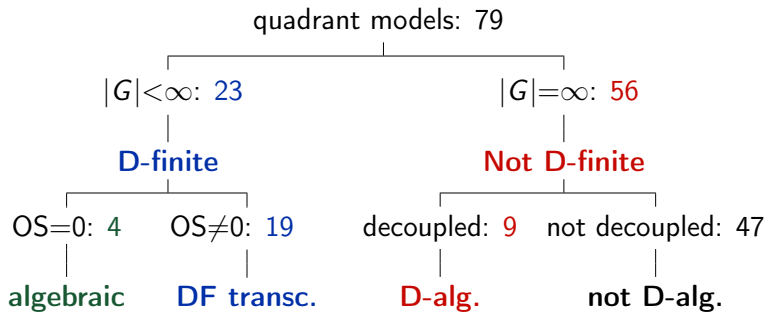
Non-D-finiteness [Bostan, Raschel, Salvy 14]

The series $Q(0, 0; t)$ is not D-finite for 51 of the 79 small step models.

Non-D-finiteness via asymptotics



Classification of quadrant walks: a variety of tools



Formal power series algebra

Complex analysis

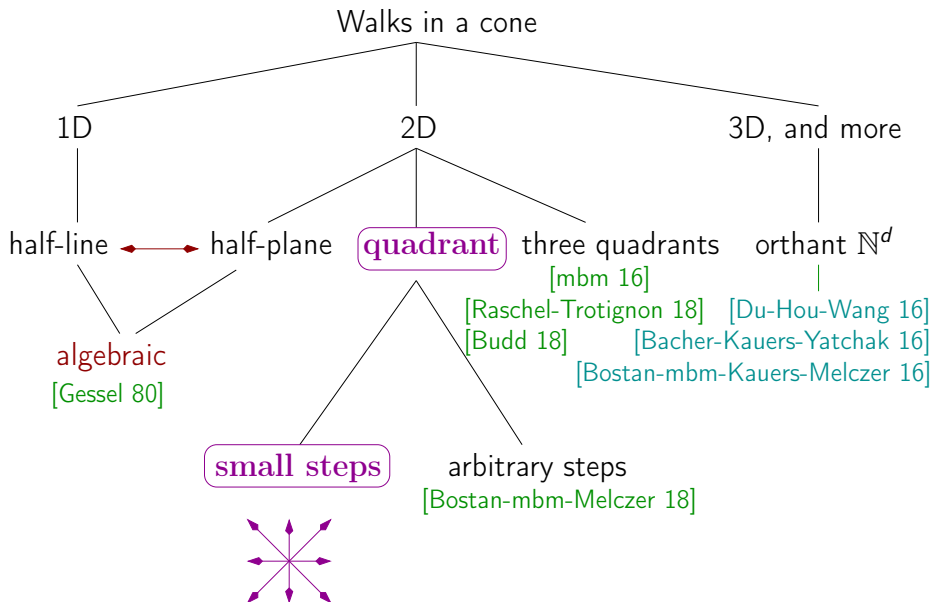
Differential Galois theory

Computer algebra

Random walks in probability

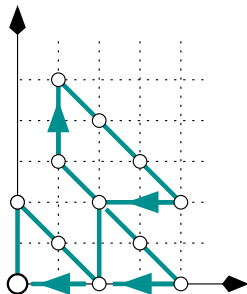
D-finite series
effective closure properties
arithmetic properties
asymptotics
G-functions

Beyond small steps in the quadrant



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- **Arbitrary steps** in the quadrant \Rightarrow equivalence between D-finiteness and finite “group”?



Beyond small steps in the quadrant

- **Arbitrary steps** in the quadrant \Rightarrow equivalence between D-finiteness and finite “group”?
- Walks **avoiding** a quadrant: same dichotomy between D-finite and non-D-finite models?
- Walks with small steps in \mathbb{N}^3 : some non-D-finite models with a finite group?

Example. The model $\{111, \bar{1}00, 0\bar{1}0, 00\bar{1}\}$ has a finite group of order 24. The orbit sum vanishes. Is it D-finite?

