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Some results on the dot-depth hierarchy

Pascal Weil*

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Résumé: Dans ce travail, nous poursuivons l'étude de la décidabilité de la hiérarchie de concaténation dite "dot-depth". Nous donnons une borne inférieure effective de la dot-depth d'un monoïde apériodique. Notre outil principal pour ce résultat est l'étude d'une certaine opération sur les variétés de monoïdes finis en termes de produit de Mal'cev. Nous prouvons aussi l'égalité de deux variétés décidables, dont on sait qu'elles contiennent tous les monoïdes de dot-depth deux. Enfin, nous restreignant aux monoïdes inversifs, nous montrons que la classe des monoïdes inversifs de dot-depth deux est localement finie.

Abstract: In this paper we pursue the study of the decidability of the dot-depth hierarchy. We give an effective lower bound for the dot-depth of an aperiodic monoid. The main tool for this is the study of a certain operation on varieties of finite monoids in terms of Mal'cev product. We also prove the equality of two decidable varieties which were known to contain all dot-depth two monoids. Finally, we restrict our attention to inverse monoids, and we prove that the class of inverse dot-depth two monoids is locally finite.

In this paper we pursue the study of the dot-depth hierarchy, a hierarchy of star-free (i.e. recognizable aperiodic) languages with connections to finite monoid theory, formal logic (Thomas [30]) and computational complexity (Barrington and Thérien [1]).

The dot-depth hierarchy was first introduced by Brzozowski and Cohen [3] in 1971 and was studied by numerous authors since. It consists in a strictly increasing sequence of classes of star-free languages whose union is the class of all star-free languages. Precise definitions are given in Section 2. The dot-depth of a star-free language L is the order of the least level of this hierarchy containing L. The main open problem regarding the dot-depth hierarchy is its decidability: for a given star-free language, can we decide at which level of the hierarchy it lies, that is, can we decide its dot-depth? This question was answered positively only for the first level of the hierarchy by Simon [25], and partial answers were given for its second level (Straubing [28], Weil [34] and Straubing and Weil [29]).

It is known that the dot-depth of a star-free language depends only on its syntactic monoid, so that the problem translates to an equivalent problem on finite aperiodic monoids. As was done in most previous papers on the subject, we rely mostly on this algebraic approach.

The main results which we wish to present here are the following. First, in Section 3, we consider an operation on **M**-varieties (or pseudo-varieties of

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finite monoids): For each monoid M and each idempotent e of M, we let M_e be the subsemigroup of M generated by the elements that are $\mathcal{J}$-greater than or equal to e. If $\mathbf{V}$ is an $\mathbf{M}$-variety, we let $\widetilde{\mathbf{V}}$ be the class of all monoids M such that $eM_ee \in \mathbf{V}$ for each idempotent e of M. Our main result on this operation is that $\widetilde{\mathbf{V}}$ is equal to the Mal'cev product $\mathbf{LV} \,ⓜ\, \mathbf{J}_1$, where $\mathbf{J}_1$ is the $\mathbf{M}$-variety of semilattice monoids. Precise definitions are given in the sequel.

Next, in Section 4, we use this result to give an effective method to compute a lower bound for the dot-depth of a given aperiodic monoid. That is, for each n, we give an effective necessary condition C_n for a monoid to have dot-depth at most n. Some effective upper bounds had already been described in the literature (Perrin [13], Pin, Straubing and Thérien [21]), but to the author's knowledge, no other general lower bound is known to date. In certain cases, our lower bound allows the exact computation of the dot-depth.

Two predecessors of this work deserve a special mention. Pin [19] gave a necessary condition P_n for a monoid to have dot-depth at most n. This necessary condition however is known to be effective only for $n = 2$. In general, our necessary condition is not better than Pin's, but it is always effective. Using Pin's result, Straubing [28] gave an apparently different effective necessary condition for a monoid to have dot-depth at most two, and went on to prove that this condition is also sufficient for 2-generated monoids, thus proving the decidability of dot-depth two for these monoids. We prove in Section 5 that Pin's condition P_2 and Straubing's condition are in fact equivalent.

Straubing's result was then used in Weil [34] and in Straubing and Weil [29] to study the special case of inverse monoids. There, it was proved that Straubing's condition is also sufficient for inverse monoids with at most 3 inverse generators, proving the decidability of dot-depth two for these monoids. In fact, we show that for inverse monoids, Straubing's condition (and hence Pin's condition P_2) is equivalent to our condition C_2. In Section 6, we use this fact to show that the class of inverse monoids of dot-depth two is locally finite.

We conclude the paper with the study of a certain 91-element monoid which can be considered as a benchmark for testing certain current conjectures on dot-depth two monoids. We also remark that the properties of this monoid disprove a conjecture of Brzozowski and Fich [4] which dates back to 1984.

Before we state precisely and prove our results, we will devote two sections to reviewing important properties of varieties of finite semigroups and monoids, and of varieties of recognizable languages.

1. Varieties of finite monoids and semigroups

Here we review some basic properties of $\mathbf{M}$-varieties, $\mathbf{S}$-varieties and Mal'cev products. In this paper, all semigroups and monoids are either free or finite.

1.1 Basic definitions

If S is a semigroup, we denote by $E(S)$ the set of idempotents of S. We say that a semigroup S *divides* a semigroup T, and we write $S < T$, if S is a quotient of a subsemigroup of T. If S and T are semigroups, a *relational morphism* $\tau: S \to T$ is a mapping from S into $\mathcal{P}(T)$, the set of subsets of T, such that

$s\tau \neq \emptyset$ for all $s \in S$, and

$(s\tau)(s'\tau) \subseteq (ss')\tau$ for all $s, s' \in S$.

Every morphism is a relational morphism. If $\tau: S \to T$ is an onto relational morphism (that is, such that $\bigcup_{s\in S} s\tau = T$) then $\tau^{-1}: T \to S$ is a relational morphism. Furthermore, the composition of relational morphisms is again a relational morphism. Let $\tau: S \to T$ be a relational morphism, and let R be the graph of τ, that is,

$$R = \{(s,t) \in S \times T \mid t \in s\tau\}.$$

Then R is a subsemigroup of $S \times T$. Let α and β be the projections of R respectively on S and T. Then α is onto and $\tau = \alpha^{-1}\beta$. This factorization of τ is called the *canonical factorization of the relational morphism* τ. For more details on relational morphisms, see [31,18].

A class of finite monoids (resp. semigroups) is called an **M**-*variety* (resp. **S**-*variety*) if it is closed under (finite) direct product and under division. A **G**-*variety* is an **M**-variety consisting only of groups.

Some important varieties for our purpose are:

A, the **M**-variety of aperiodic monoids. Recall that a semigroup is *aperiodic* if it satisfies the identity $x^n = x^{n+1}$ for some $n \geq 1$. Equivalently, **A** is the class of all monoids, all of whose subgroups are trivial;

I, the trivial **M**-variety, consisting only of the 1-element monoid;

J, the **M**-variety of $\mathcal{J}$-trivial monoids;

$\mathbf{J}_1$, the **M**-variety of commutative and idempotent monoids, that is, of semilattice monoids.

If **V** is an **M**-variety, then **LV** is the class of all semigroups S such that $eSe \in \mathbf{V}$ for each idempotent e of S. Equivalently, $S \in \mathbf{LV}$ if each submonoid of S is in **V**. **LV** is an **S**-variety.

If **H** is a **G**-variety, then $\overline{\mathbf{H}}$ is the **M**-variety of all monoids all of whose subgroups are in **H**. For instance $\overline{\mathbf{I}} = \mathbf{A}$. If **H** is a **G**-variety, $\mathbf{D}\overline{\mathbf{H}}$ is the class of all monoids in which each regular $\mathcal{D}$-class is a subsemigroup and in which each subgroup is in **H**. $\mathbf{D}\overline{\mathbf{H}}$ is an **M**-variety. The variety $\mathbf{DA} = \mathbf{D}\overline{\mathbf{I}}$ is thus the class of all monoids in which each regular $\mathcal{D}$-class is an aperiodic subsemigroup.

Let A be a finite set, or *alphabet*. Then A^* is the free monoid on A and A^+ is the free semigroup on A. An **M**-variety **V** has a *free object* on A if there exists an onto morphism $\sigma: A^* \to F_{\mathbf{V}}(A)$ satisfying the following two conditions:

(i) $F_{\mathbf{V}}(A) \in \mathbf{V}$;

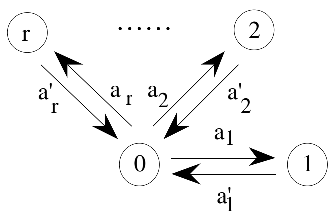
(ii) for each onto morphism $\mu: A^* \to M$ with $M \in \mathbf{V}$, there exists a morphism $\pi: F_{\mathbf{V}}(A) \to M$ such that $\mu = \sigma\pi$.

The morphism σ is then unique up to isomorphism.

We say that **V** is *locally finite* if it has free objects on each alphabet. Note that **V** is locally finite if and only if it has finitely many A-generated elements for each alphabet A. Locally finite **S**-varieties are defined similarly. An important class of locally finite **M**- and **S**-varieties is the following: if **V** is generated by a single monoid (or semigroup), then **V** is locally finite.

We say that an **M**- (resp. **S**-) variety is *decidable* if, for each finite monoid (resp. semigroup) S, one can decide whether $S \in \mathbf{V}$.

Finally, we will use the Brandt aperiodic semigroups BA_r ($r \geq 2$). Recall that BA_{r+1} is the transition semigroup of the following automaton.



The $\mathcal{D}$-class structure of, say, BA_3 is:

$^*a_1a_1^{-1}$	a_1	a_2
a_1^{-1}	$^*a_1^{-1}a_1$	$a_1^{-1}a_2$
a_2^{-1}	$a_2^{-1}a_1$	$^*a_2^{-1}a_2$

^{*0}

1.2 Mal'cev products

Let $\mathbf{V}$ be an $\mathbf{S}$-variety and let $\mathbf{W}$ be an $\mathbf{M}$- (resp. $\mathbf{S}$-) variety. We let $\mathbf{V} \textcircled{m} \mathbf{W}$ be the class of all the quotients of the monoids (resp. semigroups) S such that there exists a morphism $\varphi\colon S \to T$ with $T \in \mathbf{W}$ and $e\varphi^{-1} \in \mathbf{V}$ for each idempotent e of T. Using the notion of the canonical factorization of a relational morphism, it is easy to verify that $\mathbf{V} \textcircled{m} \mathbf{W}$ is defined equivalently as the class of all monoids (resp. semigroups) such that there exists a relational morphism $\tau\colon S \to T$ with $T \in \mathbf{W}$ and $e\varphi^{-1} \in \mathbf{V}$ for each idempotent e of T. Finally, one can verify that $\mathbf{V} \textcircled{m} \mathbf{W}$ is an $\mathbf{M}$- (resp. $\mathbf{S}$-) variety.

As a consequence of a result of Brown [2], we have

Theorem 1.1. *Let $\mathbf{V}$ be an $\mathbf{S}$-variety and let $\mathbf{W}$ be an $\mathbf{S}$- (resp. $\mathbf{M}$-) variety. If $\mathbf{V}$ and $\mathbf{W}$ are locally finite, then $\mathbf{V} \textcircled{m} \mathbf{W}$ is locally finite.*

If $\mathbf{W}$ is locally finite and if its free objects are "well understood", we can say more. Let $\mathbf{W}$ be a locally finite $\mathbf{M}$- (resp. $\mathbf{S}$-) variety and let $\sigma\colon A^* \to F_{\mathbf{W}}(A)$ (resp. $\sigma\colon A^+ \to F_{\mathbf{V}}(A)$) be its free object over A. We say that $\mathbf{W}$ is *effectively locally finite* if there exists an algorithm producing for each alphabet A and for each $x \in F_{\mathbf{W}}(A)$ a regular expression for the language $x\sigma^{-1}$.

Theorem 1.2. *Let $\mathbf{V}$ be an $\mathbf{S}$-variety and let $\mathbf{W}$ be an $\mathbf{S}$- (resp. $\mathbf{M}$-) variety. If $\mathbf{V}$ is decidable and $\mathbf{W}$ is effectively locally finite, then $\mathbf{V} \textcircled{m} \mathbf{W}$ is decidable.*

This last result is an immediate consequence of the following proposition.

Proposition 1.3. *Let $\mathbf{V}$ be a locally finite $\mathbf{M}$-variety and let $\sigma\colon A^* \to F_{\mathbf{V}}(A)$ be the free object of $\mathbf{V}$ over an alphabet A. Let $\mathbf{W}$ be an $\mathbf{S}$-variety and let $\mu\colon A^* \to M$ be an onto morphism. Then $M \in \mathbf{W} \textcircled{m} \mathbf{V}$ if and only if $e\sigma^{-1}\mu \in \mathbf{W}$ for each idempotent e of $F_{\mathbf{V}}(A)$.*

Proof. Let $\tau = \mu^{-1}\sigma$. Then τ is an onto relational morphism $\tau\colon M \to F_{\mathbf{V}}(A)$ and $\tau^{-1} = \sigma^{-1}\mu$. Thus, if $e\sigma^{-1}\mu = e\tau^{-1} \in \mathbf{W}$ for each $e \in E\big(F_{\mathbf{V}}(A)\big)$, then $M \in \mathbf{W} \textcircled{m} \mathbf{V}$.

Conversely let us assume that $M \in \mathbf{W} \textcircled{m} \mathbf{V}$. Then there exist onto morphisms $\pi\colon N' \to M$ and $\varphi\colon N' \to N$ such that $N \in \mathbf{V}$ and $e\varphi^{-1} \in \mathbf{W}$ for each idempotent e of N. By the universal property of the free monoid there exists a morphism $\nu\colon A^* \to N'$ such that $\mu = \nu\pi$. Replacing N' by $A^*\nu$ and N by $A^*\nu\varphi$, we may assume all morphisms to be onto. Now $N \in \mathbf{V}$ so there exists $\psi\colon F_{\mathbf{V}}(A) \to N$ such that $\sigma\psi = \nu\varphi$.

$$\begin{tikzcd}
N & N' \arrow[l, "\varphi"'] \arrow[dr, "\pi"] & \\
F_{\mathbf{V}}(A) \arrow[u, "\psi"] & A^* \arrow[l, "\sigma"'] \arrow[u, "\nu"] \arrow[r, "\mu"] & M
\end{tikzcd}$$

Let e be an idempotent of $F_{\mathbf{V}}(A)$. Note that $e\sigma^{-1} \subseteq e\psi\psi^{-1}\sigma^{-1}$ and that $e\psi \in E(N)$. Therefore, since $\mu = \nu\pi$ and $\sigma\psi = \nu\varphi$, we have

$$e\sigma^{-1}\mu \subseteq e\psi\psi^{-1}\sigma^{-1}\mu = e\psi\varphi^{-1}\nu^{-1}\nu\pi = e\psi\varphi^{-1}\pi.$$

But $e\psi\varphi^{-1} \in \mathbf{W}$, so $e\psi\varphi^{-1}\pi \in \mathbf{W}$ and hence $e\sigma^{-1}\mu \in \mathbf{W}$. ∎

We will also use the following technical lemma.

Lemma 1.4. *Let* $\mathbf{V}_1$, $\mathbf{V}_2$ *and* $\mathbf{V}_3$ *be varieties, with* $\mathbf{V}_1$ *and* $\mathbf{V}_2$ $\mathbf{S}$*-varieties. Then*

$$\mathbf{V}_1 ⓜ (\mathbf{V}_2 ⓜ \mathbf{V}_3) \subseteq (\mathbf{V}_1 ⓜ \mathbf{V}_2) ⓜ \mathbf{V}_3.$$

Proof. Let $\tau_1\colon M_1 \to M_2$ be a relational morphism with $M_2 \in \mathbf{V}_2 ⓜ \mathbf{V}_3$ and $e\tau_1^{-1} \in \mathbf{V}_1$ for each idempotent e of M_2. There exists a relational morphism $\tau_2\colon M_2 \to M_3$ with $M_3 \in \mathbf{V}_3$ and $e\tau_2^{-1} \in \mathbf{V}_2$ for each idempotent e of M_3. Let $\tau\colon M_1 \to M_3$ be the composition $\tau = \tau_1\tau_2$ and let $e \in E(M_3)$. Then $e\tau^{-1} = (e\tau_2^{-1})\tau_1^{-1}$. But $e\tau_2^{-1} \in \mathbf{V}_2$, so $e\tau^{-1} \in \mathbf{V}_1 ⓜ \mathbf{V}_2$. ∎

2. Varieties of languages and the dot-depth hierarchy

We give here a very brief overview of the main features of the theory of recognizable languages that are directly relevant to our purpose. Details about these results (and about many others!) can be found in [7,12,18].

2.1 Varieties of languages

Let A be an alphabet. A *language* on A is a subset of A^*. We say that a language $L \subseteq A^*$ is *recognized* by a morphism $\varphi\colon A^* \to M$ into a (finite) monoid M if $L = L\varphi\varphi^{-1}$. We say that L is recognized by a monoid M if it is recognized by some morphism from A^* into M, and we say that L is recognizable if it is recognized by some monoid. By Kleene's theorem [10], the class of recognizable languages is the least class of languages containing the finite languages and closed under Boolean operations, product and star.

If $L \subseteq A^*$, the *syntactic* congruence of L is the congruence $\sim_L$ on A^* defined by

$$u \sim_L v \qquad \text{if and only if} \qquad xuy \in L \Longleftrightarrow xvy \in L \quad \text{for all } x, y \in A^*.$$

The syntactic monoid of L is $M(L) = A^*/\sim_L$. One can verify that L is recognizable if and only if $M(L)$ is finite. Moreover, if L is recognizable, then L is recognized by a monoid M if and only if $M(L) < M$.

Let $\mathbf{V}$ be an $\mathbf{M}$-variety. For each alphabet A, let $A^*\mathcal{V}$ be the class of all languages L such that $M(L) \in \mathbf{V}$. The collection $\mathcal{V} = (A^*\mathcal{V})$ (indexed by the finite alphabets A) is called a *variety of (recognizable) languages*. Eilenberg [7] proved that $\mathbf{V}$ is the $\mathbf{M}$-variety generated by the syntactic monoids of the languages in the $A^*\mathcal{V}$, so that the correspondence $\mathbf{V} \longmapsto \mathcal{V}$ is one-to-one and onto between the $\mathbf{M}$-varieties and varieties of recognizable languages. This correspondence is known as *Eilenberg's correspondence*.

A parallel set of definitions can be given for languages contained in A^+ and for their recognizability by semigroups. In particular, and with similar definitions, there is an Eilenberg correspondence between $\mathbf{S}$-varieties and varieties of recognizable languages in free semigroups.

2.2 Products of languages and the dot-depth hierarchy

The class of *star-free* languages on an alphabet A is the least class containing the finite languages and closed under Boolean operations and product. Schützenberger [23] proved that this class is a strict subclass of the recognizable languages and that a language L is star-free if and only if $M(L) \in \mathbf{A}$. In other words, the star-free languages constitute a variety of languages and the corresponding $\mathbf{M}$-variety is $\mathbf{A}$. In particular, and since $\mathbf{A}$ is decidable, it is decidable whether a recognizable language is star-free.

The *dot-depth hierarchy* is the following increasing sequence of classes of star-free languages: $A^*\mathcal{V}_0 = \{\emptyset, A^*\}$ and, for $n \geq 1$, $A^*\mathcal{V}_n$ is the Boolean closure of the languages of the form $L_0a_1L_1 \cdots a_kL_k$ where $k \geq 0$, $a_1, \ldots, a_k \in A$ and $L_0, \ldots, L_k \in A^*\mathcal{V}_{n-1}$. The union of the $A^*\mathcal{V}_n$ is the class of all star-free languages on A. Furthermore, for each n, $\mathcal{V}_n = (A^*\mathcal{V}_n)$ is a variety of languages. Let $\mathbf{V}_n$ be the $\mathbf{M}$-variety associated to $\mathcal{V}_n$ in Eilenberg's correspondence. Then $\mathbf{A} = \bigcup_n \mathbf{V}_n$ and the containments $\mathbf{V}_n \subset \mathbf{V}_{n+1}$ are strict (Brzozowski and Knast [5], Straubing [26]). The dot-depth $\mathrm{dd}(L)$ (resp. $\mathrm{dd}(M)$) of a star-free language L (resp. of an aperiodic monoid M) is the least n such that $L \in A^*\mathcal{V}_n$ (resp. $M \in \mathbf{V}_n$). It is easy to see that $\mathbf{V}_0 = \mathbf{I}$. Simon [25] proved that $\mathbf{V}_1 = \mathbf{J}$. In particular, $\mathbf{V}_0$ and $\mathbf{V}_1$ are decidable. It is still an open problem whether $\mathbf{V}_n$ ($n \geq 2$) is decidable.

The original definition of the dot-depth hierarchy (Brzozowski and Cohen [3]) was a bit different and dealt with languages in the free semigroup. The corresponding sequence of $\mathbf{S}$-varieties is denoted $(\mathbf{B}_n)_n$. It was proved (Straubing [27]) that the decidability problems for both hierarchies are equivalent. The $\mathbf{S}$-variety $\mathbf{B}_1$ plays an important role in the sequel. Its effective characterization was first given by Knast [11] and is the following: a semigroup S is in $\mathbf{B}_1$ if and only if, for some $n \leq |S|$,

$$(esfte)^n esfve(eufve)^n = (esfte)^n(eufve)^n \qquad \text{for all } e, f \in E(S) \text{ and } s, t, u, v \in S.$$

This characterization is closely related to the characterization of glabally $\mathcal{J}$-trivial categories, which plays an important role in Section 5. Note that $\mathbf{LJ}_1 \subset \mathbf{B}_1 \subset \mathbf{LJ}$, with both inclusions strict.

We say that a product of languages $L_0a_1L_1 \cdots a_kL_k$ is *unambiguous* if the L_i are non empty and if each word w in $L_0a_1L_1 \cdots a_kL_k$ has exactly one factorization $w = u_0a_1u_1 \cdots a_ku_k$ such that $u_i \in L_i$ ($0 \leq i \leq k$). If $\mathbf{V}$ is an $\mathbf{M}$-variety and if $\mathcal{V}$ is the corresponding variety of languages, let $A^*\mathcal{W}$ be the Boolean closure of the languages of the form $L_0a_1L_1 \ldots a_kL_k$ with $k \geq 0$, $a_1, \ldots, a_k \in A$ and $L_0, \ldots, L_k \in A^*\mathcal{V}$, and let let $A^*\mathcal{W}'$ be the the class of all unions of unambiguous products of the same form. Then both $\mathcal{W}$ and $\mathcal{W}'$ are varieties of languages, and the corresponding $\mathbf{M}$-varieties $\mathbf{W}$ and $\mathbf{W}'$ are $\mathbf{W} = \Diamond\mathbf{V}$, where $\Diamond$ is the Schützenberger product (see [26,16]) and $\mathbf{W}' = \mathbf{LI} \textcircled{m} \mathbf{V}$ [17,21]. A special case of this last operation is $\mathbf{DA} = \mathbf{LI} \textcircled{m} \mathbf{J}_1 = \mathbf{LI} \textcircled{m} \mathbf{DA}$ [24,21]. Using the language characterizations of these varieties, it is not difficult to verify that

$$\mathbf{V} \subseteq \mathbf{LI} \textcircled{m} \mathbf{V} \subseteq \Diamond\mathbf{V} \quad \text{and} \quad \Diamond\mathbf{V} = \Diamond(\mathbf{LI} \textcircled{m} \mathbf{V}).$$

In particular $\mathbf{V}_{n+1} = \Diamond\mathbf{V}_n$ and $\mathbf{V}_2 = \Diamond\mathbf{J}_1 = \Diamond\mathbf{J} = \Diamond\mathbf{DA}$ [20]. Furthermore, Pin [19] proved that $\Diamond\mathbf{V} \subseteq \mathbf{B}_1 \textcircled{m} \mathbf{V}$. In particular, $\mathbf{V}_{n+1} \subseteq \mathbf{B}_1 \textcircled{m} \mathbf{V}_n$ and $\mathbf{V}_2 \subseteq \mathbf{B}_1 \textcircled{m} \mathbf{J}_1$. Straubing [28], and later Weil [34] and Straubing and Weil [29], used this last property of $\mathbf{V}_2$ to obtain some partial decidability results for $\mathbf{V}_2$. We will come back to these results more precisely in Sections 5 and 6.

3. An operation on M-varieties

Let M be a monoid and let $e \in E(M)$. We let M_e be the subsemigroup of M generated by

$$P_e = \{m \in M \mid e \leq_{\mathcal{J}} m\} = \{m \in M \mid e = xmy \text{ for some } x, y \in M\}.$$

We will be considering the submonoids of M of the form eM_ee ($e \in E(M)$). Note that when e is the identity of M, then eM_ee is the group of invertible elements of M. Dually, if e is in the minimal ideal of M, then eM_ee is the maximal subgroup of M containing e. Also, it is easy to verify that eM_ee is strictly contained in M if and only if M is not a group.

Let $\mathbf{V}$ be an $\mathbf{M}$-variety. We let $\widetilde{\mathbf{V}}$ be the class of all monoids M such that $eM_ee \in \mathbf{V}$ for all $e \in E(M)$. First we prove that $\widetilde{\mathbf{V}}$ is an $\mathbf{M}$-variety (see also Pin [15]).

Lemma 3.1. *Let $\varphi: S \to T$ be a monoid morphism.*

(1) If $e \in E(S)$ and $f = e\varphi$, then $S_e\varphi \subseteq T_f$.

(2) If φ is onto and $f \in E(T)$, then there exists $e \in E(S)$ such that $e\varphi = f$ and $S_e\varphi = T_f$.

Proof. (1) $S_e\varphi$ is generated by $P_e\varphi = \{s\varphi \mid e \leq_{\mathcal{J}} s\}$. Since $e \leq_{\mathcal{J}} s$ implies $f \leq_{\mathcal{J}} s\varphi$, we have $P_e\varphi \subseteq P_f$ and hence $S_e\varphi \subseteq T_f$.

(2) Let e be an idempotent in the minimal ideal of the subsemigroup $f\varphi^{-1}$ of S. In particular, $e\varphi = f$, so $S_e\varphi \subseteq T_f$ by (1). Let now $t \in T$ such that $f \leq_{\mathcal{J}} t$. Then $f = utv$ for some $u, v \in T$. Let x, s and y be elements respectively of $u\varphi^{-1}$, $t\varphi^{-1}$ and $v\varphi^{-1}$. Then $(xsy)\varphi = f$ so $e \leq_{\mathcal{J}} xsy \leq_{\mathcal{J}} s$. Therefore $s \in S_e$ and hence $t = s\varphi \in S_e\varphi$. Thus $P_f \subseteq S_e\varphi$ and $T_f \subseteq S_e\varphi$. ■

Proposition 3.2. *Let $\mathbf{V}$ be an $\mathbf{M}$-variety. Then $\widetilde{\mathbf{V}}$ is an $\mathbf{M}$-variety.*

Proof. Let M and N be monoids and let $(e, f) \in E(M \times N) = E(M) \times E(N)$. It is easy to verify that $P_{(e,f)} = P_e \times P_f$, so that $(e, f)M_{(e,f)}(e, f) = eM_ee \times fN_ff$. Thus $\widetilde{\mathbf{V}}$ is closed under direct product.

Now, if N is a submonoid of M and $e \in E(N)$, then $e \in E(M)$ and $eN_ee \subseteq eM_ee$. So $\widetilde{\mathbf{V}}$ is closed under taking submonoids.

Finally, if $\varphi: M \to N$ is an onto morphism and $e \in E(N)$, then $eN_ee = (fM_ff)\varphi$ for some idempotent f of M according to Lemma 3.1. So $\widetilde{\mathbf{V}}$ is closed under quotient. ■

This construction was examined in certain special cases by Brzozowski and Fich [8,4], who proved the following result [8].

Proposition 3.3. *If $\mathbf{H}$ is a $\mathbf{G}$-variety, then $\widetilde{\mathbf{H}} = \mathbf{D}\overline{\mathbf{H}}$. In particular, $\widetilde{\mathbf{I}} = \mathbf{DA}$.*

Proof. Let $M \in \widetilde{\mathbf{H}}$ and let $e \in E(M)$. We want to show that the $\mathcal{J}$-class J of e is a simple semigroup whose $\mathcal{H}$-classes are groups in H. Let $s \in J$. Then $s \in P_e \subseteq M_e$. Therefore $(ese)^n = e$ for some $n \geq 1$. This implies that $(es)^n$ and $(se)^n$ are idempotent and that $e \mathrel{\mathcal{R}} (es)^n \mathrel{\mathcal{L}} s$ and $e \mathrel{\mathcal{L}} (se)^n \mathrel{\mathcal{R}} s$. Since this is true for each $s \in J$, each $\mathcal{H}$-class in the $\mathcal{R}$-class and the $\mathcal{L}$-class of e is a group. Repeating this reasoning for each idempotent in, say, the $\mathcal{L}$-class of e shows that each $\mathcal{H}$-class is a group and hence that J is a simple semigroup. To show that these groups are in H, we notice that eM_ee contains (and in fact is equal to) the $\mathcal{H}$-class of e.

Conversely, let us consider $M \in \mathbf{D\overline{H}}$ and let $e \in E(M)$. Let $s, t \in P_e$. Then $e = xsy = wtz$ for some $x, y, w, z \in M$. Since $e = exsywtze$, we have $exs \;\mathcal{R}\; e \;\mathcal{L}\; tze$. But the $\mathcal{D}$-class of e is a subsemigroup of M, so $e \;\mathcal{H}\; exstze$ and hence $e \leq_{\mathcal{J}} st$. In particular, this proves that $M_e = P_e$, so $s \in eM_ee$ if and only if $e \leq_{\mathcal{J}} s \leq_{\mathcal{H}} e$, that is, if and only if $s \;\mathcal{H}\; e$. Thus eM_ee is a group in $\mathbf{H}$ and $M \in \widetilde{\mathbf{H}}$. ∎

We now give another description of the operation $\mathbf{V} \mapsto \widetilde{\mathbf{V}}$.

Theorem 3.4. *Let* $\mathbf{V}$ *be an* $\mathbf{M}$*-variety. Then* $\widetilde{\mathbf{V}} = \mathbf{LV} \circledm \mathbf{J}_1 = \mathbf{LV} \circledm \mathbf{DA}$.

Proof. Let A be an alphabet, let $\mu\colon A^* \to M$ be an onto morphism and let $\alpha\colon A^* \to 2^A$ be the alphabet morphism, defined by $a\alpha = \{a\}$ for each $a \in A$. It is well-known that $\mathbf{J}_1$ is locally finite and that α is its free object over A.

Let us suppose first that $M \in \widetilde{\mathbf{V}}$ and let $X \subseteq A$. Also let $e \in E(X\alpha^{-1}\mu)$. Then $e = (a_1 \cdots a_r)\mu$ with $X = \{a_1, \ldots, a_r\}$. In particular, for each $a_i \in X$, $e \leq_{\mathcal{J}} a_i\mu$ so $a_i\mu \in M_e$. Therefore $X\alpha^{-1}\mu \subseteq M_e$. So $e(X\alpha^{-1}\mu)e \subseteq eM_ee$ and hence $X\alpha^{-1}\mu \in \mathbf{V}$, thus proving that $M \in \mathbf{LV} \circledm \mathbf{J}_1$ by Proposition 1.3.

Conversely let us assume that $X\alpha^{-1}\mu \in \mathbf{LV}$ for each $X \subseteq A$, and let $e \in E(M)$. For each $a \in A$ such that $e \leq_{\mathcal{J}} a\mu$, there exist $u_a, v_a \in A^*$ such that $e = (u_a a v_a)\mu$. In particular, if $X = P_e\mu^{-1} \cap A = \{a_1, \ldots, a_r\}$, then $e = (u_{a_1}a_1v_{a_1} \cdots u_{a_r}a_rv_{a_r})\mu$ and $(u_{a_1}a_1v_{a_1} \cdots u_{a_r}a_rv_{a_r})\alpha = X$, so $e \in X\alpha^{-1}\mu$. Furthermore, for each $s \in M_e$, $s = w\mu$ for some $w \in X^*$ such that $w\alpha \subseteq X$, and hence $ese \in X\alpha^{-1}\mu$. So we have proved that $eM_ee \subseteq X\alpha^{-1}\mu$ and hence $eM_ee \in \mathbf{LV}$. But eM_ee is a monoid, so $eM_ee \in \mathbf{V}$ and $M \in \widetilde{\mathbf{V}}$. Thus $\widetilde{\mathbf{V}} = \mathbf{LV} \circledm \mathbf{J}_1$.

Since $\mathbf{J}_1 \subseteq \mathbf{DA}$, we have $\widetilde{\mathbf{V}} \subseteq \mathbf{LV} \circledm \mathbf{DA}$. Conversely, Lemma 1.4 shows that

$$\mathbf{LV} \circledm \mathbf{DA} = \mathbf{LV} \circledm (\mathbf{LI} \circledm \mathbf{J}_1) \subseteq (\mathbf{LV} \circledm \mathbf{LI}) \circledm \mathbf{J}_1 = \mathbf{LV} \circledm \mathbf{J}_1$$

since $\mathbf{LV} \circledm \mathbf{LI} = \mathbf{LV}$. ∎

Note that this reproves a well-known result (see [24]).

Corollary 3.5. *Let* $\mathbf{H}$ *be a* $\mathbf{G}$*-variety. Then* $\mathbf{D\overline{H}} = \mathbf{LH} \circledm \mathbf{J}_1$. *In particular,* $\mathbf{DA} = \mathbf{LI} \circledm \mathbf{J}_1$.

Another important corollary of Theorem 3.4 and of Theorem 1.1 is the following.

Corollary 3.6. *Let* $\mathbf{V}$ *be an* $\mathbf{M}$*-variety. If* $\mathbf{LV}$ *is locally finite, then* $\widetilde{\mathbf{V}}$ *is locally finite.*

4. A lower bound for the dot-depth

We define an increasing sequence $(\mathbf{W}_n)_n$ of $\mathbf{M}$-varieties by letting

$$\mathbf{W}_0 = \mathbf{I} \qquad \text{and} \qquad \mathbf{W}_{n+1} = \widetilde{\mathbf{W}_n}.$$

By Proposition 3.3, $\mathbf{W}_1 = \mathbf{DA}$. The proof of the following proposition is immediate by induction.

Proposition 4.1. *Let* $n \geq 0$. $\mathbf{W}_n$ *is the class of all aperiodic monoids such that, for each chain* $M_0 = M \supset M_1 \supset \cdots \supset M_n$ *with, for all* $1 \leq i \leq n$, $M_i = e(M_{i-1})_e e$ *for some* $e \in E(M_{i-1})$, *we have* $M_n = 1$.

It is clear that $\mathbf{W}_n \subset \mathbf{W}_{n+1}$ for all $n \geq 0$ and that $\mathbf{A} = \bigcup_n \mathbf{W}_n$.

Proposition 4.2. *For all $n \geq 0$, we have $\mathbf{LI} \mathbin{\textcircled{m}} \mathbf{W}_n = \mathbf{W}_n$ and $\mathbf{LDA} \mathbin{\textcircled{m}} \mathbf{W}_n \subseteq \mathbf{W}_{n+1}$.*

Proof. We have $\mathbf{W}_0 = \mathbf{LI} \mathbin{\textcircled{m}} \mathbf{W}_0$ and $\mathbf{W}_1 = \mathbf{DA} = \mathbf{LDA} \mathbin{\textcircled{m}} \mathbf{W}_0$ (since $\mathbf{LI} \mathbin{\textcircled{m}} \mathbf{I}$ and $\mathbf{LDA} \mathbin{\textcircled{m}} \mathbf{I}$ are classes of monoids). Let us assume that $\mathbf{LI} \mathbin{\textcircled{m}} \mathbf{W}_n = \mathbf{W}_n$ and $\mathbf{LDA} \mathbin{\textcircled{m}} \mathbf{W}_n \subseteq \mathbf{W}_{n+1}$ for $n \geq 0$, and let $\varphi: M \to N$ be an onto morphism with $N \in \mathbf{W}_{n+1}$ and $e\varphi^{-1} \in \mathbf{LI}$ (resp. $\mathbf{LDA}$) for each $e \in E(N)$. Let now $e \in E(M)$. Then $(eM_e e)\varphi \subseteq e\varphi N_{e\varphi} e\varphi$ by Lemma 3.1. But $N \in \mathbf{W}_{n+1}$ so $e\varphi N_{e\varphi} e\varphi \in \mathbf{W}_n$, and hence $eM_e e \in \mathbf{LI} \mathbin{\textcircled{m}} \mathbf{W}_n$ (resp. $\mathbf{LDA} \mathbin{\textcircled{m}} \mathbf{W}_n$). So $eM_e e \in \mathbf{W}_n$ (resp. $\mathbf{W}_{n+1}$) for each $e \in E(M)$ and hence $M \in \mathbf{W}_{n+1}$ (resp. $\mathbf{W}_{n+2}$). ∎

Corollary 4.3. *For $n \geq 0$, $\mathbf{LI} \mathbin{\textcircled{m}} \mathbf{V}_n \subseteq \mathbf{W}_n$.*

Proof. For $n = 0$, $\mathbf{V}_0 = \mathbf{I} = \mathbf{W}_0$. Let us now assume that $\mathbf{V}_n \subseteq \mathbf{W}_n$ for some $n \geq 0$. Then $\mathbf{V}_{n+1} \subseteq \mathbf{B}_1 \mathbin{\textcircled{m}} \mathbf{V}_n \subseteq \mathbf{B}_1 \mathbin{\textcircled{m}} \mathbf{W}_n$.

But $\mathbf{B}_1 \subseteq \mathbf{LJ} \subseteq \mathbf{LDA}$ so $\mathbf{V}_{n+1} \subseteq \mathbf{LDA} \mathbin{\textcircled{m}} \mathbf{W}_n \subseteq \mathbf{W}_{n+1}$. ∎

As a consequence, we have

Corollary 4.4. *Let M be an aperiodic monoid. If $M \notin \mathbf{W}_n$ then $M \notin \mathbf{LI} \mathbin{\textcircled{m}} \mathbf{V}_n$. In particular, the dot-depth of M is greater than or equal to $n+1$.*

It is essential to note that the varieties $\mathbf{W}_n$ are decidable, so that the above criterion gives an effective lower bound for the dot-depth of an aperiodic monoid. In this section we will see an example of an exact computation of the dot-depth made possible by this lower bound. Note that for $n = 1$ we know that $\mathbf{W}_1 = \mathbf{LI} \mathbin{\textcircled{m}} \mathbf{V}_1$. Also we will see (Section 6) that an inverse monoid M lies in $\mathbf{LI} \mathbin{\textcircled{m}} \mathbf{V}_n$ if and only if it lies in $\mathbf{V}_n$. This was shown to be equivalent to $M \in \mathbf{W}_n$ when $n = 2$ and M has two [34] or three [29] inverse generators. It is therefore tempting to conjecture that $\mathbf{W}_n = \mathbf{LI} \mathbin{\textcircled{m}} \mathbf{V}_n$ for all $n \geq 0$.

The following remarks, which are easily derived from Lemma 1.4, are interesting in this respect.

Proposition 4.5. *Let $\mathbf{V}$ be an $\mathbf{M}$-variety. Then*

$$\mathbf{LI} \mathbin{\textcircled{m}} (\mathbf{LDA} \mathbin{\textcircled{m}} \mathbf{V}) = \mathbf{LDA} \mathbin{\textcircled{m}} \mathbf{V}$$

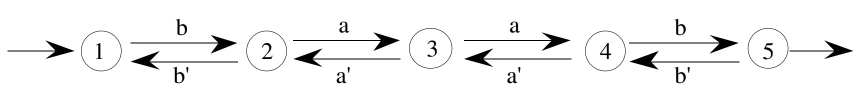
and in particular, for $n \geq 0$, $\mathbf{LI} \mathbin{\textcircled{m}} \mathbf{V}_{n+1} \subseteq \mathbf{LDA} \mathbin{\textcircled{m}} \mathbf{V}_n = \mathbf{LDA} \mathbin{\textcircled{m}} (\mathbf{LI} \mathbin{\textcircled{m}} \mathbf{V}_n)$.

Before we consider a practical example of computation, let us say a few words about upper bounds. A close examination of the proof of Schützenberger's theorem (as it is given in [13] for instance) shows that the dot-depth of an aperiodic monoid is less than or equal to $\mathcal{J}\mathrm{d}(M)$ if M has a zero, and to $1 + \mathcal{J}\mathrm{d}(M)$ otherwise. Here, $\mathcal{J}\mathrm{d}(M)$ is the $\mathcal{J}$-depth of M, which is defined by

$$\mathcal{J}\mathrm{d}(M) = \max\{n \geq 0 \mid \exists\, s_0, \ldots, s_n \in M,\ s_n <_{\mathcal{J}} \cdots <_{\mathcal{J}} s_0\}.$$

Another approach to an upper bound for the dot-depth appears in [21]. Both approaches are reviewed in [33].

Example. Let L be the language recognized by the following automaton



with initial state 1 and terminal state 5. This automaton is the minimal automaton of L and the $\mathcal{D}$-class structure of the syntactic monoid of L is

^{*1}

$^*aa'$	a
a'	$^*a'a$

$^*bb'$	b
b'	$^*b'b$

$^*baa'b'$	baa'	ba	baa	$baab$
$aa'b'$	$^*a^2a'^2$	a^2a'	a^2	a^2b
$a'b'$	aa'^{-2}	$^*a'a^2a'$	$a'a^2$	ab
a'^2b'	a'^2	a'^2a	$^*a'abb'$	$a'ab$
$b'a'^2b'$	$b'a'^2$	$b'a'$	$b'a'a$	$^*b'a'ab$

^{*0}

Since $\mathcal{J}\mathrm{d}(M) = 3$, $\mathrm{dd}(M) \leq 3$. Now, for $e = aa'$, M_e is the monoid generated by a and a', and eM_ee is isomorphic to BA_2^1 and is represented by

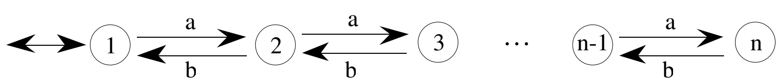
$^*aa'$

$^*a^2a'^2$	a^2a'
$a'a^2a'^2$	$^*a'a^2a'$

^{*0}

Now $eM_ee \notin \mathbf{W}_1 = \mathbf{DA}$ and one can verify that $eM_ee \in \mathbf{W}_2$. Therefore $M \notin \mathbf{W}_2$ and hence $\mathrm{dd}(M) \geq 3$, so that $\mathrm{dd}(L) = 3$.

In [33], this criterion was used to reprove the classical result that the language D_n described below has dot-depth exactly n. D_n is the language recognized by the following automaton, with initial and terminal state 1.



5. The special case of dot-depth two

For $n = 2$, the containment proved in Proposition 4.2 is an equality. In fact, Theorem 3.4 proves

$$\mathbf{W}_2 = \mathbf{LDA} \circledm \mathbf{J}_1 = \mathbf{LDA} \circledm \mathbf{DA}.$$

By definition, $\mathbf{V}_2 = \Diamond\mathbf{J}$ and we know that $\mathbf{V}_2 = \Diamond\mathbf{J}_1 = \Diamond\mathbf{J} = \Diamond\mathbf{DA}$ and $\mathbf{V}_2 \subseteq \mathbf{B}_1 \circledm \mathbf{J}_1$. An important result of Straubing proves that it is decidable whether a 2-generated monoid lies in $\mathbf{V}_2$. We will see that this result implies that the two varieties $\mathbf{V}_2$ and $\mathbf{B}_1 \ⓜ \mathbf{J}_1$ coincide on their 2-generated elements.

Formally, Straubing proved the following necessary condition for a monoid to be in $\mathbf{V}_2$ [28]. Let $\mu\colon A^* \to M$ be an onto morphism and let $\alpha\colon A^* \to 2^A$ be the alphabet morphism. A category $C(M,\mu)$ is defined as follows: Its set of objects is $\{(e,X) \mid e \in E(M),\ X \subseteq A \text{ and } e \in X\alpha^{-1}\mu\}$. There is an arrow $((e,X),w,(f,Y))$ from object (e,X) to object (f,Y) if $w \in A^*$ and $X \cup w\alpha = Y$. Two such arrows, $((e,X),w,(f,Y))$ and $((e,X),w',(f,Y))$ are identified if $e\,w\mu\,f = e\,w'\mu\,f$. Finally the product of consecutive arrows is given by

$$((e,X),w,(f,Y))((f,Y),w',(g,Z)) = ((e,X),ww_fw',(g,Z))$$

where w_f is a word such that $w_f\alpha = Y$ and $w_f\mu = f$.

We say that the category $C(M,\mu)$ is *globally $\mathcal{J}$-trivial* (see [11] and [32] for a justification of this terminology) if the following condition is satisfied: There exists an integer $n \geq 1$ such that, for each pair of objects (e,X) and (f,Y) and for each choice of arrows x and u from (e,X) to (f,Y) and y and v from (f,Y) to (e,X), we have $(xy)^n xv(uv)^n = (xy)^n(uv)^n$ in $C(M,\mu)$.

It was verified in [29] that if μ and ν are onto morphisms from a free monoid onto M, then $C(M,\mu)$ is globally $\mathcal{J}$-trivial if and only if $C(M,\nu)$ is globally $\mathcal{J}$-trivial. It was also verified there that the class $\mathbf{CJ}$ of all finite monoids such that $C(M,\mu)$ is globally $\mathcal{J}$-trivial for some morphism μ from a free monoid onto M, is an $\mathbf{M}$-variety.

Straubing proved the following theorem [29].

Theorem 5.1. $\mathbf{V}_2 \subseteq \mathbf{CJ}$. *Furthermore, if $M \in \mathbf{CJ}$ and M has two generators, then $M \in \mathbf{V}_2$.*

It is important to notice that the $\mathbf{M}$-variety $\mathbf{CJ}$ is decidable. Thus Theorem 5.1 gives an effective necessary condition for a monoid to be in $\mathbf{V}_2$, and proves the decidability of $\mathbf{V}_2$ for 2-generated monoids.

In the rest of this section, we will prove that this necessary condition is in fact a special case of the fact that $\Diamond\mathbf{V} \subseteq \mathbf{B}_1 \ⓜ \mathbf{V}$ and of the equality $\mathbf{V}_2 = \Diamond\mathbf{J}_1$. More precisely, we prove:

Proposition 5.2. $\mathbf{CJ} = \mathbf{B}_1 \ⓜ \mathbf{J}_1$.

Proof. Let $\mu\colon A^* \to M$ be an onto morphism and let $\alpha\colon A^* \to 2^A$ be the alphabet morphism. We first assume that $M \in \mathbf{CJ}$. Let $X \subseteq A$ and let $e,f,x,y,u,v \in X\alpha^{-1}\mu$ with e and f idempotents. Let w_e, w_f, w_x, w_y, w_u and w_v be words of A^* respectively in $X\alpha^{-1} \cap e\mu^{-1}$, $X\alpha^{-1} \cap f\mu^{-1}$, $X\alpha^{-1} \cap x\mu^{-1}$, $X\alpha^{-1} \cap y\mu^{-1}$, $X\alpha^{-1} \cap u\mu^{-1}$ and $X\alpha^{-1} \cap v\mu^{-1}$. Then (e,X) and (f,X) are objects of $C(M,\mu)$ and $a_x = ((e,X),w_x,(f,X))$, $a_u = ((e,X),w_u,(f,X))$, $a_y = ((f,X),w_y,(e,X))$ and $a_v = ((f,X),w_v,(e,X))$ are arrows of $C(M,\mu)$. Since $M \in \mathbf{CJ}$, $C(M,\mu)$ is globally $\mathcal{J}$-trivial and hence there exists n such that

$$(a_xa_y)^n a_xa_v(a_ua_v)^n = (a_xa_y)^n(a_ua_v)^n.$$

By definition of $C(M,\mu)$ this implies

$$(exfye)^n exfve(eufve)^n = (exfye)^n(eufve)^n.$$

So $X\alpha^{-1}\mu \in \mathbf{B}_1$ and $M \in \mathbf{B}_1 \ⓜ \mathbf{J}_1$.

Conversely let us assume that $M \in \mathbf{B}_1 \ⓜ \mathbf{J}_1$. Then by Proposition 1.3, $X\alpha^{-1}\mu \in \mathbf{B}_1$ for each $X \subseteq A$. Let (e,X) and (f,Y) be objects of $C(M,\mu)$, let

$a_x = \big((e,X), w_x, (f,Y)\big)$, and $a_u = \big((e,X), w_u, (f,Y)\big)$ be arrows from (e,X) to (f,Y), and let $a_y = \big((f,Y), w_y, (e,X)\big)$ and $a_v = \big((f,Y), w_v, (e,X)\big)$ be arrows from (f,Y) to (e,X). Let also w_e and w_f in $X\alpha^{-1} \cap e\mu^{-1}$ and $Y\alpha^{-1} \cap f\mu^{-1}$. The existence of arrows in both directions between (e,X) and (f,Y) shows that $X = Y$. Furthermore, a_x is identified with $\big((e,X), w_e w_x w_f, (f,X)\big)$, so we may assume that $w_x\alpha$ (and similarly $w_y\alpha$, $w_u\alpha$ and $w_v\alpha$) is equal to X itself. Finally let $x = w_x\mu$, $u = w_u\mu$, $y = w_y\mu$ and $v = w_v\mu$. There exists n such that $(exfye)^n exfve(eufve)^n = (exfye)^n(eufve)^n$ since $X\alpha^{-1}\mu \in \mathbf{B}_1$. It is then easy to verify that this shows the equality in $C(M,\mu)$ of the arrows $(a_x a_y)^n a_x a_v (a_u a_v)^n$ and $(a_x a_y)^n (a_u a_v)^n$. ∎

6. The inverse case

Recall that a semigroup S is inverse if and only if, for each $s \in S$ there exists a unique element, denoted s^{-1}, such that $ss^{-1}s = s$ and $s^{-1}ss^{-1} = s^{-1}$. Equivalently a semigroup is inverse if and only if it is regular and its idempotents commute. Some important properties of inverse semigroups are summarized in the following proposition (see [9,14]).

Proposition 6.1. *Let S be an inverse semigroup.*

(1) For all $s, t \in S$, $(st)^{-1} = t^{-1}s^{-1}$. Also ss^{-1} is the unique idempotent which is $\mathcal{R}$-equivalent to s and $s^{-1}s$ is the unique idempotent which is $\mathcal{L}$-equivalent to s.

(2) S is faithfully represented by right multiplication on its $\mathcal{R}$-classes. That is, if $s, t \in S$ and if

$$\forall\, x \in S, \qquad xs \,\mathcal{R}\, x \text{ or } xt \,\mathcal{R}\, x \quad \Longrightarrow \quad xs = xt$$

then $s = t$.

We say that a class of finite inverse monoids (resp. semigroups) is an **IM**- (resp. **IS**-) variety if it is closed under finite direct product, quotient and taking of inverse submonoid (resp. subsemigroup). **IM**- and **IS**-varieties are dealt with like **M**- and **S**-varieties. In particular, Mal'cev products $\mathbf{V} \,ⓜ\, \mathbf{W}$ are defined like their **S**- and **M**-variety counterparts, replacing everywhere the words semigroup and monoid by inverse semigroup and inverse monoid. Let us note in particular that the definition of a free object for an **IM**-variety **V** on an alphabet A involves an onto morphism σ from the free inverse monoid $FIM(A)$ onto an appropriate $F_{\mathbf{V}}(A)$. Recall that $FIM(A)$ is the quotient of the free involutive monoid on A, $(A \cup \bar{A})^*$ by the Vagner congruence ρ.

If **V** is an **M**- (resp. **S**-) variety, we let $\mathbf{V}^{inv}$ be the class of inverse monoids (resp. semigroups) in **V**. Then $\mathbf{V}^{inv}$ is an **IM**- (resp. **IS**-) variety. An important example is the equality $\mathbf{V}^{inv} = \mathbf{J}_1$ if $\mathbf{J}_1 \subseteq \mathbf{V} \subseteq \mathbf{DA}$.

Lemma 6.2. *Let M be an inverse semigroup and let $e \in E(M)$. Then eMe and eM_ee are inverse submonoids of M.*

Proof. Let $s \in M$. Then $(ese)^{-1} = es^{-1}e$, so eMe is closed under taking inverses, which implies easily that eMe is inverse. If $s \in M_e$, then $s = s_1 \cdots s_r$ with $e \leq_{\mathcal{J}} s_i$ for $1 \leq i \leq r$. But $s_i \;\mathcal{J}\; s_i^{-1}$, so $e \leq_{\mathcal{J}} s_i^{-1}$ and hence $s^{-1} = s_r^{-1} \cdots s_1^{-1} \in M_e$. Therefore M_e is an inverse subsemigroup of M and hence eM_ee is inverse. ∎

This allows us to extend the operations $\mathbf{V} \mapsto \mathbf{LV}$ and $\mathbf{V} \mapsto \widetilde{\mathbf{V}}$ to **IM**-varieties: If **V** is an **IM**-variety, then **LV** is the class of all inverse semigroups S such that $eSe \in \mathbf{V}$ for each $e \in E(S)$ and $\widetilde{\mathbf{V}}$ is the class of all inverse monoids M such that $eM_ee \in \mathbf{V}$ for each $e \in E(M)$. The above lemma implies the following corollary.

Corollary 6.3. *Let* $\mathbf{V}$ *be an* $\mathbf{M}$*-variety. Then* $(\mathbf{LV})^{inv} = \mathbf{L}(\mathbf{V}^{inv})$ *and* $(\widetilde{\mathbf{V}})^{inv} = \widetilde{\mathbf{V}^{inv}}$*. In particular,* $\mathbf{W}_{n+1}^{inv} = \widetilde{\mathbf{W}_n^{inv}}$*.*

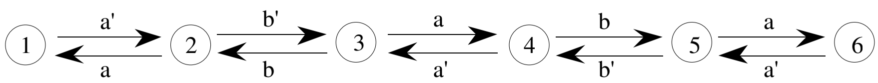
Let us note the following interesting property of the $\mathbf{IM}$-variety $\mathbf{W}_n^{inv}$.

Proposition 6.4. *Let* $n \geq 0$ *and let* $M \in \mathbf{W}_n^{inv}$*. Then* M *satisfies the identity* $x^n = x^{n+1}$*.*

Proof. We proceed by induction on n. The assertion is trivial for $n = 0$. Now let M be an inverse monoid in $\mathbf{W}_n$ with $n \geq 1$. Let $x \in M$ and let $e = xx^{-1}$. Then $eM_ee \in \mathbf{W}_{n-1}$ and hence satisfies the identity $x^{n-1} = x^n$. But $x \mathrel{\mathcal{J}} e$, so $x \in M_e$ and hence $(exe)^{n-1} = (exe)^n$. Now we have $(exe)^k = (x^2x^{-1})^k$ and we can verify by induction on k that $(x^2x^{-1})^k = x^{k+1}x^{-1}$. So $x^nx^{-1} = x^{n+1}x^{-1}$. Multiplying on the right by x, we get $x^{n+1} = x^n$. ■

Proposition 6.4 provides us with an easy test that may show that a given aperiodic inverse monoid M has dot-depth at least n. Indeed, if M does not satisfy $x^{n-1} = x^n$, then $M \notin \mathbf{W}_{n-1}$ and hence $M \notin \mathbf{V}_{n-1}$, so that $\mathrm{dd}(M) \geq n$. Note however that while $\mathbf{W}_n^{inv}$ coincides with the $\mathbf{IM}$-variety of all inverse monoids satisfying $x^n = x^{n+1}$ for $n = 0$ and $n = 1$, this is false for larger values of n.

Example. Let M be the transition monoid of the following automaton.



One can verify that M has 5 $\mathcal{J}$-classes whose poset is linearly ordered. Furthermore, M satisfies $x^2 = x^3$. However, if $e = b\bar{b}$, then $e \in E(M)$ and eM_ee is isomorphic to BA_2^1. But $BA_2^1 \in \mathbf{W}_2 \setminus \mathbf{W}_1$, so $M \notin \mathbf{W}_2$. In fact, one can verify that $M \in \mathbf{W}_3$. Note that the results mentioned in Section 4 show that $3 \leq \mathrm{dd}(M) \leq 4$. The exact dot-depth of this monoid is not known to the author.

We now consider the action of the operation $\textcircled{m}$ on $\mathbf{IM}$- and $\mathbf{IS}$-varieties. It is immediate that if $\mathbf{V}$ is an $\mathbf{S}$-variety and if $\mathbf{W}$ is an $\mathbf{M}$- (resp. $\mathbf{S}$-) variety, then

$$\mathbf{V}^{inv} \mathbin{\textcircled{m}} \mathbf{W}^{inv} \subseteq (\mathbf{V} \mathbin{\textcircled{m}} \mathbf{W})^{inv}.$$

However, equality holds in certain cases as is shown in the following propositions.

Proposition 6.5. *If* $\mathbf{V}$ *is an* $\mathbf{S}$*-variety, then* $\mathbf{V}^{inv} \mathbin{\textcircled{m}} \mathbf{J}_1 = (\mathbf{V} \mathbin{\textcircled{m}} \mathbf{J}_1)^{inv}$*.*

Remark. Note that in the left hand term of this equality, $\mathbf{J}_1$ is considered as an $\mathbf{IM}$-variety, while it is considered as an $\mathbf{M}$-variety in the right hand term.

Proof. Let M be an inverse monoid in $\mathbf{V} \mathbin{\textcircled{m}} \mathbf{J}_1$. There exists an onto morphism $\mu: FIM(A) \rightarrow M$ for some alphabet A. Let $\alpha: (A \cup \bar{A})^* \rightarrow 2^{A \cup \bar{A}}$ be the alphabet morphism, that is, the free object of the $\mathbf{M}$-variety $\mathbf{J}_1$ on $A \cup \bar{A}$. Let also $\beta: FIM(A) \rightarrow 2^A$ be the morphism defined by $a\beta = \bar{a}\beta = \{a\}$ for each $a \in A$.

Note that β is the free object of the **IM**-variety $\mathbf{J}_1$ on A.

$$\begin{tikzcd}
2^{A\cup\bar{A}} & (A\cup\bar{A})^* \arrow[l, "\alpha"'] \arrow[d, "\rho"] & \\
M & FIM(A) \arrow[l, "\mu"'] \arrow[r, "\beta"] & 2^A
\end{tikzcd}$$

By Proposition 1.3 we know that $X\alpha^{-1}\rho\mu \in \mathbf{V}$ for each $X \subseteq A\cup\bar{A}$. By the inverse analogue of the same proposition, it suffices to show that $X\beta^{-1}\mu \in \mathbf{V}$ for each $X \subseteq A$.

We claim that if $X \subseteq A$, then $X\beta^{-1} = (X\cup\bar{X})\alpha^{-1}\rho$. Let indeed $m \in (X\cup\bar{X})\alpha^{-1}\rho$. Then $m = w\rho$ with $w\alpha = X\cup\bar{X}$. Since $a\rho\beta = \bar{a}\rho\beta = \{a\}$ for each $a \in A$, we have $w\rho\beta = X$ and hence $m \in X\beta^{-1}$. Conversely, let $m \in X\beta^{-1}$. Then $m = w\rho$ where $w\rho\beta = X$. But $m = (w\bar{w}w)\rho$ and $(w\bar{w}w)\alpha = X\cup\bar{X}$, so $m \in (X\cup\bar{X})\alpha^{-1}\rho$.

Thus $X\beta^{-1}\mu = (X\cup\bar{X})\alpha^{-1}\rho\mu \in \mathbf{V}$. ∎

Proposition 6.6. *Let $\pi: M \to N$ be a morphism with N inverse. For each onto morphism $\varphi: M \to V$ such that $e\varphi^{-1} \in \mathbf{LI}$ for each idempotent e of V, there exists a morphism $\psi: V \to N$ such that $\varphi\psi = \pi$. In particular, if $\mathbf{V}$ is an $\mathbf{M}$- or an $\mathbf{S}$-variety, then $(\mathbf{LI} ⓜ \mathbf{V})^{inv} = \mathbf{V}^{inv}$.*

Proof. Let $x, y \in M$ such that $x\varphi = y\varphi$. We need to show that $x\pi = y\pi$. By Proposition 6.1, this can be done by showing that the actions of $x\pi$ and $y\pi$ by right multiplication coincide on each $\mathcal{R}$-class of N. Let then $s \in N$ be such that $s \mathrel{\mathcal{R}} s(x\pi)$. Let $e = ss^{-1}$ be the idempotent of the $\mathcal{R}$-class of s.

Let e' be an idempotent in the minimal ideal of the subsemigroup $e\pi^{-1}$ of M, and let $s' \in s\pi^{-1}$. Let $z' \in s^{-1}\pi^{-1}$: then $e' \leq_{\mathcal{J}} s'z' \leq_{\mathcal{J}} s'$. Up to replacing s' by $e's'$, we may therefore assume that $s' \mathrel{\mathcal{R}} e'$ in M. Similarly, we have $e' \mathrel{\mathcal{R}} s'x$. Let now $t' \in M$ such that $e' = s'xt'$. Again, up to replacing t' by $t'e'$, we may assume $e' \mathrel{\mathcal{L}} t'$. Then $e'\varphi = (s'xt')\varphi = (s'yt')\varphi$ since $x\varphi = y\varphi$. Now $s'ut' \leq_{\mathcal{J}} e'$. But $e'\varphi\varphi^{-1} \in \mathbf{LI}$, that is, the minimal ideal of $e'\varphi\varphi^{-1}$ is equal to $E(e'\varphi\varphi^{-1})$ and is a rectangular band, so $s'yt'$ is idempotent and $s'yt' \mathrel{\mathcal{J}} s'xt' = e'$. In particular, $s'yt' \in R_{s'} \cap L_{t'} = R_{e'} \cap L_{e'}$, so $s'yt'$ is an idempotent $\mathcal{H}$-equivalent to e', that is, $s'yt' = e'$. Furthermore, $e' \mathrel{\mathcal{R}} s'y$ and there exist idempotents f and g respectively in $R_{t'} \cap L_{s'x}$ and $R_{t'} \cap L_{s'y}$. The situation in the $\mathcal{D}$-class of e' can be represented as follows.

$\overset{*}{e'}$	s'	$\cdots$	$s'x$	$s'y$
$\vdots$				
t'			$\overset{*}{f}$	$\overset{*}{g}$

Now $f\pi \mathrel{\mathcal{R}} g\pi$, so $f\pi = g\pi$. Thus $s(x\pi) \mathrel{\mathcal{H}} s(y\pi)$ and $e = s(x\pi)(t'\pi) = s(y\pi)(t'\pi)$. But Green's lemma asserts that the right translation by $t'\pi$ is one-to-one on the $\mathcal{H}$-class of $s(x\pi)$. So $s(x\pi) = s(y\pi)$. ∎

Note that if $\mathbf{V}$ is an $\mathbf{S}$-variety such that $\mathbf{LJ}_1 \subseteq \mathbf{V} \subseteq \mathbf{LDA}$ (for instance for $\mathbf{V} = \mathbf{B}_1$), then $\mathbf{V}^{inv} = \mathbf{LJ}_1^{inv}$ by Corollary 6.3. Therefore the above results imply the following.

Proposition 6.7. *For each $n \geq 0$, we have*

$$\begin{aligned}(\mathbf{LI} ⓜ \mathbf{V}_{n+1})^{inv} &= \mathbf{V}_{n+1}^{inv} \subseteq (\mathbf{B}_1 ⓜ \mathbf{V}_n)^{inv}.\\ \mathbf{LJ}_1^{inv} ⓜ \mathbf{W}_n^{inv} &\subseteq \mathbf{W}_{n+1}^{inv} = \mathbf{LW}_n^{inv} ⓜ \mathbf{J}_1.\\ \mathbf{LJ}_1^{inv} ⓜ \mathbf{J}_1 &= \mathbf{W}_2^{inv}.\end{aligned}$$

We now consider the **IS**-variety $\mathbf{LJ}_1^{inv} = \mathbf{B}_1^{inv} = \mathbf{LDA}^{inv}$.

Proposition 6.8. $\mathbf{LJ}_1^{inv}$ *is the* **IS**-*variety generated by* BA_2.

Proof. Let $\mathbf{V}$ be the **IS**-variety generated by BA_2. It is clear that $BA_2 \in \mathbf{LJ}_1$, so $\mathbf{V} \subseteq \mathbf{LJ}_1^{inv}$. To prove the converse inclusion, let us consider $S \in \mathbf{LJ}_1^{inv}$. It suffices to show that for each $\mathcal{R}$-class R of S, the representation of S by right translations of R, S_R, is in $\mathbf{V}$. In fact we will show that if $|R| = r$, then S_R is isomorphic to BA_r, by showing that each non-zero right translation of R is a transformation of rank 1, and that $BA_r \in \mathbf{V}$.

Let R be an $\mathcal{R}$-class of S, let r and r' in R and let $a \in S$ such that $ra, r'a \in R$. Then $ra = raa^{-1}a$, so $raa^{-1} \in R$ and, similarly, $r'aa^{-1} \in R$. In particular $raa^{-1}r^{-1} = r'aa^{-1}r'^{-1}$, so

$$raa^{-1} = rr^{-1}raa^{-1} = raa^{-1}r^{-1}raa^{-1} = r'aa^{-1}r'^{-1}raa^{-1}.$$

But $aa^{-1}r'^{-1}raa^{-1}$ is an idempotent and lies in the $\mathcal{D}$-class of r and r', so $aa^{-1}r'^{-1}raa^{-1} \mathrel{\mathcal{L}} raa^{-1}$, that is, $aa^{-1}r'^{-1}raa^{-1} = aa^{-1}r^{-1}raa^{-1}$ and hence $raa^{-1} = r'aa^{-1}r^{-1}raa^{-1} = r'r^{-1}raa^{-1} = r'r'^{-1}r'aa^{-1} = r'aa^{-1}$. Thus $ra = raa^{-1}a = r'aa^{-1}a = r'a$.

It now remains to show that $BA_r \in \mathbf{V}$. It is easy to verify that BA_r is a subsemigroup of $BA_{r'}$ if $r \leq r'$, so we consider a semigroup of the form BA_{2^r} $(r \geq 1)$. Let $S_1, \ldots, S_r$ be r copies of BA_2. Then $S = S_1 \times \cdots \times S_r \in \mathbf{V}$. Let $J = (S_1 \setminus \{0\}) \times \cdots \times (S_r \setminus \{0\})$. Then J is the unique maximal $\mathcal{D}$-class of S, so that $I = S \setminus J$ is an ideal, and one verifies that $S/I = BA_{2^r}$. ∎

This result, together with the fact that $\mathbf{W}_2^{inv} = \mathbf{LJ}_1^{inv} ⓜ \mathbf{J}_1$, shows that $\mathbf{W}_2^{inv}$ is a Mal'cev product of two locally finite varieties. By Theorem 1.1, we then have

Theorem 6.9. $\mathbf{W}_2^{inv}$ *is a locally finite* **IM**-*variety.*

Corollary 6.10. $\mathbf{V}_2^{inv}$ *is a locally finite* **IM**-*variety.*

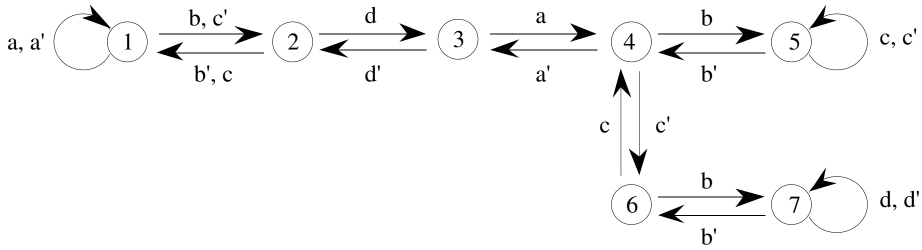
Proof. Recall that a variety is locally finite if and only if it has finitely many elements with a given number of generators. So each subvariety of a locally finite variety is also locally finite. ∎

The above two results, like Theorem 1.1, unfortunately do not give an effective construction of the free objects of $\mathbf{W}_2^{inv}$ and $\mathbf{V}_2^{inv}$. However, we constructed in [34] all the inverse monoids in $\mathbf{W}_2^{inv}$ with 2 inverse generators, and proved that each of them was in fact in $\mathbf{V}_2^{inv}$, thus proving the decidability of $\mathbf{V}_2$ for inverse monoids with 2 inverse generators. This result was later extended with Straubing [29] to inverse monoids with 3 inverse generators, using a different method.

Theorem 6.11. [34,29] *Let M be an inverse monoid with at most 3 inverse generators. Then $M \in \mathbf{V}_2$ if and only if $M \in \mathbf{W}_2$.*

The result for 3 inverse generators was obtained by using another tool, namely the known containment $\mathbf{J} ** \mathbf{DA} \subseteq \mathbf{V}_2$ (where $**$ denotes the two-sided semidirect product, see [22]). In fact it was shown that an inverse monoid with at most 3 inverse generators is in $\mathbf{V}_2$ if and only if it is in $\mathbf{W}_2$, if and only if it is in $\mathbf{J} ** \mathbf{DA}$. However we gave in [29] an example of an inverse monoid with 4 inverse generators, in $\mathbf{W}_2$ but not in $\mathbf{J} ** \mathbf{DA}$. We don't know whether this monoid is in $\mathbf{V}_2$ or not. We conclude this paper by giving a smaller monoid M with the same properties, which can hence be viewed as an interesting test: if $M \in \mathbf{V}_2$, then the containment $\mathbf{J} ** \mathbf{DA} \subseteq \mathbf{V}_2$ is strict, and if $M \notin \mathbf{V}_2$, then the containments $\mathbf{V}_2^{inv} \subseteq \mathbf{W}_2^{inv}$ and $\mathbf{LI} \textcircled{m} \mathbf{V}_2 \subseteq \mathbf{W}_2$ are strict.

M is the transition monoid of the following automaton.



The proof that $M \in \mathbf{W}_2$ and $M \notin \mathbf{J} ** \mathbf{DA}$ can be obtained by the same techniques as in [29]. The $\mathcal{D}$-class structure of M is given below.

* 1

* bb'	b
b'	* $b'b$

* cc'	c
c'	* $c'c$

* aa'	a	ab	abc
a'	* $a'a$	$a'ab$	$a'abc$
$b'a'$	$b'a'a$	* $b'a'ab$	$b'a'abc$
$c'b'a'$	$c'b'a'a$	$c'b'a'ab$	* $c'b'a'abc$

* $d'd$	d'	$d'b'$	$d'b'c'$
d	* dd'	$dd'b'$	$dd'b'c'$
bd	bdd'	* $bdd'b'$	$bdd'b'c'$
cbd	$cbdd'$	$cbdd'b'$	* $cbdb'd'c'$

* a^2	ac'	$c'd$	bda	$bdab$	$bdac$	$bdacb$
ca	* cab	$cc'd$	da	dab	dac	$dacb$
$d'c$	$aa'd'$	* $acc'a'$	acc'	abc'	ac	acb
$a'd'c$	$a'd'$	$cc'a'$	* bcb'	bc'	$bb'c$	cbd'
$b'a'd'c$	$b'a'd'$	$cb'a'$	cb'	* c^2	$b'c$	$b'cb$
$c'a'd'c$	$c'a'd'$	$c'a'$	$c'bb'$	$c'b$	* bdb'	bd'
$b'c'a'd'c$	$b'c'a'd'$	$b'c'a'$	$db'c'$	$b'c'b$	db'	* d^2

* 0

Remark. In [4], Brzozowski and Fich conjectured that $\widetilde{\mathbf{J}_1} = \mathbf{J}_1 * \mathbf{L}$ where $\mathbf{L}$ is the $\mathbf{M}$-variety of $\mathcal{L}$-trivial monoids. This conjecture is proved false by the above monoid since it is in $\mathbf{W}_2^{inv}$, which is contained in $\widetilde{\mathbf{J}_1}$, and it is not in $\mathbf{J} ** \mathbf{DA}$, which contains $\mathbf{J}_1 * \mathbf{L}$. The monoid with the same properties presented by Straubing and Weil [29] also disproved the conjecture, as well as a monoid given earlier by the author in [33], which was in $\mathbf{W}_2^{inv}$ and not in $\mathbf{J}_1 ** \mathbf{DA}$.

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LITP – Institut Blaise Pascal
4 place Jussieu
75252 Paris Cedex 05 – France
email: weil@litp.ibp.fr