

# Graph Transformations and Logic

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The general field of this research could be entitled :

## Graph structuring and Monadic Second-order logic

*History* : The confluence of 4 independent research directions, now intimately related :

1. **Polynomial algorithms** for NP-complete and other hard problems on particular classes of graphs, and especially hierarchically structured ones : series-parallel graphs, cographs, partial k-trees, graphs or hypergraphs of tree-width  $< k$ , graphs of clique-width  $< k$ .
2. **Excluded minors** and related notions of forbidden configurations (matroid minors, « vertex-minors »).
3. **Decidability of Monadic Second-Order logic** on classes of finite graphs, and on infinite graphs.
4. Extension to graphs and hypergraphs of the main concepts of **Formal Language Theory** : grammars, recognizability, transductions, decidability questions.

# Summary

## 1. Introduction

*Extension of Formal Language Theory notions*

2. Recognizability, an algebraic notion.

3. Equational (context-free) sets.

4. The graph algebras VR and HR ; the role of quantifier-free definable unary operations.

*An essential tool :*

5. Monadic Second-Order Transductions

*Monadic second-order logic and combinatorics :*

6. Seese's conjecture

7. Open questions

# 1. Introduction

## Key concepts of FLT and their extensions

<i>Languages</i>	<i>Graphs</i>
Algebraic structure : monoid $(X^*, *, \varepsilon)$	Algebras based on graph operations : $\oplus, \otimes, //$ <b>quantifier-free definable operations</b> Algebras : HR, VR, QF
Context-free languages : Equational subsets of $(X^*, *, \varepsilon)$	Equational sets of the algebras HR, VR, QF
Regular languages : Finite automata $\equiv$ Finite congruences $\equiv$ Regular expressions $\equiv$	Recognizable sets of the algebras HR, VR, QF defined by congruences
$\equiv$ Monadic Second-order definable sets of words or terms	$\cup$ Monadic Second-order definable sets of graphs
Rational and other types of transductions	<b>Monadic Second-order transductions</b>

## Relationships between algebraic and logical notions for sets of graphs (and hypergraphs)

Algebraic notions	Algebraic characterizations	Logical characterizations	Closure properties
EQ			union, $\cap$ Rec
	equation systems	MS-trans(Trees)	homo
	$Val(REC(Terms))$		MS-trans
REC			Boolean opns
	congruences	MS-def $\subset$ REC	homo <sup>-1</sup>
			MS-trans <sup>-1</sup>

### Signatures for graphs and hypergraphs :

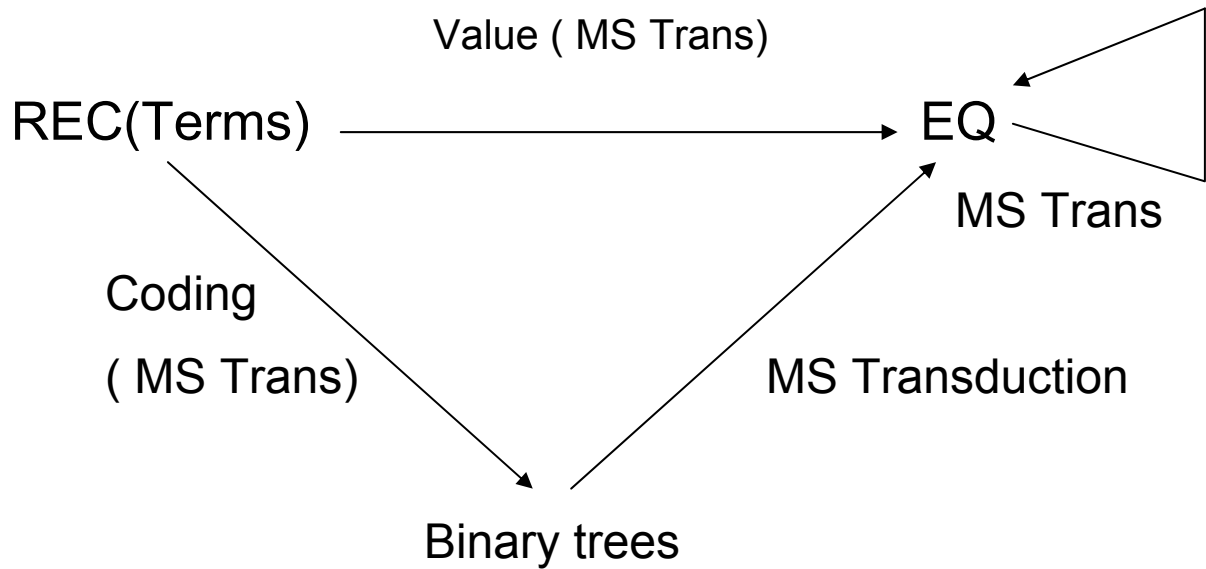
$HR$  : graphs and hypergraphs with “sources”

$VR$  : graphs with vertex labels (“ports”)

$VR^+$  :  $VR$  with quantifier-free operations  
(ex. edge complement)

$QF$  : hypergraphs ( $\equiv$  relational structures) :  
disjoint union  $\oplus$ , and quantifier-free definable  
unary operations

*Another picture :*



Equational sets = MS-Trans(Binary Trees)

*Compare :*

Context-free languages = images of the Dyck language  
(which encodes trees) under rational transductions

Since MS transductions are closed under composition, the class of equational sets is closed under MS transductions

## 2. Recognizable sets : algebraic definition

$F$  : a finite set of operations with (fixed) arity.

$\mathbf{M} = \langle M, (f_M)_{f \in F} \rangle$  : an  $F$ -algebra.

*Definition* :  $L \subseteq M$  is *(F-)recognizable* if it is a union of equivalence classes for a finite congruence  $\approx$  on  $\mathbf{M}$  (*finite* means that  $M / \approx$  is finite).

Equivalently,  $L = h^{-1}(D)$  for a homomorphism  $h : \mathbf{M} \rightarrow \mathbf{A}$ , where  $\mathbf{A}$  is a finite  $F$ -algebra,  $D \subseteq A$ .

**REC( $\mathbf{M}$ )** is the set of recognizable subsets of  $M$ ,  
with respect to the algebra  $\mathbf{M}$ .

*Closure properties* : REC( $\mathbf{M}$ ) contains  $M$  and  $\emptyset$ , and is closed under union, intersection and difference.

*The many-sorted case with infinitely many sorts*

$S$  : the countable set of sorts.

$F$  : an  $S$ -signature (means that each  $f$  in  $F$  has a type

$$s_1 s_2 \dots s_k \rightarrow s, \text{ with } s, s_i \in S)$$

$\mathbf{M} = \langle (M_s)_{s \in S}, (f_{\mathbf{M}})_{f \in F} \rangle$   $F$ -algebra,  $M_s \cap M_t = \emptyset$ , if  $s \neq t$

$$\text{where } f_{\mathbf{M}} : M_{s_1} \times M_{s_2} \times \dots \times M_{s_k} \rightarrow M_s$$

*Definition* :  $L \subseteq M_s$  is *(F-) recognizable* if it is a union of equivalence classes for a congruence  $\approx$  on  $\mathbf{M}$  such that equivalent elements are of the same sort and there are finitely many classes of each sort.

### 3. Equational (context-free) sets

Equation systems = Context-Free (Graph) Grammars  
in an algebraic setting

In the case of words, the set of context-free rules

$$S \rightarrow a S T ; \quad S \rightarrow b ; \quad T \rightarrow c T T T ; \quad T \rightarrow a$$

is equivalent to the system of two set equations:

$$S = a S T \quad \cup \quad \{ b \}$$

$$T = c T T T \quad \cup \quad \{ a \}$$

where  $S$  is the language generated by  $S$  (idem for  $T$  and  $T$ ).

For graphs we consider similarly systems of equations like:

$$S = f( k( S ), T ) \quad \cup \quad \{ b \}$$

$$T = f( T , f( g( T ), m( T ))) \cup \{ a \}$$

where  $f$  is a binary operation,  $g, k, m$  are unary operations on graphs,  $a, b$  are basic graphs.

An *equational set* is a component of the least (unique) solution of such an equation system. This is well-defined in any algebra.



# Closure properties and algebraic characterizations

## General algebraic properties

Algebraic notions	Algebraic characterizations	Closure properties
EQ	equation systems	union, $\cap$ Rec
	$Val(REC(Terms))$	homomorphisms
REC	congruences	Boolean operations
		homomorphisms <sup>-1</sup>

**Theorem** (Mezei and Wright) :

1) In an algebra of terms  $\mathbf{T}(F)$  :

$$EQ(\mathbf{T}(F)) = REC(\mathbf{T}(F))$$

2) In an F-algebra  $\mathbf{M}$  :

$$EQ(\mathbf{M}) = Val_{\mathbf{M}}(REC(\mathbf{T}(F)))$$

where  $Val_{\mathbf{M}} : \mathbf{T}(F) \longrightarrow \mathbf{M}$  is the **evaluation mapping**,  
the unique homomorphism.

## 4. The graph algebras VR and HR

### *HR operations*

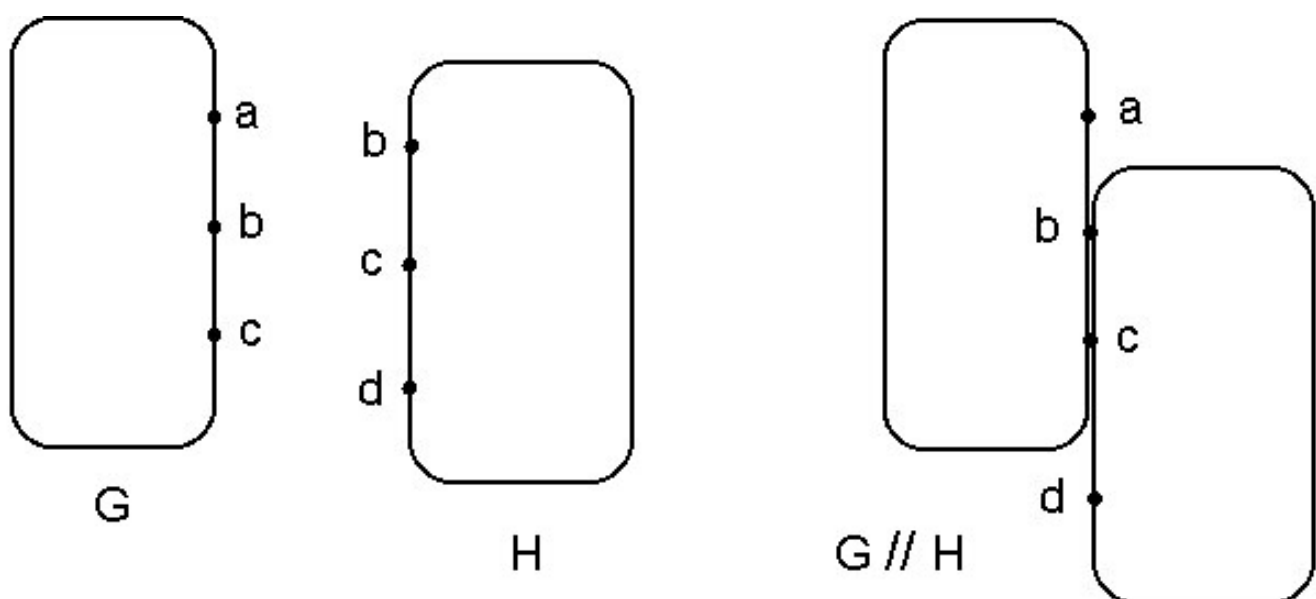
(Origin : **H**yperedge **R**eplacement hypergraph grammars ;  
associated complexity measure : **tree-width**)

Graphs have distinguished vertices called *sources*,  
pointed to by labels from a set of size  $k$  :  $\{a, b, c, \dots, h\}$ .

Binary operation(s) : *Parallel composition*

$G // H$  is the disjoint union of  $G$  and  $H$   
and sources with same  
label are **fused**.

(If  $G$  and  $H$  are not disjoint, one  
first makes a copy of  $H$  disjoint  
from  $G$ .)



Unary operations :

*Forget a source label*

$Forget_a(G)$  is  $G$  without  $a$ -source :  
the source is no longer distinguished ; it is made "internal".

*Source renaming :*

$Rena_{a,b}(G)$  renames  $a$  into  $b$  (assuming  $b$  is not a source label in  $G$ ).

Nullary operations denote *basic graphs* : the connected graphs with at most one edge.

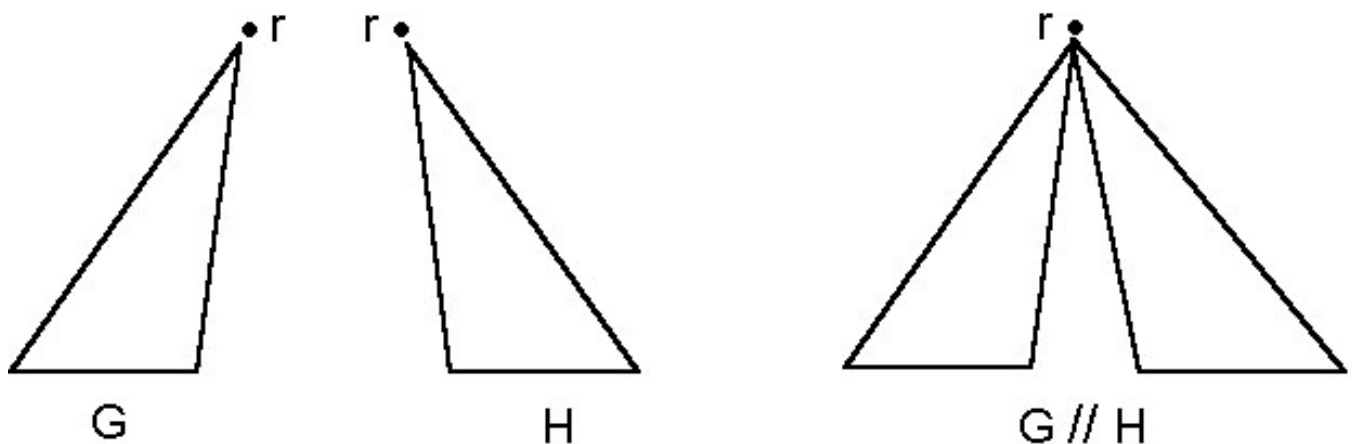
*For dealing with hypergraphs one takes more nullary symbols for denoting hyperedges.*

The proper algebraic framework : a **many sorted algebra** where **each finite set of source labels is a sort**. The above operations are overloaded.

*Proposition:* A graph has **tree-width  $\leq k$**  if and only if it can be constructed from basic graphs with  $\leq k+1$  labels by using the operations  $//$ ,  $Ren_{a,b}$  and  $Forget_a$ .

*Example :* Trees are of tree-width 1, constructed with two source labels,  $r$  (root) and  $n$  (new root):

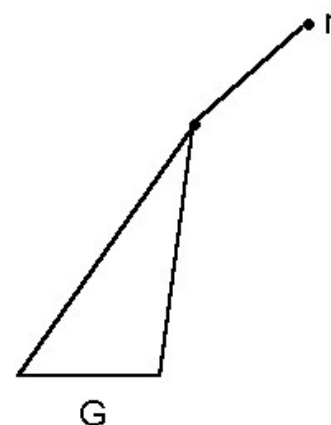
Fusion of two trees at their roots :



Extension of a tree by parallel composition with a new edge, forgetting the old root, making the "new root" as current root :

$$E = r \bullet \text{---} \bullet n$$

$$Ren_{n,r}(Forget_r(G // E))$$



# VR operations

(origin : **V**ertex **R**eplacement graph grammars ; associated complexity measure : **clique-width** )

Graphs are simple, directed or not.

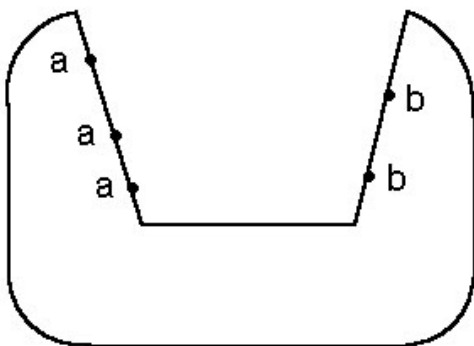
k labels :  $a, b, c, \dots, h$ .

Each vertex has one and only one label ; a label  $p$  may label several vertices, called the  $p$ -ports.

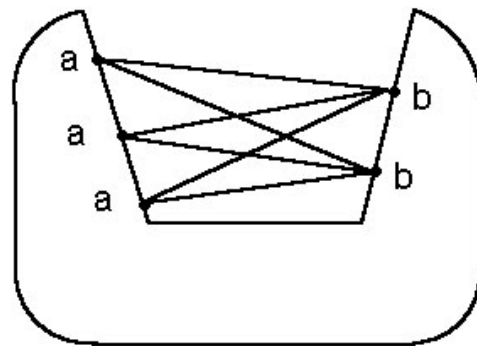
Binary operation: disjoint union  $\oplus$

Unary operations: Edge addition denoted by  $Add\text{-}edg_{a,b}$

$Add\text{-}edg_{a,b}(G)$  is  $G$  augmented with (un)directed edges from every  $a$ -port to every  $b$ -port.



G



$Add\text{-}edg_{a,b}(G)$

$Relab_{a,b}(G)$  is  $G$  with every vertex labelled by  $a$  relabelled into  $b$

*Basic graphs* are those with a single vertex.

**Definition:** A graph  $G$  has **clique-width  $\leq k$**

$\Leftrightarrow$  it can be constructed from basic graphs by means of  $k$  labels and the operations  $\oplus$ ,  $Add-edge_{a,b}$  and  $Relab_{a,b}$

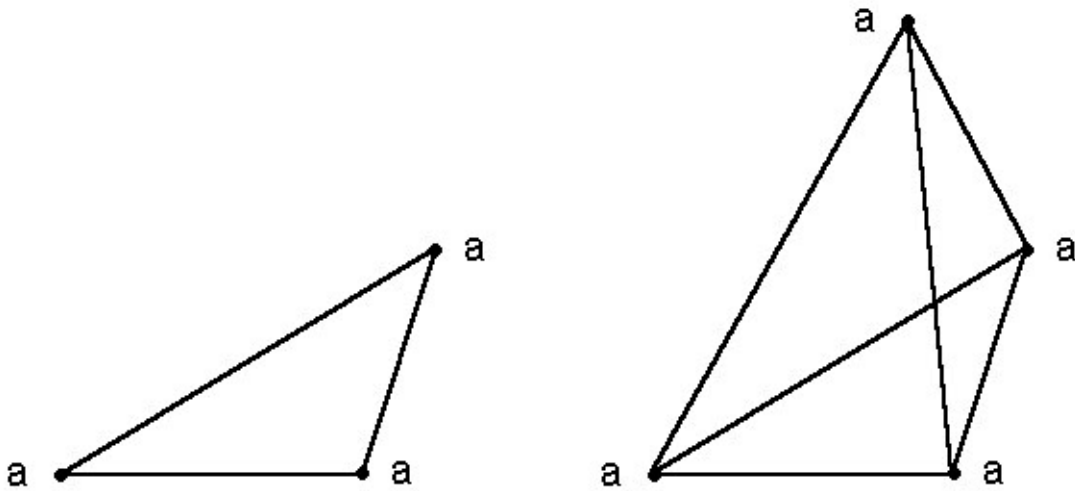
Its (exact) clique-width,  **$cwd(G)$** , is the smallest such  $k$ .

**Proposition :** (1) If a set of simple graphs has bounded tree-width, it has bounded clique-width, but not *vice versa*.

(2) Unlike tree-width, clique-width is sensible to edge directions : Cliques have clique-width 2, tournaments have unbounded clique-width.

(3) "Clique-width  $\leq 3$ " is polynomial. The complexity (polynomial or NP-complete) of "Clique-width = 4" is unknown.

*Example* : Cliques have clique-width 2.



$K_n$  is defined by  $t_n$

$$t_{n+1} = \text{Relab}_{b,a}(\text{Add-edg}_{a,b}(t_n \oplus \mathbf{b}))$$

*Another example* : *Cographs*

Cographs are generated by  $\oplus$  and  $\otimes$  defined by :

$$G \otimes H = \text{Relab}_{b,a}(\text{Add-edg}_{a,b}(G \oplus \text{Relab}_{a,b}(H)))$$

=  $G \oplus H$  with “all edges” between  $G$  and  $H$ .

## *Algorithmic applications*

On graphs of clique-width  $\leq k$  ,  
each monadic second-order property, (ex. 3-colorability)  
each monadic second-order optimization function, (ex. distance)  
each monadic second-order counting function, (ex. # of paths)  
is evaluable :

in linear time on graphs given by a term in  $T(VR)$ ,  
in time  $O(n^9 \cdot \log(n))$  otherwise (S. Oum, P. Seymour).

This is possible in linear time on graphs of tree-width  $\leq k$   
(Bodlaender)



## Equivalent signatures

If  $F \subset G$  are two signatures on  $M$  :

$$EQ(F) \subseteq EQ(G)$$

$$REC(G) \subseteq REC(F)$$

$F$  and  $G$  are **equivalent** ( $\equiv$ ) if we have equalities.

### Extending HR :

$HR^{Fuse}$  = HR augmented with the (quantifier-free definable) unary operations :

$Fuse_{a,b}$  fuses the  $a$ -source and the  $b$ -source,  
and forgets  $b$ .

*Proposition* :  $HR^{Fuse} \equiv HR$

*Proof* : For  $REC(HR^{Fuse}) = REC(HR)$  , we use :

$$Fuse_{a,b}(G // H) = Fuse_{a,b}(G) // Fuse_{a,b}(H)$$

$$Fuse_{a,b}(Forget_c(G)) = Forget_c(Fuse_{a,b}(G))$$

$$Fuse_{a,b}(Ren_{c,a}(G)) = Ren_{c,a}(Fuse_{c,b}(G))$$

*Proposition* :  $HR = \{ //, Forget, Ren \} \equiv \{ \oplus, Fuse, Forget, Ren \}$

## Extending VR :

We enrich VR by new unary operations of several types

(1) *Quantifier-free definable operations, giving VR<sup>+</sup>*

*Example* : the *edge complement* for loop-free undirected graphs

If  $G = \langle V_G, \text{edg}_G(\dots) \rangle$  then  $\text{Compl}(G) = G' = \langle V_G, \text{edg}_{G'}(\dots) \rangle$

where :  $\text{edg}_{G'}(x,y) : \Leftrightarrow x \neq y \wedge \neg \text{edg}_G(x,y)$ .

*Proposition* :  $\text{VR} \equiv \text{VR}^+$

(2) *Fusion* : a non-quantifier-free definable operation:

$\text{Fuse}_a(G)$  is obtained from  $G$  by fusing all  $a$ -ports  
into a single vertex.

*Proposition* :  $\text{VR}^+ + \text{Fusion} \equiv \text{VR}$

(A. Blumensath, B.C., J.Makowsky, P.Weil)

*Significance of these results*

*Robustness of the formal framework*

## Why quantifier-free unary operations ?

To have the inclusion :

Monadic Second Order  $\subseteq$  Recognizable

(for appropriate precise definitions).

## Monadic Second-Order (MS) Logic

= First-order logic on power-set structures

= First-order logic extended with (quantified) variables denoting subsets of the domains.

## MS properties :

transitive closure, properties of paths, connectivity,

planarity (via Kuratowski, uses connectivity),

k-colorability.

Examples of formulas for  $G = \langle V_G, \text{edg}_G(.,.) \rangle$ , undirected

### Non connectivity :

$$\exists X \left( \exists x \in X \ \& \ \exists y \notin X \ \& \right. \\ \left. \forall u,v (u \in X \ \& \ \text{edg}(u,v) \Rightarrow v \in X) \right)$$

### 2-colorability (i.e. G is bipartite) :

$$\exists X \left( \forall u,v (u \in X \ \& \ \text{edg}(u,v) \Rightarrow v \notin X) \ \& \right. \\ \left. \forall u,v (u \notin X \ \& \ \text{edg}(u,v) \Rightarrow v \in X) \right)$$

*Definition* : A set  $L$  of words, of trees, of graphs is **Monadic Second-Order (MS) definable** iff

$$L = \{ S \mid S \models \varphi \} \text{ for an MS formula } \varphi$$

**Theorem** : (1) A language or a set of finite terms is recognizable  $\Leftrightarrow$  it is MS definable

(2) A set of finite graphs is VR- or VR<sup>+</sup>-recognizable  $\Leftarrow$  it is MS definable

*Basic facts for (2)* :

Let  $F$  consist of  $\oplus$  and unary **quantifier-free definable operations**  $f$ .

For every MS formula  $\varphi$  of quantifier-height  $k$ , we have

(a) for every  $f$ , one can construct a formula  $f^\#(\varphi)$

such that :  $f(S) \models \varphi \Leftrightarrow S \models f^\#(\varphi)$

(b)(Fefermann-Vaught-Shelah) one can construct formulas

$\psi_1, \dots, \psi_n, \theta_1, \dots, \theta_n$  such that :

$$S \oplus T \models \varphi \Leftrightarrow \text{for some } i, S \models \psi_i \ \& \ T \models \theta_i$$

where  $f^\#(\varphi), \psi_1, \dots, \psi_n, \theta_1, \dots, \theta_n$  have **quantifier-height  $\leq k$** .

(c) Up to equivalence, there are finitely many formulas of quantifier-height  $\leq k$  forming a set  $\Phi_k$ . One builds an automaton with states the subsets of  $\Phi_k$  (the MS-theories of quantifier-height  $\leq k$  of the graphs defined by the subterms of the term to be processed).

## 5. Monadic Second-Order Transductions

$\text{STR}(\Sigma)$ : the set of finite  $\Sigma$ -relational structures (or finite directed ranked  $\Sigma$ -hypergraphs).

**MS transductions** are multivalued mappings

$$\tau : \text{STR}(\Sigma) \rightarrow \text{STR}(\Gamma)$$

$$S \longmapsto T = \tau(S)$$

where  $T$  is :

- a) defined by MS formulas
- b) inside the structure:  $S \oplus S \oplus \dots \oplus S$   
(fixed number of disjoint "marked" copies of  $S$ )
- c) in terms of "parameters" i.e. subsets  $X_1, \dots, X_p$  of the domain of  $S$

**Proposition** : The composition of two

MS transductions is an MS transduction.

**Remark** : For each tuple of parameters  $X_1, \dots, X_p$

satisfying an MS property,  $T$  is uniquely defined.

$\tau$  is multivalued by the different choices of parameters.

**Examples** :  $(G, \{x\}) \longmapsto$  the connected component containing  $x$ .

$(G, X, Y) \longmapsto$  the minor of  $G$  resulting from contraction of edges in  $X$  and deletion of edges and vertices in  $Y$ .

## Example of an MS transduction (without parameters)

The *square* mapping  $\delta$  on words:  $u \mapsto uu$

We let  $u = \text{aac}$

S

.	→	.	→	.
a		a		c

$S \oplus S$

.	→	.	→	.		.	→	.	→	.
a		a		c		a		a		c
$p_1$		$p_1$		$p_1$		$p_2$		$p_2$		$p_2$

$\delta(S)$

.	→	.	→	.	→	.	→	.	→	.
a		a		c		a		a		c

In  $\delta(S)$  we redefine Suc (i.e.,  $\rightarrow$ ) as follows :

$\text{Suc}(x,y) : \Leftrightarrow p_1(x) \ \& \ p_1(y) \ \& \ \text{Suc}(x,y)$   
 $\vee \ p_2(x) \ \& \ p_2(y) \ \& \ \text{Suc}(x,y)$   
 $\vee \ p_1(x) \ \& \ p_2(y) \ \& \ \text{"x has no successor"}$   
 $\ \& \ \text{"y has no predecessor"}$

We also remove the "marker" predicates  $p_1, p_2$ .

The fundamental property of MS transductions :

$$\begin{array}{ccc} S & \longrightarrow & \tau(S) \\ \tau^\#(\psi) & \longleftarrow & \psi \end{array}$$

Every MS formula  $\psi$  has an effectively computable *backwards translation*  $\tau^\#(\psi)$ , an MS formula, such that :

$$S \models \tau^\#(\psi) \quad \text{iff} \quad \tau(S) \models \psi$$

The verification of  $\psi$  in the object structure  $\tau(S)$  **reduces** to the **verification** of  $\tau^\#(\psi)$  in the given structure  $S$ .

*Intuition* :  $S$  contain all necessary information to describe  $\tau(S)$  ; the MS properties of  $\tau(S)$  are expressible by MS formulas in  $S$

*Consequence* : If  $L \subseteq \text{STR}(\Sigma)$  has a **decidable MS satisfiability problem**, so has its image under an MS transduction.

## Other results

1) A set of graphs is VR-equational iff it is the image of **(all)** binary trees under an MS transduction.

VR-equational sets are stable under MS-transductions.

A set of graphs has bounded **clique-width** iff it is the image of **a** set of binary trees under an MS transduction.

2) A set of graphs is HR-equational iff it is the image of **(all)** binary trees under an **MS<sub>2</sub> transduction**. (**MS<sub>2</sub>** = MS with edge set quantifications).

HR-equational sets are stable under **MS<sub>2</sub>**-transductions.

A set of graphs has bounded **tree-width** iff it is the image of **a** set of binary trees under an **MS<sub>2</sub> transduction**.

3) A set of hypergraphs is QF-equational iff it is the image of **(all)** binary trees under an MS-transduction.

QF-equational sets are stable under MS-transductions.

4) (A.B.,B.C.,2004) : QF-recognizable sets are preserved under **inverse MS transductions**.



## Relationships between algebraic and logical notions

Algebraic notions	Algebraic characterizations	Logical characterizations	Closure properties
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	equation systems	MS-trans(Trees)	homo
	$Val(REC(Terms))$		MS-trans
REC			Boolean opns
	congruences	MS-def $\subset$ REC	homo <sup>-1</sup>
			MS-trans <sup>-1</sup>

### Signatures for graphs and hypergraphs :

$HR$  : graphs and hypergraphs with “sources”

$VR$  : graphs with vertex labels (“ports”)

$VR^+$  :  $VR$  with quantifier-free operations  
(ex. edge complement)

$QF$  : hypergraphs, i.e., relational structures  
(disjoint union  $\oplus$  and quantifier-free definable  
unary operations)

Theorem (A. Blumensath, B.C.,2004) : QF-recognizable sets are preserved under *inverse* MS transductions.

This is already known (1) for MS definable sets and (2) for the recognizable sets of graphs of bounded tree-width.

*Proof sketch* : Every MS transduction is the composition of MS transductions of 3 types :

- Copy<sub>k</sub>
- Guess of unary relations
- Parameterless noncopying MS transductions.

For each of them we establish the closure result by a specific proof.

1) Copy<sub>k</sub> (a kind of homomorphism) :

$$S \quad | \rightarrow \quad S \oplus S \oplus \dots \oplus S \quad (k \text{ times})$$

augmented with binary relations  $Y_{i,j}$  for  $1 \leq i < j \leq k$

defined as

$$\{(x,y) \mid x \text{ is the } i\text{-copy, } y \text{ is the } j\text{-copy of some } u \text{ in } D_S\}$$

This step handles the “domain extension” possibility.

The class of recognizable sets is closed under inverse homomorphisms.

## 2) Guessing unary relations

« Guessing » the relations in  $\Pi$  can be done by means of **parameters** of an MS transduction.

*Lemma* : Its inverse, “forgetting” unary relations preserves recognizability.

We let  $\text{fgt}_{\Pi} : \text{STR}(\Sigma \cup \Pi) \rightarrow \text{STR}(\Sigma)$  « forget »  $\Pi$ , where  $\Pi$  is a finite set of unary relations.

This step handles the use of parameters in MS transductions.

## 3) Parameterless noncopying MS transductions.

*Lemma* : If  $L \in \text{REC}(\text{QF})_{\Gamma}$ , and  $\tau$  is a parameterless noncopying MS transduction :  $\text{STR}(\Sigma) \rightarrow \text{STR}(\Gamma)$ , then  $\tau^{-1}(L) \in \text{REC}(\text{QF})_{\Sigma}$

(See the slides of my DLT lecture for a proof sketch)

## 6. Links between MS logic and combinatorics : Seese's Conjecture

*Theorem* (Seese 1991) :

If a set of graphs has a decidable  $MS_2$  satisfiability problem, it has bounded **tree-width**.

*Conjecture* (Seese 1991) :

If a set of graphs has a decidable  $MS$  satisfiability problem, it is the image of a set of trees under an  $MS$  transduction, equivalently, has bounded **clique-width**.

*Theorem* (B.C., S. Oum 2004) :

If a set of graphs has a decidable  $C_2MS$  satisfiability problem, it has bounded clique-width.

$MS$  = (Basic)  $MS$  logic **without** edge quantifications

$MS_2$  =  $MS$  logic **with** edge quantifications

$C_2MS$  =  $MS$  logic with **even cardinality** set predicates

A set  $C$  has a **decidable  $L$  satisfiability problem** if one can decide for every formula in  $L$  whether it is satisfied by some graph in  $C$

### Proof of Seese's Theorem :

A) If a set of graphs  $C$  has **unbounded tree-width**, the set of its **minors includes all  $k \times k$ -grids** (Robertson, Seymour)

B) If a set of graphs **contains all  $k \times k$ -grids**, its  $MS_2$  satisfiability problem is **undecidable**

C) If  $C$  has **decidable  $MS_2$  satisfiability** problem, so has  $\text{Minors}(C)$ , because  $C \longrightarrow \text{Minors}(C)$  is an  $MS_2$  transduction.

Hence, if  $C$  has **unbounded tree-width** and a **decidable  $MS_2$  satisfiability** problem, we have a contradiction for the **decidability of the  $MS_2$  satisfiability** problem of  $\text{Minors}(C)$ .

### Proof of Courcelle-Oum's Theorem :

D) Equivalence between the cases of all (directed and undirected) graphs and bipartite undirected graphs.

A') If a set of bipartite graphs  $C$  has **unbounded clique-width**, the set of its **vertex-minors contains all " $S_k$ " graphs**

C') If  $C$  has **decidable  $C_2MS$  satisfiability** problem, so has  $\text{Vertex-Minors}(C)$ , because  $C \longrightarrow \text{Vertex-Minors}(C)$  is a  **$C_2MS$**  transduction.

E) An  **$MS$**  transduction transforms  **$S_k$**  into the  **$k \times k$ -grid**.

## Definitions and facts

**Local complementation** of  $G$  at vertex  $v$

$G * v = G$  with edge complementation of  $G[n_G(v)]$ ,

the subgraph induced by the **neighbours** of  $v$

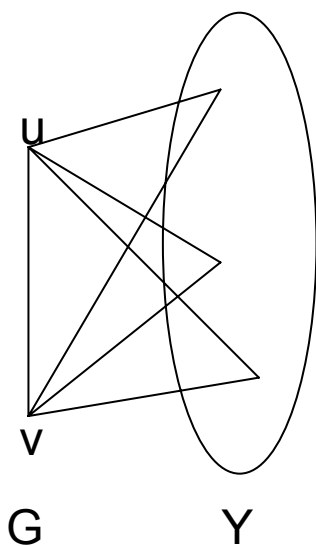
**Local equivalence** ( $\approx_{loc}$ ) = transitive closure of local complementation (at all vertices)

Vertex-minor relation :

$H \leq_{VM} G : \Leftrightarrow H$  is an induced subgraph of some  $G' \approx_{loc} G$

*Proposition* : The mapping associating with  $G$  its locally equivalent graphs is a **C<sub>2</sub>MS** transduction.

Why is the even cardinality set predicate necessary ?



Consider  $G * X$  for  $X \subseteq Y$  :

$u$  is linked to  $v$  in  $G * X$

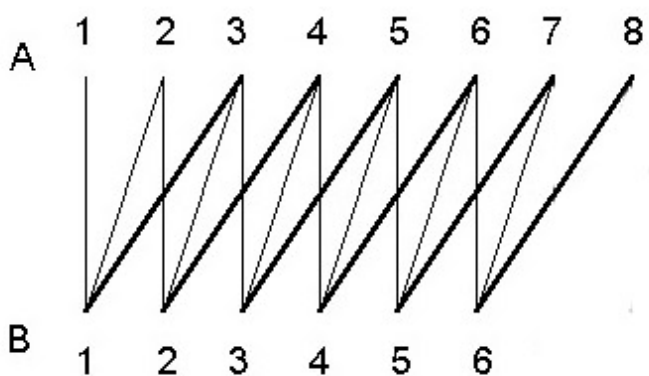
$\Leftrightarrow \text{Card}(X)$  is even

*Definition of*  $S_k$  :

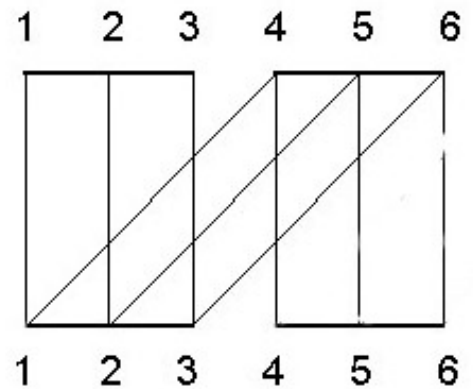
bipartite :  $A = \{1, \dots, (k+1)(k-1)\}$  ,  $B = \{1, \dots, k(k-1)\}$

for  $j \in A, i \in B$  :  $\text{edg}(i,j) \Leftrightarrow i \leq j \leq i+k-1$

From  $S_k$  to  $\text{Grid}_{k \times k}$  by an MS transduction



$S_3$



(folded)  $\text{Grid}_{3 \times 4}$

1) One can define the orderings of A and B :

$$x, y \text{ are consecutive} \Leftrightarrow \text{Card}(n_G(x) \Delta n_G(y)) = 2$$

2) One can identify the edges from  $i \in B$  to  $i \in A$ , and

from  $i \in B$  to  $i+k-1 \in A$  (thick edges on the left drawing)

3) One can create edges (e.g. from  $1 \in A$  to  $2 \in A$ , from  $2 \in A$  to  $3 \in A$  etc...and similarly for B, and from  $1 \in B$  to  $4 \in A$ , etc...) and delete others (from  $4 \in B$  to  $6 \in A$  etc...), and vertices like 7,8 in A, to get a grid containing  $\text{Grid}_{k \times k}$

*Corollary* : If a set of directed acyclic graphs having Hamiltonian directed paths has a decidable MS satisfiability problem, then :

it has bounded clique-width,

it is the image of a set of trees under an MS transduction.

*Proof* : Since on these graphs a linear order is MS definable, MS and  $C_2MS$  are equivalent.

The previously known techniques for similar results (in particular for line graphs or interval graphs, B.C. 2004) do not work in this case.



## 7. Some open questions

*Question 1* : Which operations, quantifier-free definable or not, yield extensions of VR, HR, QF that are equivalent ?

*Question 2* : Under which operations, quantifier-free definable or not, are REC(VR) and REC(HR) closed ?

(The case of REC(HR) is considered in B.C.: *(HR-)Recognizable sets of graphs, equivalent definitions and closure properties*, 1994. It is not hard to see that REC(VR) is closed under  $\oplus$  (disjoint union) but not under the operations Add-edg<sub>a,b</sub>.)

*Question 3* : Is it true that the decidability of the MS (and not the  $C_2$ MS) satisfiability problem for a set of graphs implies bounded clique-width, as conjectured by D. Seese ?