On Timed Languages

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based on joint work with Paul Caspi, Oded Maler, Cătălin Dima, Matthieu Moy

VERIMAG

The Problem

Build a good (natural, simple, and usable) theory of timed languages

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Applying classical recipes

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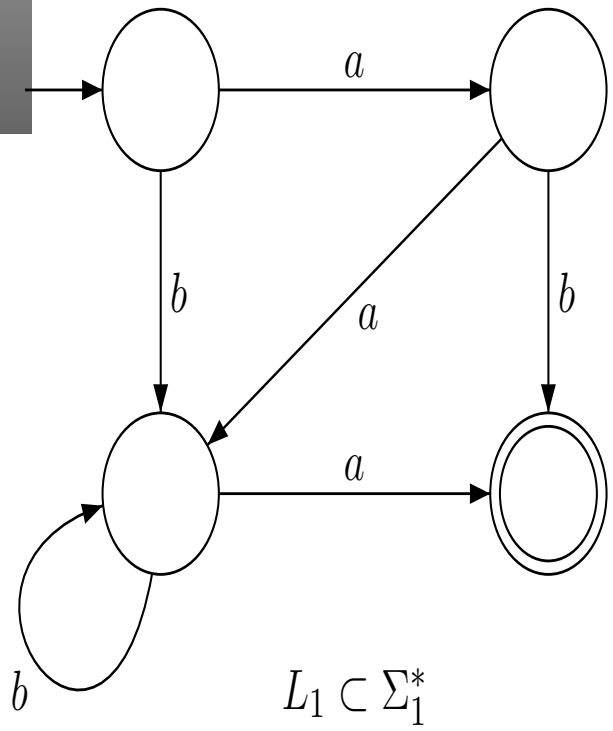
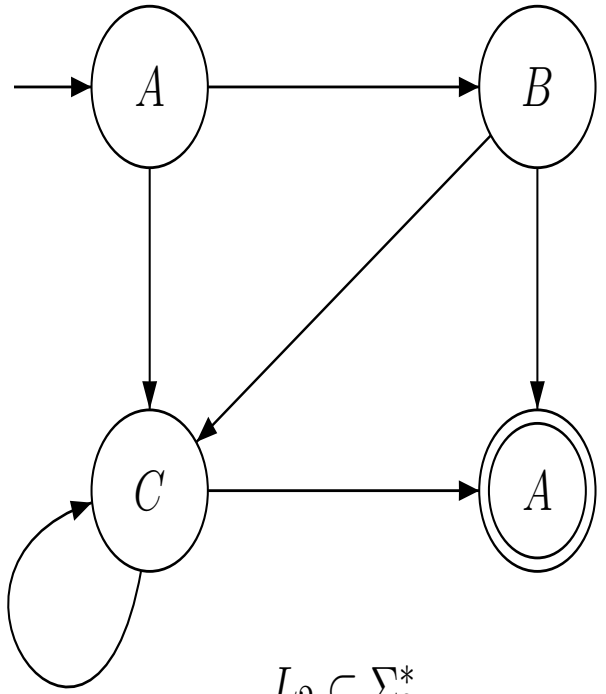
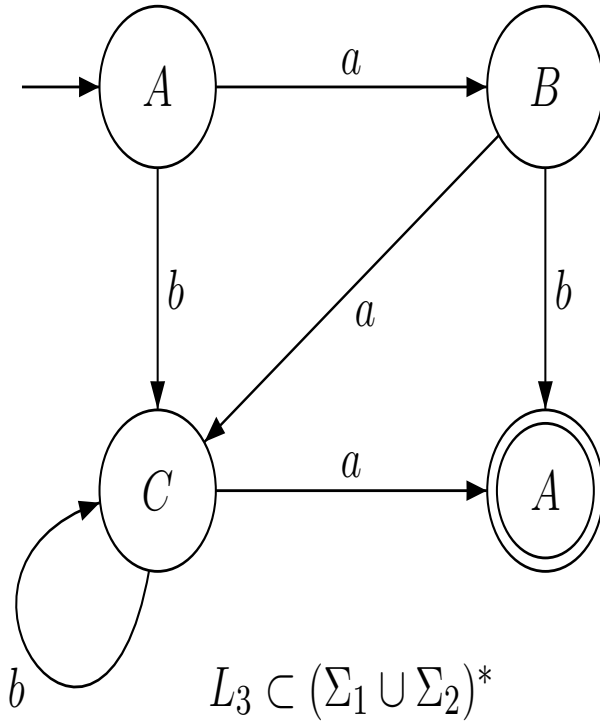
Untimed case: one type of monoids, one class of languages



$L_1 \subset \Sigma_1^*$



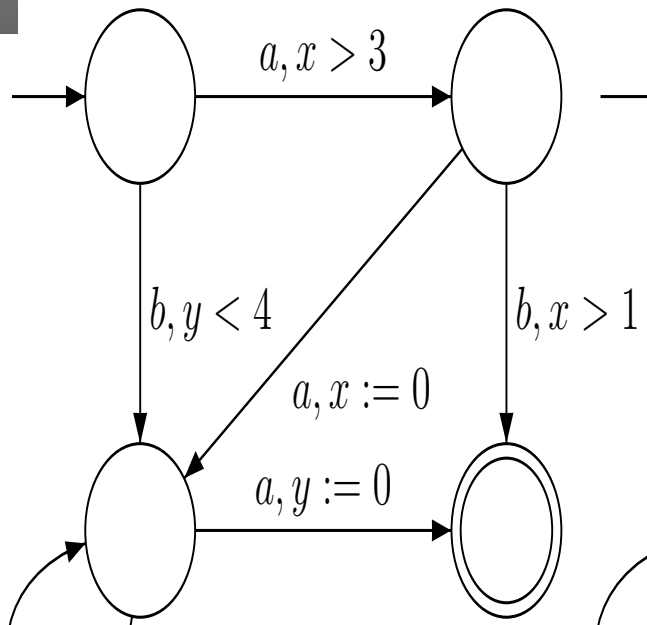
$L_2 \subset \Sigma_2^*$



$L_3 \subset (\Sigma_1 \cup \Sigma_2)^*$

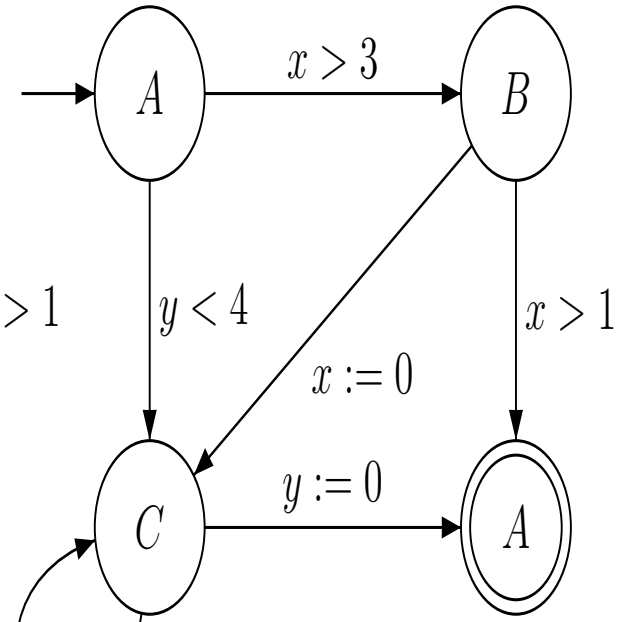
Timed case: 3 types of monoids

Time-event sequences



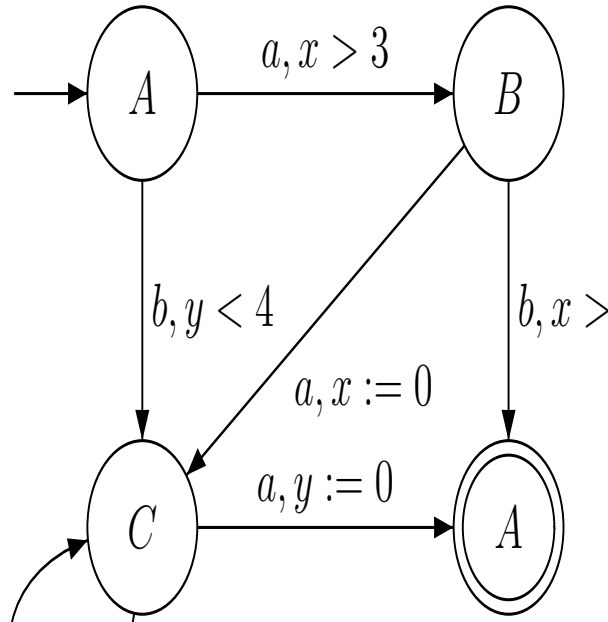
$L_1 \subset \{a, b, c\}^* \oplus \mathbb{R}_{\geq 0}$

Signals



$L_2 \subset \mathbb{R}_{\geq 0 A} \oplus \mathbb{R}_{\geq 0 B} \oplus \mathbb{R}_{\geq 0 C}$

"Sigacts"



$L_3 \subset \{a, b, c\}^* \oplus \mathbb{R}_{\geq 0 A} \oplus \mathbb{R}_{\geq 0 B} \oplus \mathbb{R}_{\geq}$

Properties of timed monoids

- They are not isomorphic. No true Moore-Mealy equivalence

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- None of the monoids above satisfies requirements from Eilenberg, Schutzenberger etc. . .
- Nevertheless we can try to apply classical recipes.

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Recognizable languages in $\Sigma^* \oplus \mathbb{R}_{\geq 0}$

L recognizable $\Leftrightarrow$ finitely many $w\backslash L$

- Is $\{5\} \subset \mathbb{R}_{\geq 0}$ recognizable?

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$$\begin{aligned} 1\backslash\{5\} &= \{4\} \\ 2\backslash\{5\} &= \{3\} \\ 2.19\backslash\{5\} &= \{2.81\} \end{aligned}$$

- Only four recognizable subsets of $\mathbb{R}_{\geq 0}$:
  $\emptyset, \{0\}, (0, \infty), \mathbb{R}_{\geq 0}$

Recognizable languages in $\Sigma^* \oplus \mathbb{R}_{\geq 0}$

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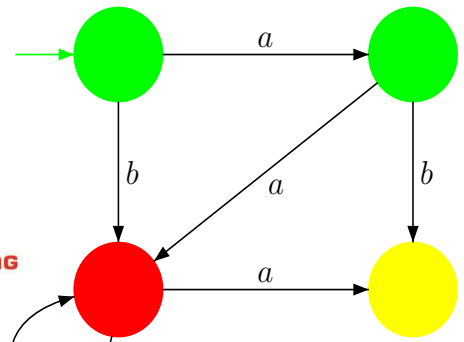
- Is $\{5\} \subset \mathbb{R}_{\geq 0}$ recognizable? No!
- Only four recognizable subsets of $\mathbb{R}_{\geq 0}$: $\emptyset, \{0\}, (0, \infty), \mathbb{R}_{\geq 0}$
- No quantitative timing! Recognizable timed languages : $(a + b\tau)^*$ etc. . . where τ stands for any positive time.

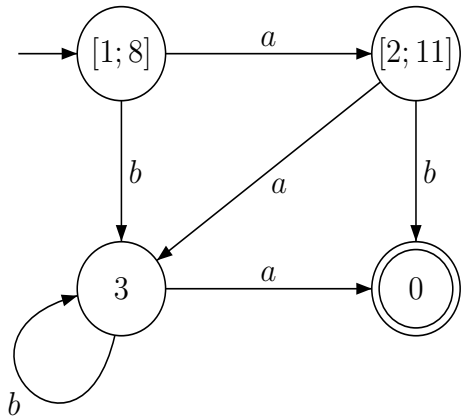


Timed rational languages

- **1st try.** Kleene algebra on $\Sigma^* \oplus \mathbb{R}_{\geq 0}$ generated by singletons: e.g. $(a^*\pi)^* + 2^*$ Known issues:
  - only countable languages. $[2;3]a$ is not in
  - even if we add complementation $[2;3]a$ is not in
  - non-computable numbers allowed

Timed rational languages

- **1st try.** Kleene algebra on $\Sigma^* \oplus \mathbb{R}_{\geq 0}$ generated by singletons: e.g. $(a^*\pi)^* + 2^*$
- **2nd try**(Cătălin's real-time languages.) Kleene algebra on $\Sigma^* \oplus \mathbb{R}_{\geq 0}$ generated by $a, b, c, \ldots, 1, (0, 1)$: e.g $(a^*[2, 3])^* + 2^*$



Conclusion for part 1

Classical recipes partly apply, but they give classes of languages too restricted to be useful

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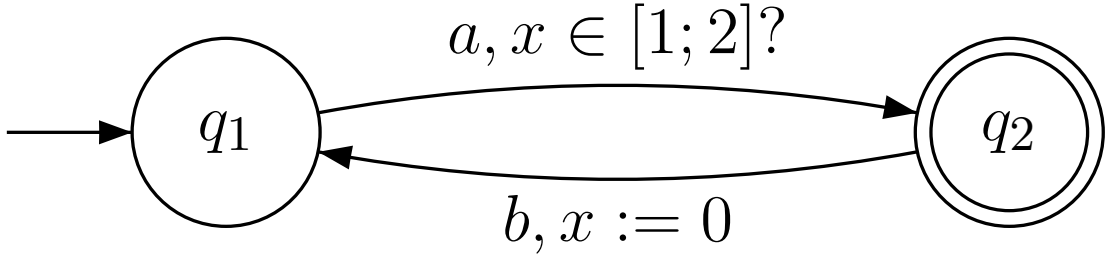
Timed automata

An automaton and its language

- Timed automaton :



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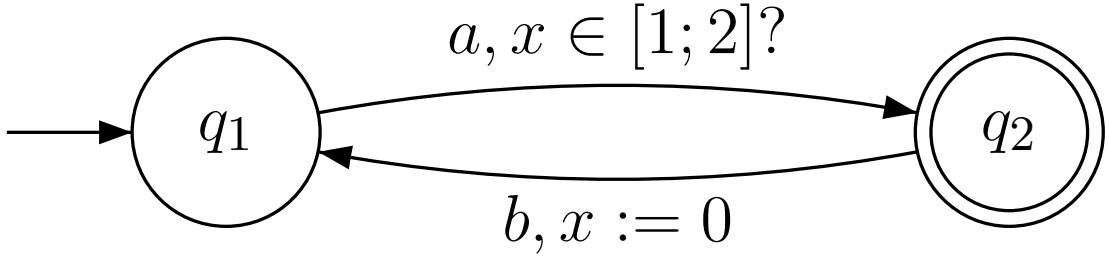

- Its language : $\{t_1\, a\, s_1\, b\, t_2\, a\, s_2\, b \ldots t_n\, a \mid \forall i.t_i \in [1;2]\}$

Reminder: some old results

- Rich enough for many real-life examples (real-time programs, scheduling, circuits . . .)
- Membership and Empty Language decidable (finite bisimulation)
- Inclusion and Universal Language undecidable
- Languages closed under $\cup, \cap$ and morphism (renaming)
- Languages non-closed under complementation.
- No determinization.
- Model-checking algorithms implemented in KRONOS and UPPAAL

Some technical problems with TA

- They are infinite state : $(q, x_1, \ldots x_k) \in Q \times {\mathbb{R}_{\geq 0}}^k$.
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  Unavoidable since recognizable languages are too poor
- There are long (or infinite) runs that never visit the same state.
  Nevertheless there are complicated pumping lemmata (D. Beauquier)
- No direct way to have Kleene Theorem
  Nevertheless there are some ways to have it - see below

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Conclusions of Part 2

They are useful, they have nice properties, they look like automata, but in the strict sense THEY ARE NOT automata.

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Regular expressions for timed languages

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Existing approaches

ACM	TRE - see below
BP99	Expressions with 2^C compositions and stars
BP	Expression with clocks
D	“Multithread” regmino expressions
AD	Balanced expressions

Our old expressions - TRE

$$E ::= 0 \mid \varepsilon \mid \underline{\mathbf{t}} \mid a \mid E+E \mid E \cdot E \mid E^* \mid \langle E \rangle_I \mid E \wedge E \mid [a \mapsto z]E$$

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Semantics:

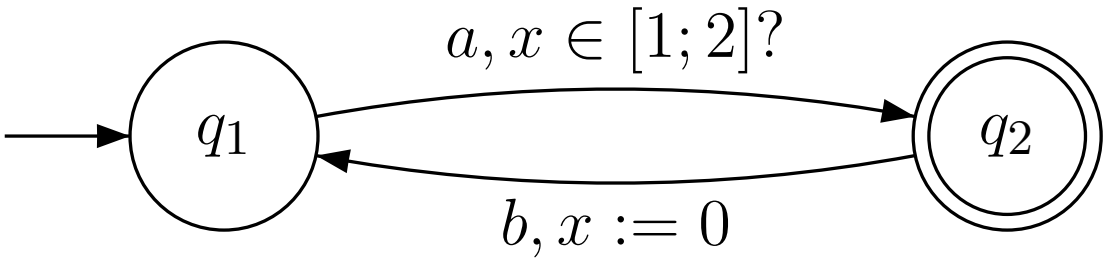
$$\|\underline{\mathbf{t}}\| = \mathbb{R}_{\geq 0} \quad \|a\| = \{a\} \qquad \|0\| = \emptyset \quad \|\varepsilon\| = \{\varepsilon\}$$

$$\|E_1 \cdot E_2\| = \|E_1\| \cdot \|E_2\| \qquad \|E_1 + E_2\| = \|E_1\| \cup \|E_2\|$$

$$\|\langle E \rangle\|_I = \{\sigma \in \|E\| \mid \ell(\sigma) \in I\} \qquad \|E^*\| = \|E\|^*$$

$$\|E_1 \wedge E_2\| = \|E_1\| \cap \|E_2\| \qquad \|[a \mapsto z]E\| = [a \mapsto z]\|E\|$$

A good example and a theorem



$$\{L = \{t_1\, a\, s_1\, b\, t_2\, a\, s_2\, b \ldots t_n\, a \mid \forall i.t_i \in [1;2]\}$$

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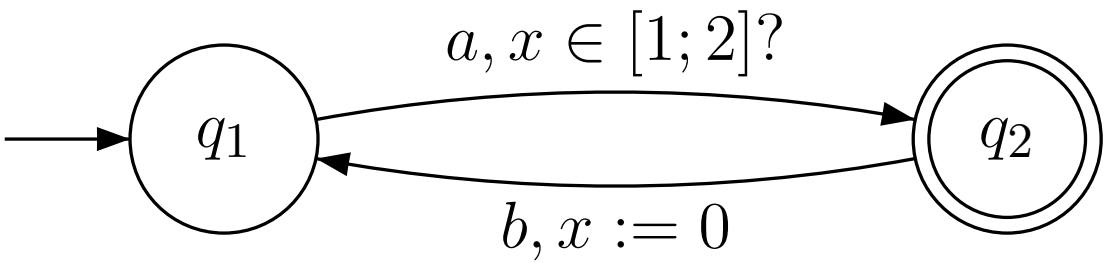

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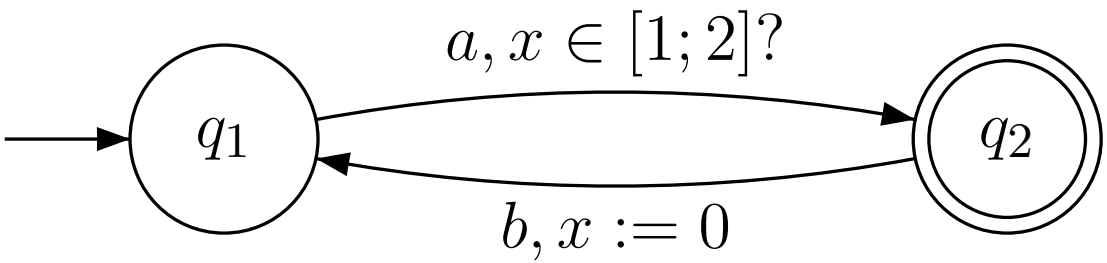

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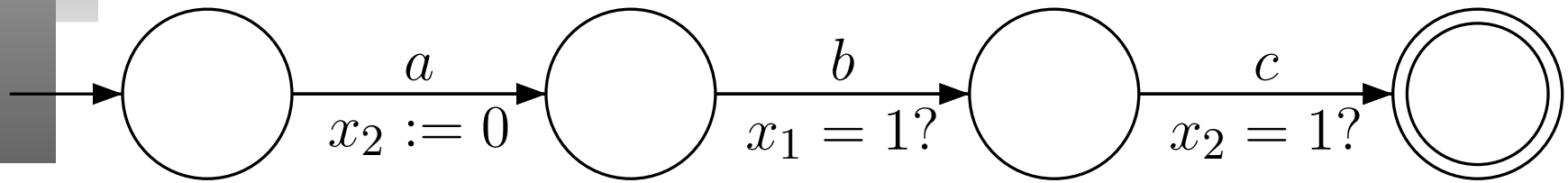
"Kleene theorem" [ACM]Timed Automata andTRE (with $\wedge$ and $[a \mapsto z]$) define the same class of timed languages

A nasty example

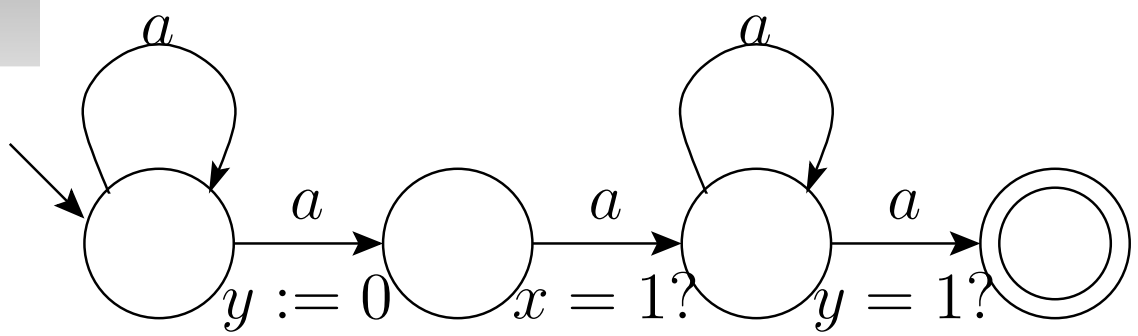
Intersection needed [ACM]



$$\{t_1 a t_2 b t_3 c \mid t_1 + t_2 = 1, t_2 + t_3 = 1\} = \underline{\mathbf{t}} a \langle \underline{\mathbf{t}} b \underline{\mathbf{t}} c \rangle_1 \wedge \langle \underline{\mathbf{t}} a \underline{\mathbf{t}} b \rangle_1 \underline{\mathbf{t}} c$$

Another nasty example

Renaming needed [Herrmann]



$$[b \mapsto a]\big((\underline{\mathbf{t}}a)^*\langle \underline{\mathbf{t}}b(\underline{\mathbf{t}}a)^*\rangle_1 \wedge \langle(\underline{\mathbf{t}}a)^*\underline{\mathbf{t}}b\rangle_1(\underline{\mathbf{t}}a)^*\big).$$

Wanted!

Timed regular expressions

- Equivalent to timed automata
- Automata-free
- Describing directly the timed words (projection-free)
- Without heavy/ugly operations
- With simple and natural semantics
- Elegant
- Ergonomic

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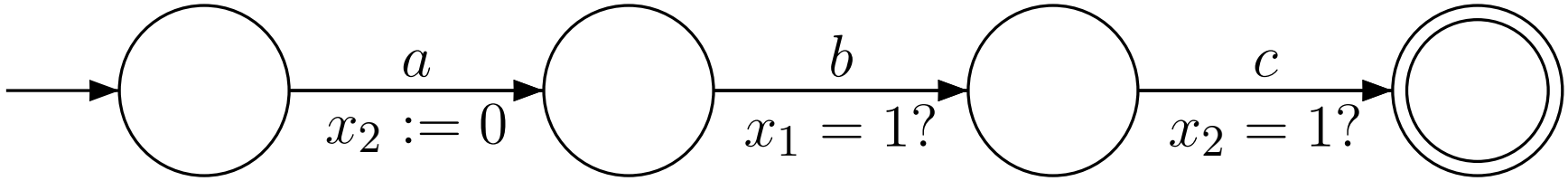
Sometimes Color may help

- Intersection not needed! $\langle \underline{\mathbf{t}} a \langle \underline{\mathbf{t}} b \rangle_1 \underline{\mathbf{t}} c \rangle_1$



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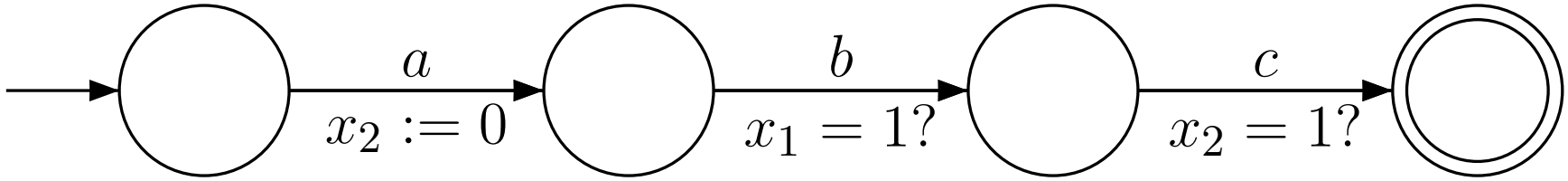

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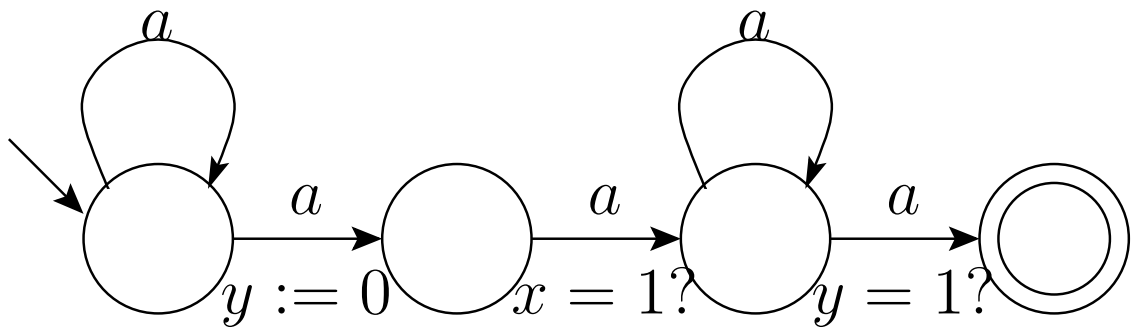

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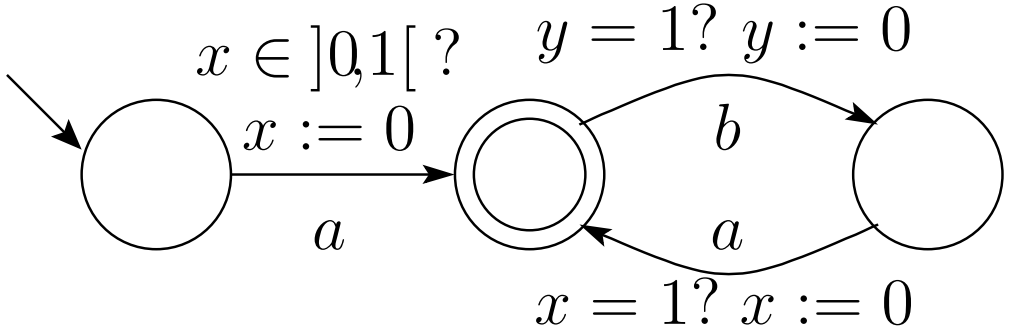

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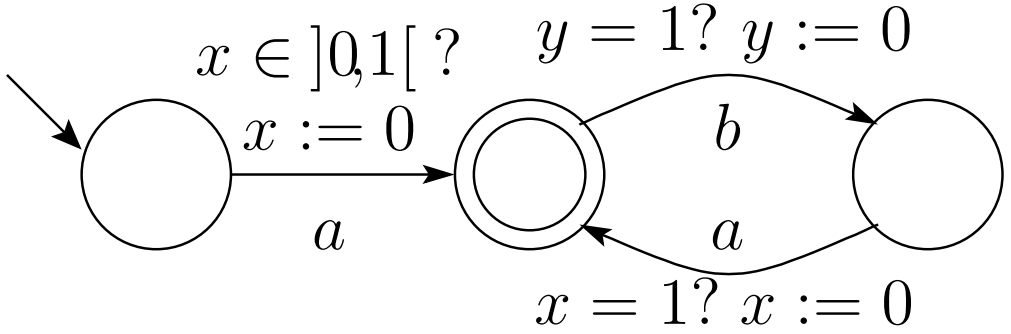

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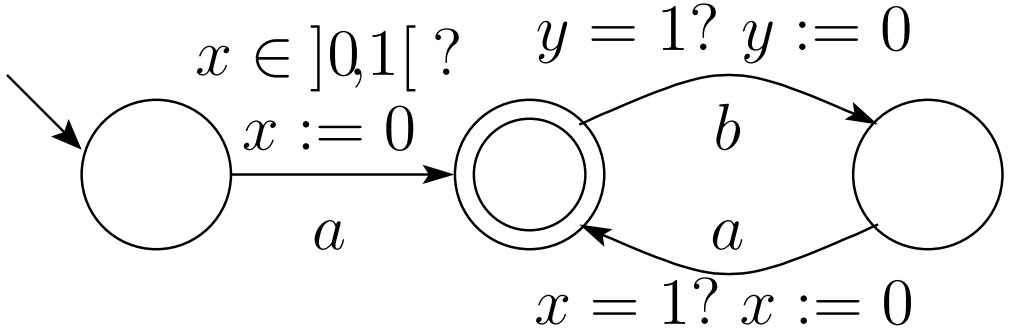
Color does not help, intersection needed:

$$\langle \underline{\mathbf{t}}a\rangle_{]0;1[} \langle \underline{\mathbf{t}}b\underline{\mathbf{t}}a\rangle_1^* \wedge \langle \underline{\mathbf{t}}a\underline{\mathbf{t}}b\rangle_1^* \underline{\mathbf{t}}a$$

Let us try again



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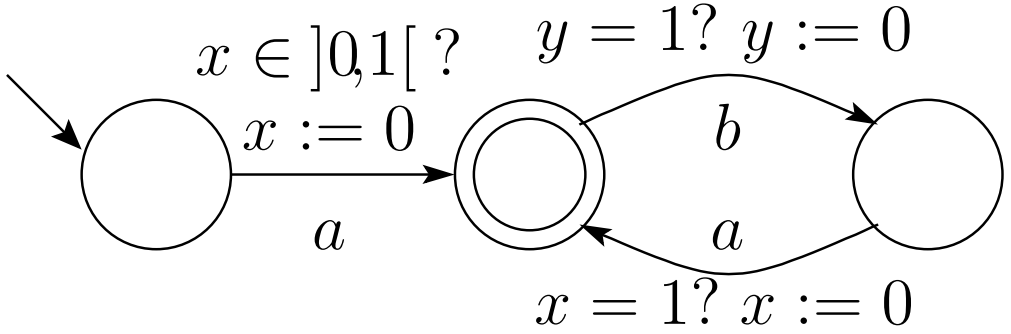

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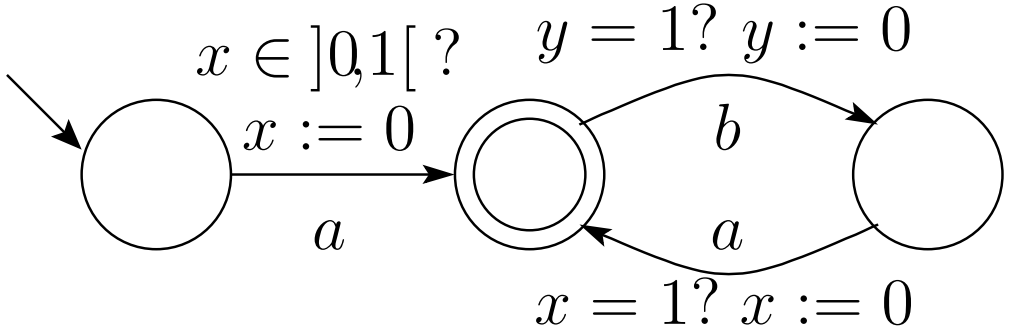

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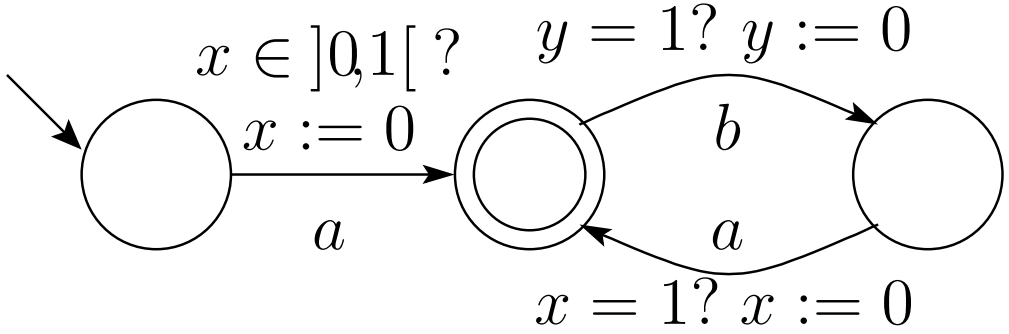

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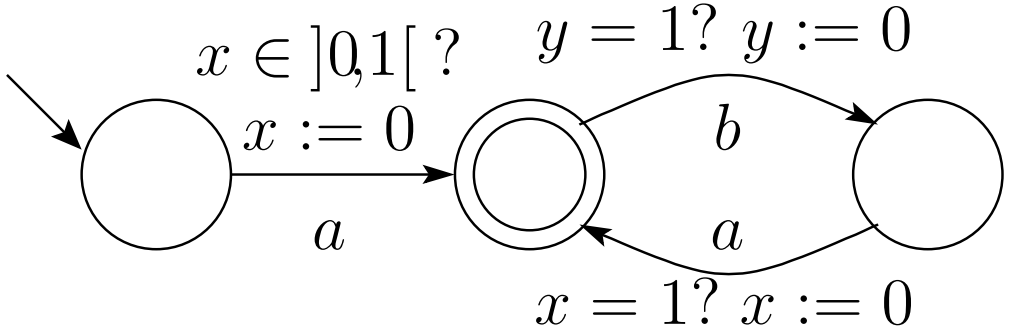

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Eureka!

Balanced Expressions

- **Syntax.** BRE - a regular expression E over $\widetilde{\Sigma} = \Sigma \cup \{\langle, \rangle_I, \langle, \rangle_I, \langle, \rangle_I, \dots\}$

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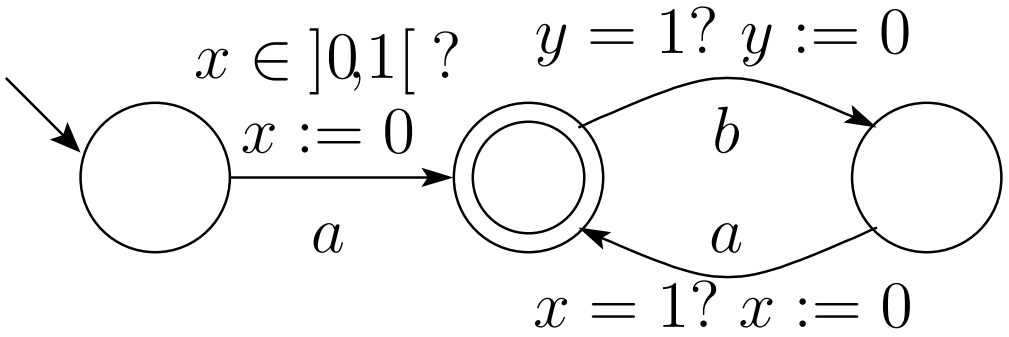
- **Restriction.** Each $w \in |E|$ should be well parenthesized wrt each color $\boxed{|E| \subset \mathsf{Bal}}$

Balanced Expressions

- **Syntax.** BRE - a regular expression E over $\widetilde{\Sigma} = \Sigma \cup \{\langle, \rangle_I, \langle, \rangle_I, \langle, \rangle_I, \dots\}$
- **Two-stage semantics**
  1. $|E|$ - untimed language over $\widetilde{\Sigma}$ generated by E. Each $w \in |E|$ can be considered as a (colored) timed regular expression!
  2. Take the union of their timed languages: $\|E\| = \bigcup \{\|w\| \mid w \in |E|\}$
- **Restriction.** Each $w \in |E|$ should be well parenthesized wrt each color $\boxed{|E| \subset \mathsf{Bal}}$

Back to the example



- $E = \langle \underline{\mathbf{t}}a \Big(\langle \underline{\mathbf{t}}b \rangle_1 \langle \underline{\mathbf{t}}a \rangle_1 \Big)^* \langle \underline{\mathbf{t}}b \rangle_1 \underline{\mathbf{t}}a \rangle_1 + \langle \underline{\mathbf{t}}a \rangle_{]0,1[}$
- $|E| = \left\{ \langle \underline{\mathbf{t}}a \rangle_{]0,1[};\ \langle \underline{\mathbf{t}}a \langle \underline{\mathbf{t}}b \rangle_1 \underline{\mathbf{t}}a \rangle_1;\ \langle \underline{\mathbf{t}}a \langle \underline{\mathbf{t}}b \rangle_1 \langle \underline{\mathbf{t}}a \rangle_1 \langle \underline{\mathbf{t}}b \rangle_1 \underline{\mathbf{t}}a \rangle_1; \ldots \right\}$
- $\|E\| = \|\langle \underline{\mathbf{t}}a \rangle_{]0,1[}\| \cup \|\langle \underline{\mathbf{t}}a \langle \underline{\mathbf{t}}b \rangle_1 \underline{\mathbf{t}}a \rangle_1\| \cup \|\langle \underline{\mathbf{t}}a \langle \underline{\mathbf{t}}b \rangle_1 \langle \underline{\mathbf{t}}a \rangle_1 \langle \underline{\mathbf{t}}b \rangle_1 \underline{\mathbf{t}}a \rangle_1\| \cup \ldots$

Kleene Theorem

Balanced Expressions and Timed Automata define the same class of timed languages

Key lemma

Any TA $\mathcal{A}$ is equivalent to a TA $\overline{\mathcal{A}}$ s.t on each accepting run each clock is checked exactly once after each reset.

Proof idea It is useless to check whether $x \in I$ million times : it should be checked only the first and the last time. This allows to check each clock at most twice...

Checking expression for balance

How to check syntactic correctness: $|E| \subset \mathsf{Bal}$?

Idea: Count unmatched parentheses (structural induction over E).

The Algorithm

$$\mathsf{eff}(a) = \{(\mathbf{0}_n, \mathbf{0}_n)\}$$

$$\mathsf{eff}(\underline{\mathbf{t}}) = \{(\mathbf{0}_n, \mathbf{0}_n)\}$$

$$\mathsf{eff}(\langle^i) = \{(e_i, \mathbf{0}_n)\}$$

$$\mathsf{eff}(\rangle_I^i) = \{(\mathbf{0}_n, e_i)\}$$

$$\mathsf{eff}(E_1 + E_2) = \begin{cases} \perp & \text{iff } \mathsf{eff}(E_1) = \perp \text{ or } \mathsf{eff}(E_2) = \perp \\ \mathsf{eff}(E_1) \cup \mathsf{eff}(E_2) & \text{otherwise} \end{cases}$$

$$\mathsf{eff}(E_1 \cdot E_2) = \begin{cases} \perp & \text{iff } \mathsf{eff}(E_1) = \perp \text{ or } \mathsf{eff}(E_2) = \perp \\ \mathsf{eff}(E_1) \ominus \mathsf{eff}(E_2) & \text{otherwise} \end{cases}$$

$$\mathsf{eff}(E^*) = \begin{cases} \perp & \text{iff } \mathsf{eff}(E) = \perp \\ \perp & \text{iff } \varphi(\mathsf{eff}(E)) \neq \{\mathbf{0}_n\} \\ \bigcup_{k \leq card(\mathsf{eff}(E))} \mathsf{eff}(E)^{k\ominus} & \text{otherwise} \end{cases}$$

Expression E is balanced iff $\mathsf{eff}(E) = \{(\mathbf{0}_n, \mathbf{0}_n)\}$

Comparing formalisms

	BP	TRE	Color	BRE
Equivalent to timed automata	✓	✓	✓	✓
Automata-free	✓	✓	✓	✓
Projection-free	✓	✓	✓	✓
No heavy/ugly operations	✓	✓	✓	✓
Simple and natural semantics	✓	✓	✓	✓
Elegant	?	?	?	?
Ergonomic	✓	✓	✓	✓
Easy syntactic analysis	✓	✓	✓	✓

Verimag

Conclusion of Part 4

We know several solutions, but not the perfect one. . .