Modular decompositions of graphs and hypergraphs, and their definitions by monadic second-order formulas

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We know from graph theory that graphs have certain canonical hierarchical decompositions. One of them is the decomposition of a graph into a forest over its 3-connected components, another one is the *modular decomposition*. These hierarchical decompositions are based on operations that construct graphs: respectively, certain "gluing operations" and substitutions of graphs for vertices.

Hierarchical decompositions are interesting because they yield in some cases, proofs of properties based on induction on the hierarchical structure of the considered graphs. Efficient algorithms, based on such induction (and the Cartesian principle of splitting a problem into smaller, easier to solve ones) can be obtained through them.

The notion of modular decomposition has been rediscovered frequently by many authors, and named in different ways. It is based on the notion of a *module* in a graph, i.e., of a set of vertices that are all linked in the same way to each vertex not in the set (*linked in the same way* means, either linked or not, and with same edge directions in case of directed graphs.) It has been considered in particular by R. Fraïssé under the name of an *interval*. The terminology based on the term *module* seems to be now the standard one.

We are interested in situations where such hierarchical graph decompositions can be defined in monadic second-order logic [1]. This gives in particular results saying that a set of graphs is *monadic second-order definable* if and only if it is *recognizable*, i.e., is a union of congruence classes having finitely many classes [2]. (The notion of congruence is relative to the operations upon which the decomposition is defined.)

The definability result holds for the decomposition of a graph in 3-connected blocks, and also for the modular decomposition under the additional assumption that the graph is given with some linear order of its set of vertices [2]. (Any two linear orders yield the same decomposition since modular decomposition is unique; we do not know how to do the construction with monadic second-order formulas without using an auxiliary linear order.)

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We will present the notion of modular decomposition for *finite undirected hypergraphs of unbounded rank* [3] (we call *rank* the maximal size of a hyperedge). Such hypergraphs can be handled as bipartite graphs, but their modular decompositions are not the ones for graphs via this representation. The trees representing modular decompositions can be defined in monadic second-order logic without needing any auxiliary ordering.

A hypergraph is *convex* if there exists a linear order on the vertices such that all hyperedges are intervals (with respect to the order, not "modules"). If it is *prime* (i.e., is a basic component of the modular decomposition), then only two linear orders witness its convexity (one and its opposite). They can be defined by monadic second-order formulas [3].

We deduce from this construction that if a set of finite convex hypergraphs has a decidable monadic second-order theory then, considered as a set of bipartite graphs, it has so-called *bounded clique-width*, or equivalently, can be obtained from finite trees by a certain graph transformation expressible in monadic second-order logic.

This provides us with a new case of validity of a conjecture made by Seese (see [1]) that sets of graphs having a decidable monadic second-order theory are definable from trees by monadic second-order formulas.

Bibliographical note

The first reference to modular decomposition is usually to the 1967 paper by Gallai [6]. However, the modular decomposition is already defined in the 1953 note by Fraïssé [4] (no name is given to the notion of a module), and for an n-ary relation, not only a binary one. In a more recent paper by Fraïssé [5], the corresponding notion of a module is called an interval.

References

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