Free profinite semigroups over semidirect products*

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Abstract

We give a general description of the free profinite semigroups over a semidirect product of pseudovarieties. More precisely, $\overline{\Omega}_A(\mathbf{V} * \mathbf{W})$ is described as a closed subsemigroup of a profinite semidirect product of the form $\overline{\Omega}_{\overline{\Omega}_A \mathbf{W} \times A} \mathbf{V} * \overline{\Omega}_A \mathbf{W}$. As a particular case, the free profinite semigroup over $\mathbf{J}_1 * \mathbf{V}$ is described in terms of the geometry of the Cayley graph of the free profinite semigroup over $\mathbf{V}$ (here $\mathbf{J}_1$ is the pseudovariety of semilattice monoids). Applications are given to the calculations of the free profinite semigroup over $\mathbf{J}_1 * \mathbf{Nil}$ and of the free profinite monoid over $\mathbf{J}_1 * \mathbf{G}$ (where $\mathbf{Nil}$ is the pseudovariety of finite nilpotent semigroups and $\mathbf{G}$ is the pseudovariety of finite groups). The latter free profinite monoid is compared with the free profinite inverse monoid, which is also calculated here.

Introduction

The operation of semidirect product plays a particularly important role in problems of composition and decomposition of finite semigroups and monoids, and ultimately in the problem of their classification. This is examplified by the Krohn-Rhodes theorem, which states that every finite semigroup divides a semidirect product of groups and aperiodic (or group-free) semigroups. The Krohn-Rhodes theorem also shows that the semidirect product of semigroups, or rather the related wreath product, models a

*Both authors gratefully acknowledge support from ESPRIT-BRA Working Group 6317 "ASMICS". The first author was also supported in part by JNICT through the project SAL (PBIC/C/CEN/1021/92). The second author was also supported in part by the Projet de Recherche Coordonnée "Mathématiques et Informatique".

natural operation on finite automata, the cascade product. Thus a better understanding of the semidirect product of semigroups would help to better understand and classify the family of rational languages, that is languages which are recognized by finite automata.

Later, work of Schützenberger and Eilenberg showed that the classification effort was best performed at a more general level than that of the individual semigroup, and that the appropriate classes to be considered are the pseudovarieties of finite semigroups or monoids. The operation of semidirect product is naturally extended to pseudovarieties, and is the object of the present paper. If $\mathbf{V}$ and $\mathbf{W}$ are pseudovarieties, their semidirect product is denoted $\mathbf{V} * \mathbf{W}$ (see [11, 19]).

One key question that can be asked about pseudovarieties is the decidability of their membership problem: given a pseudovariety of finite semigroups $\mathbf{V}$ and a finite semigroup S, does S lie in $\mathbf{V}$? In the context of the study of the semidirect product, an important question is to find out how the membership problem of $\mathbf{V} * \mathbf{W}$ relates to those of $\mathbf{V}$ and $\mathbf{W}$.

Pseudovarieties can be conveniently described by pseudoidentities. Pseudoidentities can be viewed as sequences of identities which must be ultimately satisfied. They can also be defined, in a richer way, as formal equalities between elements of relatively free profinite semigroups (or semigroups of implicit operations), which are compact completions of free semigroups, and are defined in Section 1 below.

In another paper, the authors discuss techniques to produce pseudoidentities for $\mathbf{V} * \mathbf{W}$ based on the knowledge of the pseudoidentities of $\mathbf{W}$ and of a certain pseudovariety of finite categories associated to $\mathbf{V}$ [8]. In that paper and in other works, it becomes apparent that, while the knowledge of pseudoidentities defining a pseudovariety $\mathbf{V}$ may help solving its membership problem, it is not sufficient in general to solve the analogous problem for pseudovarieties built from $\mathbf{V}$ (e.g. using semidirect product or Malcev product). Additional information on $\mathbf{V}$ lies in the structure of the relatively free profinite semigroups over $\mathbf{V}$.

The question addressed in this paper is therefore that of the structure of the free profinite semigroups and monoids over semidirect products. We give a general description of these semigroups in Section 2. A similar result was already known for varieties of semigroups, and for pseudovarieties admitting finite free objects on each finite alphabet (Almeida [3]). In order to obtain our result, we need to extend the usual consideration of finitely generated relatively free profinite semigroups and monoids, and to consider profinite sets of generators.

In the case of semidirect products of the form $\mathbf{J}_1 * \mathbf{V}$ ($\mathbf{J}_1$ is the pseudovariety of finite semilattice monoids), our description can be interpreted in a nice geometrical way, involving certain closed subgraphs of the Cayley graphs of the free profinite monoids over $\mathbf{V}$ (Section 3). These subgraphs are profinite analogues of directed paths in a discrete graph. Analogous results can also be given for two-sided semidirect products (Section 5).

The first application of our results given here, more like an exercise, is a complete study of the pseudovariety $\mathbf{J}_1 * \mathbf{Nil}$ ($\mathbf{Nil}$ is the pseudovariety of finite nilpotent semigroups). By complete study, we mean that we give pseudoidentities defining $\mathbf{J}_1 * \mathbf{Nil}$, but we also give a description of the corresponding free profinite semigroups, solve the word problem for them, and use this solution to give a description of the variety of rational languages associated with $\mathbf{J}_1 * \mathbf{Nil}$ (Section 4).

Our other application (Section 6) is more theoretical and deals with the pseudovariety $\mathbf{J}_1 * \mathbf{G}$, where $\mathbf{G}$ is the pseudovariety of all finite groups. It is known that $\mathbf{J}_1 * \mathbf{G}$ is generated by the finite inverse monoids (Margolis and Pin [20]). We construct the free profinite inverse monoid on an alphabet A, $\widehat{FIM}(A)$, and we show how the free profinite monoid over $\mathbf{J}_1 * \mathbf{G}$ on A can be identified with a closed submonoid of $\widehat{FIM}(A)$. Both $\widehat{FIM}(A)$ and the free profinite monoid over $\mathbf{J}_1 * \mathbf{G}$ are described in terms of the geometry of the Cayley graph of the free profinite group.

1 Preliminaries

A pseudovariety of monoids (resp. semigroups) is a class of finite monoids (resp. semigroups) which is closed under taking submonoids (resp. subsemigroups), quotients and finite direct products. For elementary notions and results on semigroups, monoids and pseudovarieties not proved here, the reader is referred to the books of Eilenberg [11], Lallement [15], Pin [19] or Almeida [4].

Whenever convenient, we identify isomorphic monoids or semigroups. With this convention, there are only countably many finite monoids and semigroups. In particular, for each pseudovariety $\mathbf{V}$, there exists a sequence of finitely generated pseudovarieties $(\mathbf{V}_k)_{k\geq 0}$ such that, for all $k \geq 0$, $\mathbf{V}_k \subseteq \mathbf{V}_{k+1}$ and $\mathbf{V} = \bigcup_{k\geq 0} \mathbf{V}_k$. Let us indeed enumerate the elements of $\mathbf{V}$ as $\mathbf{V} = \{S_0, S_1, \ldots, S_k, \ldots\}$. It suffices to let $\mathbf{V}_k$ be the pseudovariety generated by $\{S_0, S_1, \ldots, S_k\}$ for each $k \geq 0$.

Definitions and results in this paper will usually be given for pseudovarieties of monoids. Up to the obvious changes, they will hold also for pseudovarieties of semigroups. We will freely navigate from the realm of monoids to that of semigroups. We will only indicate briefly the few differences in approach when we get to them.

1.1 Projective limits

Projective limits can be defined for any type of algebras (e.g. semigroups, monoids, groups, etc.) or partial algebras (e.g. graphs). In this section we will be interested only in semigroups and monoids, but graphs will play an important role in section 3. For this reason, this subsection on the basic properties of projective limits, is written in terms of algebras in general.

Recall that a partially ordered set (poset) I is *directed* if any two elements of I admit a common upper bound. A *directed system of algebras* $(S_i)_{i\in I}$ is a family of algebras indexed by a directed poset I such that, for each pair (i,j) of elements of I verifying $i \geq j$, there exists a homomorphism $\varphi_{i,j}\colon S_i \to S_j$, and such that, for all $i \geq j \geq k$ in I, we have $\varphi_{i,i} = \mathrm{id}_{S_i}$ and $\varphi_{j,k} \circ \varphi_{i,j} = \varphi_{i,k}$.

By definition, the *projective limit* of a directed system of algebras $(S_i)_{i\in I}$ is the following subalgebra of the direct product $\prod_{i\in I} S_i$:

$$\varprojlim (S_i)_{i\in I} = \{(x_i)_{i\in I} \in \prod_{i\in I} S_i : \forall i \geq j,\ \varphi_{i,j} x_i = x_j\}.$$

Example. Let $I = \mathbb{N}$ be the set of natural numbers. For each $n \geq 0$, let $S_n = \langle a; a^n = a^{n+1}\rangle = \{1, a, \dots, a^n\}$ with

$$a^i a^j = \begin{cases} a^{i+j} & \text{if } i+j \leq n \\ a^n & \text{if } i+j > n. \end{cases}$$

For each $m \geq n \geq 0$, let $\varphi_{m,n}\colon S_m \to S_n$ be the morphism given by $\varphi_{m,n} a^i = a^{\min(i,n)}$. The morphisms $\varphi_{m,n}$ make $(S_n)_n$ into a directed system of monoids, and $\varprojlim (S_n)_n$ is isomorphic to the free monoid generated by a with a zero added, that is, $\varprojlim (S_n)_n$ is the set $\{1, a, a^2, \dots, a^n, \dots\} \cup \{0\}$.

The *canonical morphisms* $\pi_i\colon \varprojlim (S_i)_{i\in I} \to S_i$ are defined to be the restrictions of the coordinate projections to $\varprojlim (S_i)_i$. Note that these morphisms are not necessarily onto.

Example. Let $I = \mathbb{N}$. For each $n, k \geq 0$, let $S_n = \mathbb{Z}$, and let $\varphi_{n+k,n}\colon \mathbb{Z} \to \mathbb{Z}$ be given by $\varphi_{n+k,n}x = 2^k x$. These data define a projective system and $\varprojlim(S_n)_n = \{0\}$.

A *directed system of morphisms* defined on an algebra S is a family of morphisms $(\rho_i\colon S \to S_i)_{i \in I}$ indexed by a directed poset I such that:

- there exist morphisms $\varphi_{i,j}\colon S_i \to S_j$ for all pairs (i, j) of elements of I verifying $i \geq j$, and these morphisms make $(S_i)_i$ a directed system of algebras,
- for all $i \geq j$ in I, we have $\rho_j = \varphi_{i,j} \circ \rho_i$.

It is not difficult to verify the following universal property of projective limits.

Proposition 1.1 *Let $(\rho_i\colon S \to S_i)_{i \in I}$ be a directed system of morphisms defined on an algebra S. Then there exists a unique morphism $\rho\colon S \to \varprojlim(S_i)_i$ satisfying $\rho_i = \pi_i \circ \rho$ for all i. Moreover ρ is defined by $\rho s = (\rho_i s)_i$.*

This property defines $\varprojlim(S_i)_i$ up to an isomorphism.

We say that the morphism ρ described above is *induced* by the directed system $(\rho_i)_i$.

Let us now assume that the algebras S_i are topological algebras, and that the morphisms $\varphi_{i,j}$ are continuous. The direct product $\prod_i S_i$ is equipped with the product topology. Then one verifies that $\varprojlim(S_i)_i$ is a closed subalgebra of $\prod_i S_i$. Thus the canonical morphisms π_i are continuous. In particular, if the S_i are compact, then so is their direct product (by the classical Tikhonov's theorem), and hence so is their projective limit.

The case where the S_i are finite algebras is particularly important for our purpose. We consider each finite algebra to be equipped with the discrete topology: each of them is compact, and any morphism between them is continuous. In this situation the topological algebra $\varprojlim(S_i)_i$ is compact and totally disconnected.

We say that an algebra is *profinite* if it is the projective limit of a directed system of finite algebras. In fact, for the most usual kinds of algebras (like groups, rings, lattices, semigroups, monoids, sets; in general for those in which syntactic congruences are characterized by a finite number of terms), one can verify that an algebra is profinite if and only if it is compact and totally disconnected (cf. [2]). We do not know, for instance, if this property holds for groupoids.

More generally, for each class $\mathcal{C}$ of algebras, we say that an algebra is *pro-$\mathcal{C}$* if it is a projective limit of algebras in $\mathcal{C}$.

Let us note the following important properties of projective limits (for Lemma 1.2 and Proposition 1.5, see Fried and Jarden [12, Chapter 1]).

Lemma 1.2 *Let $(\rho_i : S \to S_i)_{i\in I}$ be a directed system of morphisms defined on S. Let $\rho : S \to \varprojlim (S_i)_i$ be the induced morphism.*

a) *ρ is one-to-one if and only if, for each $s \in S$, $\{s\} = \bigcap_{i\in I} \rho_i^{-1}(\rho_i s)$.*

b) *If S is compact and if, for each $i \in I$, ρ_i is onto, then ρ is onto.*

Proof. (a) is immediate by definition of projective limits.

In order to prove (b), for each element $(s_i)_i$ of $\varprojlim (S_i)_i$, we consider the family of closed sets $(\rho_i^{-1}(s_i))_i$. Since the system of morphisms $(\rho_i)_i$ is directed, each finite intersection of $\rho_i^{-1}(s_i)$ is non-empty. But S is compact, so $\bigcap_{i\in I} \rho_i^{-1}(s_i)$ is non-empty, which concludes the proof.□

The following lemma is an immediate but useful observation. It shows that, when we are given a pro-$\mathcal{C}$ algebra S where $\mathcal{C}$ is a class closed under taking closed subalgebras, we may always assume that S is the projective limit of a directed system $(S_i)_{i\in I}$ of algebras in $\mathcal{C}$ in such a way that the canonical morphisms $\pi_i : S \to S_i$ are onto.

Lemma 1.3 *Let $S = \varprojlim (S_i)_i$ and let $\pi_i : S \to S_i$ be the canonical morphisms. For each i, $\pi_i S$ is a subalgebra of S_i and $S = \varprojlim (\pi_i S)_i$.*□

The next lemma shows that the finite continuous homomorphic images of a projective limit $\varprojlim (S_i)_i$ of compact algebras are exactly the images of the S_i.

Lemma 1.4 *Let $S = \varprojlim (S_i)_{i\in I}$ be a projective limit of compact algebras and let $\pi_i : S \to S_i$ ($i \in I$) be the canonical morphisms, which we assume to be onto. Let T be a finite algebra and let $\varphi : S \to T$ be a continuous morphism. Then there exists $i \in I$ and a continuous morphism $\varphi_i : S_i \to T$ such that $\varphi = \varphi_i \circ \pi_i$.*

Proof. The $\pi_i^{-1}(V)$ ($i \in I$, V open subset of S_i) constitute a basis of open sets for the topology of S. Since φ is continuous, for each $s \in S$, there exists an element i_s of I and an open neighborhood V_s of $\pi_{i_s}s$ in S_{i_s} such that $s \in \pi_{i_s}^{-1}(V_s)$ and $\pi_{i_s}^{-1}(V_s) \subseteq \varphi^{-1}(\varphi s)$. Now S is compact and $S = \bigcup_s \pi_{i_s}^{-1}(V_s)$. So there exists an integer n, elements $s_1, \ldots, s_n \in S$, elements $i_1, \ldots, i_n \in I$ and open sets $V_1, \ldots, V_n$ respectively of $S_{i_1}, \ldots, S_{i_n}$ such that

$$S = \bigcup_{h=1}^{n} \pi_{i_h}^{-1}(V_h) \qquad \text{with } s_h \in \pi_{i_h}^{-1}(V_h) \text{ and } \pi_{i_h}^{-1}(V_h) \subseteq \varphi^{-1}(\varphi s_h) \text{ for all } h.$$

By hypothesis, I is a directed poset, so we may consider an element i of I which is greater than or equal to $i_1, \ldots, i_n$. For each $s \in S$, there exists $1 \leq h \leq n$ such that $s \in \pi_{i_h}^{-1}(V_h)$. Therefore $\pi_i^{-1}(\pi_i s) \subseteq \pi_{i_h}^{-1}(\pi_{i_h} s) \subseteq \pi_{i_h}^{-1}(V_h) \subseteq \varphi^{-1}(\varphi s_h)$. In particular $\varphi s_h = \varphi s$, so

$$\text{for each } s \in S, \quad \pi_i^{-1}(\pi_i s) \subseteq \varphi^{-1}(\varphi s).$$

Thus, letting $\varphi_i(\pi_i s) = \varphi s$ defines a mapping φ_i from S_i into T which is trivially a morphism and satisfies $\varphi = \varphi_i \circ \pi_i$.

There remains to see that φ_i is continuous. For this it suffices to verify that $\varphi_i^{-1}(t)$ is closed in S_i for each $t \in T$. By hypothesis, $\varphi^{-1}(t)$ is closed in S, and hence is compact. Since π_i is continuous, it follows that $\varphi_i^{-1}(t) = \pi_i(\varphi^{-1}(t))$ is compact, which concludes the proof.□

We conclude this section with the proof of the following properties of pro-$\mathcal{C}$ algebras.

Proposition 1.5 *Let $\mathcal{C}$ be a class of algebras closed under taking subalgebras, homomorphic images and finitary direct product. Pro-$\mathcal{C}$ algebras enjoy the following closure properties.*

1. *A closed sub-algebra of a pro-$\mathcal{C}$ algebra is pro-$\mathcal{C}$.*
2. *If $\mathcal{C}$ consists of compact algebras, then a direct product of pro-$\mathcal{C}$ algebras is pro-$\mathcal{C}$.*
3. *If $\mathcal{C}$ consists of compact algebras, then a projective limit of pro-$\mathcal{C}$ algebras is pro-$\mathcal{C}$.*

Proof. Let T be a closed subalgebra of $S = \varprojlim(S_i)_i$, where the S_i are algebras in $\mathcal{C}$. For each i, let $T_i = \pi_i T$, where π_i is the canonical morphism from S into S_i. It is easy to verify that the system of algebras $(T_i)_{i\in I}$ is directed and that $T \subseteq \varprojlim(T_i)_i$. By definition of the topology on S, T is dense in $\varprojlim(T_i)_i$, so $T = \varprojlim(T_i)_i$, thus proving statement 1.

Let now $S = \prod_{i\in I}(S_i)$, where the S_i are pro-$\mathcal{C}$ algebras. For each i, $S_i = \varprojlim(S_{i,k})_{k\in K_i}$ where the $S_{i,k}$ are compact algebras. For each finite subset J of I and for each element $(k_j)_{j\in J} \in \prod_{j\in J} K_j$, the definition of the S_i as projective limits yields a natural morphism from S into the compact algebra $\prod_{j\in J} S_{i,k_j}$. The set of indices of such products, namely the set of all pairs $(J,(k_j)_{j\in J})$ where J is a finite subset of I and $(k_j)_j \in \prod_{j\in J} K_j$ can be equipped with a structure of directed poset as follows: let $m = (J,(k_j)_{j\in J})$ and $m' = (H,(k'_h)_{h\in H})$. We write $m \geq m'$ if $J \supseteq H$ and $k_h \geq k'_h$ for all $h \in H$. We may now use Lemma 1.2 to conclude that S is the projective limit of these algebras of $\mathcal{C}$.

Statement 3 is an immediate consequence of statements 1 and 2, since $\varprojlim(S_i)$ is a closed sub-algebra of $\prod_i S_i$. $\square$

1.2 Relatively free profinite monoids and semigroups

Let A be a profinite set and let $\mathbf{V}$ be a pseudovariety. An *A-generated profinite* (resp. *pro-$\mathbf{V}$*) *monoid* is a continuous mapping $\mu\colon A \to M$ into a profinite (resp. pro-$\mathbf{V}$) monoid M such that the submonoid generated by μA is dense in M. By extension, we will say that the profinite (resp. pro-$\mathbf{V}$) monoid M is A-generated if there exists a mapping $\mu\colon A \to M$ as above.

A-generated profinite monoids are the objects of a category whose morphisms are defined as follows: Let $\mu\colon A \to M$ and $\nu\colon A \to N$ be A-generated profinite monoids. A morphism of profinite monoids (that is, a monoid morphism which is continuous) $\varphi\colon M \to N$ is a morphism of A-generated profinite monoids from μ to ν if $\nu = \varphi \circ \mu$. In particular, φ must be onto.

Let $\mathbf{V}$ be a pseudovariety of finite monoids and let A be a profinite set. The class $\mathbf{V}_A$ of all A-generated profinite monoids in $\mathbf{V}$, together with all the morphisms of A-generated profinite monoids between them, constitutes a category and a directed system. By definition, we let $\overline{\Omega}_A\mathbf{V}$ be its projective limit.

In the rest of Section 1, we review some important properties of the $\overline{\Omega}_A\mathbf{V}$. For proofs omitted here and for additional information, see [1, 4, 6].

For each A-generated profinite monoid $\mu\colon A \to M$ with $M \in \mathbf{V}$, we may consider the natural extension of μ to a morphism $\mu: A^* \to M$. Since M is finite, we have $A^*\mu = M$. By the universal property of projective limits, these morphisms μ induce a morphism $\iota: A^* \to \overline{\Omega}_A\mathbf{V}$, which we call the *natural morphism* from A^* into $\overline{\Omega}_A\mathbf{V}$. We let ι_A be the restriction of ι to A. Note that ι need not be injective. However, if $\mathbf{V}$ is not trivial, then ι_A is one-to-one.

Proposition 1.6 *Let A be a profinite set and let $\mathbf{V}$ be a pseudovariety of monoids. Then $\overline{\Omega}_A\mathbf{V}$ (together with the mapping ι_A) is the free pro-$\mathbf{V}$ monoid over the profinite set A.*

Furthermore, an A-generated finite monoid is in $\mathbf{V}$ if and only if it is a continuous homomorphic image of $\overline{\Omega}_A\mathbf{V}$.

Proof. We first verify that ιA^* is dense in $\overline{\Omega}_A\mathbf{V}$. For each A-generated profinite monoid in $\mathbf{V}$, that is, for each onto morphism $\mu: A^* \to M$ with $M \in \mathbf{V}$, let $\pi_\mu: \overline{\Omega}_A\mathbf{V} \to M$ be the corresponding canonical morphism. Then $\pi_\mu \circ \iota = \mu$. By definition of the topology on $\overline{\Omega}_A\mathbf{V}$, a basis of neighborhoods of an element $x \in \overline{\Omega}_A\mathbf{V}$ consists of all $\pi_\mu^{-1}(\pi_\mu x)$. But $\mu A^* = M$ for each μ, so if $\pi_\mu x = \mu u$ for some $u \in A^*$, then $\iota u \in \pi_\mu^{-1}(\pi_\mu x)$. Thus ιA^* is dense in $\overline{\Omega}_A\mathbf{V}$.

Let now S be an A-generated pro-$\mathbf{V}$ monoid: there exists a continuous mapping $\sigma: A \to S$ such that the submonoid generated by σA is dense in S, and there exists a directed system $(S_i)_i$ of monoids in $\mathbf{V}$ such that $S = \varprojlim (S_i)_i$. Let $\pi_i: S \to S_i$ $(i \in I)$ be the corresponding canonical morphisms, which, by Lemma 1.3, can be assumed to be onto. By definition of projective limits, each element x of S is of the form $x = (\pi_i x)_i$. Let $\sigma_i = \pi_i \circ \sigma: A \to S_i$. Then the σ_i are A-generated profinite monoids in $\mathbf{V}$. Therefore, considering the canonical morphisms associated with the projective limit defining $\overline{\Omega}_A\mathbf{V}$, we find that there exist continuous morphisms $\varphi_i: \overline{\Omega}_A\mathbf{V} \to S_i$ such that $\varphi_i \circ \iota_A = \sigma_i = \pi_i \circ \sigma$. We use the universal property of projective limits once more to see that these morphisms induce an onto continuous morphism $\varphi: \overline{\Omega}_A\mathbf{V} \to S$ such that $\pi_i \circ \varphi = \varphi_i$. So $\pi_i \circ \varphi \circ \iota_A = \pi_i \circ \sigma$ for each i, and hence $\varphi \circ \iota_A = \sigma$.

The proof of the last statement of the proposition is an immediate consequence of the definition of $\overline{\Omega}_A\mathbf{V}$ and of Lemma 1.4.□

The following corollary of Proposition 1.6 will turn out to be rather useful.

Corollary 1.7 *Let $\mathbf{V}$ and $\mathbf{W}$ be pseudovarieties of monoids such that $\mathbf{W} \subseteq \mathbf{V}$, let A and B be profinite sets, let ι be the natural map from A^* into $\overline{\Omega}_A\mathbf{V}$, and let ι_A be the restriction of ι to A. If $\varphi: A \to \overline{\Omega}_B\mathbf{W}$ is a continuous mapping, then φ induces a unique continuous morphism $\bar{\varphi}: \overline{\Omega}_A\mathbf{V} \to \overline{\Omega}_B\mathbf{W}$ such that $\bar{\varphi} \circ \iota_A = \varphi$.*

In particular, the identity function of A induces an onto continuous morphism from $\overline{\Omega}_A\mathbf{V}$ onto $\overline{\Omega}_A\mathbf{W}$.$\square$

The situation is particularly simple when A is finite and $\mathbf{V}$ admits a finite free object over A. This is the case for instance when $\mathbf{V}$ is generated by a finite number of monoids. Then, $\overline{\Omega}_A\mathbf{V}$ is the free object of $\mathbf{V}$ over A.

Some information regarding the passage from this special case to the general one is given in the following two lemmas. Recall that each pseudovariety $\mathbf{V}$ is the directed union of a sequence of pseudovarieties admitting free objects over each finite alphabet.

Proposition 1.8 *Let A be a profinite set and let $(\mathbf{V}_k)_{k\geq 0}$ be a sequence of pseudovarieties such that $\mathbf{V}_k \subseteq \mathbf{V}_{k+1}$ for all $k \geq 0$. Let $\mathbf{V} = \bigcup_k \mathbf{V}_k$. Then $\overline{\Omega}_A\mathbf{V} = \varprojlim(\overline{\Omega}_A\mathbf{V}_k)_k$.*

Proof. According to Corollary 1.7, we may consider the continuous morphisms π_k from $\overline{\Omega}_A\mathbf{V}$ onto $\overline{\Omega}_A\mathbf{V}_k$ induced by the identity mapping of A^*. These morphisms are easily seen to constitute a directed system.

By Lemma 1.2, it suffices to show that, for all $x, y \in \overline{\Omega}_A\mathbf{V}$, if $\pi_k x = \pi_k y$ for all k, then $x = y$. But, for each A-generated monoid M in $\mathbf{V}$, there exists $k \geq 0$ such that $M \in \mathbf{V}_k$. So the canonical morphism π_M from $\overline{\Omega}_A\mathbf{V}$ onto M factors through π_k, and hence $\pi_k x = \pi_k y$ implies $\pi_M x = \pi_M y$. Since this holds for each M in $\mathbf{V}$, and since $\overline{\Omega}_A\mathbf{V}$ is the projective limit of the A-generated elements of $\mathbf{V}$, this implies $x = y$.$\square$

Proposition 1.9 *Let $\mathbf{V}$ be a pseudovariety of monoids and let A be a profinite set, $A = \varprojlim(A_i)_{i\in I}$, where the sets A_i are profinite. For each i, let $\pi_i: A \to A_i$ be the canonical projection, which we assume to be onto. Then $\overline{\Omega}_A\mathbf{V} = \varprojlim(\overline{\Omega}_{A_i}\mathbf{V})_i$.*

Proof. By Corollary 1.7, each π_i induces a morphism of profinite monoids $\bar{\pi}_i$ from $\overline{\Omega}_A\mathbf{V}$ into $\overline{\Omega}_{A_i}\mathbf{V}$. Note that $\bar{\pi}_i$ is onto since $A_i = \pi_i A$ generates $\overline{\Omega}_{A_i}\mathbf{V}$ as a profinite monoid.

Let now $x \in \overline{\Omega}_A\mathbf{V}$ and let $y \in \bigcap_i \bar{\pi}_i^{-1}(\bar{\pi}_i x)$. Let M be a monoid in $\mathbf{V}$, let $\varphi: A \to M$ be a continuous mapping, and let $\bar{\varphi}: \overline{\Omega}_A\mathbf{V} \to M$ be its canonical extension. By Lemma 1.4, there exists $i \in I$ and there exists a continuous mapping $\varphi_i: A_i \to M$ such that $\varphi = \varphi_i \circ \pi_i$. Let $\bar\varphi_i: \overline{\Omega}_{A_i} \to M$ be the canonical extension of φ_i. We note that $\bar\varphi$ and $\bar\varphi_i \circ \bar\pi_i$ coincide on the elements of A, so $\bar\varphi = \bar\varphi_i \circ \bar\pi_i$, and hence $\bar\varphi x = \bar\varphi y$. Since this holds for each choice of φ, we have $x = y$ and hence $\overline{\Omega}_A\mathbf{V} = \varprojlim(\overline{\Omega}_{A_i}\mathbf{V})_i$ by Lemma 1.2. □

Example. Let $\mathbf{J}_1$ be the pseudovariety of all finite commutative and idempotent monoids. It is well-known that $\mathbf{J}_1$ is generated by the 2-element monoid $\{0, 1\}$. Let also $\mathbf{Nil}$ and $\mathbf{K}$ be the pseudovarieties of finite semigroups defined by

$$S \in \mathbf{Nil} \iff \exists k,\ \forall s_1, \ldots, s_k, t \in S,\ s_1 \cdots s_k t = s_1 \cdots s_k = t s_1 \cdots s_k$$
$$S \in \mathbf{K} \iff \exists k,\ \forall s_1, \ldots, s_k, t \in S,\ s_1 \cdots s_k t = s_1 \cdots s_k.$$

If A is a finite set, then the free object of $\mathbf{J}_1$ over A is the set of all subsets of A, $\mathcal{P}(A)$, under union. It is also known that $\overline{\Omega}_A\mathbf{Nil} = A^+ \cup \{0\}$ and $\overline{\Omega}_A\mathbf{K} = A^+ \cup A^{\mathbb{N}}$, the set of all finite and infinite words over the alphabet A (see Almeida [4]).

Let now $A = \varprojlim(A_i)_i$ be a projective limit of finite sets. For each $i \geq j$ in I, let $\pi_{i,j}$ be the mapping from A_i to A_j. By proposition 1.9, $\overline{\Omega}_A\mathbf{Nil} = \varprojlim(\overline{\Omega}_{A_i}\mathbf{Nil})_i = \varprojlim(A_i^+ \cup \{0\})_i$, where the canonical morphism from $A_i^+ \cup \{0\}$ onto $A_j^+ \cup \{0\}$ (when $i \geq j$) is the natural extension of $\pi_{i,j}$. Since the restriction of this extension to A_i^+ is length-preserving, it is now easy to see that $\overline{\Omega}_A\mathbf{Nil} = A^+ \cup \{0\}$. One verifies similarly that $\overline{\Omega}_A\mathbf{K} = A^+ \cup A^{\mathbb{N}}$.

We leave it to the reader to verify that $\overline{\Omega}_A\mathbf{J}_1$ is equal to $\bar{\mathcal{P}}(A)$, the set of all closed subsets of A, under union.

In the sequel, we will use the following notion of Cayley graph of a monoid with respect to a certain set: If M is a monoid, A is a set and $\mu: A \to M$ is a mapping, the *Cayley graph of M with respect to A* is the directed graph with vertex set M and edge set $M \times A$, where (m, a) is an edge from m to $m(\mu a)$. If $\mathbf{V}$ is a pseudovariety of monoids, the Cayley graph of $\overline{\Omega}_A\mathbf{V}$ with respect to A is written $\Gamma_A\mathbf{V}$ (where the mapping from A into $\overline{\Omega}_A\mathbf{V}$ is the mapping ι_A described in the proof of Proposition 1.6). It is not difficult to see that $\Gamma_A\mathbf{V}$ is the projective limit of the Cayley graphs of the A-generated elements of $\mathbf{V}$ with respect to A, so that $\Gamma_A\mathbf{V}$ is a profinite graph.

If $\mathbf{V}$ is a pseudovariety of semigroups, we write again $\Gamma_A\mathbf{V}$ for the Cayley graph of the monoid $(\overline{\Omega}_A\mathbf{V})^1$ (obtained by adding an identity to the free pro-$\mathbf{V}$ semigroup over A) with respect to A.

1.3 Implicit operations and pro-V monoids

The theory of implicit operations constitutes another way to approach relatively free profinite monoids. Informally speaking, an implicit operation on $\mathbf{V}$ is any operation that makes sense on each element of $\mathbf{V}$ and is preserved under morphisms between elements of $\mathbf{V}$. More precisely, if A is a finite set, an *A-ary implicit operation π on* $\mathbf{V}$ is a family $\pi = (\pi_S)$ indexed by the elements S of $\mathbf{V}$, such that π_S is a mapping from S^A into S, and such that, for each morphism $\varphi\colon S \to T$ between elements of $\mathbf{V}$, the following diagram is commutative,

$$\begin{tikzcd} S^A \arrow[r, "\pi_S"] \arrow[d, "\varphi^A"] & S \arrow[d, "\varphi"] \\ T^A \arrow[r, "\pi_T"] & T \end{tikzcd}$$

that is, such that $\varphi\pi_S((s_a)_a) = \pi_T((\varphi s_a)_a)$ for all $(s_a)_{a\in A}$ in S^A.

If A is an n-letter alphabet, we sometimes call A-ary operations n-ary. An important class of A-ary implicit operations is given by the words on the alphabet A. For instance, the word $u = x_1x_2^2x_3x_1$ defines a 3-ary operation u_S on any monoid S, by letting, for all s_1, s_2 and s_3 in S, $u_S(s_1, s_2, s_3) = s_1s_2^2s_3s_1$. It is not difficult to verify that, for any pseudovariety $\mathbf{V}$, $u = (u_S)_{S\in\mathbf{V}}$ is an implicit operation. Such implicit operations, which are defined by words, are called *explicit operations*, and their set is denoted $\Omega_A\mathbf{V}$.

In general, there are implicit operations which are not explicit. The easiest to describe is the ω-power. Let S be a finite monoid. For each $s \in S$, let s^ω be the unique idempotent in the subsemigroup of S generated by s. It is not difficult to verify that, for any pseudovariety $\mathbf{V}$, $(\ )^\omega$ is a unary implicit operation on $\mathbf{V}$.

For each finite set A, pointwise multiplication makes the set of implicit operations a monoid: If π and π' are A-ary implicit operations on $\mathbf{V}$, the implicit operation $\pi\pi'$ is defined, for each $S \in \mathbf{V}$ and each $(s_a)_{a\in A} \in S^A$, by $(\pi\pi')_S((s_a)_a) = \pi_S((s_a)_a)\ \pi'_S((s_a)_a)$. We denote this monoid by $I_A(\mathbf{V})$. Its unit element is the explicit operation induced by the empty word.

Proposition 1.10 *Let $\mathbf{V}$ be a pseudovariety and let A be a finite set. Then the monoids $I_A(\mathbf{V})$ and $\overline{\Omega}_A\mathbf{V}$ are naturally isomorphic. In this isomorphism,* *the A-ary explicit operations on $\mathbf{V}$ are mapped onto ιA^*, where ι is the natural morphism from A^* into $\overline{\Omega}_A\mathbf{V}$.*

Thus the study of implicit operations on a pseudovariety of monoids $\mathbf{V}$ is equivalent to the study of the relatively free profinite monoids $\overline{\Omega}_A\mathbf{V}$. This point of view is the one which is adopted, for instance, in [4]. We will mostly retain from it the use of the distinction between explicit and non-explicit operations.

Example. In view of Proposition 1.10 and of the results of Section 1.2, each implicit operation is a limit of explicit operations. For instance, the ω-power $(\ )^\omega$ is the limit of the sequence of explicit operations $(a^{n!})_n$. Indeed, for each monoid S and each element s of S, the idempotent s^ω is equal to $s^{n!}$ for all $n \geq |S|$.

1.4 Reiterman's theorem

Let $\mathbf{V}$ be a pseudovariety and let A be a finite set. A *pseudoidentity* on $\mathbf{V}$ on the set A (or, in $|A|$ variables) is a pair of elements (u, v) of the free pro-$\mathbf{V}$ monoid $\overline{\Omega}_A\mathbf{V}$ (we also say *the pseudoidentity* $u = v$). It is said to be *nontrivial* if the elements u and v of $\overline{\Omega}_A\mathbf{V}$ are distinct. Let now S be a pro-$\mathbf{V}$ monoid. Recall that each mapping $\mu\colon A \to S$ induces a continuous morphism $\mu\colon \overline{\Omega}_A\mathbf{V} \to S$. We say that S *satisfies the pseudoidentity* (u, v), and we write $S \models u = v$ if, for each map $\mu\colon A \to S$, we have $\mu u = \mu v$.

Note that this is in analogy with the notion of identity. Since each element of $\overline{\Omega}_A\mathbf{V}$ is the limit of a sequence of words, or rather of a sequence of explicit operations, we may view any pseudoidentity as the limit of a sequence of identities.

Example. If $S = \varprojlim(S_i)_{i\in I}$, then S satisfies all pseudoidentities which are satisfied by the S_i. In particular, the free pro-$\mathbf{V}$ monoid $\overline{\Omega}_A\mathbf{V}$ satisfies all pseudoidentities which are satisfied by the A-generated elements of $\mathbf{V}$.

Let now Σ be a set of pseudoidentities on $\mathbf{V}$ (not necessarily involving a bounded number of variables). We let $[\![\Sigma]\!]_{\mathbf{V}}$ be the class of all elements of $\mathbf{V}$ satisfying all pseudoidentities in Σ. For instance, if $\mathbf{S}$, $\mathbf{M}$ and $\mathbf{G}$ denote respectively the pseudovarieties of all semigroups, monoids and groups, and if **LI** is the pseudovariety of locally trivial semigroups, then

$$\begin{aligned}
\mathbf{J}_1 &= [\![xy = yx, x^2 = x]\!]_{\mathbf{M}},\\
\mathbf{LI} &= [\![x^\omega y x^\omega = x^\omega]\!]_{\mathbf{S}},\\
\mathbf{K} &= [\![x^\omega y = x^\omega]\!]_{\mathbf{S}},\\
\mathbf{Nil} &= [\![x^\omega y = x^\omega, yx^\omega = x^\omega]\!]_{\mathbf{S}} = [\![yx^\omega = x^\omega]\!]_{\mathbf{K}},\\
\mathbf{G} &= [\![x^\omega y = y, yx^\omega = y]\!]_{\mathbf{M}}.
\end{aligned}$$

Reiterman's theorem below states that every pseudovariety can be defined in this fashion [22].

Theorem 1.11 *Let* $\mathbf{V}$ *be a pseudovariety and let* $\mathbf{W} \subseteq \mathbf{V}$ *be a subclass of* $\mathbf{V}$. *Then* $\mathbf{W}$ *is a sub-pseudovariety if and only if there exists a class* Σ *of pseudoidentities on* $\mathbf{V}$ *such that* $\mathbf{W} = [\![\Sigma]\!]_{\mathbf{V}}$.

1.5 Languages recognized by a pseudovariety V

There is a very interesting connection between the topology of $\overline{\Omega}_A\mathbf{V}$ and the variety of languages of A^* which are recognized by the elements of $\mathbf{V}$. We refer the reader to [11, 19] for background on varieties of recognizable languages.

Let $\mathbf{V}$ be a pseudovariety such that $\Omega_A\mathbf{V} = A^*$, that is, such that $\mathbf{V}$ satisfies no non-trivial identity. It is known (see Almeida [2, 4]) that a language $L \subseteq A^*$ is recognized by a monoid in $\mathbf{V}$ if and only if the topological closure $\bar{L}$ of L in $\overline{\Omega}_A\mathbf{V}$ is clopen and $L = \bar{L} \cap A^*$. Furthermore, the collection of all $\bar{L}$ ($L \subseteq A^*$ recognized by $\mathbf{V}$) is a basis of clopen sets for the topology of $\overline{\Omega}_A\mathbf{V}$.

It is often an important question to characterize the variety of recognizable languages associated with a given peudovariety $\mathbf{V}$. On each finite alphabet, these languages form a Boolean algebra. In this context, we look for simple generating sets of this Boolean algebra, and the following observation (due to Almeida [4]) turns out to be very useful.

We say that a family $\mathcal{X}$ of subsets of $\overline{\Omega}_A\mathbf{V}$ *separates* the points of $\overline{\Omega}_A\mathbf{V}$ if, for each pair of distinct elements x and y in $\overline{\Omega}_A\mathbf{V}$, there exists an element X of $\mathcal{X}$ such that either $x \in X$ and $y \notin X$, or $x \notin X$ and $y \in X$.

Lemma 1.12 *Let* A *be an alphabet, let* $\mathbf{V}$ *be a pseudovariety satisfying no nontrivial identity and let* $A^*\mathcal{V}$ *be the Boolean algebra of languages over*

A recognized by $\mathbf{V}$. Let $\mathcal{L}$ be a subset of $A^*\mathcal{V}$ and let $\bar{\mathcal{L}}$ be the set of the topological closures in $\overline{\Omega}_A\mathbf{V}$ of the elements of $\mathcal{L}$.

The Boolean algebra $A^*\mathcal{V}$ is generated by $\mathcal{L}$ if and only if the points of $\overline{\Omega}_A\mathbf{V}$ are separated by $\bar{\mathcal{L}}$.

Example. Using the examples developed in section 1.2, we recover the well-known characterization of the variety of languages associated with **Nil**: For each finite alphabet A, the Boolean algebra of all **Nil**-recognizable languages of A^+ is generated by the singleton languages.

2 Implicit operations on semidirect products

Let $\mathbf{V}$ and $\mathbf{W}$ be pseudovarieties of monoids. The main result of this section is the following theorem, which describes the profinite monoids $\overline{\Omega}_A(\mathbf{V} * \mathbf{W})$.

Theorem 2.1 *Let $\mathbf{V}$ and $\mathbf{W}$ be pseudovarieties of monoids, and let A be a profinite set. The action of $\overline{\Omega}_A\mathbf{W}$ on the set $\overline{\Omega}_A\mathbf{W} \times A$ given by $x \cdot (y,a) = (xy,a)$ (for $x, y \in \overline{\Omega}_A\mathbf{W}$ and $a \in A$) can be naturally extended to a continuous action of $\overline{\Omega}_A\mathbf{W}$ on $\overline{\Omega}_{\overline{\Omega}_A\mathbf{W}\times A}\mathbf{V}$. The resulting semidirect product $\overline{\Omega}_{\overline{\Omega}_A\mathbf{W}\times A}\mathbf{V} * \overline{\Omega}_A\mathbf{W}$ is profinite.*

Moreover, letting $\theta a = ((1,a),a)$ defines a unique continuous morphism θ from $\overline{\Omega}_A(\mathbf{V} * \mathbf{W})$ into $\overline{\Omega}_{\overline{\Omega}_A\mathbf{W}\times A}\mathbf{V} * \overline{\Omega}_A\mathbf{W}$ which is one-to-one.

Note. This result was given in [3, 4] in the case where A is finite and $\mathbf{W}$ has a finite free object over A. Our present proof holds without any restriction on the pseudovarieties $\mathbf{V}$ and $\mathbf{W}$ and on the profinite set A.

Proof of Theorem 2.1. If $\mathbf{W}$ is trivial (that is, $\mathbf{W}$ consists only of the 1-element monoid), then the result is immediate. We now assume that $\mathbf{W}$ is non trivial.

For each $x \in \overline{\Omega}_A\mathbf{W}$, let λ_x be the continuous mapping from $\overline{\Omega}_A\mathbf{W} \times A$ into itself defined by $\lambda_x(y,a) = (xy,a)$. By Corollary 1.7, λ_x induces a morphism of profinite monoids from $\overline{\Omega}_{\overline{\Omega}_A\mathbf{W}\times A}\mathbf{V}$ into itself. If $x, y \in \overline{\Omega}_A\mathbf{W}$, then $\lambda_x \circ \lambda_y$ and λ_{xy} coincide on $\overline{\Omega}_A\mathbf{W} \times A$, so they coincide on $\overline{\Omega}_{\overline{\Omega}_A\mathbf{W}\times A}\mathbf{V}$ as well. Thus the action of $\overline{\Omega}_A\mathbf{W}$ on $\overline{\Omega}_A\mathbf{W} \times A$ given by $x \cdot (y,a) = (xy,a)$ for all $x, y \in \overline{\Omega}_A\mathbf{W}$ and $a \in A$ induces a unique continuous action of $\overline{\Omega}_A\mathbf{W}$ on $\overline{\Omega}_{\overline{\Omega}_A\mathbf{W}\times A}\mathbf{V}$.

First we verify that the resulting semidirect product $\overline{\Omega}_{\overline{\Omega}_A\mathbf{W}\times A}\mathbf{V} * \overline{\Omega}_A\mathbf{W}$ is profinite. Let $(\mathbf{V}_k)_{k\geq 0}$ and $(\mathbf{W}_\ell)_{\ell\geq 0}$ be increasing sequences of finitely generated pseudovarieties such that $\mathbf{V} = \bigcup_k \mathbf{V}_k$ and $\mathbf{W} = \bigcup_\ell \mathbf{W}_\ell$. Let also $(A_m)_{m\geq 0}$ be a sequence of finite sets forming a directed system such that $A = \varprojlim(A_m)_m$. It follows from Propositions 1.8 and 1.9 that

$$\overline{\Omega}_A\mathbf{W} = \varprojlim(\overline{\Omega}_A\mathbf{W}_k)_k \quad \text{and} \quad \overline{\Omega}_A\mathbf{W}_k = \varprojlim(\overline{\Omega}_{A_\ell}\mathbf{W}_k)_\ell.$$

As in the proof of Proposition 1.5, this implies that $\overline{\Omega}_A\mathbf{W} = \varprojlim(\overline{\Omega}_{A_\ell}\mathbf{W}_k)_{k,\ell}$. For the same reasons,

$$\begin{aligned}\overline{\Omega}_{\overline{\Omega}_A\mathbf{W}\times A}\mathbf{V} &= \varprojlim(\overline{\Omega}_{\overline{\Omega}_{A_\ell}\mathbf{W}_k\times A_\ell}\mathbf{V}_m)_{k,\ell,m} \qquad \text{and}\\ \overline{\Omega}_A\mathbf{W}\times\overline{\Omega}_{\overline{\Omega}_A\mathbf{W}\times A}\mathbf{V} &= \varprojlim(\overline{\Omega}_{A_\ell}\mathbf{W}_k\times\overline{\Omega}_{\overline{\Omega}_{A_\ell}\mathbf{W}_k\times A_\ell}\mathbf{V}_m)_{k,\ell,m}.\end{aligned}$$

Note that for all $k,\ell,m\geq 0$, $\overline{\Omega}_{A_\ell}\mathbf{W}_k$ and $\overline{\Omega}_{\overline{\Omega}_{A_\ell}\mathbf{W}_k\times A_\ell}\mathbf{V}_m$ are finite. We now consider for each k,ℓ,m the following diagram,

$$\begin{array}{ccc}\overline{\Omega}_A\mathbf{W}\times\overline{\Omega}_{\overline{\Omega}_A\mathbf{W}\times A}\mathbf{V} & \longrightarrow & \overline{\Omega}_{\overline{\Omega}_A\mathbf{W}\times A}\mathbf{V}\\ \downarrow & & \downarrow\\ \overline{\Omega}_{A_\ell}\mathbf{W}_k\times\overline{\Omega}_{\overline{\Omega}_{A_\ell}\mathbf{W}_k\times A_\ell}\mathbf{V}_m & \longrightarrow & \overline{\Omega}_{\overline{\Omega}_{A_\ell}\mathbf{W}_k\times A_\ell}\mathbf{V}_m\end{array}$$

where the vertical arrows represent the canonical projection morphisms and the horizontal arrows represent the actions described above. By definition of those actions, these diagrams commute. Therefore, the action mapping from $\overline{\Omega}_A\mathbf{W}\times\overline{\Omega}_{\overline{\Omega}_A\mathbf{W}\times A}\mathbf{V}$ into $\overline{\Omega}_{\overline{\Omega}_A\mathbf{W}\times A}\mathbf{V}$ is continuous, and $\overline{\Omega}_{\overline{\Omega}_A\mathbf{W}\times A}\mathbf{V} * \overline{\Omega}_A\mathbf{W}$ is a topological monoid. Moreover, this implies that the corresponding mappings

$$\overline{\Omega}_{\overline{\Omega}_A\mathbf{W}\times A}\mathbf{V} * \overline{\Omega}_A\mathbf{W} \longrightarrow \overline{\Omega}_{\overline{\Omega}_{A_\ell}\mathbf{W}_k\times A_\ell}\mathbf{V}_m * \overline{\Omega}_{A_\ell}\mathbf{W}_k$$

are continuous morphisms. Since they are sufficient to separate the elements of $\overline{\Omega}_{\overline{\Omega}_A\mathbf{W}\times A}\mathbf{V} * \overline{\Omega}_A\mathbf{W}$, it follows from Lemma 1.2 that

$$\overline{\Omega}_{\overline{\Omega}_A\mathbf{W}\times A}\mathbf{V} * \overline{\Omega}_A\mathbf{W} = \varprojlim(\overline{\Omega}_{\overline{\Omega}_{A_\ell}\mathbf{W}_k\times A_\ell}\mathbf{V}_m * \overline{\Omega}_{A_\ell}\mathbf{W}_k)_{k,\ell,m},$$

and hence $\overline{\Omega}_{\overline{\Omega}_A\mathbf{W}\times A}\mathbf{V} * \overline{\Omega}_A\mathbf{W}$ is a profinite monoid.

Let now $S * T$ be a semidirect product with $S \in \mathbf{V}$ and $T \in \mathbf{W}$, and let $\varphi\colon A \to S * T$ be a continuous mapping. Say, $\varphi a = (\varphi_1 a, \varphi_2 a)$ for all $a \in A$. Then, both φ_1 and φ_2 are continuous mappings. So there exists a continuous morphism $\bar\varphi_2\colon \overline{\Omega}_A\mathbf{W} \to T$ which coincides with φ_2 on A. Also the mapping from $\overline{\Omega}_A\mathbf{W} \times A$ into S which maps (x, a) to $\bar\varphi_2 x \cdot (\varphi_1 a)$ is continuous and hence can be extended to a continuous morphism $\bar\varphi_1\colon \overline{\Omega}_{\overline{\Omega}_A\mathbf{W}\times A}\mathbf{V} \to S$. Note that, for all $x, y \in \overline{\Omega}_A\mathbf{W}$ and $a \in A$, $\bar\varphi_1(x \cdot (y, a)) = \bar\varphi_2 x \cdot \bar\varphi_1(y, a)$, so that $\bar\varphi_1(x \cdot z) = \bar\varphi_2 x \cdot \bar\varphi_1 z$ for all $z \in \overline{\Omega}_{\overline{\Omega}_A\mathbf{W}\times A}\mathbf{V}$. Let now

$$\begin{array}{rcl}\bar\varphi\colon \overline{\Omega}_{\overline{\Omega}_A\mathbf{W}\times A}\mathbf{V} * \overline{\Omega}_A\mathbf{W} & \longrightarrow & S * T \\ (x, y) & \longmapsto & (\bar\varphi_1 x, \bar\varphi_2 y).\end{array}$$

Then $\bar\varphi$ is a continuous mapping and

$$\begin{aligned}\bar\varphi(x, y)\bar\varphi(x', y') &= (\bar\varphi_1 x, \bar\varphi_2 y)(\bar\varphi_1 x', \bar\varphi_2 y') \\ &= ((\bar\varphi_1 x)\ (\bar\varphi_2 y)(\bar\varphi_1 x'),\ \bar\varphi_2(yy')) \\ &= (\bar\varphi_1(x(y \cdot x')),\ \bar\varphi_2(yy')) \\ &= \bar\varphi((x, y)(x', y')).\end{aligned}$$

That is, $\bar\varphi$ is a morphism of profinite monoids.

Let now $\theta\colon A \to \overline{\Omega}_{\overline{\Omega}_A\mathbf{W}\times A}\mathbf{V} * \overline{\Omega}_A\mathbf{W}$ be given by $\theta a = ((1, a), a)$ for all $a \in A$. Note that θ is one-to-one on A since $\mathbf{W}$ is non trivial. Let $R = \overline{(\theta A)^*}$ be the profinite submonoid of $\overline{\Omega}_{\overline{\Omega}_A\mathbf{W}\times A}\mathbf{V} * \overline{\Omega}_A\mathbf{W}$ generated by θA. For each $a \in A$, $\bar\varphi\theta a = (\bar\varphi_1(1, a), \bar\varphi_2 a) = (\varphi_1 a, \varphi_2 a) = \varphi a$.

Thus, φ can be extended to a continuous morphism defined on R, and this extension is unique since R is generated by θA. Therefore, θ extends to an isomorphism from $\overline{\Omega}_A(\mathbf{V} * \mathbf{W})$ to R. $\square$

To illustrate this general result, we give a simple proof of the following fact (which was first established by Weil [28] by a different method).

Proposition 2.2 *Let* $\mathbf{V}$ *be a pseudovariety of semigroups. Then* $\mathbf{Nil} * \mathbf{V} = \mathbf{Nil} \vee \mathbf{V}$.

Proof. By Theorem 2.1, if θ is given by $\theta a = ((1, a), a)$ for each $a \in A$, then $\overline{\Omega}_A(\mathbf{Nil} * \mathbf{V})$ is the closure of θA^+ in $\overline{\Omega}_{\overline{\Omega}_A\mathbf{V}\times A}\mathbf{Nil} * \overline{\Omega}_A\mathbf{V}$.

If $x = \theta(a_1 \cdots a_r)$ $(a_i \in A)$, then the first component of x is

$$(1, a_1)(a_1, a_2) \cdots (a_1 \cdots a_{r-1}, a_r).$$

So θ is one-to-one on A^+. On the other hand, if $x \notin \theta A^+$, then $x = (0, \pi x)$, where π denotes the canonical projection of $\overline{\Omega}_A(\mathbf{Nil} * \mathbf{V})$ onto $\overline{\Omega}_A\mathbf{V}$. That is, π is one-to-one on the complement of θA^+. Thus $\overline{\Omega}_A(\mathbf{Nil} * \mathbf{V})$ is isomorphic to the product of $\overline{\Omega}_A\mathbf{Nil}$ and $\overline{\Omega}_A\mathbf{V}$ (in the category of A-generated profinite monoids), which is exactly $\overline{\Omega}_A(\mathbf{Nil} \vee \mathbf{V})$. $\square$

3 Implicit operations on pseudovarieties of the form $\mathbf{J}_1 * \mathbf{V}$

The above description of the implicit operations on a semidirect product takes on an interesting form when applied to semidirect products of the form $\mathbf{J}_1 * \mathbf{V}$. In order to describe the implicit operations on such products, we first introduce a new notion on the geometry of profinite graphs.

3.1 Profinitely quasi-linear graphs

Let $\Gamma = (V, E)$ be a profinite graph and let v and w be vertices of Γ. We say that w is *profinitely accessible from* v and we write $v \preceq w$ if, in each finite continuous quotient of Γ, there exists a directed path from the image of v to the image of w.

The relation $\preceq$ on V is easily seen to be a quasi-order. The classes of the associated equivalence relation, or the subgraphs generated by these classes are called the *profinitely strongly connected components* of Γ.

Example. If Γ is finite, then $v \preceq w$ if and only if there exists a directed path from v to w. The profinitely strongly connected components of Γ are exactly its strongly connected components.

Let $\mathbf{V}$ be a pseudovariety of monoids and let A be a profinite set. Let $\Gamma = \Gamma_A\mathbf{V}$ and let $v, w \in \overline{\Omega}_A\mathbf{V}$ be vertices of Γ. Then $v \preceq w$ if and only if, for each continuous morphism $\varphi\colon \overline{\Omega}_A\mathbf{V} \to M$ onto a finite monoid M (which is then in $\mathbf{V}$), we have $\varphi w = \varphi v \varphi u$ for some word $u \in A^*$. This comes from the fact that the continuous images of $\Gamma_A\mathbf{V}$ are exactly the Cayley graphs of the A-generated monoids in $\mathbf{V}$. Thus $v \preceq w$ if and only if $w \leq_{\mathcal{R}} v$ in $\overline{\Omega}_A\mathbf{V}$.

Let us note the following elementary properties of the relation $\preceq$.

Proposition 3.1 *Let $\Gamma = (V, E)$ be a profinite graph. The relation $\preceq$ on Γ enjoys the following properties.*

1. *$\preceq$ is a closed quasi-order. That is, if $v = \lim_n v_n$ and $w = \lim_n w_n$ in V, and if $v_n \preceq w_n$ for all n, then $v \preceq w$.*

2. *The profinitely strongly connected components of Γ are profinite graphs.*

3. *In the ordered set of profinitely strongly connected components of Γ, every element is dominated by a maximal (resp. dominates a minimal) element.*

4. *Let Δ be another profinite graph, and let $\varphi\colon \Gamma \to \Delta$ be a continuous graph morphism. If $v \preceq w$ in Γ, then $\varphi v \preceq \varphi w$. Furthermore, if X is a profinitely strongly connected component of Γ, then φX is a profinite subgraph of a profinitely strongly connected component of Δ.*

5. *If Γ is the projective limit of a directed system $(\Gamma_i)_{i\in I}$ of profinite graphs with canonical morphisms $\pi_i\colon \Gamma \to \Gamma_i$, and if $v, w \in V$, then $v \preceq w$ if and only if $\pi_i v \preceq \pi_i w$.*

Proof. Let us first consider two converging sequences in V, $v = \lim_n v_n$ and $w = \lim_n w_n$. For each continuous morphism $\varphi\colon \Gamma \to \Delta$ into a finite graph, there exists an integer k such that $\varphi v = \varphi v_n$ and $\varphi w = \varphi w_n$ for all $n \geq k$. If we assume that $v_n \preceq w_n$ for all n, then there exists a directed path in Δ from φv to φw, that is, $v \preceq w$. This proves statement 1. As an immediate consequence, the profinitely strongly connected components of Γ are closed subgraphs of Γ by Proposition 1.5, and hence are profinite graphs themselves, thus proving statement 2.

In order to prove statement 3, it suffices to verify that each sequence $(v_n)_n$ of vertices such that $v_n \preceq v_{n+1}$ for all $n \geq 0$ admits an upper bound in V. Indeed, by compactness, the sequence $(v_n)_n$ admits a converging subsequence, say (v_{k_n}). Let v be the limit of that subsequence. By statement 1, $v_{k_n} \preceq v$ for all n, and hence $v_n \preceq v$ for all n.

The proof of statement 4 is immediate. Statement 5 is an immediate consequence of the fact that each finite continuous homomorphic image of Γ is a homomorphic image of some Γ_i (Lemma 1.4). □

We say that the profinite graph $\Gamma = (V, E)$ is *profinitely quasi-linear* if the quasi-order $\preceq$ is total. In that case, Γ admits a unique $\preceq$-maximal (resp. minimal) profinitely strongly connected component denoted $\max \Gamma$ (resp. $\min \Gamma$). If Γ is finite, this is equivalent to the condition that there exists a directed path in Γ which visits all the vertices.

Proposition 3.2 *Let Γ be a profinite graph. The following assertions are equivalent.*

1. *Γ is profinitely quasi-linear.*
2. *The finite continuous homomorphic images of Γ are profinitely quasi-linear.*
3. *The continuous homomorphic images of Γ are profinitely quasi-linear.*
4. *Γ is the projective limit of a directed system of profinitely quasi-linear finite graphs.*
5. *Γ is the projective limit of a directed system of profinitely quasi-linear graphs.*

Proof. We already know (Proposition 3.1) that if $\preceq$ is total on Γ, then it is total also on the continuous homomorphic images of Γ. Therefore, (1) implies (2), (3), (4) and (5).

It is also immediate that (3) implies (2), (2) implies (4) and (4) implies (5). Let us now assume that (5) holds. Then Γ is the projective limit of a directed system of projective limits of finite profinitely quasi-linear graphs. As in the proof of Proposition 1.5, it follows that Γ is the projective limit of a directed system of finite profinitely quasi-linear graphs, that is, (5) implies (4). It now suffices to show that (4) implies (1).

Let us assume that $\Gamma = \varprojlim(\Gamma_i)_{i\in I}$, where the Γ_i are finite, profinitely quasi-linear graphs and let $(\pi_i)_i$ be the system of canonical morphisms. It follows from Lemma 1.4 that $\preceq$ is total on the set of vertices of Γ. $\square$

Example. The profinite Cayley graph $\Gamma_A\mathbf{G}$ is profinitely quasi-linear, and consists of a single profinitely strongly connected component.

$\Gamma_A\mathbf{Nil}$ is not profinitely quasi-linear. For instance, the vertices a and b (if a and b are distinct letters of A) are not $\preceq$-comparable. Let indeed Δ be the following graph.

$$\begin{tikzcd}
1 \arrow[r, "a"] \arrow[d, "b"'] & a \arrow[d, "{a,b}"] \\
b \arrow[r, "{a,b}"] & 0
\end{tikzcd}$$

Δ is the Cayley graph of a monoid in **Nil**, so it is a continuous image of $\Gamma_A\mathbf{Nil}$. In Δ one cannot go from a to b, or from b to a, by a positive path. In fact, the profinitely strongly connected components of $\Gamma_A\mathbf{Nil}$ all consist of a single vertex.

Note. Gildenhuys and Ribes [13] defined the notion of profinite connectedness for profinite graphs (see also Zalesskiĭ and Mel'nikov [30], or Almeida and Weil [6, 7]). In particular, a profinite graph is profinitely connected if and only if its finite continuous homomorphic images are connected in the usual sense. In fact, if one replaces everywhere profinitely quasi-linear by profinitely connected, Proposition 3.2 still holds. Note that profinite quasi-linearity implies profinite connectedness. The notion of profinite connectedness plays an important role in Section 6.

3.2 Implicit operations on $\mathbf{J}_1 * \mathbf{V}$

Let $\mathbf{V}$ be a pseudovariety of monoids, let A be a profinite set and let θ be the injective morphism from $\overline{\Omega}_A(\mathbf{J}_1 * \mathbf{V})$ into $\overline{\Omega}_{\overline{\Omega}_A\mathbf{V}\times A}\mathbf{J}_1 * \overline{\Omega}_A\mathbf{V}$ such that $\theta a = ((1,a),a)$ for all $a \in A$ (Theorem 2.1). Let also $\pi_{\mathbf{V}}$ be the projection on the second component, $\pi_{\mathbf{V}}\colon \overline{\Omega}_{\overline{\Omega}_A\mathbf{V}\times A}\mathbf{J}_1 * \overline{\Omega}_A\mathbf{V} \to \overline{\Omega}_A\mathbf{V}$. Note that $\pi_{\mathbf{V}} \circ \theta$ is the canonical projection from $\overline{\Omega}_A(\mathbf{J}_1 * \mathbf{V})$ onto $\overline{\Omega}_A\mathbf{V}$.

Recall that for any profinite set B, $\overline{\Omega}_B\mathbf{J}_1$ is the set $\bar{\mathcal{P}}(B)$ of closed subsets of B, a monoid under union. If M is a profinite monoid, and $\mu\colon A \to M$ is a continuous function, we identify elements of $\overline{\Omega}_{M\times A}\mathbf{J}_1 = \bar{\mathcal{P}}(M \times A)$ with the corresponding sets of edges of the Cayley graph of M with respect to A, or with the corresponding closed subgraphs of this graph (by adding the relevant vertices. By convention, the empty subset of $M\times A$ is identified with the subgraph consisting of the sole vertex 1). Note that M acts naturally on the left on $\bar{\mathcal{P}}(M \times A)$ by

$$m \cdot X = \{(mn, a) : (n, a) \in X\}.$$

Let us first assume that A is a finite set. Since $\overline{\Omega}_A\mathbf{V}$ is the projective limit of the system $\mathbf{V}_A$ of all A-generated elements of $\mathbf{V}$, Proposition 1.9 shows that $\overline{\Omega}_{\overline{\Omega}_A\mathbf{V}\times A}\mathbf{J}_1 = \varprojlim\,(\bar{\mathcal{P}}(M \times A))_{M\in\mathbf{V}_A}$. We verified in the proof of Proposition 2.1 that $\overline{\Omega}_{\overline{\Omega}_A\mathbf{V}\times A}\mathbf{J}_1 * \overline{\Omega}_A\mathbf{V}$ is the projective limit of the finite semidirect products $(\bar{\mathcal{P}}(M \times A) * M)_{M\in\mathbf{V}_A}$. For each $M \in \mathbf{V}_A$, let π_M be the canonical morphism from $\overline{\Omega}_{\overline{\Omega}_A\mathbf{V}\times A}\mathbf{J}_1 * \overline{\Omega}_A\mathbf{V}$ onto $\bar{\mathcal{P}}(M \times A) * M$, and let $\theta_M = \pi_M \circ \theta$. For each $x \in \overline{\Omega}_A(\mathbf{J}_1 * \mathbf{V})$, there exists a word $u_M \in A^*$ such that $\theta_M x = \theta_M u_M$. But, for all $a \in A$, $\theta_M a = (\{(1,a)\}, a)$, so that, if μ is the second component of π_M and $u_M = a_1a_2\cdots a_r$, then

$$\theta_M x = \theta_M u_M = (\{(1,a_1),(\mu a_1, a_2),\ldots,(\mu(a_1\cdots a_{r-1}), a_r)\}, \mu(a_1\cdots a_r)).$$

Using the canonical identification of $\bar{\mathcal{P}}(M\times A)$ with the set of closed subgraphs of the Cayley graph Γ_M of M with respect to A, we have $\theta_M x = \theta_M u_M = (X_M, x_M)$, where $x_M = \mu u_M$ and X_M is the path in Γ_M starting at 1 and labelled u_M. In particular, X_M is profinitely quasi-linear, 1 is a vertex of $\min X_M$ and x_M is a vertex of $\max X_M$.

Now $\theta x = (X, \pi_{\mathbf{V}}\theta x)$ is the collection $(\theta_M x)_{M\in\mathbf{V}_A}$, so X is the projective limit of the graphs $(X_M)_{M\in\mathbf{V}_A}$ and hence X is profinitely quasi-linear. Furthermore, 1 is a vertex of $\min X$ and $\pi_{\mathbf{V}}\theta x = (x_M)_{M\in\mathbf{V}_A}$ is a vertex of $\max X$.

Conversely, let us show that each pair (X, y), consisting of a profinitely quasi-linear subgraph X of $\Gamma_A\mathbf{V}$ with 1 in $\min X$ and of a vertex y of $\max X$, is an element of $\theta\overline{\Omega}_A(\mathbf{J}_1 * \mathbf{V})$. We know that the finite continuous images of $\Gamma_A\mathbf{V}$ are the Cayley graphs Γ_M of the monoids M in $\mathbf{V}_A$. For each such M, let ρ_M be the canonical morphism from $\overline{\Omega}_A\mathbf{V}$ onto M. We also write ρ_M for its extension to a morphism of profinite graphs from $\Gamma_A\mathbf{V}$ onto Γ_M. Then $\rho_M X$ is profinitely quasi-linear with 1 in $\min\rho_M X$ and y in $\max\rho_M X$. Let u_M be a word of A^* such that $\rho_M u_M = \rho_M y$ and the path form 1 to $\rho_M y$ labelled u_M uses exactly the edges of $\rho_M X$. Note that $\theta_M u_M = (\rho_M X, \rho_M u_M) = (\rho_M X, \rho_M y)$. Therefore $(\theta_M u_M)_{M\in\mathbf{V}_A} \in \varprojlim(\bar{\mathcal{P}}(M\times A) * M)_{M\in\mathbf{V}_A} = \overline{\Omega}_{\overline{\Omega}_A\mathbf{V}\times A}\mathbf{J}_1 * \overline{\Omega}_A\mathbf{V}$.

It now suffices to verify that $\bigcap_M \theta_M^{-1}(\theta_M u_M)$ is a singleton in $\overline{\Omega}_A(\mathbf{J}_1 * \mathbf{V})$. Each finite intersection

$$\theta_{M_1}^{-1}(\theta_{M_1}u_{M_1}) \cap \theta_{M_2}^{-1}(\theta_{M_2}u_{M_2}) \cap \cdots \cap \theta_{M_r}^{-1}(\theta_{M_r}u_{M_r})$$

is non-empty since $\mathbf{V}_A$ constitutes a directed system. Then, $\bigcap_M \theta_M^{-1}(\theta_M u_M)$ is non-empty by compactness. Finally, $\bigcap_M \theta_M^{-1}(\theta_M u_M)$ is a singleton since θ is one-to-one (Lemma 1.2).

So we have shown that if A is finite, then $\overline{\Omega}_A(\mathbf{J}_1 * \mathbf{V})$ is isomorphic to the submonoid of $\bar{\mathcal{P}}(\overline{\Omega}_A\mathbf{V}\times A) * \overline{\Omega}_A\mathbf{V}$ which consists of all pairs (X, x) where $x \in \overline{\Omega}_A\mathbf{V}$ and where X is a profinitely quasi-linear closed subgraph of $\Gamma_A\mathbf{V}$ such that 1 is in $\min X$ and x is in $\max X$.

Let now $(A_i)_{i\in I}$ be a directed system of finite sets such that $A = \varprojlim(A_i)_i$. Using again Proposition 1.9 and Theorem 2.1, we find that

$$\begin{aligned}\overline{\Omega}_A(\mathbf{J}_1 * \mathbf{V}) &= \varprojlim\,(\overline{\Omega}_{A_i}(\mathbf{J}_1 * \mathbf{V}))_i \quad \text{and}\\ \bar{\mathcal{P}}(\overline{\Omega}_A\mathbf{V}\times A) * \overline{\Omega}_A\mathbf{V} &= \varprojlim\,(\bar{\mathcal{P}}(\overline{\Omega}_{A_i}\mathbf{V}\times A_i) * \overline{\Omega}_{A_i}\mathbf{V})_i.\end{aligned}$$

So $\overline{\Omega}_A(\mathbf{J}_1 * \mathbf{V})$ is isomorphic to the submonoid of $\bar{\mathcal{P}}(\overline{\Omega}_A\mathbf{V}\times A) * \overline{\Omega}_A\mathbf{V}$ which consists of all pairs (X, x) such that $X = (X_i)_i$, $x = (x_i)_i$ and $(X_i, x_i) \in \overline{\Omega}_{A_i}(\mathbf{J}_1 * \mathbf{V})$ for each i. Therefore X_i is profinitely quasi-linear, $1 \in \min X_i$ and $x_i \in \max X_i$ for each i. By Propositions 3.1 and 3.2, it follows that X is profinitely quasi-linear, $1 \in \min X$ and $x \in \max X$.

Thus we have shown the following general result.

Theorem 3.3 *Let* $\mathbf{V}$ *be a pseudovariety of monoids and let* A *be a profinite set. Then* $\overline{\Omega}_A(\mathbf{J}_1 * \mathbf{V})$ *is isomorphic to the submonoid of* $\bar{\mathcal{P}}(\overline{\Omega}_A\mathbf{V}\times A) * \overline{\Omega}_A\mathbf{V}$ *which consists of all pairs* (X, x) *where* $x \in \overline{\Omega}_A\mathbf{V}$ *and where* X *is a profinitely quasi-linear closed subgraph of* $\Gamma_A\mathbf{V}$ *such that* 1 *is in* $\min X$ *and* x *is in* $\max X$.

Note. The same result holds if $\mathbf{V}$ is a pseudovariety of semigroups. We recall that in this case, the graph $\Gamma_A\mathbf{V}$ is defined to be the Cayley graph of the monoid $(\overline{\Omega}_A\mathbf{V})^1$ with respect to A (Section 1.2).

4 An example

We illustrate this new technique by computing the pseudovarieties $\mathbf{J}_1 * \mathbf{Nil}$ and $\mathbf{J}_1 * \mathbf{K}$. We use the following notation: If A is a profinite alphabet and u is a finite or infinite word over A, we let $c(u)$ be the closure of the set of letters of A occurring in u. If $u \in A^{\mathbb{N}}$, we let $c_\infty(u)$ be the set of cluster points in A of the sequence u. In the particular case where A is finite, $c(u)$ is exactly the set of letters of A occurring in u and $c_\infty(u)$ is the set of those letters occurring infinitely often in u.

Theorem 4.1 $\mathbf{J}_1 * \mathbf{Nil} = \mathbf{J}_1 * \mathbf{K} = [\![x^\omega y^2 = x^\omega y,\ x^\omega yz = x^\omega zy]\!]$. *Let* A *be a profinite set. Then*

$$\overline{\Omega}_A(\mathbf{J}_1 * \mathbf{Nil}) = A^+ \cup \{(v, B) \in A^{\mathbb{N}} \times \bar{\mathcal{P}}(A) : c_\infty(v) \subseteq B\},$$

and the product is given, for all $u, u' \in A^+$ *and* $(v, B), (v', B') \in A^{\mathbb{N}} \times \bar{\mathcal{P}}(A)$ *such that* $c_\infty(v) \subseteq B, c_\infty(v') \subseteq B'$ *by:*

$$\begin{aligned} u \cdot u' &= uu' \\ u \cdot (v, B) &= (uv, B) \\ (v, B) \cdot u &= (v, B \cup c(u)) \\ (v, B) \cdot (v', B') &= (v, B \cup c(v') \cup B'). \end{aligned}$$

In the case where A *is finite, each implicit operation* x *of* $\overline{\Omega}_A(\mathbf{J}_1 * \mathbf{Nil})$ *admits a factorization of the form* $x = u_0$ *or* $x = u_0 x_1 u_1$ *where* $u_0, u_1 \in A^*$, $u_0 \neq 1$ *if* $x = u_0$, x_1 *is idempotent, the last letter of* u_0 *is not in* $c(x_1)$ *(if* u_0 *is not empty), the word* u_1 *is linear and* $c(u_1) \cap c(x_1) = \emptyset$*. Furthermore, this factorization is unique up to a permutation of* u_1*.*

Finally, if A *is finite and* $\mathcal{V}$ *is the variety of languages associated with* $\mathbf{J}_1 * \mathbf{Nil}$*, then* $A^+\mathcal{V}$ *is the Boolean algebra generated by the finite languages and by the languages of the form* uB^* *where* $u \in A^+$ *and* $B \subseteq A$*.*

Proof. We know that $\Gamma_A\mathbf{K}$ is the tree of A^*, with added vertices in bijection with the set of infinite words over A, and around each of these vertices, a set of loops labelled by the letters of A. In particular, the profinitely strongly quasi-linear subgraphs of $\Gamma_A\mathbf{K}$ contain at most one vertex. Thus, a closed profinitely quasi-linear subgraph X with $1 \in \min X$ is either a finite path, with label $u \in A^+$ (and $u = \max X$), or an infinite path with label $v \in A^{\mathbb{N}}$, together with the vertex v (which is the limit of the vertices of the infinite path), and with a closed set of loops around vertex v containing the loops labelled by $c_\infty(v)$. In that case, $\max X = v$. This establishes a bijection between $\overline{\Omega}_A(\mathbf{J}_1 * \mathbf{K})$ and the set $A^+ \cup \{(v, B) \in A^{\mathbb{N}} \times \bar{\mathcal{P}}(A) : c_\infty(v) \subseteq B\}$, and the induced product on this set is as described in the statement of Theorem 4.1.

We also know that $\Gamma_A\mathbf{Nil}$ is the tree of A^*, with a single isolated vertex, and around this vertex, a set of loops labelled by the letters of A. The closed profinitely quasi-linear subgraphs of $\Gamma_A\mathbf{Nil}$ are in bijection with those of $\Gamma_A\mathbf{K}$, and hence $\overline{\Omega}_A(\mathbf{J}_1 * \mathbf{Nil}) = \overline{\Omega}_A(\mathbf{J}_1 * \mathbf{K})$. Since this holds for all finite sets A, we have $\mathbf{J}_1 * \mathbf{Nil} = \mathbf{J}_1 * \mathbf{K}$.

Let $\mathbf{V} = [\![x^\omega y^2 = x^\omega y,\ x^\omega yz = x^\omega zy]\!]$. In order to prove that $\mathbf{J}_1 * \mathbf{Nil} \subseteq \mathbf{V}$, it suffices to show that the semigroup $\overline{\Omega}_A(\mathbf{J}_1 * \mathbf{Nil})$ computed above satisfies the two pseudo-identities which define $\mathbf{V}$. This is immediate once we notice that, for all $u \in A^+$, $v \in A^{\mathbb{N}}$ and $B \in \bar{\mathcal{P}}(A)$ such that $c_\infty(v) \subseteq B$, we have $(v, B)^\omega = (v, B \cup c(v))$ and $u^\omega = (u^\infty, c(u))$. In this last equality,

u^∞ denotes the infinite word obtained by infinitely repeating the finite word u.

Now we notice that $\mathbf{V} \subseteq \mathbf{DA}$. Indeed

$$\mathbf{V} \quad \models \quad (xy)^\omega (yx)^\omega (xy)^\omega = (xy)^\omega xy = (xy)^\omega.$$

Therefore, by [5, Theorem 4.9] (or [4, Theorem 8.1.11]), each implicit operation x on $\mathbf{S}$ admits a factorization of the form $x = u_0 x_1 u_1 \cdots x_r u_r$ where $r \geq 0$, the x_i are idempotent in $\mathbf{V}$, $u_i \in A^*$ (if $r = 0$, then $u_0 \neq 1$), and if $u_i \neq 1$, then the last letter of u_i does not lie in $c(x_{i+1})$. If $r \neq 0$, then

$$\mathbf{V} \quad \models \quad x = u_0 x_1 u_1 \cdots x_r u_r = u_0 x_1 v_1$$

where v_1 is a linear word with content $c(u_1 x_2 \cdots x_r u_r) \setminus c(x_1)$. This proves the existence of factorizations of the required form for all the elements of $\overline{\Omega}_A \mathbf{V}$. Furthermore, $\mathbf{V}$ satisfies $x = u_0 x_1 v_1'$ for each permutation v_1' of v_1. Such factorizations are said to be reduced.

In order to conclude this part of the proof, we consider, for each finite alphabet A, the canonical projection φ from $\overline{\Omega}_A \mathbf{V}$ onto $\overline{\Omega}_A(\mathbf{J}_1 * \mathbf{Nil})$. If $x = u_0 x_1 u_1$ is a reduced factorization of $x \in \overline{\Omega}_A \mathbf{V}$, then $x\varphi = (u_0\varphi)(x_1\varphi)(u_1\varphi) = u_0(v, c(v))u_1$, where $v \in A^{\mathbb{N}}$. Therefore, $x\varphi = (u_0 v, c(v) \cup c(u_1))$. Conversely, given an element (v, B) of $A^{\mathbb{N}} \times \bar{\mathcal{P}}(A)$ and a reduced factorization $u_0 x_1 u_1$ such that $(v, B) = (u_0 x_1 u_1)\varphi$, then $c(u_1)$ is uniquely determined as $B \setminus c_\infty(v)$, u_0 is uniquely determined as the shortest prefix u of v such that $c(u^{-1}v) = c_\infty(v)$ and $x_1 = (u_0^{-1}v, c_\infty(v))$. This proves both the required unique factorization statement for elements of $\overline{\Omega}_A \mathbf{V}$, and the fact that φ is an isomorphism. But this isomorphism holds for all finite alphabets A, so $\mathbf{V} = \mathbf{J}_1 * \mathbf{Nil} = \mathbf{J}_1 * \mathbf{K}$.

Finally, by Lemma 1.12, it suffices to show that the subsets of the form $\overline{\{u\}}$ and $\overline{uB^*}$ separate the elements of $\overline{\Omega}_A(\mathbf{J}_1 * \mathbf{Nil})$. Note that

$$\overline{\{u\}} = \{u\} \quad \text{and} \quad \overline{uB^*} = uB^* \cup \{(uv, C) : v \in B^{\mathbb{N}} \text{ and } c_\infty(v) \subseteq C \subseteq B\}.$$

It is clear that these sets are sufficient to separate the elements of A^+. Let now $u \in A^+$, let (v, B) be a non-explicit element of $\overline{\Omega}_A(\mathbf{J}_1 * \mathbf{Nil})$, and let v_1 be a finite prefix of v such that $v = v_1 v_2$ for some infinite word v_2 using only letters of $c_\infty(v)$. We may choose v_1 in such a way that v_1 is not a prefix of u. Then $(v, B) \in \overline{v_1 B^*}$ and $u \notin \overline{v_1 B^*}$. Let us now consider another non-explicit operation (v', B'). If $v \neq v'$, we may choose v_1 so that it is not a prefix of v'. Then $(v, B) \in \overline{v_1 B^*}$ and $(v', B') \notin \overline{v_1 B^*}$. Finally, if $v = v'$, we may assume without loss of generality that $B' \not\subseteq B$. Note that $c_\infty(v) \subseteq B \cap B'$. With v_1 as above, $(v, B') \notin \overline{v_1 B^*}$ and $(v, B) \in \overline{v_1 B^*}$. This concludes the proof. $\Box$

The effect of the operation $\mathbf{V} \mapsto \mathbf{J}_1 * \mathbf{V}$ on the associated variety of languages was explored by Straubing. It is a consequence of Straubing's "principle of the semidirect product" [27] that, if $\mathcal{V}$ is the variety of languages associated to $\mathbf{V}$, then, for each finite alphabet A, the class of languages of A^+ recognized by semigroups in $\mathbf{J}_1 * \mathbf{V}$ is the Boolean algebra generated by the languages of $A^+\mathcal{V}$ and by the languages of the form LaA^*, where $L \in A^+\mathcal{V}$ and $a \in A$. If $\mathbf{V} = \mathbf{Nil}$, it is known that $A^+\mathcal{V}$ consists of the finite and cofinite languages in A^+. So Theorem 4.1 above gives us a different description of the languages of $\mathbf{J}_1 * \mathbf{Nil}$ than a direct application of Straubing's principle would.

5 Implicit operations on two-sided semidirect products

Two-sided semidirect products were introduced by Rhodes and Tilson [23] and they play an important role in the study of decompositions of finite monoids (see also Rhodes and Weil [24, 25], Pin, Straubing and Thérien [21], Weil [29], etc.)

Let S and T be monoids and let us assume that a right and a left action of T on S are given in such a way that these actions commute, that is, $t \cdot (s \cdot t') = (t \cdot s) \cdot t'$ for all $s \in S$ and $t, t' \in T$. The two-sided semidirect product $S{*}{*}T$ is the set $S \times T$ with the operation $(s,t)(s',t') = (s \cdot t' + t \cdot s', tt')$. Here the operation of S is written additively. We notice that, when the right action of T on S is trivial, then $S ** T$ is in fact a semidirect product. The reader is referred to [23, 25, 29] for further definitions and background.

Using the same methods as in the previous sections, it is not difficult to verify the following sequence of results.

Theorem 5.1 *Let $\mathbf{V}$ and $\mathbf{W}$ be pseudovarieties of monoids, and let A be a profinite set. The right and left actions of $\overline{\Omega}_A\mathbf{W}$ on the set $\overline{\Omega}_A\mathbf{W} \times A \times \overline{\Omega}_A\mathbf{W}$ given by $x \cdot (y,a,z) = (xy,a,z)$ and $(y,a,z) \cdot x = (y,a,zx)$ (for $x,y,z \in \overline{\Omega}_A\mathbf{W}$ and $a \in A$) can be naturally extended to continuous actions of $\overline{\Omega}_A\mathbf{W}$ on $\overline{\Omega}_{\overline{\Omega}_A\mathbf{W} \times A \times \overline{\Omega}_A\mathbf{W}}\mathbf{V}$. The resulting two-sided semidirect product $\overline{\Omega}_{\overline{\Omega}_A\mathbf{W} \times A \times \overline{\Omega}_A\mathbf{W}}\mathbf{V} ** \overline{\Omega}_A\mathbf{W}$ is a profinite monoid.*

Moreover, letting $\theta a = ((1,a,1),a)$ defines a unique continuous morphism θ from $\overline{\Omega}_A(\mathbf{V} ** \mathbf{W})$ into $\overline{\Omega}_{\overline{\Omega}_A\mathbf{W} \times A \times \overline{\Omega}_A\mathbf{W}}\mathbf{V} ** \overline{\Omega}_A\mathbf{W}$ which is one-to-one.

We now define the *two-sided Cayley graph* $\Gamma_A^2\mathbf{V}$ of a pseudovariety of monoids $\mathbf{V}$ with respect to A. The vertex set of $\Gamma_A^2\mathbf{V}$ is $\overline{\Omega}_A\mathbf{V} \times \overline{\Omega}_A\mathbf{V}$ and its edge set is $\overline{\Omega}_A\mathbf{V} \times A \times \overline{\Omega}_A\mathbf{V}$, where (x, a, y) is an edge from (x, ay) to (xa, y):

$$(x, ay) \xrightarrow{(x,a,y)} (xa, y)$$

Like $\Gamma_A\mathbf{V}$, $\Gamma_A^2\mathbf{V}$ is a profinite graph.

In the case where $\mathbf{V}$ is a pseudovariety of semigroups, we construct similarly a profinite graph $\Gamma_A^2\mathbf{V}$ with vertex set $(\overline{\Omega}_A\mathbf{V})^1 \times (\overline{\Omega}_A\mathbf{V})^1$.

Theorem 5.2 *Let $\mathbf{V}$ be a pseudovariety of monoids and let A be a profinite set. Then $\overline{\Omega}_A(\mathbf{J}_1 ** \mathbf{V})$ is isomorphic to the submonoid of $\bar{\mathcal{P}}(\overline{\Omega}_A\mathbf{V} \times A \times \overline{\Omega}_A\mathbf{V}) ** \overline{\Omega}_A\mathbf{V}$ which consists of all pairs (X, x) where $x \in \overline{\Omega}_A\mathbf{V}$ and where X is a profinitely quasi-linear closed subgraph of $\Gamma_A^2\mathbf{V}$ such that $(1, x)$ is in $\min X$ and $(x, 1)$ is in $\max X$.*

We then obtain a description of, say, $\mathbf{J}_1 ** \mathbf{Nil}$ analogous to that of Section 4. If A is a profinite set, we let ${}^{\mathbb{N}}\!A$ be the set of words over A that are "infinite to the left", that is, the set of all sequences of elements of A indexed by $-\mathbb{N}$. The notations $c(u)$ and $c_\infty(u)$ introduced in Section 4 are extended to the elements of ${}^{\mathbb{N}}\!A$.

Theorem 5.3 $\mathbf{J}_1 ** \mathbf{Nil} = [\![x^\omega y^2 t^\omega = x^\omega y t^\omega,\ x^\omega yzt^\omega = x^\omega zyt^\omega]\!]$.

Let A be a profinite set. Then

$$\overline{\Omega}_A(\mathbf{J}_1 ** \mathbf{Nil}) = A^+ \cup \{(v, B, w) \in A^{\mathbb{N}} \times \bar{\mathcal{P}}(A) \times {}^{\mathbb{N}}\!A : c_\infty(v) \cup c_\infty(w) \subseteq B\},$$

and the product is given, for all $u, u' \in A^+$ and for all $(v, B, w), (v', B', w')$ in $A^{\mathbb{N}} \times \bar{\mathcal{P}}(A) \times {}^{\mathbb{N}}\!A$ such that $c_\infty(v) \cup c_\infty(w) \subseteq B, c_\infty(v') \cup c_\infty(w') \subseteq B'$ by:

$$\begin{aligned}
u \cdot u' &= uu' \\
u \cdot (v, B, w) &= (uv, B, w) \\
(v, B, w) \cdot u &= (v, B, wu) \\
(v, B, w) \cdot (v', B', w') &= (v, B \cup c(w) \cup c(v') \cup B', w').
\end{aligned}$$

In case A is finite, each implicit operation x of $\overline{\Omega}_A(\mathbf{J}_1 ** \mathbf{Nil})$ admits a factorization of the form $x = u_0$, $x = u_0x_1u_2$ or $x = u_0x_1u_1y_2u_2$ where $u_0, u_2 \in A^*$, $u_0 \neq 1$ if $x = u_0$, x_1, x_2 are idempotent, the last letter of u_0 is not in $c(x_1)$ (if u_0 is not empty) and the first letter of u_2 is not in $c(x_2)$

(if u_2 is not empty), the word u_1 is linear and $c(u_1) \cap (c(x_1) \cup c(x_2)) = \emptyset$. Furthermore, this factorization is unique up to a permutation of u_1.

Finally, if A is finite and $\mathcal{V}$ is the variety of languages associated to $\mathbf{J}_1 ** \mathbf{Nil}$*, then $A^+\mathcal{V}$ is the Boolean algebra generated by the finite languages and by the languages of the form uB^*v where $u, v \in A^+$ and $B \subseteq A$.*

Proof. Recall that $\overline{\Omega}_A\mathbf{Nil} = A^+ \cup \{0\}$. In $\Gamma_A^2\mathbf{Nil}$, the vertices of the form $(1, x)$ and $(x, 1)$ are alone in their profinitely strongly connected components. So, the profinitely strongly quasi-linear subgraphs X such that $(1, x) \in \min X$ and $(x, 1) \in \max X$ are of the following form:

- If x is finite, that is, $x = a_1 \cdots a_n \in A^+$, then X is the finite path labelled by x, visiting the vertices of the form $(a_1 \cdots a_i, a_{i+1} \cdots a_n)$ $(0 \leq i < n)$.
- If x is not finite, that is $x = 0$, then X consists of an infinite path, with label $v = a_1a_2 \cdots \in A^{\mathbb{N}}$, visiting the vertices $(a_1 \cdots a_i, 0)$ $(i \geq 0)$, an infinite path with label $w = \cdots b_2b_1 \in {}^{\mathbb{N}}A$ visiting the vertices $(0, b_i \cdots b_1)$ $(i \geq 0)$, the vertex $(0, 0)$, and around it, a closed set of loops, containing the loops labelled by letters of $c_\infty(v) \cup c_\infty(w)$.

This establishes a set-theoretic bijection between $\overline{\Omega}_A(\mathbf{J}_1 ** \mathbf{Nil})$ and

$$A^+ \cup \{(v, B, w) \in A^{\mathbb{N}} \times \bar{\mathcal{P}}(A) \times {}^{\mathbb{N}}A : c_\infty(v) \cup c_\infty(w) \subseteq B\}.$$

The rest of the proof is as in Theorem 4.1, noticing that the ω-power of the elements of $\overline{\Omega}_A(\mathbf{J}_1 ** \mathbf{Nil})$ are given by

$$(v, B, w)^\omega = (v, B \cup c(v) \cup c(w), w) \quad \text{and} \quad u^\omega = (u^\infty, c(u), {}^\infty u) \text{ if } u \in A^+.$$

Here ${}^\infty u$ designates the element of ${}^{\mathbb{N}}A$ obtained by repeating u infinitely often “to the left”. □

There exists a two-sided analogue of Straubing’s “principle of the semidirect product” mentioned in Section 4 (Weil [29]). According to it, if $\mathcal{V}$ is the variety of languages associated to a pseudovariety $\mathbf{V}$, and if A is a finite alphabet, then the class of languages recognized by $\mathbf{J}_1 ** \mathbf{V}$ is the Boolean algebra generated by the languages in $A^+\mathcal{V}$ and by the languages of the form LaL', where $a \in A$ and $L, L' \in A^+\mathcal{V}$. Here again, we have found a description of the languages of $\mathbf{J}_1 ** \mathbf{Nil}$ different from that provided by this general method.

Note that, contrary to the situation of Section 4, $\mathbf{J}_1 ** \mathbf{Nil}$ is not equal to $\mathbf{J}_1 ** \mathbf{K}$. For each $n \geq 1$, recall that

$$\begin{aligned} \mathbf{LI}_n &= [\![x_1 \cdots x_n y z_1 \cdots z_n = x_1 \cdots x_n z_1 \cdots z_n]\!], \\ \mathbf{K}_n &= [\![x_1 \cdots x_n x_{n+1} = x_1 \cdots x_n]\!] \text{ and} \\ \mathbf{D}_n &= \mathbf{K}_n^{\mathrm{r}} = [\![x_{n+1} x_n \cdots x_1 = x_n \cdots x_1]\!]. \end{aligned}$$

We let $\mathbf{D} = \mathbf{K}^{\mathrm{r}}$. In particular, $\mathbf{LI} = \bigcup_n \mathbf{LI}_n$, $\mathbf{K} = \bigcup_n \mathbf{K}_n$ and $\mathbf{D} = \bigcup_n \mathbf{D}_n$.

Theorem 5.4 *For each $n \geq 1$, $\mathbf{J}_1 ** \mathbf{LI}_n = \mathbf{J}_1 * \mathbf{D}_{2n+1}$. Consequently,*

$$\mathbf{J}_1 * \mathbf{D} = \mathbf{J}_1 ** \mathbf{D} = \mathbf{J}_1 ** \mathbf{K} = \mathbf{J}_1 ** \mathbf{LI} = \mathbf{LJ}_1.$$

Proof. For each word $w \in A^+$, we let $p_n w$ (resp. $s_n w$) be equal to w if w is of length less than n, and to its prefix (resp. suffix) of length n otherwise. Recall that $\overline{\Omega}_A \mathbf{LI}_n$ can be identified with the set $A^{<n} \cup (A^n \times A^n)$, with product given, for all words u, v of length less than n and for all $w, w' \in A^n$, by

$$\begin{aligned} u \cdot v &= \begin{cases} uv & \text{if } |uv| < n \\ (p_n(uv), s_n(uv)) & \text{if } |uv| \geq n \end{cases} \\ u \cdot (w, w') &= (p_n(uw), w') \\ (w, w') \cdot u &= (w, s_n(w'u)). \end{aligned}$$

The natural morphism ι from A^+ onto $\overline{\Omega}_A \mathbf{LI}_n$ is given by

$$u\iota = \begin{cases} u & \text{if } |u| < n \\ (p_n(u), s_n(u)) & \text{if } |u| \geq n. \end{cases}$$

By Theorem 5.2, $\overline{\Omega}_A(\mathbf{J}_1 ** \mathbf{LI}_n)$ is the set of all pairs (X, x) where $x \in \overline{\Omega}_A \mathbf{LI}_n$, X is a profinitely quasi-linear subgraph of $\Gamma_A^2 \mathbf{LI}_n$, $(1, x) \in \min X$ and $(x, 1) \in \max X$. But $\Gamma_A^2 \mathbf{LI}_n$ is finite, and the vertices of the form $(1, x)$ and $(x, 1)$ are alone in their strongly connected components, so X must be a (finite) path from $(1, x)$ to $(x, 1)$.

More precisely, let $\theta_n \colon A^+ \to \overline{\Omega}_A(\mathbf{J}_1 ** \mathbf{LI}_n)$ be given by

$$\theta_n a = (\{(1, a, 1)\}, a).$$

Then θ_n is onto. If $u = a_1 \cdots a_k$ $(a_i \in A)$, then

$$\begin{aligned} \theta_n u = \Big(&\{(1, a_1, \iota(a_2 \cdots a_k)), \cdots, (\iota(a_1 \cdots a_{i-1}), a_i, \iota(a_{i+1} \cdots a_k)), \cdots \\ &\cdots, (\iota(a_1 \cdots a_{k-1}), a_k, 1)\},\ \iota(a_1 \cdots a_k)\Big). \end{aligned}$$

Thus, $\theta_n u$ is entirely determined by the prefix and suffix of u of length $2n$ and its set of factors of length $2n+1$.

Let now η_{2n+1} be the natural morphism from A^+ onto $\overline{\Omega}_A(\mathbf{J}_1 * \mathbf{D}_{2n+1})$. This morphism can be computed in a similar fashion, and one can verify that $\eta_{2n+1}u$ is also determined by the prefix and the suffix of u of length $2n$ and by its set of factors of length $2n+1$ (this is a well known result, see [10, 11]). Thus, for all words $u, v \in A^+$, we have $\eta_{2n+1}u = \eta_{2n+1}v$ if and only if $\theta_n u = \theta_n v$, and hence $\overline{\Omega}_A(\mathbf{J}_1 * \mathbf{D}_{2n+1})$ and $\overline{\Omega}_A(\mathbf{J}_1 ** \mathbf{LI}_n)$ are isomorphic. Thus, $\mathbf{J}_1 ** \mathbf{LI}_n = \mathbf{J}_1 * \mathbf{D}_{2n+1}$.

Since $\mathbf{J}_1 * \mathbf{D} = \bigcup_n \mathbf{J}_1 * \mathbf{D}_n$ and $\mathbf{J}_1 ** \mathbf{LI} = \bigcup_n \mathbf{J}_1 ** \mathbf{LI}_n$, it follows that $\mathbf{J}_1 * \mathbf{D} = \mathbf{J}_1 ** \mathbf{LI}$. This implies that $\mathbf{J}_1 ** \mathbf{D} = \mathbf{J}_1 ** \mathbf{LI}$ since the semidirect product is a special case of two-sided semidirect product and since $\mathbf{D} \subset \mathbf{LI}$. Now Brzozowski, Simon and McNaughton proved that $\mathbf{J}_1 * \mathbf{D} = \mathbf{LJ}_1$, the class of all finite semigroups S such that $eSe \in \mathbf{J}_1$ for each idempotent e of S [10, 17]. Finally, we observe that $(\mathbf{LJ}_1)^{\mathrm{r}} = \mathbf{LJ}_1$, $\mathbf{J}_1^{\mathrm{r}} = \mathbf{J}_1$, $\mathbf{D}^{\mathrm{r}} = \mathbf{K}$ and $(\mathbf{V} ** \mathbf{W})^{\mathrm{r}} = \mathbf{V}^{\mathrm{r}} ** \mathbf{W}^{\mathrm{r}}$ for all $\mathbf{V}$, $\mathbf{W}$. So $\mathbf{LJ}_1 = \mathbf{J}_1 ** \mathbf{K}$. $\Box$

6 Implicit operations on $\mathbf{J}_1 * \mathbf{G}$ and profinite inverse monoids

The pseudovariety $\mathbf{J}_1 * \mathbf{G}$ received considerable attention in recent years. It is the pseudovariety generated by inverse monoids (Margolis and Pin [20]), and Ash proved that it is exactly the class of monoids in which the idempotents commute [9].

According to Theorem 3.3, $\overline{\Omega}_A(\mathbf{J}_1 * \mathbf{G})$ is the submonoid of $\bar{\mathcal{P}}(\overline{\Omega}_A\mathbf{G} \times A) * \overline{\Omega}_A\mathbf{G}$ consisting of all pairs (X, g) where $g \in \overline{\Omega}_A\mathbf{G}$ and X is a profinitely quasi-linear closed subgraph of the profinite Cayley graph of the free profinite group, such that $1 \in \min X$ and $g \in \max X$.

The geometry of this particular class of profinite graphs was studied in Gildenhuys and Ribes [13], Zalesskiĭ and Mel'nikov [30], Ribes and Zalesskiĭ [26] and Almeida and Weil [6, 7].

This construction of $\overline{\Omega}_A(\mathbf{J}_1 * \mathbf{G})$ is reminiscent of the general construction of certain relatively free E-unitary inverse monoids due to Margolis and Meakin [16]. This section is devoted to giving a profinite version of their results.

Recall that a monoid M is inverse if, for each $x \in M$, there exists a unique element $x^{-1} \in M$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. This is equivalent to requesting M to be regular and to have commuting idempotents. For more details about inverse monoids, see Howie [14] or Petrich [18]. If M is an inverse monoid, we let σ_M be the minimum group congruence of M, that is, the intersection of all congruences ρ on M such that M/ρ is a group. Then σ_M is generated by the pairs (e, f) of idempotents of M, and $x \,\sigma_M\, y$ if and only if $ex = fy$ for some idempotents e and f of M. We write also σ_M for the canonical projection of M onto $\sigma(M) = M/\sigma_M$, the maximal group image of M. Then σ is easily seen to be a functor from the category of inverse monoids into the category of groups.

An important class of inverse monoids is that of E-unitary inverse monoids. An inverse monoid M is *E-unitary* if, for each idempotent e and for each element x of M, ex is idempotent if and only if x is idempotent. This is equivalent to the fact that $E(M)$ constitutes a σ_M-class. Let us observe that if M is E-unitary, then any two elements x and y of M are σ_M-related if and only if xy^{-1} is idempotent. Indeed, if $x \,\sigma_M\, y$, there exist idempotents e and f such that $ex = fy$, so $exy^{-1} = fyy^{-1}$ is idempotent, and hence xy^{-1} is idempotent. Conversely, if xy^{-1} is idempotent, then $xy^{-1} = (xy^{-1})^{-1} = yx^{-1}$, so $xy^{-1}x = xy^{-1}yx^{-1}x = xx^{-1}xy^{-1}y = xy^{-1}y$, and hence $x \,\sigma_M\, y$.

This observation leads to the following proposition.

Proposition 6.1 *A projective limit of E-unitary inverse monoids is E-unitary. Moreover, the restriction of the functor σ to the category of E-unitary inverse monoids commutes with projective limits.*

Proof. Let $M = \varprojlim (M_i)_i$ where the M_i are E-unitary. Let $e = (e_i)_i$ be an idempotent of M, and let $x = (x_i)_i \in M$ be such that ex is idempotent. Then each $e_i x_i$ is idempotent, so $x_i = x_i^2$ and hence $x^2 = x$. Thus M is E-unitary.

If $x = (x_i)_i$ and $y = (y_i)_i$ are in M, then $x \,\sigma_M\, y$ if and only if xy^{-1} is idempotent, that is, if and only if $x_i y_i^{-1} \in E(M_i)$ for each i, and hence

$$x \,\sigma_M\, y \qquad \Longleftrightarrow \qquad \forall i, \quad x_i \,\sigma_{M_i}\, y_i.$$

Thus $\sigma(M) = \varprojlim \left(\sigma(M_i)\right)_i$ and $\sigma_M\colon M \to \sigma(M)$ is given, for each $x = (x_i)_i \in M$, by $\sigma_M x = (\sigma_{M_i} x_i)_i$. $\square$

Another important situation where σ commutes with projective limits is that of profinite inverse monoids. More precisely, we have the following.

Proposition 6.2 *Let M be a profinite inverse monoid. Then M has a minimal ideal G which is a profinite group, $\sigma(M) = G$ and the corresponding morphism $\sigma_M : M \to G$ is given by $\sigma_M x = ex$. Here e is the idempotent of the group G.*

Furthermore, when restricted to profinite inverse monoids, the functor σ commutes with projective limits.

Proof. Let us first consider the case where M admits a minimal ideal G (e.g. M is finite). It is easy to verify that G is a group. Let e be the idempotent of G. Then e is a zero in the semilattice $E(M)$, so that two elements x and y of M are σ_M-related if and only if $ex = ey$. Therefore $\sigma(M) = G$ and the morphism $\sigma_M : M \to G$ is given by $\sigma_M x = ex$.

Let now $M = \varprojlim(M_i)_i$ where the M_i are profinite inverse monoids, each admitting a minimal ideal G_i, with idempotent e_i (e.g. the M_i are finite). Let us assume also that the canonical morphisms $\pi_i : M \to M_i$ are onto. We observe that any onto morphism from M_i onto M_j (for some i, j) maps G_i onto G_j. Therefore $e = (e_i)_i$ is an idempotent of M and $G = \varprojlim(G_i)_i$ is a profinite subgroup of M. Finally, if $x = (x_i)_i \in M$, then $ex = (e_i x_i)_i \in G$, so G is the minimal ideal of M. Furthermore, by the above remark, $\sigma(M) = G = \varprojlim\left(\sigma(M_i)\right)_i$ and $\sigma_M(x_i)_i = (\sigma_{M_i} x_i)_i$. $\Box$

In [16], Margolis and Meakin introduced for each (discrete) set A, the following functor, from the category $\mathcal{G}(A)$ of A-generated groups to the category $\mathcal{M}(A)$ of A-generated E-unitary inverse monoids. If $\gamma : A \to G$ is an A-generated group (that is, γ is a function and γA generates the group G), $M_A(\gamma)$ is the subset of $\mathcal{P}(G \times A) \times G$ consisting of all pairs (X, g) such that X is a finite connected subgraph of the Cayley graph of G with respect to γA, containing 1 and g. Here, similarly to Section 3.2, we identify the subgraphs of a graph with the corresponding sets of edges. Margolis and Meakin proved that this construction is functorial, and that it is left adjoint to the restriction of σ to $\mathcal{M}(A)$ [16, Theorems 2.1 and 3.11].

In an analogous fashion, for each profinite set A, we consider $\hat{\mathcal{M}}(A)$, the category of A-generated profinite E-unitary inverse monoids, and $\hat{\mathcal{G}}(A)$, the category of A-generated profinite groups. If $\gamma : A \to G$ is an object of $\hat{\mathcal{G}}(A)$ (that is, γ is a continuous mapping and the subgroup generated by γA is dense in G), we let $\hat{M}_A(\gamma)$ be the subset of $\bar{\mathcal{P}}(G \times A) \times G$ consisting of all pairs (X, g), where X is a profinitely connected subgraph of the Cayley graph of G with respect to γA, containing 1 and g. Thus $\hat{M}_A(\gamma)$ is a subset of the semidirect product $\bar{\mathcal{P}}(G \times A) * G$ described in Section 3.2. Note that this construction coincides with Margolis and Meakin's $M_A(\gamma)$ in the case where G is finite.

Proposition 6.3 *Let A be a profinite set and let $\gamma: A \to G$ be an A-generated profinite group. Then $\hat{M}_A(\gamma)$ is an E-unitary profinite inverse monoid. Its maximum group image is G and the morphism $\sigma_{\hat{M}(\gamma)}$ from $\hat{M}_A(\gamma)$ onto G is the projection π given by $\pi(X,g) = g$.*

Furthermore, $\hat{M}_A$ is a functor from $\hat{\mathcal{G}}(A)$ into $\hat{\mathcal{M}}(A)$ which commutes with projective limits.

Proof. The first part of the proof is readily adapted from Meakin and Margolis's [16, Theorem 2.1].

Multiplication in $\hat{M}_A(\gamma)$ is given by $(X,g)(Y,h) = (X \cup g \cdot Y, gh)$. X and $g \cdot Y$ are two profinitely connected subgraphs of the Cayley graph of G with respect to γA, and $X \cap g \cdot Y \neq \emptyset$ (since it contains g). So the union $X \cup g \cdot Y$ is profinitely connected [30, Proposition 1.18]. Thus $\hat{M}_A(\gamma)$ is a monoid.

It is not difficult to verify that $\hat{M}_A(\gamma)$ is inverse with

$$(X,g)^{-1} = (g^{-1} \cdot X, g^{-1}),$$

and that the idempotents are the elements of the form $(X,1)$, so that $\hat{M}_A(\gamma)$ is E-unitary. In particular, if $(X,g),(Y,h) \in \hat{M}_A(\gamma)$, then $(X,g)\,\sigma_{\hat{M}_A(\gamma)}\,(Y,h)$ if and only if $(X,g)(Y,h)^{-1}$ is idempotent, that is, if and only if $g = h$. Thus G is the maximal group image of $\hat{M}_A(\gamma)$ and $\sigma_{\hat{M}_A(\gamma)} = \pi$.

Let now $\delta: A \to H$ be another object of $\hat{\mathcal{G}}(A)$ and let $\varphi: G \to H$ be a morphism of $\hat{\mathcal{G}}(A)$. That is, $\varphi \circ \gamma = \delta$, and hence φ is onto. Now φ induces an onto continuous morphism from the Cayley graph of G with respect to γA to the Cayley graph of H with respect to δA. But the continuous homomorphic images of profinitely connected graphs are profinitely connected, so φ induces an onto continuous morphism $\hat{M}_A(\varphi): \hat{M}_A(\gamma) \to \hat{M}_A(\delta)$, given by $\hat{M}_A(\varphi)(X,g) = (\varphi X, \varphi g)$. It is now easy to verify that if φ and ψ are consecutive morphisms of $\hat{\mathcal{G}}(A)$, then $\hat{M}_A(\psi \circ \varphi) = \hat{M}_A(\psi) \circ \hat{M}_A(\varphi)$.

We now assume that $(\gamma_i: A \to G_i)_i$ is a directed system of objects of $\hat{\mathcal{G}}(A)$, and that $(\gamma: A \to G) = \varprojlim(\gamma_i: A \to G_i)_i$. In particular, $G = \varprojlim(G_i)_i$, and the canonical morphisms $\pi_i: G \to G_i$ are onto and such that $\pi_i \circ \gamma = \gamma_i$. Each π_i induces an onto continuous morphism $\hat{M}_A(\pi_i): \hat{M}_A(\gamma) \to \hat{M}_A(\gamma_i)$, and these morphisms, in turn, induce an onto continuous morphism $\varphi: \hat{M}_A(\gamma) \to \varprojlim\,(\hat{M}_A(\gamma_i))_i$. Let now $(X,g) \in \hat{M}_A(\gamma)$ and let $(Y,h) \in$

$\bigcap_i \hat{M}_A(\pi_i)^{-1}(\hat{M}_A(\pi_i)(X,g))$. Then $h \in \bigcap_i \pi_i^{-1}(\pi_i g)$, so $h = g$. Moreover, for each i, $\pi_i Y = \pi_i X$. Let then $y = ((g_i)_i, a) \in Y$. For each j there exists $(g^{(j)}, b^{(j)}) = ((g_i^{(j)})_i, b^{(j)}) \in X$ such that $g_j = g_j^{(j)}$ and $b^{(j)} = a$. Using the definition of the topology on $G = \varprojlim(G_i)_i$, one verifies that $y = ((g_i)_i, a)$ is a cluster point of the set $\{(g^{(j)}, a)\}$. But X is closed, so $y \in X$. Symmetrically, one would prove that $X \subseteq Y$, so that $X = Y$. This proves that φ is an isomorphism, that is $\hat{M}_A(\gamma) = \varprojlim\,(\hat{M}_A(\gamma_i))_i$.

Now Margolis and Meakin proved that if $\gamma\colon A \to G$ is a finite A-generated group, then $\hat{M}_A(\gamma)$ is generated, as an inverse monoid, by the set of all $\hat{\mu}a = (\{(1,a)\}, \gamma a)$ $(a \in A)$ and that $\sigma(\hat{M}_A(\gamma)) = G$ [16, Theorem 2.1]. The fact that the functors σ and $\hat{M}_A$ between the categories $\hat{\mathcal{M}}(A)$ and $\hat{\mathcal{G}}(A)$ commute with projective limits allows us to carry over these results to the profinite case. $\square$

[16, Theorem 2.2] also carries over to the profinite case. More specifically, let $\gamma\colon A \to G$ be a fixed A-generated profinite group. Let $\hat{\mathcal{M}}(A,\gamma)$ be the category of all A-generated E-unitary profinite inverse monoids with maximum group image γ. We just verified that $\hat{\mu}\colon A \to \hat{M}_A(\gamma)$, given by $\hat{\mu}a = (\{(1,a)\}, \gamma a)$ is an object of $\hat{\mathcal{M}}(A,\gamma)$.

Proposition 6.4 *Let $\gamma\colon A \to G$ be an A-generated profinite group. Then $\hat{\mu}\colon A \to \hat{M}_A(\gamma)$ is the initial object of $\hat{\mathcal{M}}(A,\gamma)$.*

Proof. Margolis and Meakin proved that the result holds when G is finite [16, Theorem 2.2]. Let now $\mu\colon A \to M$ be the projective limit of a directed system $(\mu_i\colon A \to M_i)_i$ of finite inverse monoids, and let us assume that $\mu\sigma_M = \gamma$. For each i, let $\gamma_i = \sigma_{M_i} \circ \mu_i\colon A \to G_i$ be the maximum group image of M_i. Then we know that $\gamma = \varprojlim(\gamma_i)_i$ and that, for each i, there exists a morphism $\varphi_i\colon \hat{M}_A(\gamma_i) \to M_i$ such that $\sigma_{M_i} \circ \varphi_i = \pi_i$. Recall that π_i is the canonical projection of $\hat{M}_A(\gamma_i)$ onto G_i. We have the following commutative diagram.

$$
\begin{tikzcd}
A \arrow[r, "\hat{\mu}"] \arrow[d, "\mu"'] \arrow[dr, "\gamma"] \arrow[rr, bend left, "\hat{\mu}_i"] \arrow[dd, bend right, "\mu_i"'] & \hat{M}_A(\gamma) \arrow[r, "\hat{M}_A(\beta_i)"] \arrow[d, "\pi"] \arrow[dl, dotted, "\varphi"] & \hat{M}_A(\gamma_i) \arrow[dd, "\pi_i"] \arrow[ddll, dotted, "\varphi_i"'] \\
M \arrow[r, "\sigma_M"'] \arrow[d, "\alpha_i"'] & G \arrow[dr, "\beta_i"] & \\
M_i \arrow[rr, "\sigma_{M_i}"'] & & G_i
\end{tikzcd}
$$

where the α_i and the β_i are the canonical projections, and $\gamma_i = \beta_i \circ \gamma$.

Since φ_i is entirely determined by the $\varphi_i \hat{\mu}_i a = \mu_i a$ $(a \in A)$, $\varphi = (\varphi_i)_i$ properly defines a morphism from $\hat{M}_A(\gamma)$ into M such that $\varphi \circ \hat{\mu} = \mu$ and $\sigma_M \circ \varphi = \pi$. $\square$

This proposition has the following consequences (see [16, Corollary 2.10 and Theorem 3.11]).

Corollary 6.5 *The functor $\hat{M}_A \colon \hat{\mathcal{G}}(A) \to \hat{\mathcal{M}}(A)$ is left adjoint to the functor σ.*

Proof. As in [16, Theorem 3.11].$\square$

Corollary 6.6 *Let $\mathbf{H}$ be a pseudovariety of groups, let A be a profinite set and let $\gamma_{\mathbf{H}} \colon A \to \overline{\Omega}_A \mathbf{H}$ be the canonical mapping. Then $\hat{M}_A(\gamma_{\mathbf{H}})$ is the free pro-$\widetilde{\mathbf{H}}$ inverse monoid, where $\widetilde{\mathbf{H}}$ is the class of all finite inverse monoids having their maximal group image in $\mathbf{H}$.*

Note. In general, $\widetilde{\mathbf{H}}$ is not a pseudovariety of inverse monoids. However, it is a class that is closed under subdirect products and homomorphic images, so that its A-generated elements form a directed system. By definition, the free pro-$\widetilde{\mathbf{H}}$ inverse monoid is the projective limit of this system.

Proof of Corollary 6.6. Let M be an A-generated element of $\widetilde{\mathbf{H}}$: M is a finite inverse monoid, with maximum group image $\sigma(M) = G \in \mathbf{H}$, and there exists a continuous mapping $\mu: A \to M$ such that μA generates M. Let $\gamma = \sigma_M \circ \mu: A \to G$. Then γA generates G and, by Proposition 6.4, there exists an onto continuous morphism $\varphi: \hat{M}_A(\gamma) \to M$ such that $\sigma_M \circ \varphi = \sigma_{\hat{M}_A(\gamma)}$. So $\sigma_M \circ \varphi \circ \hat{\mu} = \gamma$ and $\hat{M}_A(\gamma)$ is an A-generated element of $\widetilde{\mathbf{H}}$ (since $\sigma(\hat{M}_A(\gamma)) = G$).

Thus, the free pro-$\widetilde{\mathbf{H}}$ inverse monoid over A is the projective limit of the system $(\hat{M}_A(\gamma_i))_i$ where $(\gamma_i)_i$ is the system of A-generated elements of $\mathbf{H}$. The result is then a consequence of Proposition 6.3 since

$$\varprojlim\, (\hat{M}_A(\gamma_i))_i = \hat{M}_A(\varprojlim(\gamma_i)_i) = \hat{M}_A(\gamma_{\mathbf{H}}).$$

□

Corollary 6.7 *Let A be a profinite set and let $\gamma: A \to \overline{\Omega}_A\mathbf{G}$ be the canonical mapping. Then $\hat{M}_A(\gamma)$ is the free profinite inverse monoid.*

This somewhat extended digression away from the realm of pseudovarieties of finite monoids has allowed us to view, for each pseudovariety of groups $\mathbf{H}$, the free pro-$(\mathbf{J}_1 * \mathbf{H})$ monoid over A as a particular topologically closed submonoid of the free pro-$\widetilde{\mathbf{H}}$ inverse monoid over A: the latter consists of all pairs (X, g) where X is a profinitely connected subgraph of $\Gamma_A\mathbf{H}$ containing 1 and g, while $\overline{\Omega}_A(\mathbf{J}_1 * \mathbf{H})$ is the set of those pairs (X, g) where in addition, X is profinitely quasi-linear, $1 \in \min X$ and $g \in \max X$.

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