Parity Games on BPA Graphs

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Plan of the talk:

Parity Games

BPA

Local Parity BPA Games

- Definitions and Example.
- Winning Strategies and Winning Sets.
- Deciding the Winner and Complexity.

Global Parity BPA Games

- Definitions and Example.
- Deciding the Winner and Complexity.

BPA Games *vs.* Pushdown Games

Parity Games: Definitions

Framework:

- Two players: I et II.
- Graph $G = (V, E)$, $V = V_I \cup V_{II}$.
- Coloring function: $\rho : V \to C$.

Play:

- Play: path in G.
- I wins $\Leftrightarrow$ the smallest infinitely repeated color is even.

Parity Games: Some Results

General Case:

- $V = W_I \cup W_{II}$.
- Positional winning strategies.

Special case of finite graphs:

- Deciding the winner: $\mathcal{NP} \cap co - \mathcal{NP}$.
- [Jurdzinski] Algorithm in:

$$\mathcal{O}\left(dm\left(\frac{n}{\lfloor d/2 \rfloor}\right)^{\lfloor d/2 \rfloor}\right)$$

Basic Process Algebra (BPA)

Definition

- Greibach normal form [Bergstra & Klop].
- $\mathcal{B} = (\Gamma, \perp, \hookrightarrow)$.
- *Push*: $a \hookrightarrow ba$, $a \in \Gamma$ and $b \in (\Gamma \setminus \{\perp\})$.
- *Pop*: $a \hookrightarrow \varepsilon$, $a \in (\Gamma \setminus \{\perp\})$.

BPA Graph: $G = (V, \rightarrow)$

- $V = (\Gamma \setminus \{\perp\})^* \perp$.
- $\rightarrow$ induced by $\hookrightarrow$.

Playing on a BPA Graph: Local Parity BPA Games

Partition of V:

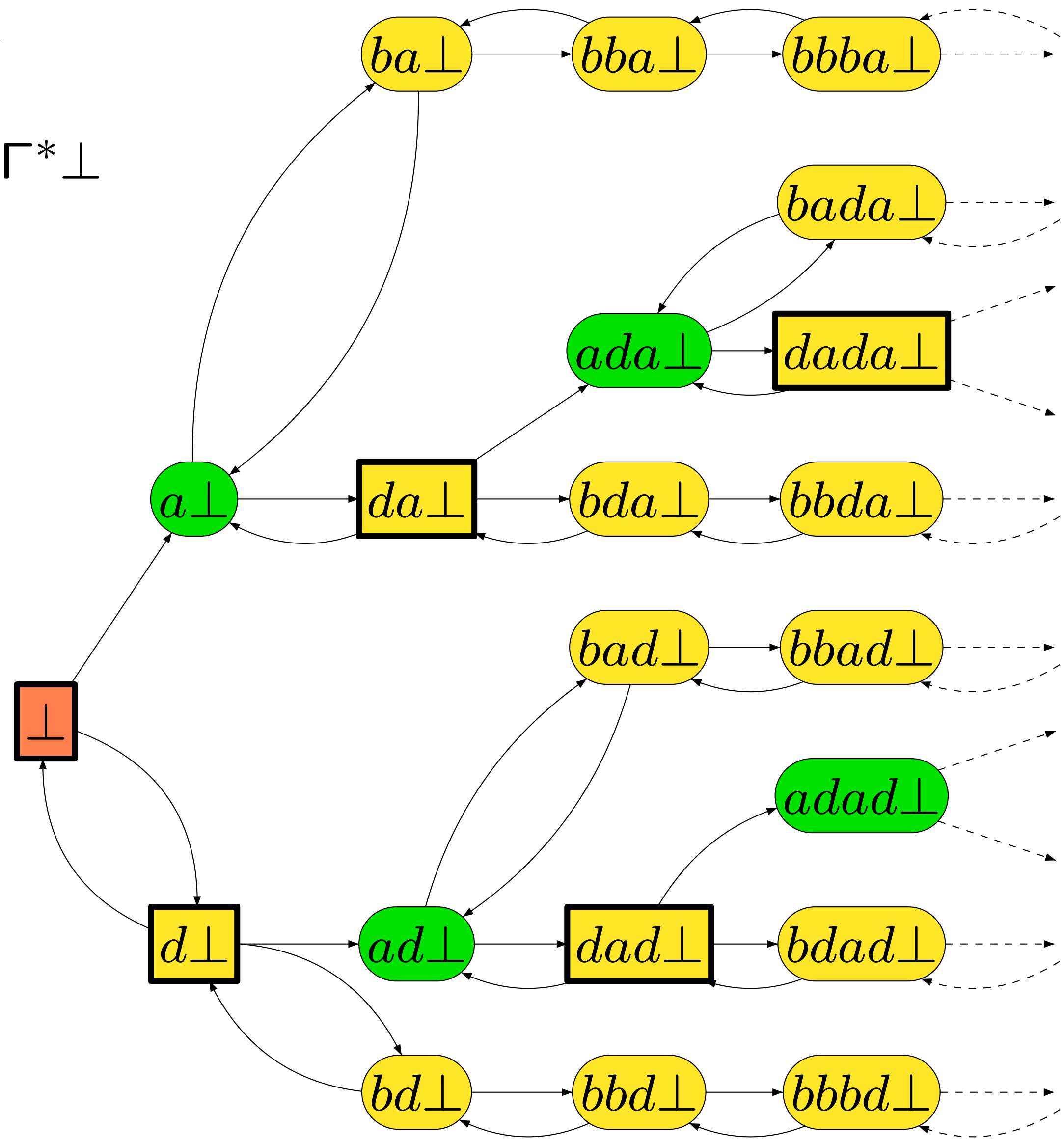
- Two players : I and II.
- $\Gamma = \Gamma_I \cup \Gamma_{II}$.
- $V_I = V \cap (\Gamma_I \Gamma^*)$ and $V_{II} = V \cap (\Gamma_{II} \Gamma^*)$.

Parity Condition:

- $\rho : \Gamma \rightarrow C$.
- $\rho : V \rightarrow C$: $\rho(v) = \rho(top(v))$: local coloring condition.
- I wins $\Leftrightarrow$ Smallest infinitely repeated color is even.

$$
\begin{aligned}
\Gamma_{win} &= \{a\}\\
\Gamma_{?} &= \{b\}\\
\Gamma_{loose} &= \{d\}
\end{aligned}
$$

$$W_I = b^*a\Gamma^*\perp$$



$$
\begin{aligned}
\mathsf{a} &\hookrightarrow \mathsf{ba}\\
&\hookrightarrow \mathsf{da}\\
\mathsf{b} &\hookrightarrow \varepsilon\\
&\hookrightarrow \mathsf{bb}\\
\mathsf{d} &\hookrightarrow \varepsilon\\
&\hookrightarrow \mathsf{ad}\\
&\hookrightarrow \mathsf{bd}\\
\perp &\hookrightarrow \mathsf{a}\\
&\hookrightarrow \mathsf{d}
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_I &= \{a, b\}\\
\Gamma_{II} &= \{d, \perp\}
\end{aligned}
$$

$$
\begin{aligned}
\rho(a) &= 0\\
\rho(b) &= 1\\
\rho(d) &= 1\\
\rho(\perp) &= 2
\end{aligned}
$$

Local Parity BPA Games: Winning Strategies

Local Strategies:

- $au \in W_I$ and φ winning strategies.
- If $\varphi(au) = push(b)$ then $\forall w$, $aw \in W_I$ and $\varphi(aw) = push(b)$.

Deciding the winner: $\mathcal{NP} \cap co - \mathcal{NP}$.

- Guess a strategy.
- Solve a "1 player" problem.

Local Parity BPA Games: Winning sets

Partition of Γ:

- $\Gamma_w = \{a \in \Gamma \mid$ from $a\perp$, I wins without popping $a\}$.
- $\Gamma_l = \{a \in \Gamma \mid$ from $a\perp$, II wins without popping $a\}$.
- $\Gamma_? = (\Gamma \setminus \{\perp\}) \setminus (\Gamma_w \cup \Gamma_l)$.

Regular Winning Set:

- $W_I = \begin{cases} \Gamma_?^* \Gamma_w (\Gamma \setminus \{\perp\})^* \perp & \text{If } \perp \notin W_I, \\ \Gamma_?^* \Gamma_w (\Gamma \setminus \{\perp\})^* \perp \cup \Gamma_?^* \perp & \text{Otherwise.} \end{cases}$

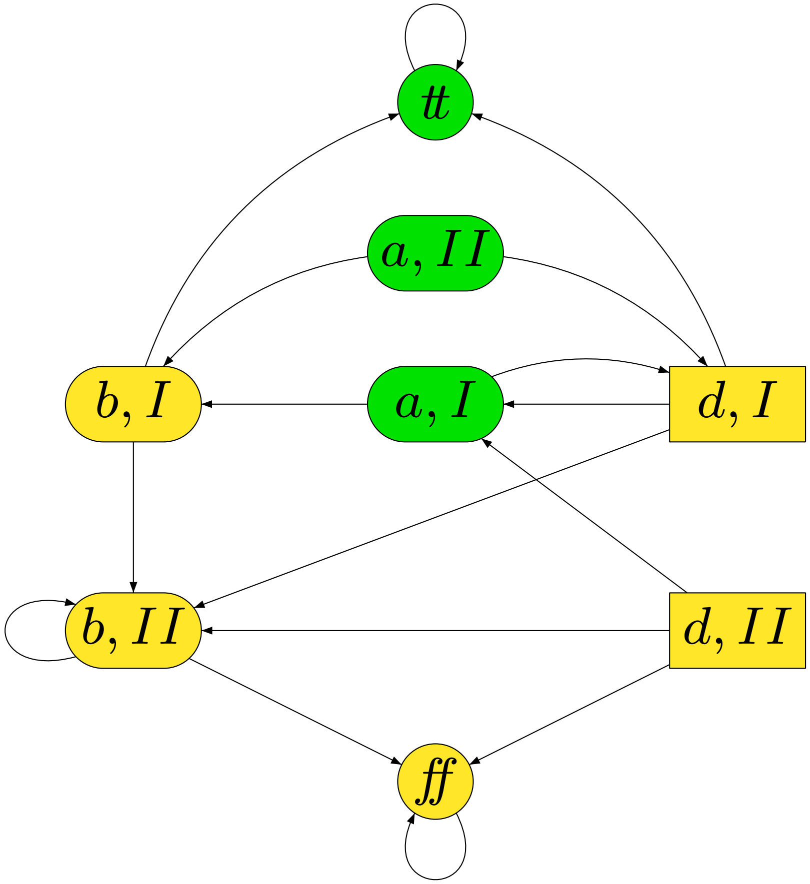
Local Parity BPA Games: Computing the Partition

Associated Finite Graph: $\widetilde{G} = (\widetilde{V}, \widetilde{E})$

- $\widetilde{V} = \Gamma \times \{I, II\} \cup \{tt, ff\}$. $\widetilde{V}_I = \Gamma_I \times \{I, II\}$
- $a \hookrightarrow \varepsilon$ in $\mathcal{B} \Rightarrow (a, I) \rightarrow tt$ and $(a, II) \rightarrow ff$.
- $a \hookrightarrow ba$ in $\mathcal{B}$ and $\min\{\rho(a), \rho(b)\}$ even $\Rightarrow (a, X) \rightarrow (b, I)$.
- $a \hookrightarrow ba$ in $\mathcal{B}$ and $\min\{\rho(a), \rho(b)\}$ odd $\Rightarrow (a, X) \rightarrow (b, II)$.
- $\widetilde{\rho}((a, X)) = \rho(a)$.
- $\widetilde{\rho}(tt) = 0$ and $\widetilde{\rho}(ff) = 1$

Deducing the partition:

- $a \in \Gamma_w \Leftrightarrow (a, II) \in \widetilde{W}_I$.
- $a \in \Gamma_? \Leftrightarrow (a, II) \notin \widetilde{W}_I$ and $(a, I) \in \widetilde{W}_I$.
- $a \in \Gamma_l \Leftrightarrow (a, I) \in \widetilde{W}_{II}$.



$$
\begin{array}{lcl}
\mathsf{a} & \hookrightarrow & \mathsf{ba} \\
 & \hookrightarrow & \mathsf{da} \\
\mathsf{b} & \hookrightarrow & \varepsilon \\
 & \hookrightarrow & \mathsf{bb} \\
\mathsf{d} & \hookrightarrow & \varepsilon \\
 & \hookrightarrow & \mathsf{ad} \\
 & \hookrightarrow & \mathsf{bd} \\
\perp & \hookrightarrow & \mathsf{a} \\
 & \hookrightarrow & \mathsf{d}
\end{array}
$$

$$
\begin{array}{lcl}
\Gamma_I & = & \{a, b\} \\
\Gamma_{II} & = & \{d, \perp\}
\end{array}
$$

$$
\begin{array}{lcl}
\rho(a) & = & 0 \\
\rho(b) & = & 1 \\
\rho(d) & = & 1 \\
\rho(\perp) & = & 2
\end{array}
$$

- $\widetilde{W}_I = \{(a, I), (a, II), (b, I), tt\}$.
- $\widetilde{W}_{II} = \{(b, II), (d, I), (d, II), ff\}$.
- $\Gamma_w = \{a\}$.
- $\Gamma_? = \{b\}$.
- $\Gamma_l = \{d\}$.

Local Parity BPA Games: Deciding whether $\perp \in W_I$

For any rule $\perp \hookrightarrow a\perp$:

(1) $a \in \Gamma_w$.

(2) $a \in \Gamma_?$ and $\min\{\rho(a), \rho(\perp)\}$ is even.

$\perp \in W_I$ iff:

- If $\perp \in \Gamma_I$, $\perp \in W_I \Leftrightarrow \exists \perp \hookrightarrow a\perp$ s.t (1) or (2) holds.
- If $\perp \in \Gamma_{II}$, $\perp \in W_I \Leftrightarrow \forall \perp \hookrightarrow a\perp$, (1) or (2) holds.

Local Parity BPA Games: Complexity

Local Strategies $\Rightarrow \mathcal{NP} \cap co - \mathcal{NP}$.

Turing Reduction to a Finite Graph.

- Effective algorithm in time

$$\mathcal{O}\left(dn^2 \left(\frac{2n+2}{\lfloor d/2 \rfloor}\right)^{\lfloor d/2 \rfloor} + l\right)$$

Going Further: Global Parity BPA Games

Global BPA:

- $\mathcal{B} = (\Gamma, \perp, (L_0, \ldots, L_{d-1}), \hookrightarrow)$.
- $(\Gamma \setminus \{\perp\})^* \perp = \bigcup_{i=0}^{d-1} L_i$: Regular partition.
- *Push*: $a \overset{i}{\hookrightarrow} ba$, $a \in \Gamma$ and $b \in (\Gamma \setminus \{\perp\})$.
- *Pop*: $a \overset{i}{\hookrightarrow} \varepsilon$, $a \in (\Gamma \setminus \{\perp\})$.

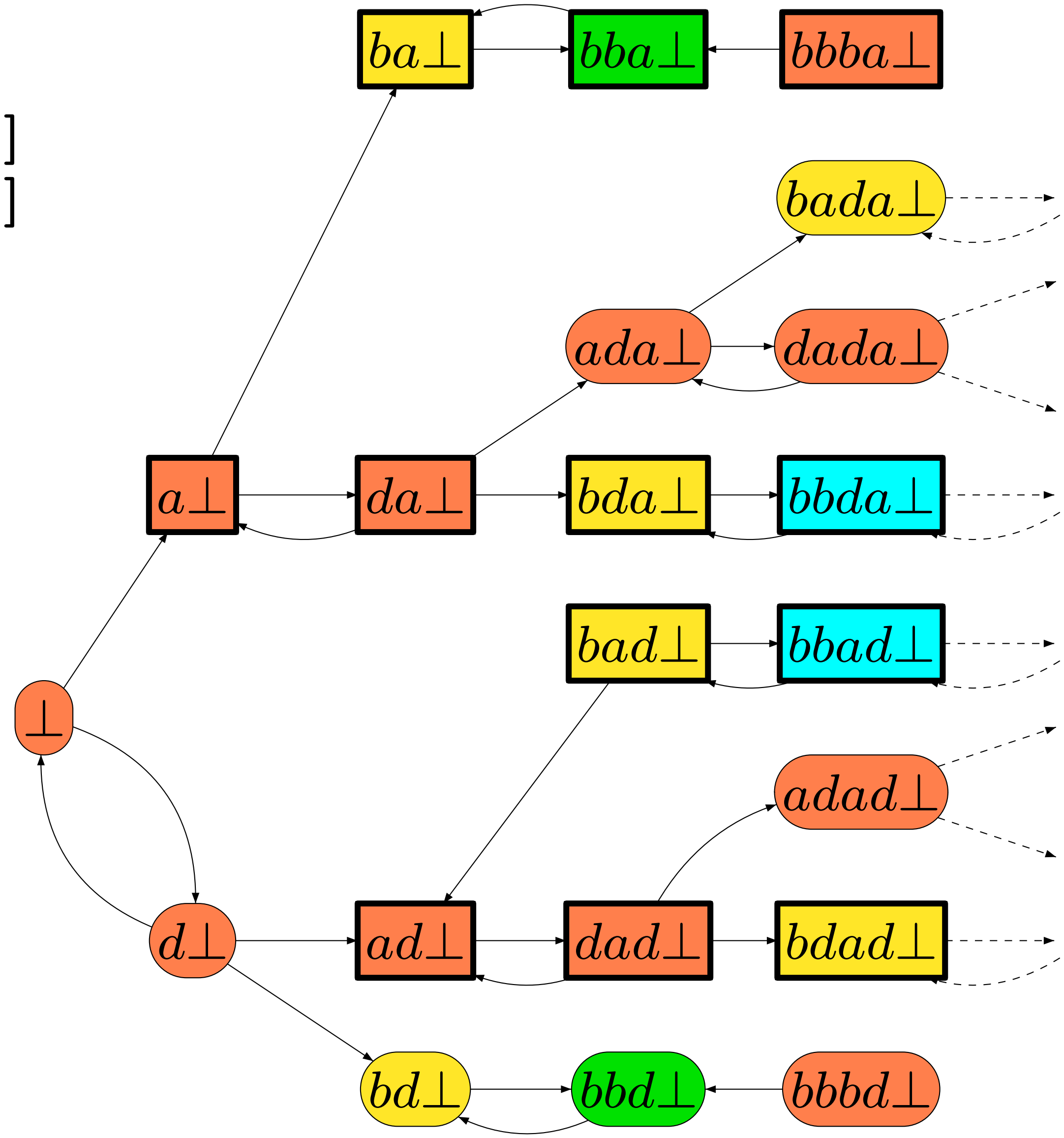
Playing on a Global BPA Graph:

- $V = (\Gamma \setminus \{\perp\})^* \perp = V_I \cup V_{II}$, V_I: Regular.
- $\rightarrow$ induced by $\hookrightarrow$.
- $\rho(u) = i \Leftrightarrow u \in L_i$.

$$v \in V_I \quad \Leftrightarrow \quad \{v \in V \mid |v|_a = 0 \ [2]\}$$

$$\begin{array}{lcl}
v \in L_1 & \Leftrightarrow & |v|_b \equiv 1 \ [3] \\
v \in L_2 & \Leftrightarrow & |v|_b \equiv 2 \ [3] \wedge |v| \equiv 0 \ [2] \\
v \in L_3 & \Leftrightarrow & |v|_b \equiv 2 \ [3] \wedge |v| \equiv 1 \ [2] \\
v \in L_4 & \Leftrightarrow & |v|_b \equiv 0 \ [3]
\end{array}$$

$$\begin{array}{lcl}
\mathsf{a} & \xhookrightarrow{4} & \mathsf{ba} \\
 & \xhookrightarrow{4} & \mathsf{da} \\
\mathsf{b} & \xhookrightarrow{2,3,4} & \varepsilon \\
 & \xhookrightarrow{1,3} & \mathsf{bb} \\
\mathsf{d} & \xhookrightarrow{4} & \varepsilon \\
 & \xhookrightarrow{4} & \mathsf{ad} \\
 & \xhookrightarrow{2,4} & \mathsf{bd} \\
\perp & \xhookrightarrow{4} & \mathsf{a} \\
 & \xhookrightarrow{4} & \mathsf{d}
\end{array}$$



Global Parity BPA Games: Deciding the winner

Reduction to Local Parity BPA Games:

- G global game $\rightsquigarrow$ G' local game with alphabet of exponential size and linear number of colors.
- Alphabet of G': $\Gamma \times Q_I \times Q_0 \times Q_1 \times \cdots \times Q_{d-1}$.
- Transition in G': update control states.

Consequences:

- $\Rightarrow$ W_I is regular.
- $\Rightarrow$ Deciding the winner is in DEXPTIME.

Global Parity BPA Games: Complexity

Deciding the winner is in DEXPTIME:

- Reduction to a local parity BPA game.

Deciding the winner is DEXPTIME-Complete:

- Simulation of alternating linear space bounded Turing machine with a global parity game.

Pushdown Parity Games *vs.* Global BPA Parity Games

Pushdown Parity Games

- [Walukiewicz] Deciding the winner is DEXPTIME-Complete.
- Graph: Arbitrary loops.
- Winning sets: Any regular languages.

Global BPA Parity Games

- Deciding the winner is DEXPTIME-Complete.
- Graph: Loops of length 2.
- Winning sets: Specific languages.

Conclusion

- Local BPA Parity games : to relate with parity games on finite graphs.
- Global BPA Parity games : to relate with parity games on pushdown graphs.
- Relation with the μ-calculus model checking for BPA?