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# Quantifier-free definable graph operations preserving recognizability

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## Abstract

We show that an operation on graphs, and more generally, on relational structures that has an inverse definable by a monadic second-order transduction preserves the family of recognizable sets.

## 1 Introduction

Several algebras of graphs, and more generally of relational structures, can be defined in terms of disjoint union as unique binary operation and of several unary operations defined by quantifier-free formulas. These algebras are the basis of the extension to graphs and hypergraphs of the *theory of formal languages* in a universal algebra setting.

In every algebra, one can define two families of subsets, the family of *equational sets* which generalizes the family of context-free languages, and the family of *recognizable sets* which generalizes the family of recognizable languages. Equational sets are defined as least solutions of systems of recursive set equations and not in terms of rewriting rules. Recognizable sets are defined in terms of finite congruences and not in terms of finite automata. These purely algebraic definitions which are due to Mezei and Wright [8] have the advantage of being applicable to every algebra, whereas rewriting systems and finite automata cannot. One obtains definitions of "context-free" sets of graphs which avoid the cumbersome analysis of the confluence of particular graph rewriting systems. The basic definitions and facts regarding these notions can be found in [2, 5, 6, 7].

Certain closure properties of the families of equational and recognizable sets are valid at the most general level. In particular, the family of equational sets of an algebra  $\mathbf{M}$  is closed under union, intersection with the recognizable sets and under the operations of this algebra. For an example, the concatenation of two equational (i.e., context-free) languages is equational. The family of recognizable sets of an algebra  $\mathbf{M}$  is closed under

union, intersection and difference, and under the inverses of *unary derived operations* (the operations defined by finite terms over the signature of  $\mathbf{M}$ ). The family of recognizable languages (alternatively called *rational* or *regular*) is also closed under concatenation, but this is not a special case of a general algebraic property, by contrast with the case of equational languages. In a general algebra, the family of recognizable sets is not always closed under the operations of the algebra. That these closure properties are true depends on particular properties of the considered algebra.

*Which properties of an algebra ensure that the family of recognizable sets is closed under the operations of the algebra?*

Two types of answers can be given : algebraic and logical answers. Algebraic answers have been given in [4], an article motivated by the study of the so-called *Hyperedge Replacement (HR) algebra of graphs and hypergraphs*, that is connected in a natural way to the notion of *tree-width* ([6]). The results of the article [4] can be applied to the case of languages in a quite simple way: the property of words that  $uv = wx$  if and only if there exists a word  $z$  such that  $u = wz$  and  $zv = x$ , or  $uz = w$  and  $v = zx$  implies that the concatenation of two recognizable languages is recognizable, by a proof that uses only finite congruences and no construction of automata.

Another important case is that of an associative and commutative operation, a useful example being the disjoint union of graphs and relational structures denoted by  $\oplus$ . The corresponding (commutative) concatenation of subsets preserves recognizability because the equality  $u \oplus v = w \oplus x$  is equivalent to the existence of  $y_1, y_2, y_3, y_4$  such that  $u = y_1 \oplus y_2$ ,  $v = y_3 \oplus y_4$ ,  $w = y_1 \oplus y_3$  and  $x = y_2 \oplus y_4$ .

The article [4] establishes that the family of HR-recognizable sets of graphs is closed under the operations of the HR-algebra. One might think that these results would extend without difficulties to the somewhat similar *Vertex Replacement (VR) algebra of graphs* (which we define below). However this is not the case as we will see in the next section.

In the present article, we do not answer the above question in full generality, but we give a sufficient condition for algebras of finite relational structures (hence also of finite graphs) whose operations are disjoint union and unary operations defined by quantifier-free formulas, that we call *quantifier-free definable operations*. We are particularly interested by these algebras because every *monadic second-order definable set* of finite relational structures is recognizable (see Theorem 3 below). Our main result (Theorem 6) is a direct consequence of a result of [2]. It relates the preservation of recognizability in the algebra of relational structures under a unary operation to the

existence an inverse for this operation that is a *monadic second-order transduction*. The present article continues the exploration done in particular in [1, 2, 3, 6, 7] of the deep links between algebraic and logical properties, more precisely here, between recognizability and monadic second-order logic.

## 2 The VR-algebra of simple graphs.

Graphs are finite, simple (without multiple edges), directed, and loop-free. Let  $C$  be a countable set of labels containing the set of nonnegative integers. A  $C$ -graph is a graph  $G$  given with a total mapping  $lab_G$  from its vertex set  $V_G$  to  $C$ . Hence  $G$  is defined as a triple  $\langle V_G, edg_G, lab_G \rangle$  where  $edg_G$  is the binary edge relation. We call  $lab_G(v)$  the *label* of a vertex  $v$ . We denote by  $\pi(G)$  the finite set  $lab_G(V_G) \subseteq C$ , and we call it the *type* of  $G$ . The operations on  $C$ -graphs are the following ones :

- (i) We define a constant  $\mathbf{1}$  to denote an isolated vertex labelled by 1.
- (ii) For  $i, j \in C$  with  $i \neq j$ , we define a unary function  $add_{i,j}$  such that:

$$add_{i,j}(\langle V_G, edg_G, lab_G \rangle) = \langle V_G, edg'_G, lab_G \rangle$$

where  $edg'_G$  is  $edg_G$  augmented with the set of pairs  $(u, v)$  such that  $lab_G(u) = i$  and  $lab_G(v) = j$ . In order to add undirected edges (considered as pairs of opposite directed edges), we take :

$$add_{i,j}(add_{j,i}(\langle V_G, edg_G, lab_G \rangle)).$$

- (iii) We let also  $ren_{i \rightarrow j}$  be the unary function such that

$$ren_{i \rightarrow j}(\langle V_G, edg_G, lab_G \rangle) = \langle V_G, edg_G, lab'_G \rangle$$

where  $lab'_G(v) = j$  if  $lab_G(v) = i$ , and  $lab'_G(v) = lab_G(v)$ , otherwise. This mapping relabels into  $j$  every vertex label  $i$ .

- (iv) Finally, we use the binary operation  $\oplus$  that makes the union of disjoint copies of its arguments. Hence the graph  $G \oplus H$  is well-defined up to isomorphism.

We denote by  $F^{VR}$  the countable set of all these operations, including the constant  $\mathbf{1}$ . The *VR-algebra* has for domain the set  $\mathcal{G}$  of all isomorphism classes of  $C$ -graphs and the operations of  $F^{VR}$ . A well-formed term  $t$  written with the symbols of  $F^{VR}$  defines a  $C$ -graph  $G = val(t)$ , actually a graph up to isomorphism. However,  $val(t)$  can be defined as a "concrete" graph with vertex set  $Occ_{\mathbf{1}}(t)$  the set of occurrences in  $t$  of the constant  $\mathbf{1}$ .

A set of  $C$ -graphs  $L$  is *VR-recognizable* if there exists an  $F^{VR}$ -congruence  $\approx$  on  $\mathcal{G}$  such that :

- (1)  $G \approx H$  implies  $\pi(G) = \pi(H)$

- (2) for each finite subset  $D$  of  $C$ , the congruence  $\approx$  has finitely many equivalence classes of graphs of type  $D$ ,
- (3)  $L$  is the union of a finite set of equivalence classes of  $\approx$ .

We will prove below that the disjoint union and the renaming operations  $ren_{i \rightarrow j}$  preserve VR-recognizability. (A more complicated proof can be based on the algebraic lemmas of [4].) However :

**Proposition 1** : The operation  $add_{a,b}$  does not preserve recognizability. The operation that deletes all edges does not either.

**Proof** : Here is a counter-example. One takes the set  $L$  of finite directed graphs  $G$  of type  $\{a, b\}$  consisting of pairwise nonadjacent edges linking one vertex labelled by  $a$  to one vertex labelled by  $b$ . Hence, we have as many  $a$ -labelled vertices as  $b$ -labelled ones. This set is definable in monadic second-order logic (and even in first-order logic) hence is VR-recognizable by a general theorem (see [3, 6], Theorem 3 below). The set  $K = add_{a,b}(L)$  consists of complete bipartite graphs  $K_{n,n}$ . And this set is not recognizable, because otherwise, so would be the set of terms of the form  $add_{a,b}([\mathbf{a} \oplus (\mathbf{a} \oplus (\dots \mathbf{a}))..]) \oplus [\mathbf{b} \oplus (\dots (\mathbf{b} \oplus \mathbf{b})..)]$  having  $n$  occurrences of  $\mathbf{a}$  defined as  $ren_{1 \rightarrow a}(\mathbf{1})$  and  $n$  occurrences of  $\mathbf{b}$  defined as  $ren_{1 \rightarrow b}(\mathbf{1})$  with  $n > 0$ . By a standard pumping argument this set is not recognizable. The proof is similar for the operation that deletes all edges. One uses the terms  $[\mathbf{a} \oplus (\mathbf{a} \oplus (\dots \mathbf{a}))..]) \oplus [\mathbf{b} \oplus (\dots (\mathbf{b} \oplus \mathbf{b})..)]$ .  $\square$

We now describe the logical setting that will help to investigate recognizability. We formulate it not only for graphs but for finite relational structures.

### 3 Relational structures and monadic second-order logic

Let  $R = \{A, B, C, \dots\}$  be a finite set of *relation symbols* each of them given with a nonnegative integer  $\rho(A)$  called its *arity*. We denote by  $STR(R)$  the set of *finite R-structures*  $S = \langle D_S, (A_S)_{A \in R} \rangle$  where  $A_S \subseteq D_S^{\rho(A)}$  if  $A \in R$  is a relation symbol, and  $D_S$  is the *domain* of  $S$ . If  $R$  consists of relation symbols of arity one or two we say that the structures in  $STR(R)$  are *binary*. Binary structures can be seen as vertex- and edge-labelled graphs. If we have several binary relations say  $A, B, C$ , the corresponding graphs have edges with labels  $A, B, C$ .

*Monadic Second-order logic* (*MS logic* for short) is the extension of *First-Order logic* with variables denoting subsets of the domains of the considered

structures and atomic formulas of the form  $x \in X$  expressing the membership of  $x$  in a set  $X$ . We will denote by  $MS(R, W)$  the set of *Monadic second-order* formulas written with the set  $R$  of relation symbols and having their free variables in a set  $W$  consisting of *first-order and set variables*.

As a typical and useful example, we give an MS formula with free variables  $x$  and  $y$  expressing that  $(x, y)$  belongs to the reflexive and transitive closure of a binary relation  $A$  :

$$\forall X(x \in X \wedge \forall u, v[(u \in X \wedge A(u, v)) \implies v \in X] \implies y \in X).$$

If the relation  $A$  is not given in the considered structures but is defined by an MS formula, then one replaces  $A(u, v)$  by this formula with appropriate substitutions of variables.

A subset of  $STR(R)$  is *MS-definable* if it is the set of finite models of a *monadic second-order sentence*, i.e., of an MS formula without free variables. Such a set is closed under isomorphism.

## 4 Monadic second-order transductions

Monadic second-order formulas can be used to define transformations of graphs and relational structures. As in language theory, a binary relation  $\mathcal{R} \subseteq \mathcal{A} \times \mathcal{B}$  where  $\mathcal{A}$  and  $\mathcal{B}$  are sets of words, graphs or relational structures is called a *transduction*:  $\mathcal{A} \rightarrow \mathcal{B}$ . An *MS transduction* is a transduction specified by MS formulas. It transforms a structure  $S$ , given with an  $n$ -tuple of subsets of its domain called the *parameters*, into a structure  $T$ , the domain of which is a subset of  $D_S \times [k]$ , (where  $[k] = \{1, \dots, k\}$ ). It is *noncopying* if  $k = 1$ . The general definition can be found in [1, 2, 6]. We only define noncopying MS transductions which are needed in this article.

We let  $R$  and  $Q$  be two finite sets of relation symbols. Let  $W$  be a finite set of set variables, called *parameters*. A  $(Q, R)$ -*definition scheme* is a tuple of formulas of the form :

$$\begin{aligned} \Delta &= (\varphi, \psi, (\theta_A)_{A \in Q}) \\ \text{where } \varphi &\in MS(R, W), \psi \in MS(R, W \cup \{x_1\}), \\ \text{and } \theta_A &\in MS(R, W \cup \{x_1, \dots, x_{\rho(A)}\}), \text{ for } A \in Q. \end{aligned}$$

These formulas are intended to define a structure  $T$  in  $STR(Q)$  from a structure  $S$  in  $STR(R)$ . Let  $S \in STR(R)$ , let  $\gamma$  be a  $W$ -assignment in  $S$ . A  $Q$ -structure  $T$  with domain  $D_T \subseteq D_S$  is *defined in*  $(S, \gamma)$  by  $\Delta$  if :

- (i)  $(S, \gamma) \models \varphi$
- (ii)  $D_T = \{d \mid d \in D_S, (S, \gamma, d) \models \psi\}$

(iii) for each  $A$  in  $Q$  :  $A_T = \{(d_1, \dots, d_t) \in D_T^t \mid (S, \gamma, d_1, \dots, d_t) \models \theta_A\}$ , where  $t = \rho(A)$ .

Since  $T$  is associated in a unique way with  $S, \gamma$  and  $\Delta$  whenever it is defined, i.e., whenever  $(S, \gamma) \models \varphi$ , we can use the functional notation  $def_\Delta(S, \gamma)$  for  $T$ . The *transduction defined by  $\Delta$*  is the binary relation :

$$def_\Delta := \{(S, T) \mid T = def_\Delta(S, \gamma) \text{ for some } W\text{-assignment } \gamma \text{ in } S\}.$$

A transduction  $f \subseteq STR(R) \times STR(Q)$  is a noncopying *MS transduction* if it is equal to  $def_\Delta$  (up to isomorphism) for some  $(Q, R)$ -definition scheme  $\Delta$ . We will also write functionally :  $def_\Delta(S) := \{def_\Delta(S, \gamma) \mid \gamma \text{ is a } W\text{-assignment in } S\}$ . A definition scheme without parameters defines a *parameterless MS transduction*, which is actually a partial function:  $STR(R) \longrightarrow STR(Q)$ .

A *quantifier-free definable operation* (a *QF operation in short*) is a parameterless noncopying MS-transduction  $\delta : STR(R) \longrightarrow STR(Q)$  defined by a scheme  $\Delta = (\varphi, \psi, (\theta_A)_{A \in Q})$  such that the formula  $\varphi$  is equivalent to *True*, and the formulas  $\theta_A$  are without quantifiers (whence also without set variables). This implies that  $\delta$  is total. Furthermore, we say that such an operation is *nondeleting* if the formula  $\psi$  is equivalent to *True*. This condition implies that the domains of  $S$  and of  $\delta(S)$  are the same.

A labelled graph  $\langle V_G, \text{edg}_G, \text{lab}_G \rangle$  of type contained in  $D$  will be represented by the relational structure  $\lfloor G \rfloor = \langle V_G, \text{edg}_G, p_{aG}, \dots, p_{dG} \rangle$  where  $D = \{a, \dots, d\}$  and  $p_{xG}(u)$  is true if and only if  $\text{lab}_G(u) = x$ . Through this representation, the unary operations  $add_{a,b}$  and  $ren_{a \rightarrow b}$  are quantifier-free. This means that for some QF operation  $\alpha$ , we have  $\alpha(\lfloor G \rfloor) = \lfloor add_{a,b}(G) \rfloor$  for all graphs  $G$  of type contained in  $D$ , and similarly for  $ren_{a \rightarrow b}$ .

The composition of two transductions is defined as their composition as binary relations. If they are both partial functions, then one obtains the composition of these functions. The *inverse image* of a set  $L \subseteq STR(Q)$  under a transduction  $\delta : STR(R) \longrightarrow STR(Q)$  is the set of elements  $S$  of  $STR(R)$  such that  $\delta(S) \cap L$  is not empty. It is denoted by  $\delta^{-1}(L)$ . (Equality of structures is understood up to isomorphism, hence  $\delta^{-1}(L)$  is closed under isomorphisms.)

**Proposition 2 ([6])** : 1) The composition of two MS transductions is an MS transduction.

2) The inverse image of an MS-definable set of structures under an MS transduction is MS-definable.

## 5 The many-sorted algebra of relational structures

We now make the family of sets  $STR(R)$  for all relational signatures  $R$  into a many-sorted algebra  $\mathbf{STR}$ , where each  $R$  is a sort and  $STR(R)$  is the corresponding domain. Here are the operations. First we define a disjoint union  $\oplus : STR(R) \times STR(Q) \longrightarrow STR(R \cup Q)$  for each pair of sorts  $(R, Q)$  (using the same notation for all of these operations). Then we also let in the signature all QF operations  $: STR(R) \longrightarrow STR(Q)$  for all pairs of sorts  $(R, Q)$ . For each pair  $(R, Q)$  there are actually only finitely many such operations (see [7], Appendix A). We take the constant  $*$  denoting the structure in  $STR(\emptyset)$  with a single element. We could actually take other constants, this would not affect the results stated below because recognizability does not depend on the set of constants. We let  $F^{QF}$  be this signature. The notation refers to the role of QF operations.

A subset of  $STR(R)$  is *QF-recognizable* if it is a (finite) union of classes of an  $F^{QF}$ -congruence on  $\mathbf{STR}$  (equivalent elements must have the same sort) that has finitely many classes in each domain  $STR(R)$ .

The labelled graphs having a type included in a finite set  $D$  are represented by relational structures  $\lfloor G \rfloor = \langle V_G, \text{edg}_G, p_a, \dots, p_d \rangle$  in  $STR(\{\text{edg}\} \cup \{p_a, \dots, p_d\})$  where  $D = \{a, \dots, d\}$ . A set of labelled graphs is VR-recognizable if and only if it is QF-recognizable, and it is VR-equational if and only if it is QF-equational ([BC] Theorem 68).

**Theorem 3 ([3, 6]) :** If a subset of  $STR(R)$  is MS-definable, then it is QF-recognizable.

**Theorem 4 ([2], Theorem 51) :** The inverse image of a QF-recognizable set of relational structures under an MS transduction is QF-recognizable.

The following definition is new.

**Definition 5 :** Let  $\theta$  be a mapping that associates with every structure  $S$  in  $STR(R)$  a structure  $T$  in  $STR(Q)$  with same domain. It is *MS-invertible* if there exists a noncopying and nondeleting MS transduction  $\xi$  with set of parameters  $W$  such that, for all structures  $S$  and  $T$  :

- (1) if  $\theta(S) = T$ , then there exists a  $W$ -assignment  $\gamma$  such that  $\xi(T, \gamma) = S$ ,
- (2) for every  $W$ -assignment  $\gamma$  such that  $\xi(T, \gamma)$  is defined, we have  $\theta(\xi(T, \gamma)) = T$ .

As an example, we can observe that the operation  $ren_{a \rightarrow b}$  is MS-invertible. Let  $H = ren_{a \rightarrow b}(G)$  be obtained from  $G$  by replacing each vertex label  $a$  by  $b$ . This means that the sets  $X$  and  $Y$  of vertices labelled by  $a$  and by  $b$  are made into a unique set  $X \cup Y$ , the set of vertices of  $H$  labelled by  $b$ . To recover  $G$  from  $H$ , it is enough to use a set parameter  $Z$  that guesses, among the vertices labelled by  $b$  those which were originally labelled by  $a$ . Clearly, for each set  $Z$  of vertices labelled by  $b$ , one obtains a graph  $G$  such that  $H = ren_{a \rightarrow b}(G)$ , and every such  $G$  is of this form.

On the contrary, the operation  $add_{a,b}$  is not MS-invertible: the inverse MS-transduction would need to guess a set of edges to be deleted. This is not possible without using edge set quantifications, which is not what we are doing here (but can be done in relation with the HR-algebra, see [1, 6]). However, the restriction of  $add_{a,b}$  to the set of graphs that have no edge from an  $a$ -labelled vertex to a  $b$ -labelled one is MS-invertible, and its inverse MS-transduction is parameterless.

**Theorem 6 :** Every MS-invertible mapping preserves QF-recognizability.

**Proof :** Let  $\theta$  be an MS-invertible mapping :  $STR(R) \rightarrow STR(Q)$  with inverse MS transduction  $\xi$ , using a set of parameters  $W$ . Let  $L \subseteq STR(R)$  be recognizable. We claim that  $\theta(L) = \xi^{-1}(L)$ , which will yield the result by Theorem 5.

If  $T = \theta(S)$ ,  $S \in L$  there exists a  $W$ -assignment  $\gamma$  such that  $\xi(T, \gamma) = S$ , hence  $T$  belongs to  $\xi^{-1}(L)$ . Conversely, if  $T \in \xi^{-1}(L)$ , then  $\xi(T, \gamma) \in L$  for some  $W$ -assignment  $\gamma$  hence  $\theta(\xi(T, \gamma)) = T$  and  $T \in \theta(L)$ .  $\square$

**Remarks :** (1) The proof of Theorem 5 in [2] uses the fact that the QF operation that deletes a unary relation preserves recognizability (Proposition 58). Such an operation is clearly MS-invertible. The proof of Proposition 58 is done with the algebraic techniques of [4]. (Since recognizability is an algebraic notion, algebraic constructions must be used somewhere.)

(2) The same proof yields that MS-invertible QF operations preserve MS-definability, whereas a QF operation like  $add_{a,b}$  does not.

**Question 7 :** Which QF operations are MS-invertible ?

It does not seem easy to give necessary and sufficient conditions. We have already given examples and counter-examples (with help of Proposition 1). The operation that relabels a binary symbol, say  $A$  into  $B$ , does not preserve recognizability. The proof is as in Proposition 1. Here is a related question.



**Question 8 :** Does there exist a QF operation that is not MS-invertible but preserves  $QF$ -recognizability.

We now consider in a similar way the disjoint union  $\oplus : STR(R) \times STR(Q) \rightarrow STR(R \cup Q)$ . Let  $mark$  be a unary relation not in  $R \cup Q$ . Let us define the *marked* disjoint union  $\oplus_{mark} : STR(R) \times STR(Q) \rightarrow STR(R \cup Q \cup \{mark\})$ , such that  $S \oplus_{mark} T = S \oplus T$  augmented with  $mark(u)$  for every  $u$  in the domain of  $T$ . It clear that there are two parameterless QF operations  $\xi_1$  and  $\xi_2$  such that for every structure  $Z$  :

- (1)  $\xi_1(Z)$  and  $\xi_2(Z)$  are defined if and only if  $Z = S \oplus_{mark} T$  for some  $S$  in  $STR(R)$  and some  $T$  in  $STR(Q)$ ,
- (2) and if this is the case  $S$  and  $T$  as in (1) are unique and

$$Z = \xi_1(Z) \oplus_{mark} \xi_2(Z) .$$

**Theorem 9 :** Disjoint union preserves  $QF$ -recognizability.

**Proof :** Let  $L \subseteq STR(R)$  and  $K \subseteq STR(Q)$  be recognizable. Let  $M = L \oplus_{mark} K$ . We claim that  $M = \xi_1^{-1}(L) \cap \xi_2^{-1}(K)$ .

If  $Z = S \oplus_{mark} T \in M$ ,  $S \in L$ ,  $T \in K$ , then  $\xi_1(Z) = S$  and  $\xi_2(Z) = T$ , hence  $Z \in \xi_1^{-1}(S)$  and  $Z \in \xi_2^{-1}(T)$ ,  $Z \in \xi_1^{-1}(L) \cap \xi_2^{-1}(K)$ . Conversely, if  $Z \in \xi_1^{-1}(L) \cap \xi_2^{-1}(K)$  then  $\xi_1(Z) = S \in L$  and  $\xi_2(Z) = T \in K$  and  $Z = S \oplus_{mark} T \in L \oplus_{mark} K = M$ . This proves the claim, and by Theorem 5,  $\xi_1^{-1}(L)$  and  $\xi_2^{-1}(K)$  are recognizable and so is their intersection  $M$ .

The image of  $M$  under the QF operation that deletes  $mark$  is recognizable by Proposition 58 of [2], and this image is  $L \oplus K$ .  $\square$

A similar proof shows that disjoint union preserves MS-definability.

The family of recognizable sets of relational structures is thus preserved under disjoint union and MS-invertible QF operations. These operations form a subsignature  $F^{inv-QF}$  of  $F^{QF}$ . From general facts discussed in depth in [2], it follows that the  $F^{inv-QF}$ -equational sets form a subfamily of the  $QF$ -equational ones, and that the  $QF$ -recognizable sets form a subfamily of the  $F^{inv-QF}$ -recognizable ones. If those two inclusions are equalities, then we say that the signatures  $F^{inv-QF}$  and  $F^{QF}$  are *equivalent*.

**Question 10 :** Is the signature  $F^{inv-QF}$  equivalent to  $F^{QF}$ ?

Let us first go back to the case of the VR-algebra. The signature  $F^{VR}$  is equivalent to the restriction to graphs of the signature  $F^{QF}$  ([3] and Theorem 4.5 of [7]). Furthermore, one can eliminate from  $F^{VR}$  the

operations  $add_{a,b}$  and replace them by *derived operations* of the form  $G \otimes_\lambda H = \lambda(G \oplus H)$  where  $\lambda$  is a composition of  $add_{a,b}$  operations and of relabellings that only create edges between  $G$  and  $H$  (and not inside  $G$  or  $H$ ). One obtains an algebra of graphs with the same recognizable sets ([7], Proposition 4.9) and the same equational sets. For each operation  $\otimes_\lambda$  a pair of inverse MS-transductions like  $\xi_1$  and  $\xi_2$  for  $\oplus$  can be defined so that the operations  $\otimes_\lambda$  preserve recognizability. In this way we can handle the problem of the non-MS-invertibility of  $add_{a,b}$ .

Could we do the same for  $F^{QF}$ ? There is another difficulty with the QF operations that delete relations of arity more than one, and those which rename them, because, as observed above, they are not MS-invertible. A subsignature of  $F^{QF}$  equivalent to it is defined in [2] but it uses these non-MS-invertible operations. We leave open Question 10.

As final comment, we observe that the result of [4] stating that the family of HR-recognizable sets of graphs and hypergraphs is closed under the operations of the HR-algebra can be proved by the tools used for Theorems 6 and 9.

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