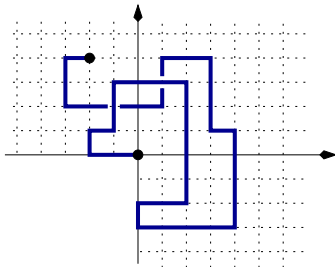


Plane lattice paths avoiding a quadrant

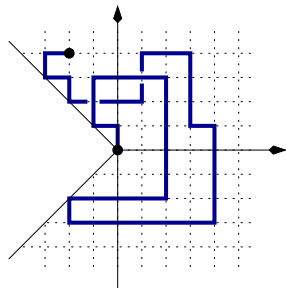
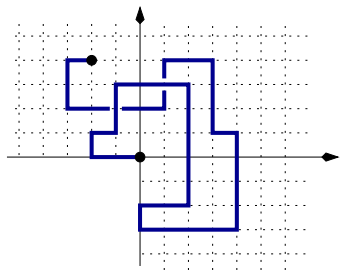
Mireille Bousquet-Mélou, CNRS, LaBRI, Bordeaux



Two problems

Question: How many n -step walks on the square lattice, starting from $(0, 0)$, avoiding a given quadrant?

Two natural choices for the quadrant:



One can also prescribe the endpoint (i, j) of the walk.

I. Counting walks in cones

Counting walks in cones

Take a starting point p_0 in \mathbb{Z}^d , a (finite) step set $\mathcal{S} \subset \mathbb{Z}^d$ and a cone \mathcal{C} .

Questions

- What is the number $c(n)$ of n -step walks starting at p_0 , taking their steps in \mathcal{S} and contained in \mathcal{C} ?
- For $i = (i_1, \dots, i_d) \in \mathcal{C}$, what is the number $c(i; n)$ of such walks that end at i ?

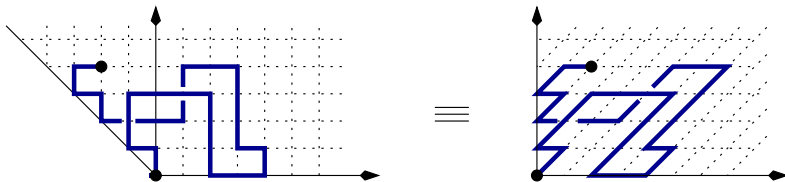
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Example: Gessel's walks



Why count walks in cones?

Many discrete objects can be encoded in that way:

- in combinatorics, statistical physics...
- in (discrete) probability theory: random walks, queuing theory...

Many contributions

Adan, Banderier, Bernardi, Bostan, Cori, Denisov, Duchon, Dulucq, Fayolle, Gessel, Fisher, Flajolet, Gouyou-Beauchamps, Guttmann, Guy, Janse van Rensburg, Johnson, Kauers, Koutschan, Krattenthaler, Kurkova, Kreweras, van Leeuwarden, MacMahon, Melczer, Mishna, Niederhausen, Petkovšek, Prellberg, Raschel, Rechnitzer, Sagan, Salvy, Viennot, Wachtel, Wilf, Yeats, Zeilberger...

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~~Specific question
Ad hoc solution~~



Systematic approach

A too ambitious question?

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$$C(t) = \sum_{n \geq 0} c(n)t^n, \quad C(x_1, \dots, x_d; t) = \sum_{\mathbf{i}, n} c(\mathbf{i}; n) \mathbf{x}^{\mathbf{i}} t^n$$

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Remarks

- $C(1, \dots, 1; t) = C(t)$
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Can one express these series? Are they rational? algebraic? D-finite?

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A hierarchy of formal power series

- The formal power series $C(t)$ is **rational** if it can be written

$$C(t) = P(t)/Q(t)$$

where $P(t)$ and $Q(t)$ are polynomials in t .

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- Nice and effective closure properties + asymptotics of the coefficients
- Extension to several variables (D-finite: **one DE per variable**)

A (very) basic cone: the full space

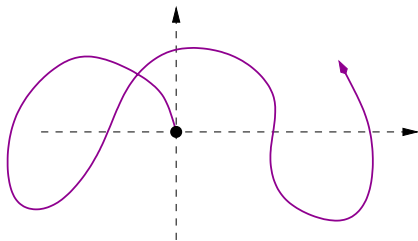
Rational series

If $S \subset \mathbb{Z}^d$ is finite and $C = \mathbb{R}^d$, then $C(x; t)$ is rational:

$$c(n) = |S|^n \Leftrightarrow C(t) = \sum_{n \geq 0} c(n)t^n = \frac{1}{1 - |S|t}$$

More generally:

$$C(x; t) = \frac{1}{1 - t \sum_{s \in S} x^s}.$$

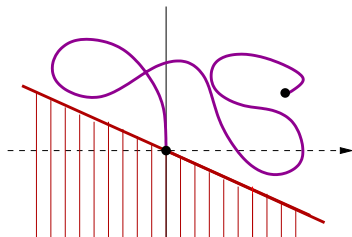


Also well-known: a (rational) half-space

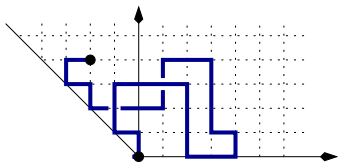
Algebraic series

If $\mathcal{S} \subset \mathbb{Z}^d$ is finite and \mathcal{C} is a rational half-space, then $C(x; t)$ is algebraic, given by an explicit system of polynomial equations.

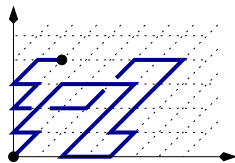
[mbm-Petkovšek 00]; [Gessel 80], [Duchon 00]...



The “next” case: intersection of two half-spaces

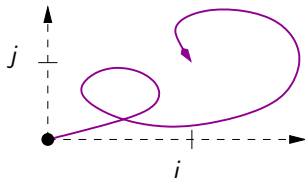


\equiv



- The quarter plane, with $p_0 = (0, 0)$

$$Q(x, y; t) = \sum_{i, j, n \geq 0} q(i, j; n) x^i y^j t^n = ?$$



Quadrant walks with *small* steps: classification

- $\mathcal{S} \subset \{\bar{1}, 0, 1\}^2 \setminus \{(0, 0)\} \Rightarrow 2^8 = 256$ step sets (or: **models**)

Quadrant walks with *small* steps: classification

- $\mathcal{S} \subset \{\bar{1}, 0, 1\}^2 \setminus \{(0, 0)\} \Rightarrow 2^8 = 256$ step sets (or: **models**)
- However, some models are equivalent:
 - to a model of walks in the full or half-plane (\Rightarrow algebraic)



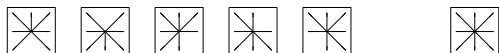
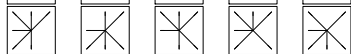
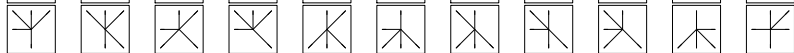
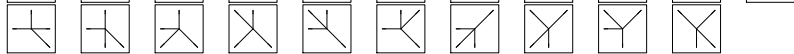
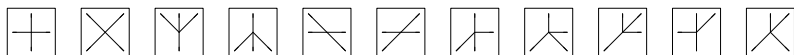
- to another model in the collection (diagonal symmetry)



- One is left with **79 interesting distinct models** [mbm-Mishna 09].



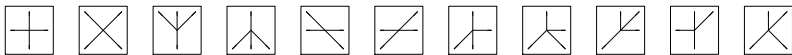
Non-singular



Singular



Non-singular



Singular



Classification of quadrant walks with small steps

Preview:

The series $Q(x, y; t)$ is D-finite
iff a certain group associated with \mathcal{S} is finite.



Example. Take $\mathcal{S} = \{\bar{1}0, 01, 1\bar{1}\}$, with **step polynomial**

$$S(x, y) = \frac{1}{x} + y + x \cdot \frac{1}{y} = \bar{x} + y + x\bar{y}$$



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Observation: $S(x, y)$ is left unchanged by the rational transformations

$$\Phi : (x, y) \mapsto (\bar{x}y, y) \quad \text{and} \quad \Psi : (x, y) \mapsto (x, x\bar{y}).$$

The group of the model



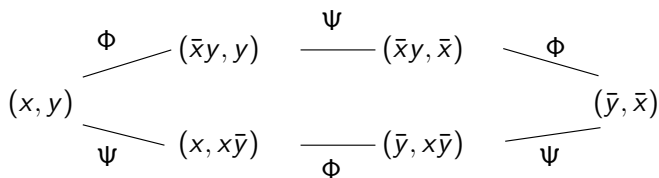
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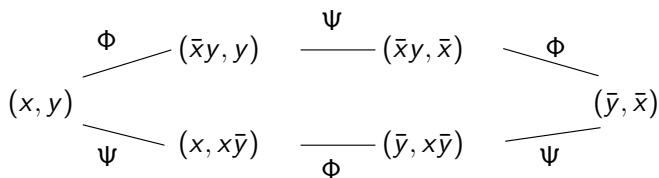
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The group can be defined for any quadrant model *with small steps*

Classification of quadrant walks with small steps

Theorem

The series $Q(x, y; t)$ is D-finite iff the group G is finite.

[mbm-Mishna 10], [Bostan-Kauers 10]

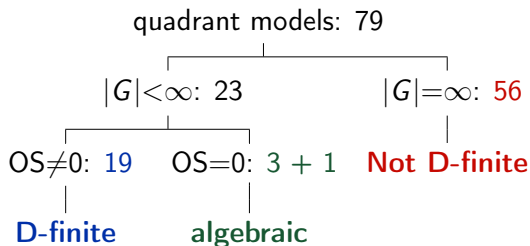
[Kurkova-Raschel 12]

[Mishna-Rechnitzer 07], [Melczer-Mishna 13]

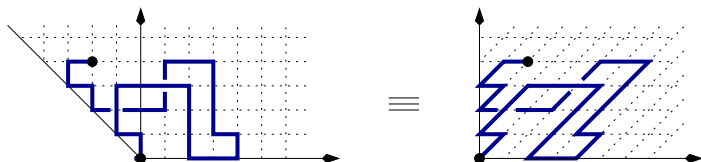
D-finite

non-singular non-D-finite

singular non-D-finite



One hard case: Gessel's walks



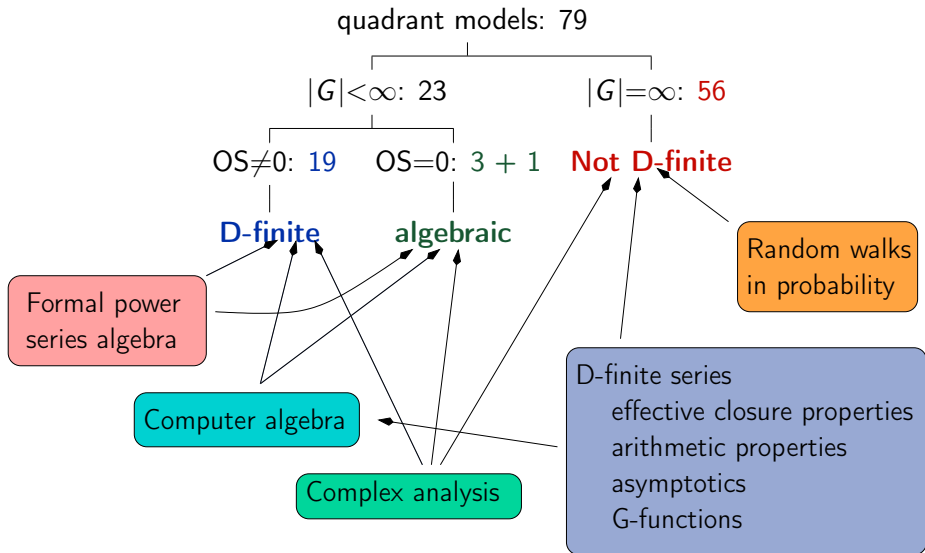
Conjecture: [Gessel, around 2000]

$$g_{0,0}(2n) = 16^n \frac{(1/2)_n (5/6)_n}{(2)_n (5/3)_n}$$

with $(a)_n = a(a+1) \cdots (a+n-1)$.

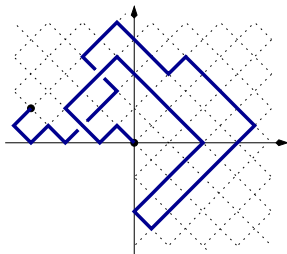
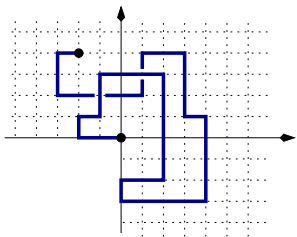
- Proved in 2009 using computer algebra [Kauers, Koutschan & Zeilberger 09]
- In fact, $Q(x, y; t)$ is **algebraic!** (degree 72...) [Bostan & Kauers 10]
- Two other proofs, one analytic, one “elementary” [Bostan, Kurkova and Raschel 13(a)], [mbm 15(a)]

Classification of quadrant walks with small steps

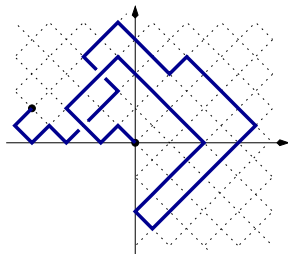
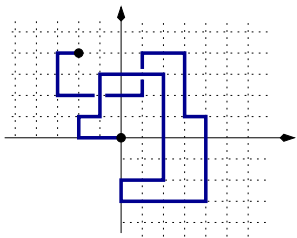


II. Walks avoiding a quadrant

The “next” case: *union* of two half-spaces!

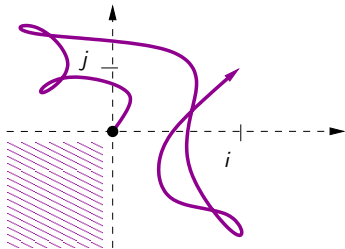


The “next” case: *union* of two half-spaces!



- Walks **avoiding** the negative quadrant, with $p_0 = (0, 0)$

$$C(x, y; t) = \sum_{i, j, n \geq 0} c(i, j; n) x^i y^j t^n = ?$$



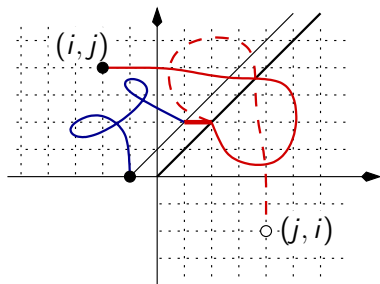
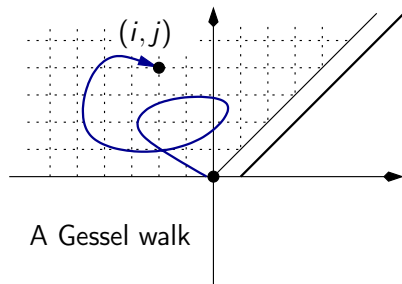
Is there a nice story to tell for walks avoiding a quadrant?

Another motivation: a new approach to Gessel's walks

Idea: count Gessel's walks by the reflection principle

If $c(i, j; n)$ counts walks starting at $(-1, 0)$ and avoiding the negative quadrant, then for $j \geq 0$ and $i < j$,

$$c_{i,j}(n) - c_{j,i}(n) = g_{i+1,j}(n).$$

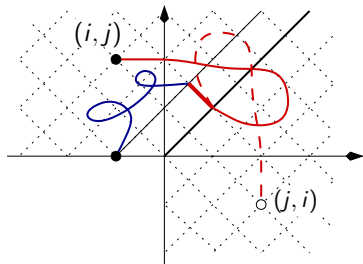
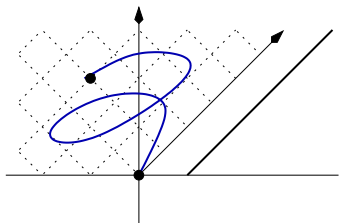


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An alternative is to count walks on the **diagonal square lattice** starting from $(-2, 0)$ and avoiding the negative quadrant.

An *a posteriori* motivation

A hard (?) problem with nice numbers!

An *a posteriori* motivation

A hard (?) problem with nice numbers!

Example: the number of square lattice walks of length $2n$ starting and ending at $(0, 0)$ and avoiding the West quadrant is

$$c(0, 0; 2n) = \frac{16^n}{9} \left(3 \frac{(1/2)_n^2}{(2)_n^2} + 8 \frac{(1/2)_n(7/6)_n}{(2)_n(4/3)_n} - 2 \frac{(1/2)_n(5/6)_n}{(2)_n(5/3)_n} \right)$$

with $(a)_n = a(a+1) \cdots (a+n-1)$.

Two known components: Quadrant walks (left) and Gessel walks (right).
Conjectured before 2007...

Asymptotically,

$$c(0, 0; 2n) \sim \frac{2^5}{3^2} \frac{\Gamma(2/3)}{\pi} \frac{4^{2n}}{(2n)^{5/3}}.$$

Results (square lattice, starting point $(0, 0)$)

- Let $Q(x, y) \equiv Q(x, y; t)$ be the generating function of square lattice walks starting from $(0, 0)$ and *confined to the first quadrant*:

$$Q(x, y) = \sum_{i, j, n \geq 0} \frac{(i+1)(j+1)}{(n+1)(n+2)} \binom{n+2}{\frac{n-i-j}{2}} \binom{n+2}{\frac{n+i-j+2}{2}} x^i y^j t^n$$

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Theorem [mbm 15(a)]

The generating function $C(x, y; t) \equiv C(x, y)$ counting walks that avoid the negative quadrant is

$$A(x, y) + \frac{1}{3} (Q(x, y) - \bar{x}^2 Q(\bar{x}, y) - \bar{y}^2 Q(x, \bar{y}))$$

where $\bar{x} = 1/x$, $\bar{y} = 1/y$ and $A(x, y)$ is algebraic. This series satisfies

$$(1 - t(x + \bar{x} + y + \bar{y})) A(x, y) = (2 + \bar{x}^2 + \bar{y}^2)/3 - t\bar{y}A_-(\bar{x}) - t\bar{x}A_-(\bar{y}),$$

where $A_-(x)$ is a series in t with coefficients in $\mathbb{Q}[x]$, algebraic of degree 24.

Results (square lattice, starting point (0, 0))

Theorem [mbm 15(a)], cont'd

$$C(x, y) = A(x, y) + \frac{1}{3} (Q(x, y) - \bar{x}^2 Q(\bar{x}, y) - \bar{y}^2 Q(x, \bar{y}))$$

where

$$(1 - t(x + \bar{x} + y + \bar{y})) A(x, y) = (2 + \bar{x}^2 + \bar{y}^2)/3 - t\bar{y}A_-(\bar{x}) - t\bar{x}A_-(\bar{y}),$$

and $A_-(x)$ is a series in t with coefficients in $\mathbb{Q}[x]$, algebraic of degree 24.

- Moreover, $A_-(tx)$ has a rational expression in terms of the (unique) series $Z = 1 + O(t)$ and $U(x) = 1 + O(t)$ satisfying:

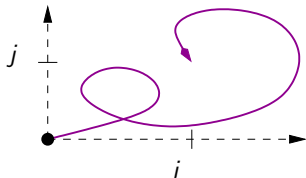
$$Z = \sqrt{T}, \quad T = 1 + 256t^2 \frac{T^3}{(T+3)^3},$$

$$16T^2(U^2 - T) = x(U + UT - 2T)(U^2 - 9T + 8TU + T^2 - TU^2).$$

III. Two solutions for... quadrant walks

- the (ordinary) kernel method
- the (algebraic) kernel method: the reflection principle in disguise

Starting point: a functional equation

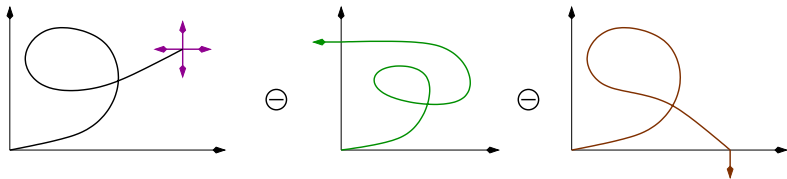


Walks in the quadrant: a functional equation



Step by step construction:

$$Q(x, y; t) \equiv Q(x, y) = 1 + t(x + \bar{x} + y + \bar{y})Q(x, y) - t\bar{x}Q(0, y) - t\bar{y}Q(x, 0)$$





Step by step construction:

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or

$$(1 - t(x + \bar{x} + y + \bar{y}))Q(x, y) = 1 - t\bar{x}Q(0, y) - t\bar{y}Q(x, 0),$$

Walks in the quadrant: a functional equation



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or

$$(1 - t(x + \bar{x} + y + \bar{y}))Q(x, y) = 1 - t\bar{x}Q(0, y) - t\bar{y}Q(x, 0),$$

or

$$(1 - t(x + \bar{x} + y + \bar{y}))xyQ(x, y) = xy - tyQ(0, y) - txQ(x, 0)$$

- The polynomial $1 - t(x + \bar{x} + y + \bar{y})$ is the **kernel** of this equation
- The equation is linear, with two **catalytic** variables x and y (tautological at $x = 0$ or $y = 0$)

The ordinary kernel method: cancel the kernel



- The equation:

$$(1 - t(x + \bar{x} + y + \bar{y}))xyQ(x, y) = xy - tyQ(0, y) - txQ(x, 0)$$

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$$(1 - t(x + \bar{x} + y + \bar{y}))xyQ(x, y) = xy - tyQ(0, y) - txQ(x, 0)$$

- Let

$$Y = \frac{1 - t(x + \bar{x}) - \sqrt{(1 - t(x + \bar{x}))^2 - 4t^2}}{2t} = t + (x + \bar{x})t^2 + O(t^3)$$

be the only power series in t such that $K(x, Y) = 0$.

The ordinary kernel method: cancel the kernel



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- Since Y is symmetric in x and \bar{x} ,

$$\bar{x}Y = tYQ(0, Y) + t\bar{X}Q(\bar{x}, 0).$$

The ordinary kernel method: cancel the kernel



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- Since Y is symmetric in x and \bar{x} ,

$$\bar{x}Y = tYQ(0, Y) + t\bar{x}Q(\bar{x}, 0).$$

- Thus

$$txQ(x, 0) - t\bar{x}Q(\bar{x}, 0) = (x - \bar{x})Y,$$

and $txQ(x, 0)$ can be obtained by extracting positive powers of x in $(x - \bar{x})Y$.

The algebraic kernel method: use the full group



- The equation reads (with $K(x, y) = 1 - t(x + \bar{x} + y + \bar{y})$):

$$K(x, y)xyQ(x, y) = xy - txQ(x, 0) - tyQ(0, y)$$

- The kernel is invariant by $x \mapsto \bar{x}$ and $y \mapsto \bar{y}$. Hence

$$K(x, y)xyQ(x, y) = xy - txQ(x, 0) - tyQ(0, y)$$

$$K(x, y)\bar{x}yQ(\bar{x}, y) = \bar{x}y - t\bar{x}Q(\bar{x}, 0) - tyQ(0, y)$$

$$K(x, y)\bar{x}\bar{y}Q(\bar{x}, \bar{y}) = \bar{x}\bar{y} - t\bar{x}Q(\bar{x}, 0) - t\bar{y}Q(0, \bar{y})$$

$$K(x, y)x\bar{y}Q(x, \bar{y}) = x\bar{y} - txQ(x, 0) - t\bar{y}Q(0, \bar{y}).$$

The algebraic kernel method: use the full group



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$$K(x, y)x\bar{y}Q(x, \bar{y}) = x\bar{y} - txQ(x, 0) - t\bar{y}Q(0, \bar{y}).$$

\Rightarrow Form the alternating sum of the four equations:

$$K(x, y)\left(xyQ(x, y) - \bar{x}yQ(\bar{x}, y) + \bar{x}\bar{y}Q(\bar{x}, \bar{y}) - x\bar{y}Q(x, \bar{y})\right) =$$

$$xy - \bar{x}y + \bar{x}\bar{y} - x\bar{y}$$

(the orbit sum).

Why is this interesting?



$$xyQ(x, y) - \bar{x}yQ(\bar{x}, y) + \bar{x}\bar{y}Q(\bar{x}, \bar{y}) - x\bar{y}Q(x, \bar{y}) = \frac{xy - \bar{x}y + \bar{x}\bar{y} - x\bar{y}}{1 - t(x + \bar{x} + y + \bar{y})}$$

- Both sides are power series in t , with coefficients in $\mathbb{Q}[x, \bar{x}, y, \bar{y}]$.

Why is this interesting?



$$xyQ(x, y) - \bar{x}yQ(\bar{x}, y) + \bar{x}\bar{y}Q(\bar{x}, \bar{y}) - x\bar{y}Q(x, \bar{y}) = \frac{xy - \bar{x}y + \bar{x}\bar{y} - x\bar{y}}{1 - t(x + \bar{x} + y + \bar{y})}$$

- Both sides are power series in t , with coefficients in $\mathbb{Q}[x, \bar{x}, y, \bar{y}]$.
- Extract the part with positive powers of x and y :

$$xyQ(x, y) = [x^{>0}y^{>0}] \frac{xy - \bar{x}y + \bar{x}\bar{y} - x\bar{y}}{1 - t(x + \bar{x} + y + \bar{y})}$$

is a D-finite series.

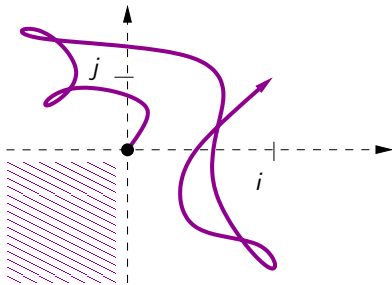
[Lipshitz 88]

$$\Rightarrow Q(x, y) = \sum_{i, j, n \geq 0} \frac{(i+1)(j+1)}{(n+1)(n+2)} \binom{n+2}{\frac{n-i-j}{2}} \binom{n+2}{\frac{n+i-j+2}{2}} x^i y^j t^n$$

IV. Some ingredients for the three quadrant problem

- the (ordinary) kernel method
- the (algebraic) kernel method
- polynomial equations with one catalytic variable

Starting point: a functional equation again



Walks avoiding a quadrant: a functional equation

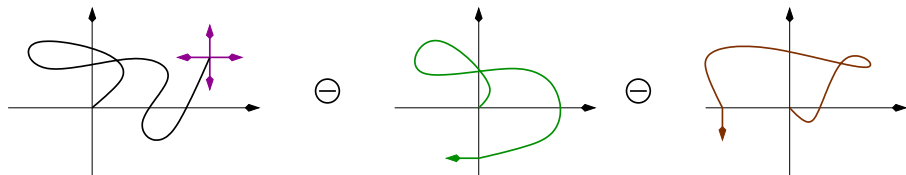


Step by step construction:

$$C(x, y; t) \equiv C(x, y) = 1 + t(x + \bar{x} + y + \bar{y})C(x, y) - t\bar{x}C_-(\bar{y}) - t\bar{y}C_-(\bar{x})$$

with

$$C_-(\bar{x}) = \sum_{i < 0, n \geq 0} c(i, 0; n) x^i t^n.$$



Walks avoiding a quadrant: a functional equation



Step by step construction:

$$C(x, y; t) \equiv C(x, y) = 1 + t(x + \bar{x} + y + \bar{y})C(x, y) - t\bar{x}C_-(\bar{y}) - t\bar{y}C_-(\bar{x})$$

with

$$C_-(\bar{x}) = \sum_{i < 0, n \geq 0} c(i, 0; n)x^i t^n.$$

or

$$(1 - t(x + \bar{x} + y + \bar{y}))C(x, y) = 1 - t\bar{x}C_-(\bar{y}) - t\bar{y}C_-(\bar{x}),$$

Walks avoiding a quadrant: a functional equation



Step by step construction:

$$C(x, y; t) \equiv C(x, y) = 1 + t(x + \bar{x} + y + \bar{y})C(x, y) - t\bar{x}C_-(\bar{y}) - t\bar{y}C_-(\bar{x})$$

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$$(1 - t(x + \bar{x} + y + \bar{y}))C(x, y) = 1 - t\bar{x}C_-(\bar{y}) - t\bar{y}C_-(\bar{x}),$$

or

$$(1 - t(x + \bar{x} + y + \bar{y}))xyC(x, y) = xy - t\bar{y}C_-(\bar{y}) - t\bar{x}C_-(\bar{x})$$

Compare with

$$(1 - t(x + \bar{x} + y + \bar{y}))xyQ(x, y) = xy - t\bar{y}Q(0, y) - t\bar{x}Q(x, 0)$$

Where the series $A(x, y)$ comes from

- In particular, $C(x, y)$ and $Q(x, y)$ have the same orbit sum:

$$xyQ(x, y) - \bar{x}yQ(\bar{x}, y) + \bar{x}\bar{y}Q(\bar{x}, \bar{y}) - x\bar{y}Q(x, \bar{y}) =$$
$$xyC(x, y) - \bar{x}yC(\bar{x}, y) + \bar{x}\bar{y}C(\bar{x}, \bar{y}) - x\bar{y}C(x, \bar{y}) = \frac{xy - \bar{x}y + \bar{x}\bar{y} - x\bar{y}}{1 - t(x + \bar{x} + y + \bar{y})}$$

$-\bar{x}^2Q(\bar{x}, y)$ and $-\bar{y}^2Q(x, \bar{y})$ (and $\bar{x}^2\bar{y}^2Q(\bar{x}, \bar{y})$) as well!

Where the series $A(x, y)$ comes from

- In particular, $C(x, y)$ and $Q(x, y)$ have the same orbit sum:

$$\begin{aligned} & xyQ(x, y) - \bar{x}yQ(\bar{x}, y) + \bar{x}\bar{y}Q(\bar{x}, \bar{y}) - x\bar{y}Q(x, \bar{y}) = \\ & xyC(x, y) - \bar{x}yC(\bar{x}, y) + \bar{x}\bar{y}C(\bar{x}, \bar{y}) - x\bar{y}C(x, \bar{y}) = \frac{xy - \bar{x}y + \bar{x}\bar{y} - x\bar{y}}{1 - t(x + \bar{x} + y + \bar{y})} \\ & -\bar{x}^2Q(\bar{x}, y) \text{ and } -\bar{y}^2Q(x, \bar{y}) \text{ (and } \bar{x}^2\bar{y}^2Q(\bar{x}, \bar{y})) \text{ as well!} \end{aligned}$$

- This leads us to introduce

$$A(x, y) := C(x, y) - \frac{1}{3} (Q(x, y) - \bar{x}^2Q(\bar{x}, y) - \bar{y}^2Q(x, \bar{y})),$$

which satisfies

$$(1 - t(x + \bar{x} + y + \bar{y})) A(x, y) = (2 + \bar{x}^2 + \bar{y}^2)/3 - t\bar{y}A_-(\bar{x}) - t\bar{x}A_-(\bar{y}),$$

with

$$A_-(\bar{x}) = \sum_{i < 0, n \geq 0} a(i, 0; n) x^i t^n.$$

The algebraic kernel method for $A(x, y)$?

The equation reads (with $K(x, y) = 1 - t(x + \bar{x} + y + \bar{y})$):

$$K(x, y)A(x, y) = (2 + \bar{x}^2 + \bar{y}^2)/3 - t\bar{y}A_-(\bar{x}) - t\bar{x}A_-(\bar{y})$$

The orbit sum vanishes:

$$xyA(x, y) - \bar{x}yA(\bar{x}, y) + \bar{x}\bar{y}A(\bar{x}, \bar{y}) - x\bar{y}A(x, \bar{y}) = 0$$

and this does not characterize A .

The ordinary kernel method for $A(x, y)$?

The equation reads (with $K(x, y) = 1 - t(x + \bar{x} + y + \bar{y})$):

$$K(x, y)A(x, y) = (2 + \bar{x}^2 + \bar{y}^2)/3 - t\bar{y}A_-(\bar{x}) - t\bar{x}A_-(\bar{y})$$

But none of the roots Y of $K(x, Y)$ can be legally substituted for y in $A(x, y)$ (due to positive and negative powers of y).

Back to series with *polynomial coefficients* in x and y

- The equation (with $K(x, y) = 1 - t(x + \bar{x} + y + \bar{y})$):

$$K(x, y)A(x, y) = (2 + \bar{x}^2 + \bar{y}^2)/3 - t\bar{y}A_-(\bar{x}) - t\bar{x}A_-(\bar{y})$$

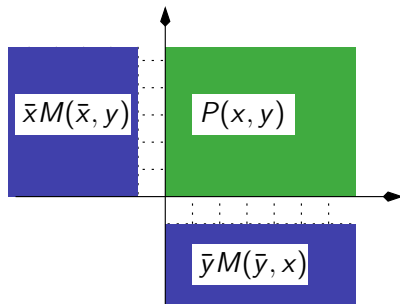
Back to series with *polynomial coefficients* in x and y

- The equation (with $K(x, y) = 1 - t(x + \bar{x} + y + \bar{y})$):

$$K(x, y)A(x, y) = (2 + \bar{x}^2 + \bar{y}^2)/3 - t\bar{y}A_-(\bar{x}) - t\bar{x}A_-(\bar{y})$$

- Split $A(x, y)$ into three parts:

$$A(x, y) = P(x, y) + \bar{x}M(\bar{x}, y) + \bar{y}M(\bar{y}, x)$$



Back to series with *polynomial coefficients* in x and y

- The equation (with $K(x, y) = 1 - t(x + \bar{x} + y + \bar{y})$):

$$K(x, y)A(x, y) = (2 + \bar{x}^2 + \bar{y}^2)/3 - t\bar{y}A_-(\bar{x}) - t\bar{x}A_-(\bar{y})$$

- Split $A(x, y)$ into three parts:

$$A(x, y) = P(x, y) + \bar{x}M(\bar{x}, y) + \bar{y}M(\bar{y}, x)$$

- Then the equation for A implies

$$P(x, y) = \bar{x}(M(x, y) - M(0, y)) + \bar{y}(M(y, x) - M(0, x)),$$

and

$$(1 - t(x + \bar{x} + y + \bar{y}))(2M(x, y) - M(0, y)) = \\ 2x/3 - 2t\bar{y}M(x, 0) + t(x - \bar{x})M(0, y) + t\bar{y}M(y, 0).$$

This looks (remotely) like a quadrant problem. Two main differences:

- $M(y, 0)$ is a new ingredient
- the r.h.s. cannot be decoupled into $F(x) + G(y)$

Some pages later: a final ingredient

Let $\Delta(x) := (1 - t(x + \bar{x}))^2 - 4t^2$,

$$S(x) = tM(0, x), \quad S_1 = [x]S(x) = tM(0, 0)$$

and

$$P_0 := [x^0] (\Delta(x)S(x)S(\bar{x})).$$

Some pages later: a final ingredient

$$\text{Let } \Delta(x) := (1 - t(x + \bar{x}))^2 - 4t^2,$$

$$S(x) = txM(0, x), \quad S_1 = [x]S(x) = tM(0, 0)$$

and

$$P_0 := [x^0](\Delta(x)S(x)S(\bar{x})).$$

Then

$$\begin{aligned} \Delta(x) (S(x)^3 + (2x + \bar{x})S(x)^2 + x(x + \bar{x})S(x)) &= t^2(x - \bar{x})(1 + S_1)^2 \\ + (2t^2S_1^2 + 2t(tx^2 + t\bar{x}^2 - x - \bar{x} + t)S_1 - P_0 + t^2(x^2 + \bar{x}^2)) (S(x) + x). \end{aligned}$$

A polynomial equation with one catalytic variable x , solvable using [mbm-Jehanne 06]. The solution is (invariably) algebraic.

Results (square lattice, starting point $(0, 0)$)

- Let $Q(x, y) \equiv Q(x, y; t)$ be the generating function of square lattice walks starting from $(0, 0)$ and *confined to the first quadrant*:

$$Q(x, y) = \sum_{i, j, n \geq 0} \frac{(i+1)(j+1)}{(n+1)(n+2)} \binom{n+2}{\frac{n-i-j}{2}} \binom{n+2}{\frac{n+i-j+2}{2}} x^i y^j t^n$$

Theorem [mbm 15(a)]

The generating function $C(x, y; t) \equiv C(x, y)$ counting walks that avoid the negative quadrant is

$$A(x, y) + \frac{1}{3} (Q(x, y) - \bar{x}^2 Q(\bar{x}, y) - \bar{y}^2 Q(x, \bar{y}))$$

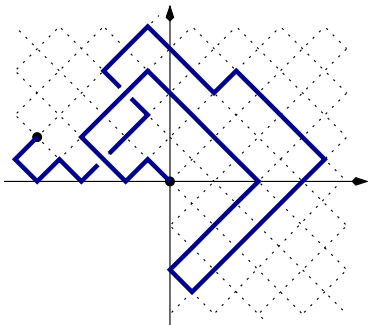
where $\bar{x} = 1/x$, $\bar{y} = 1/y$ and $A(x, y)$ is algebraic. This series satisfies

$$(1 - t(x + \bar{x} + y + \bar{y})) A(x, y) = (2 + \bar{x}^2 + \bar{y}^2)/3 - t\bar{y}A_-(\bar{x}) - t\bar{x}A_-(\bar{y}),$$

where $A_-(x)$ is a series in t with coefficients in $\mathbb{Q}[x]$, algebraic of degree 24.

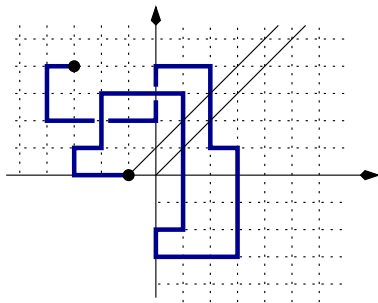
More results

- Walks on the diagonal square lattice avoiding the negative cone
- Walks on the square lattice starting at $(-1, 0)$ (entirely algebraic!)
- Walks on the diagonal square lattice starting at $(-2, 0)$
- The solution of Gessel's model in a 135° cone then comes for free



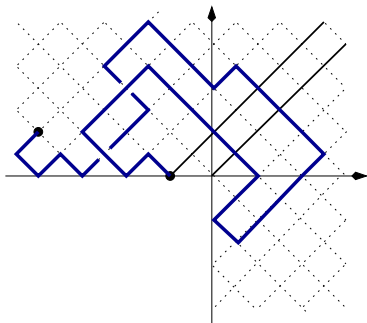
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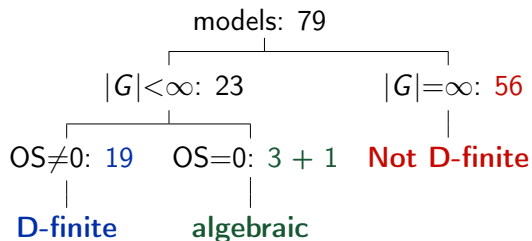
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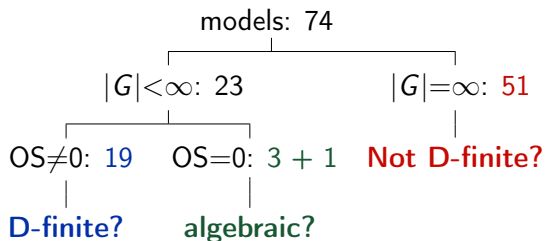
Perspectives: classification

Walks with small steps **confined to the positive quadrant**:



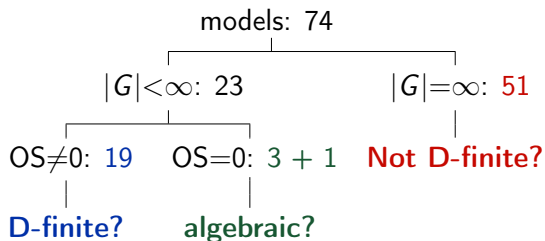
Perspectives: classification

Walks with small steps **avoiding the negative quadrant:**



Perspectives: classification

Walks with small steps **avoiding the negative quadrant**:



Conjecture: the generating function of walks starting and ending at $(0,0)$ and avoiding the negative quadrant is algebraic for the four models for which $Q(x,y)$ (the quadrant GF) is algebraic:

