Algebraic and Topological Theory of Languages*

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Résumé: On dit qu'un langage est de torsion (resp. de torsion bornée, apériodique, apériodique borné) si son monoïde syntaxique est de torsion (resp. de torsion bornée, apériodique, apériodique borné). Nous généralisons les théorèmes sur les langages rationnels de Kleene, Schützenberger et Straubing pour décrire les classes des langages de torsion, de torsion bornée, apériodiques et apériodiques bornés. Ces descriptions imposent la considération de limites de suites de langages et d'automates pour certaines topologies définies par des filtrations du monoïde libre. Nous donnons également un théorème concernant les langages arbitraires sur des alphabets finis.

Abstract: A language is torsion (resp. bounded torsion, aperiodic, bounded aperiodic) if its syntactic monoid is torsion (resp. bounded torsion, aperiodic, bounded aperiodic). We generalize the regular language theorems of Kleene, Schützenberger and Straubing to describe the classes of torsion, bounded torsion, aperiodic and bounded aperiodic languages. These descriptions involve taking limits of sequences of languages and automata for certain topologies defined by filtrations of the free monoid. A theorem for arbitrary languages over finite alphabets is also stated and proved.

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Introduction

The aim of this paper is to generalize the central results of the theory of rational, or recognizable languages (the languages which are recognized by finite automata) to a much wider class of languages over finite alphabets.

We rely in part on the powerful algebraic methods whose use is well-established for recognizable languages. In that more restrained framework, the relevant algebraic objects are the finite monoids. A standard algebraic way of generalizing finiteness is the concept of torsion: an algebraic object is torsion if each of its elements has only finitely many distinct powers, and it is bounded-torsion if the number of distinct powers of its elements is uniformly bounded above by some fixed integer. Around 1900 Burnside conjectured that all finitely generated bounded-torsion groups were finite. This was proved false 70 years later by Adjan-Novikov when the exponent is large and odd. More recently a shorter proof was given by Ol'shanskiĭ using small-cancellation diagrams (see [12]). More recently also, some important results on bounded-torsion monoids were obtained by McCammond [11], de Luca and Varricchio [2,3] and Pereira do Lago [14]. Our main theorems deal with bounded torsion and torsion languages, that is, languages which are recognized by bounded torsion or torsion monoids. Such monoids were already considered by Rhodes [17,18].

Another essential tool of our work reveals interesting connections with topology. Let X be a finite alphabet. We say that a sequence $\mathcal{I} = (I_n)_n$ of subsets of the free monoid X^* is a filtration if $I_0 = X^*$, $I_{n+1} \subseteq I_n$ for all $n \geq 0$ and $\bigcap_n I_n = \emptyset$. Now a filtration $\mathcal{I}$ gives rise to a topology on $X^* \cup \{\infty\}$ for which a basis of open sets is $\big\{\{w\} \bigm| w \in X^*\big\} \cup \big\{I_n \cup \{\infty\} \bigm| n \geq 0\big\}$. In particular a sequence of words $(w_n)_n$ tends to ∞ if and only if, for each $n \geq 0$, all but a finite number of the w_k lie in I_n. This topology can then be extended to define the convergence of sequences of languages in X^* and of sequences of automata over X: a sequence of languages $(L_n)_n$ tends to a language L modulo $\mathcal{I}$ if for each $n \geq 0$, there exists $k \geq 0$ such that $L_m \setminus I_n = L \setminus I_n$ for all $m \geq k$. In an analogous fashion, a sequence of automata $(A_n)_n$ tends to an automaton A modulo $\mathcal{I}$ if for each $n \geq 0$, there exists $k \geq 0$ such that, whenever $m \geq k$ and $u, v \notin I_n$, $(q_0^m \cdot u = q_0^m \cdot v) \Leftrightarrow (q_0 \cdot u = q_0 \cdot v)$ (where q_0^m is the initial state of A_m and q_0 the initial state of A).

Note that this notion of convergence modulo a filtration arises in an intuitive fashion when one considers classical machine models such as, say, Turing machines. For this model, we can consider bounding the amount of time or space or any suitable function of time and space, which we can call "stuff", made available to the machine. Any Turing machine M, when restricted to using at most n units of stuff, is equivalent to a finite-state machine M_n, and it is natural to try and view the language recognized by M as "a limit" of the (rational) languages recognized by the M_n. If we let, for $n \geq 0$, I_n be the set of all words $w \in X^*$ such that M cannot make any decision using less than n units of stuff upon reading w, then $\mathcal{I} = (I_n)_n$ is a filtration and the language of M is the limit of the languages of the M_n with respect to $\mathcal{I}$. This idea is only intuitively presented here, but it is illustrated by a surprising result proved in the second appendix of this paper.

Our main results are characterizations of the classes $\mathcal{L}_{\mathrm{tor}}$, $\mathcal{L}_{\mathrm{btor}}$, $\mathcal{L}_{\mathrm{ap}}$ and $\mathcal{L}_{\mathrm{bap}}$, respectively of all torsion, bounded-torsion, aperiodic and bounded-aperiodic languages. (A monoid S is aperiodic if for each $s \in S$ there exists $n \geq 1$ such that $s^n = s^{n+1}$ and a language is aperiodic if it is recognized by an aperiodic monoid.) These characterizations generalize the thoerems of Kleene, Schützenberger and Straubing on recognizable languages. Recall that these theorems state that the classes of rational (resp. rational aperiodic, rational subgroup-solvable) languages are the least classes containing the finite languages and closed under certain language operations (such as union, product, star, etc.) Our results characterize the classes $\mathcal{L}_{\mathrm{tor}}$, $\mathcal{L}_{\mathrm{btor}}$, $\mathcal{L}_{\mathrm{ap}}$ and $\mathcal{L}_{\mathrm{bap}}$ as the least classes containing the finite languages, closed under some of these languages operations, and closed under taking certain limits with respect to certain filtrations.

Of course these results hint at a possible generalization of Eilenberg's variety (or stream) theorem to classes of arbitrary languages (not just rational), and we explore and prove this generalization in our third appendix.

Part of the proof of these main results relies on combinatorial and algebraic methods more or less of the same flavor as the techniques used classically to deal with rational languages. The second part of these proofs uses in a crucial way the properties of the finitely generated Burnside monoids, recently established by McCammond [11] and Pereira do Lago [14], and the properties of two semigroup expansions which were studied by Birget, Rhodes and Henckell [1,17,6]. One of these expansions, $S \mapsto \widehat{S}^{(3)}$ is particularly interesting to illustrate another point of view on the generalization of finiteness, a concept underlying all of this paper. Given any finitely generated monoid S, a finitely generated monoid $\widehat{S}^{(3)}$ can be constructed along with an onto morphism π from $\widehat{S}^{(3)}$ onto S such that:

(1) $\widehat{S}^{(3)}$ is "close" to S (technically, the inverse image by π of each idempotent of

S satisfies the identity $x^5 = x^6$), and

(2) $\widehat{S}^{(3)}$ contains a sequence of ideals $(J_n)_n$ such that $J_0 = \widehat{S}^{(3)}$, $J_{n+1} \subseteq J_n$ for all n and $\bigcap_n J_n = \emptyset$, and such that each Rees quotient $\widehat{S}^{(3)}/J_n$ is *finite*.

This sequence of *ideals* $(J_n)_n$ is of course reminiscent of our definition of a filtration over a free monoid. Then, if we consider the topology on $\widehat{S}^{(3)} \cup \{\infty\}$ defined as above (with $\{\{s\} \mid s \in \widehat{S}^{(3)}\} \cup \{J_n \cup \{\infty\} \mid n \geq 0\}$ as a basis of open sets), then $\widehat{S}^{(3)} \cup \{\infty\}$ is *compact*. Note that compactness is another natural generalization of finiteness. In fact, the result which we present in our second appendix is essentially an application of this construction, and we give its full details in that appendix.

The precise organization of this paper is the following. In Section 1, we present rapidly the basic definitions and properties of automata, semigroups and syntactic monoids, and we review the statements of the theorems of Kleene, Schützenberger and Straubing. Section 2 is devoted to exploring the first properties of torsion and aperiodic monoids and languages. In Section 3 we introduce the notions of convergence of a sequence of languages and of a sequence of accessible automata modulo a given filtration. Our main theorems are stated in Section 4 and proved in Section 5.

We then consider in three subsequent appendices some connected results. The first one gives a variant of our results in terms of convergent sequences of onto morphisms. The second one is the description of a rational filtration which can be canonically associated to an arbitrary congruence on X^*, and the last one is the generalization to *arbitrary* classes of languages (over finite alphabets) of Eilenberg's variety theorem.

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1 Preliminaries

In this section we will review briefly the definition and basic properties of the objects that we will be dealing with, namely languages and automata. In particular we will remind the reader of the concept of recognizability of a language by a monoid, and we will recall some of the fundamental results of the theory of rational languages. For a more detailed presentation of the various aspects of the theory of languages and automata, the reader is referred to [13,4,10,16].

1.1 Languages and automata

Throughout this paper, X will denote a *finite* non-empty set called the *alphabet*. Its elements are called *letters*. Finite (possibly empty) sequences of letters are called *words*. The set X^* of all words over the alphabet X is a monoid under concatenation. Its identity is the empty word, denoted 1. The monoid X^* is the free monoid over X. A *language* over X is any subset of X^*.

An *automaton* over X is a 5-tuple $A = (Q, X, q_0, \lambda, F)$ where Q is a countable (*not necessarily finite*) set called the *state set*, $q_0 \in Q$ is called the *initial state*, $\lambda\colon Q \times X \to Q$ is the *transition function* and $F \subseteq Q$ is called the set of *final states*. When there is no ambiguity as to which automaton is being discussed, we write $\lambda(q, x) = q \cdot x$ ($q \in Q$ and $x \in X$). The function λ is extended to $\lambda\colon Q \times X^* \to Q$ by letting

$$\begin{aligned} q \cdot 1 &= q && \text{for all } q \in Q, \\ q \cdot wx &= (q \cdot w) \cdot x && \text{for all } q \in Q,\ x \in X \text{ and } w \in X^*. \end{aligned}$$

We will always suppose our automata to be *accessible*, that is, they satisfy $q_0 \cdot X^* = Q$. The *language recognized by* A is $L(A) = \{w \in X^* \mid q_0 \cdot w \in F\}$.

Let $L \subseteq X^*$ be a language. The *translates* of L are the languages

$$\begin{aligned} u^{-1}L &= \{v \in X^* \mid uv \in L\} \quad (u \in X^*), \\ Lu^{-1} &= \{v \in X^* \mid vu \in L\} \quad (u \in X^*) \text{ and} \\ u^{-1}Lv^{-1} &= (u^{-1}L)v^{-1} = u^{-1}(Lv^{-1}) \quad (u, v \in X^*). \end{aligned}$$

It is easy to verify that, for each $u, v \in X^*$, then $u^{-1}(v^{-1}L) = (vu)^{-1}L$. Let us note also the following simple remark. If $A_0 = (Q, X, q_0, \lambda, F)$ is an automaton, if $u \in X^*$ and $q_1 = q_0 \cdot u$, let $A_1 = (Q, X, q_1, \lambda, F)$. Then $L(A_1) = u^{-1}L(A_0)$.

To each language $L \subseteq X^*$ we associate a canonical automaton $A(L) = (Q, X, q_0, \lambda, F)$ in the following way:

$$\begin{aligned} &Q = \{u^{-1}L \mid u \in X^*\}, \qquad q_0 = L = 1^{-1}L,\\ &F = \{u^{-1}L \mid u \in X^* \text{ and } 1 \in u^{-1}L\},\\ &\lambda(u^{-1}L, x) = (ux)^{-1}L \quad \text{for } u \in X^* \text{ and } x \in X. \end{aligned}$$

In this automaton, for all $w \in X^*$, we have

$$q_0 \cdot w = (1^{-1}L) \cdot w = w^{-1}L.$$

Therefore $w \in L(A(L))$ if and only if $1 \in w^{-1}L$, that is, if and only if $w \in L$. Thus $L(A(L)) = L$. This shows in particular that every language is recognized by some automaton. In general, a language can be recognized by several different automata. However, $A(L)$ is the *minimal automaton* of L in the following sense.

Proposition 1.1 *Let $L \subseteq X^*$, let $A(L) = (Q_L, X, L, \lambda_L, F_L)$ and let $A = (Q, X, q_0, \lambda, F)$ be any automaton recognizing L. Then there exists a surjective mapping $\pi: Q \to Q_L$ such that $q_0\pi = L$, $F\pi = F_L$ and $q\pi \cdot u = (q \cdot u)\pi$ for all $q \in Q$ and $u \in X^*$. (We say that A* reduces *to $A(L)$.) Furthermore, if B is another automaton such that each automaton recognizing L reduces to B, then B is isomorphic to $A(L)$.*

Proof. Well-known. □

In the sequel it will be convenient to consider automata with unspecified set of final states, that is, of the form (Q, X, q_0, λ). We still call these objects automata and we let $\mathcal{A}(X)$ be the class of all automata with unspecified set of final states over X. If $A \in \mathcal{A}(X)$, we let $\mathcal{L}(A)$ be the set of all the languages that are recognized by A (when the set of final states assumes all possible values).

1.2 Syntactic monoids

We first recall a few basic notions on semigroups. Readers are referred to [4,10,16,8] for more details. A *semigroup* is a pair $(S, \cdot)$ where S is a set and $\cdot$ is an associative binary operation on S. In a semigroup S, an *idempotent* is an element e such that $e^2 = e$. An *identity* (resp. *zero*) is an idempotent e such that $es = se = s$ (resp. $es = se = e$) for all $s \in S$. An identity (resp. zero) is usually denoted 1 (resp. 0). Any semigroup has at most one identity (resp. zero), but it may have an unrestricted number of idempotents. We say that S is a *monoid* if it has an identity. If S is a semigroup we define S^1 to be the monoid equal to S if S is a monoid, and to $S \cup \{1\}$ otherwise (where 1 is an adjoined identity).

If $A, B \subseteq S$, we let $AB = \{ab \mid a \in A,\ b \in B\}$. Then the power set of S, $\mathcal{P}(S)$, is a semigroup. If $T \subseteq S$ and $T^2 \subseteq T$, we say that T is a *subsemigroup* of S. Let $A \subseteq S$. The subsemigroup $\langle A \rangle$ *generated by* A is the least subsemigroup of S containing A, that is, the set $\bigcup_{n \geq 1} A^n$ of all finite products of elements of A. We say that A generates S if $S = \langle A \rangle$ and that S is *finitely generated* if S is generated by a finite set.

An *ideal* of a semigroup S is a subset I of S such that $S^1 I S^1 = I$. If I is a non-empty ideal of S, a new semigroup $(S/I, *)$ is constructed, called the *Rees quotient* of S by I, by letting $S/I = S \setminus I \cup \{0\}$ and, for $s, s' \in S/I$,

$$s * s' = \begin{cases} ss' & \text{if } ss' \notin I, \\ 0 & \text{otherwise.} \end{cases}$$

If S and T are semigroups, a *morphism* $\varphi\colon S \to T$ is a mapping such that $(ss')\varphi = (s\varphi)(s'\varphi)$ for all $s, s' \in S$. When S and T are both monoids, we will implicitly assume that $1\varphi = 1$. We say that T *divides* S if there exists a surjective morphism from a subsemigroup of S onto T. If I is a non-empty ideal of S, then the canonical projection $\pi\colon S \to S/I$ defined by

$$s\pi = \begin{cases} s & \text{if } s \notin I, \\ 0 & \text{otherwise.} \end{cases}$$

is an onto morphism. A *congruence* on a semigroup S is an equivalence relation $\sim$ such that, for all $s, s' \in S$ and $u, v \in S^1$, $s \sim s'$ implies $usv \sim us'v$. If $\varphi\colon S \to T$ is a morphism and if we define $\sim_\varphi$ on S by $s \sim_\varphi s'$ if and only if $s\varphi = s'\varphi$, then $\sim_\varphi$ is a congruence. Conversely, if $\sim$ is a congruence, then the set $S/\!\sim$ of $\sim$-classes is naturally equipped with a semigroup structure given by $[s] \cdot [s'] = [ss']$ for all $s, s' \in S$ (where $[s]$ is the $\sim$-class of s). The canonical projection from S onto $S/\!\sim$ is a morphism.

Morphisms on free monoids will be crucial in the sequel. Let S be a monoid and let X be an alphabet. Recall that, for any mapping $\varphi\colon X \to S$, there exists a unique morphism from X^* into S extending φ.

Let $A = (Q, X, q_0, \lambda) \in \mathcal{A}(X)$ be an automaton. For each word $u \in X^*$, let $u\mu$ be the function from Q to Q given by $q \mapsto q \cdot u$. Then μ defines a morphism from X^* into the monoid of functions from Q into itself (a monoid under the composition of functions). We say that μ is the *transition morphism* of A and we denote the range of μ by $S(A)$, the *transition monoid* of A. Note that μ and $S(A)$ depend only on Q, X and λ. Let L be a language recognized by A with set of final states F, and let P be the set of functions from Q to Q which map q_0 to an element of F. Then $L = P\mu^{-1}$.

In general we say that a morphism $\varphi: X^* \to S$ *recognizes* a language $L \subseteq X^*$ if there exists a subset P of S such that $L = P\varphi^{-1}$. Thus, if an automaton recognizes L, then its transition morphism recognizes L as well. Conversely, let $\varphi: X^* \to S$ be an onto morphism recognizing a language L. Then $L = P\varphi^{-1}$ for some $P \subseteq S$. We define an automaton $A(S) = (S^1, X, 1, \lambda) \in \mathcal{A}(X)$ by letting $\lambda(s, x) = s\,(x\varphi)$ ($s \in S^1$ and $x \in X$). If we choose P for the set of final states, then this automaton recognizes L.

Let $L \subseteq X^*$. The *syntactic congruence* $\equiv_L$ of L in X^* is the largest (coarsest) congruence for which X is a union of classes. One can easily prove (see for instance [4,10]) that $\equiv_L$ is given by

$$w_1 \equiv_L w_2 \quad \text{if and only if} \quad \forall u, v \in X^*,\ uw_1v \in L \Longleftrightarrow uw_2v \in L.$$

The quotient monoid $X^*/\equiv_L$ is called the *syntactic monoid* of L and is denoted $S(L)$. The *syntactic morphism* of L is the canonical projection from X^* onto $S(L)$. By definition of $\equiv_L$, the language L is recognized by its syntactic morphism. Furthermore, the following results are well-known (see [4,16]).

Proposition 1.2 *Let $L \subseteq X^*$ and let $\eta_L: X^* \to S(L)$ be its syntactic morphism.*
(1) η_L is the transition morphism of the minimal automaton $A(L)$ of L.
(2) If a morphism $\varphi: X^* \to S$ recognizes L, then $S(L) < S$.

Let $L_1, \ldots, L_n$ be finitely many languages in X^*. We define $BT(L_1, \ldots, L_n)$ to be the least family of languages of X^* containing the languages of the form $u^{-1}L_iv^{-1}$ ($1 \leq i \leq n$ and $u, v \in X^*$) and closed under complement and arbitrary unions and intersections.

Proposition 1.3 (**Little boxes theorem**) *Let $L_1, \ldots, L_n$ be finitely many languages in X^* and let $\eta_i: X^* \to S(L_i)$ be their respective syntactic morphisms. Let η be the morphism from X^* into $S(L_1) \times \cdots \times S(L_n)$ defined by $w\eta = (w\eta_1, \ldots, w\eta_n)$. Then $L \in BT(L_1, \ldots, L_n)$ if and only if L is recognized by η.*

Proof. The proof relies on the following important remark: Let $w \in X^*$. Then

$$w\eta_i\eta_i^{-1} = \bigcup u^{-1}L_iv^{-1} \setminus (\bigcup x^{-1}L_iy^{-1})$$

where the first union runs over all pairs of words (u, v) such that $uwv \in L_i$, and the other union runs over all pairs (x, y) such that $xwy \notin L_i$. This is immediate by the characterization of the syntactic congruence.

Now, if $w \in X^*$, then $w\eta\eta^{-1} = \bigcap_{i=1}^n w\eta_i\eta_i^{-1}$. So the languages recognized by η are all in $BT(L_1, \ldots, L_n)$. Conversely let us remark that if $w\eta_i = w'\eta_i$ and $w \in u^{-1}L_iv^{-1}$ for some i, u and v, then $w' \in u^{-1}L_iv^{-1}$. That is, each $u^{-1}L_iv^{-1}$ is recognized by the morphism η_i and hence by the morphism η. On the other hand, any Boolean combination (using complement and arbitrary unions and intersections) of languages recognized by η is also recognized by η. So every language in $BT(L_1, \ldots, L_n)$ is recognized by η. □

Corollary 1.4 *The set of languages recognized by the minimal automaton of a language L is $BT(L)$.*

1.3 Rational languages

If $L, L' \subseteq X^*$, we define the *product* LL' and the *star* L^* by

$$LL' = \{ww' \mid w \in L,\ w' \in L\} \text{ and}$$
$$L^* = \{1\} \cup \{w_1 \cdots w_n \mid n \geq 1,\ w_i \in L \text{ for all } i\}.$$

The class of *rational* languages over X is the least class of languages containing the finite subsets of X^* and closed under union, product and star. The following theorem, due to Kleene [9] is a fundamental result of the theory of rational languages.

Theorem 1.5 **(Kleene's theorem)** *Let $L \subseteq X^*$. The following are equivalent.*
(1) L is rational.
(2) L is recognized by a finite automaton.
(3) L is recognized by a finite monoid.
(4) $A(L)$ is finite.
(5) $S(L)$ is finite.

Proposition 1.3 implies immediately the following corollary.

Corollary 1.6 *If $L_1, \cdots, L_n \subseteq X^*$ are rational languages, then $BT(L_1, \cdots, L_n)$ is a finite set of languages, all of which are rational.*

An important subclass of the rational languages is the class of *star-free* languages: it is the least class containing the finite sets and closed under Boolean operations and product. Schützenberger [19] gave the following characterization of the star-free languages. We say that a monoid S is *aperiodic* if

$$\forall s \in S,\ \exists n \geq 1, \qquad s^{n+1} = s^n.$$

Theorem 1.7 **(Schützenberger's theorem)** *Let $L \subseteq X^*$. The following are equivalent.*

(1) L is star-free.

(2) L is recognized by a finite aperiodic monoid.

(3) $S(L)$ is finite and aperiodic.

Again we have

Corollary 1.8 *If $L_1, \cdots, L_n \subseteq X^*$ are star-free languages, then all the languages in $BT(L_1, \cdots, L_n)$ are star-free.*

Because of Schützenberger's theorem, star-free languages are also called *rational aperiodic* languages. It is not difficult to verify that a finite monoid is aperiodic if and only if its subgroups are trivial.

Two other subclasses of the rational languages will be of interest for us. We say that a language L is *rational subgroup-cyclic* (resp. *rational subgroup-solvable*) if each subgroup of its syntactic monoid is cyclic (resp. solvable). In particular each star-free language is rational subgroup-cyclic and each rational subgroup-cyclic language is rational subgroup-solvable. If $L \subseteq X^*$, if p is prime and $0 \leq q < p$, we define $\langle L, p, q\rangle$ to be the set of all words w having a number of prefixes in L congruent to q mod p. Straubing [20] proved the following characterization of rational subgroup-solvable languages.

Theorem 1.9 **(Straubing's theorem)** *Let $L \subseteq X^*$. The following are equivalent.*

(1) L can be obtained from the finite subsets of X^* using only Boolean operations, products and operations of the form $L \mapsto \langle L, p, q\rangle$ where p is prime and $0 \leq q < p$.

(2) L is recognized by a finite monoid all of whose subgroups are solvable.

(3) L is rational subgroup-solvable.

Corollary 1.10 *If $L_1, \cdots, L_n \subseteq X^*$ are rational subgroup-solvable, then all the languages in $BT(L_1, \cdots, L_n)$ are rational subgroup-solvable.*

2 Torsion and aperiodic languages and semigroups

Let S be a semigroup. We say that S is *torsion* if

$$\forall s \in S,\ \exists n \geq 1, \quad s^n \text{ is idempotent.}$$

For $n \geq 1$, we say that S is *n-bounded-torsion* if s^n is idempotent for all s in S. Finally we say that S is *bounded-torsion* if S is n-bounded-torsion for some n. We denote by $\mathbf{Tor}$, $\mathbf{bTor}$ and $\mathbf{bTor}_n$ the classes of torsion, bounded-torsion and n-bounded-torsion semigroups. All finite semigroups are bounded-torsion.

Recall that S is *aperiodic* if

$$\forall s \in S,\ \exists n \geq 1, \quad s^n = s^{n+1}.$$

We say that S is *n-bounded-aperiodic* for $n \geq 1$ if $s^n = s^{n+1}$ for all $s \in S$. Finally we say that S is *bounded-aperiodic* if S is n-bounded-aperiodic for some n. We denote by $\mathbf{Ap}$, $\mathbf{bAp}$ and $\mathbf{bAp}_n$ the classes of aperiodic, bounded-aperiodic and n-bounded-aperiodic semigroups. Notice that finite aperiodic semigroups are bounded-aperiodic. The following important properties are easily verified.

Proposition 2.1 *The following strict containments hold:*

$$\begin{array}{lll} \mathbf{bAp}_n \subset \mathbf{bAp}_{n+1}, & \mathbf{bTor}_n \subset \mathbf{bTor}_{n+1}, & \mathbf{bTor}_n \cap \mathbf{Ap} = \mathbf{bAp}_n \\ \mathbf{bAp}_n \subset \mathbf{bTor}_n, & \mathbf{bAp} \subset \mathbf{bTor}, & \mathbf{Ap} \subset \mathbf{Tor}. \end{array}$$

If T divides S and if S is in $\mathbf{Tor}$ (resp. $\mathbf{Ap}$, $\mathbf{bTor}$, $\mathbf{bAp}$, $\mathbf{bTor}_n$, $\mathbf{bAp}_n$), then so is T.

The classes $\mathbf{Tor}$, $\mathbf{bTor}$, $\mathbf{Ap}$ and $\mathbf{bAp}$ are closed under finite direct product and the classes $\mathbf{bTor}_n$ and $\mathbf{bAp}_n$ ($n \geq 1$) are closed under arbitrary direct product.

We will also use the following property of the classes $\mathbf{Tor}$, $\mathbf{Ap}$, $\mathbf{bTor}$, $\mathbf{bAp}$, $\mathbf{bTor}_n$ and $\mathbf{bAp}_n$. Let $\pi\colon S \to T$ be a morphism. We say that π is *torsion* (resp. *aperiodic, n-bounded-torsion, n-bounded-aperiodic*) if, for each idempotent e of T, the semigroup $e\pi^{-1}$ is in $\mathbf{Tor}$ (resp. $\mathbf{Ap}$, $\mathbf{bTor}_n$, $\mathbf{bAp}_n$). We say that π is *bounded-torsion* (resp. *bounded-aperiodic*) if there exists $n \geq 1$ such that π is n-bounded-torsion (resp. n-bounded-aperiodic). Note that this is different from requiring that $e\pi^{-1} \in \mathbf{bTor}$ (resp. $\mathbf{bAp}$) for each idempotent e of T.

Proposition 2.2 *Let $\pi: S \to T$ be a morphism.*

(1) If $T \in \mathbf{Tor}$ (resp. $\mathbf{Ap}$) and π is a torsion (resp. aperiodic) morphism, then $S \in \mathbf{Tor}$ (resp. $\mathbf{Ap}$).

(2) Let $n, n' \geq 1$. If $T \in \mathbf{bTor}_n$ (resp. $\mathbf{bAp}_n$) and π is a n'-bounded-torsion (resp. n'-bounded-aperiodic) morphism, then $S \in \mathbf{bTor}_{nn'}$ (resp. $\mathbf{bAp}_{n+n'}$).

(3) If $T \in \mathbf{bTor}$ (resp. $\mathbf{bAp}$) and π is a bounded-torsion (resp. bounded-aperiodic) morphism, then $S \in \mathbf{bTor}$ (resp. $\mathbf{bAp}$).

Proof. We prove (1) for torsion semigroups and morphisms. The other proofs are similar. Let $\pi: S \to T$ be a torsion morphism with $T \in \mathbf{Tor}$. Let $s \in S$. Then $(s\pi)^n = (s\pi)^{2n}$ for some $n \geq 1$. Thus $e = s^n\pi$ is an idempotent of T and $s^n \in e\pi^{-1}$. Since $e\pi^{-1} \in \mathbf{Tor}$, there exists $n' \geq 1$ such that $(s^n)^{n'} = (s^n)^{2n'}$, that is $s^{nn'} = s^{2nn'}$. So $S \in \mathbf{Tor}$. □

We now turn to languages. We say that a language $L \subseteq X^*$ is *torsion* (resp. *aperiodic, bounded-torsion, bounded-aperiodic, n-bounded-torsion, n-bounded-aperiodic* ($n \geq 1$)) if it is recognized by some automaton A such that $S(A) \in \mathbf{Tor}$ (resp. $\mathbf{Ap}$, $\mathbf{bTor}$, $\mathbf{bAp}$, $\mathbf{bTor}_n$, $\mathbf{bAp}_n$). We denote these classes of languages by $\mathcal{L}_{\rm tor}$, $\mathcal{L}_{\rm ap}$, $\mathcal{L}_{\rm btor}$, $\mathcal{L}_{\rm bap}$, $\mathcal{L}_{\rm btor}^{(n)}$, $\mathcal{L}_{\rm bap}^{[n]}$. Of course we have

$$\mathcal{L}_{\rm bap}^{[n]} \subset \mathcal{L}_{\rm bap}^{[n+1]}, \qquad \mathcal{L}_{\rm btor}^{(n)} \subset \mathcal{L}_{\rm btor}^{(n+1)}, \qquad \mathcal{L}_{\rm btor}^{(n)} \cap \mathcal{L}_{\rm ap} = \mathcal{L}_{\rm bap}^{[n]}$$
$$\mathcal{L}_{\rm bap}^{[n]} \subset \mathcal{L}_{\rm btor}^{(n)}, \qquad \mathcal{L}_{\rm bap} \subset \mathcal{L}_{\rm btor}, \qquad \mathcal{L}_{\rm ap} \subset \mathcal{L}_{\rm tor}.$$

Since rational languages are recognized by finite automata, all rational languages are bounded-torsion and all star-free languages are bounded-aperiodic. Note also the following proposition, which is immediate using Propositions 1.2 and 2.1.

Proposition 2.3 *Let $L \subseteq X^*$. L is torsion (resp. bounded-torsion, n-bounded-torsion, aperiodic, bounded-aperiodic, n-bounded-aperiodic) if and only if it is recognized by a monoid in $\mathbf{Tor}$ (resp. $\mathbf{bTor}$, $\mathbf{bTor}_n$, $\mathbf{Ap}$, $\mathbf{bAp}$, $\mathbf{bAp}_n$), if and only if $S(L)$ is in $\mathbf{Tor}$ (resp. $\mathbf{bTor}$, $\mathbf{bTor}_n$, $\mathbf{Ap}$, $\mathbf{bAp}$, $\mathbf{bAp}_n$).*

It is well-known that not all torsion languages are aperiodic. For instance the language of all words of even length on an alphabet X is rational but not aperiodic.

The set of all square-free words over a 3-letter alphabet is an aperiodic language which is not rational. More precisely, this language is in $\mathcal{L}_{\rm ap}^{[2]}$. This is a consequence of the existence of an infinite square-free word over a 3-letter alphabet.

Torsion and bounded torsion groups have received considerable attention. Examples of finitely generated torsion groups which are not bounded-torsion were exhibited by Golod and Shafarevitch (see Herstein [7]), by Grigorchuk [5], etc. These provide examples of torsion languages which are not bounded-torsion. The existence of infinite (Burnside) groups with two generators and fixed exponent k (for k large enough and odd) provides other examples of languages that are bounded-torsion and not rational. See [12].

Finally, not all languages are torsion. For instance the Dyck language D over two letters, defined by

$$D_0 = \{1\}, \qquad D_{n+1} = (bD_n^* a)^* \text{ and } \qquad D = \bigcup_{n \geq 0} D_n$$

is not a torsion language.

Let w be a non-empty word. Recall that $w^* = \{w^n \mid n \geq 0\}$ and that w is *primitive* if $w \in u^*$ implies $w = u$. In particular each non-empty word is a power of a primitive word. For $L \subseteq X^*$ we define $K(L, w)$ by $K(L, w) = \{k \geq 0 \mid w^k \in L\}$. Let $n \geq 1$. We say that a set K of integers is *ultimately n-periodic* if there exists $t \geq 0$ such that $k \geq t$ and $k \in K$ implies $k + n \in K$. We say that K is *ultimately periodic* if it is ultimately n-periodic for some n. When $n = 1$ we speak of an *ultimately aperiodic* set of integers. The least t for which the above implication holds is called the *threshold*.

We have the following characterization of the classes $\mathcal{L}_{\text{tor}}$, $\mathcal{L}_{\text{ap}}$, $\mathcal{L}_{\text{tor}}^{(n)}$ and $\mathcal{L}_{\text{ap}}^{[n]}$.

Proposition 2.4 *Let $L \subseteq X^*$ and let $n \geq 1$.*

(1) $L \in \mathcal{L}_{\text{tor}}$ if and only if, for all $L' \in BT(L)$ and for all $w \in X^*$, $L' \cap w^*$ is rational, which is equivalent to requesting that $L' \cap w^* \in \mathcal{L}_{\text{tor}}$ or that $K(L', w)$ be ultimately periodic.

(2) $L \in \mathcal{L}_{\text{ap}}$ if and only if, for all $L' \in BT(L)$ and for all $w \in X^*$, $L' \cap w^*$ is rational and aperiodic, which is equivalent to requesting that $L' \cap w^* \in \mathcal{L}_{\text{ap}}$ or that $K(L', w)$ be ultimately aperiodic.

(3) $L \in \mathcal{L}_{\text{tor}}^{(n)}$ if and only if, for all $L' \in BT(L)$ and for all $w \in X^*$, $K(L', w)$ is ultimately n-periodic with threshold n.

(4) $L \in \mathcal{L}_{\text{ap}}^{[n]}$ if and only if, for all $L' \in BT(L)$ and for all $w \in X^*$, $K(L', w)$ is ultimately aperiodic with threshold n.

(5) Statements (1)–(4) above still hold if we restrict the words w to being primitive.

Proof. Let us prove (1) and the corresponding assertion of (5). Let us first assume that $L \in \mathcal{L}_{\rm tor}$, and let $L' \in BT(L)$ and $w \in X^*$. By Corollary 1.4, L' is recognized by the syntactic morphism of L and by Proposition 2.3 this implies that $L' \in \mathcal{L}_{\rm tor}$. Therefore there exists $n \geq 1$ such that $w^n \equiv w^{2n}$ where $\equiv$ is the syntactic congruence of L', and hence $w^k \equiv w^{k+n}$ for all $k \geq n$. Thus $K(L', w)$ is ultimately n-periodic with threshold n. Therefore

$$L' \cap w^* = \{w^k \mid k < n \text{ and } w^k \in L'\} \cup \bigcup_{\substack{n \leq k < 2n \\ w^k \in L'}} w^k(w^n)^*$$

is rational (and hence torsion).

For the converse let us assume that $L \notin \mathcal{L}_{\rm tor}$ and let η_L be its syntactic morphism. Then there exists $w \in X^*$ such that the $w^n\eta_L$ are pairwise distinct. We can further assume that w is primitive. Let now K be a non-ultimately periodic set of integers, say K is the set of all primes, and let $P = \{w^k\eta_L \mid k \in K\}$. Let $L' = P\eta_L^{-1}$. Then L' is recognized by η_L, so that $L' \in BT(L)$. But $K(L', w) = K$, which is not ultimately periodic.

The first part of this proof can be easily adapted to show that if $L \in \mathcal{L}_{\rm ap}$ (resp. $\mathcal{L}_{\rm btor}^{(n)}$, $\mathcal{L}_{\rm bap}^{[n]}$), then $K(L', w)$ is ultimately aperiodic (resp. ultimately n-periodic with threshold n, ultimately aperiodic with threshold n) for all $L' \in BT(L)$ and $w \in X^*$, that is, the direct part of statements (2) to (4). For the converse part in statement (2), we consider $L \notin \mathcal{L}_{\rm ap}$ and η_L its syntactic morphism. We already know that if $L \notin \mathcal{L}_{\rm tor}$ then we can find $L' \in BT(L)$ and w a primitive word such that $K(L', w)$ is not ultimately periodic. Let us now assume that $L \in \mathcal{L}_{\rm tor} \setminus \mathcal{L}_{\rm ap}$. Then there exists a word w in X^* (which can be assumed to be primitive) such that $w^*\eta_L$ consists of exactly $a + b$ elements $1, w\eta_L, \ldots, w^a\eta_L, w^{a+1}\eta_L, \ldots, w^{a+b-1}\eta_L$ with $w^a\eta_L = w^{a+b}\eta_L$, $a \geq 0$ and $b \geq 2$. Let $P = w^a\eta_L$ and $L' = P\eta_L^{-1}$. Then L' is recognized by η_L and hence $L' \in BT(L)$. Furthermore $K(L', w) = \{a + bk \mid k \geq 0\}$, which is not ultimately aperiodic.

Similarly, for the converse part of statement (3) (resp. (4)), it is sufficient to consider $L \in \mathcal{L}_{\rm tor} \setminus \mathcal{L}_{\rm btor}^{(n)}$ (resp. $\mathcal{L}_{\rm ap} \setminus \mathcal{L}_{\rm bap}^{[n]}$) and η_L its syntactic morphism. Then there exists $w \in X^*$ (which can be assumed to be primitive) such that $w^n\eta_L \neq w^{2n}\eta_L$ (resp. $w^n\eta_L \neq w^{n+1}\eta_L$). If we let $P = w^n\eta_L$ and $L' = P\eta_L^{-1}$, then $L' \in BT(L)$ and $K(L', w)$ contains n but not $2n$ (resp. $n+1$), so $K(L', w)$ is not ultimately n-periodic (resp. ultimately aperiodic) with threshold n. $\square$

Finally we note the following characterization of automata whose transition monoid is torsion or aperiodic.

Proposition 2.5 *Let $A = (Q, X, q_0, \lambda) \in \mathcal{A}(X)$. Then $\mathcal{L}(A) \subseteq \mathcal{L}_{\text{tor}}$ (resp. $\mathcal{L}_{\text{ap}}$) if and only if $S(A) \in$ **Tor** (resp. **Ap**).*

Proof. If $S(A) \in$ **Tor** (resp. **Ap**), then $\mathcal{L}(A) \subseteq \mathcal{L}_{\text{tor}}$ (resp. $\mathcal{L}_{\text{ap}}$) by definition. We prove the converse statement concerning torsion languages. This proof can easily be modified to prove the aperiodic case. Let $\mu: X^* \to S(A)$ be the transition morphism of A. We assume that $S(A) \notin$ **Tor** and we will prove that $\mathcal{L}(A) \not\subseteq \mathcal{L}_{\text{tor}}$. Since $S(A) \notin$ **Tor**, there exists $w \in X^*$ such that the $w^n\mu$ are pairwise distinct.
First case. $\exists q \in Q,\ \forall a, b \geq 1,\ q \cdot w^a \neq q \cdot w^{a+b}$.
Let $F = \{q \cdot w^p \mid p \text{ prime}\}$ and let L be recognized by A with F as the set of final states. Let $u \in X^*$ be such that $q_0 \cdot u = q$ and let $L' = u^{-1}L$. Then $K(L', w) = \{k \geq 1 \mid uw^k \in L\}$ is the set of prime numbers, which is not ultimately periodic. So Proposition 2.4 shows that L is not torsion.
Second case. $\forall q \in Q,\ \exists a, b \geq 1,\ q \cdot w^a = q \cdot w^{a+b}$.
For each $q \in Q$ let C_q be the "cyclic" part of $q \cdot w^*$, that is,

$$C_q = \{q' \in Q \mid \exists a, b \geq 1,\ q \cdot w^a = q \cdot w^{a+b} = q'\}.$$

Let $a_q \geq 0$ be minimum such that $q \cdot w^{a_q} \in C_q$ and let $b_q \geq 1$ be minimum such that $q \cdot w^{a_q} = q \cdot w^{a_q + b_q}$. Note that $C_q = \{q \cdot w^n \mid n \geq a_q\}$ has cardinality b_q. Furthermore the sets C_q are pairwise disjoint. Since $S(A) \notin$ **Tor**, either the set $\{a_q \mid q \in Q\}$ or the set $\{b_q \mid q \in Q\}$ is unbounded. Therefore we can choose a sequence $(q_n)_n$ of states such that we have

$$\begin{aligned} \text{either}\ &: \quad a_{q_1} < a_{q_2} < \cdots < a_{q_n} < \cdots \\ \text{or}\ &: \quad b_{q_1} < b_{q_2} < \cdots < b_{q_n} < \cdots \end{aligned}$$

Let us first assume that $a_{q_1} < a_{q_2} < \cdots$ Let L be the language recognized by A with set of final states $\bigcup_{n \geq 1} C_{q_n}$ and let $\equiv_L$ be its syntactic congruence. Note that if $u \in X^*$ and $q_0 \cdot u = q_n$, then $uw^m \in L$ if and only if $m \geq a_{q_n}$. If L is torsion there exists $a, b \geq 1$ such that $w^a \equiv_L w^{a+b}$. Let us choose n and m such that $a < a_{q_n} < a + mb$, and let $u \in X^*$ be such that $q_0 \cdot u = q_n$. Then $uw^a \notin L$ and $uw^{a+mb} \in L$, thus contradicting $w^a \equiv_L w^{a+b}$. So L is not torsion.

Let us now assume that $b_{q_1} < b_{q_2} < \cdots$ Let L be the language recognized by A with set of final states $\{q_n \cdot w^{a_{q_n}} \mid n \geq 1\}$ and let $\equiv_L$ be its syntactic congruence. We notice that if $u \in X^*$ and $q_0 \cdot u = q_n \cdot w^{a_{q_n}}$ then $uw^m \in L$ if and only if $m \equiv 0 \pmod{b_{q_n}}$. As before, if L is torsion there exists $a, b \geq 1$ such that $w^a \equiv_L w^{a+b}$.

Let n be such that $a+b < b_{q_n}$ and let $c = b_{q_n} - (a+b)$. Then $a+b+c \equiv 0 \pmod{b_{q_n}}$ but $a+2b+c \not\equiv 0 \pmod{b_{q_n}}$. Therefore if $q_0 \cdot u = q_n \cdot w^{a_{q_n}}$ $(u \in X^*)$ then $uw^{a+b+c} \in L$ and $uw^{a+2b+c} \notin L$, thus contradicting $w^a \equiv_L w^{a+b}$. So L is not torsion. □

3 Filtrations and limits of sequences of languages and morphisms

This section is devoted to the concept of a topology on X^* defined by a filtration and to the notions of convergence of sequences of languages and of sequences of automata over X.

3.1 Topology induced by a filtration

Let X be a finite alphabet. A *filtration* $\mathcal{I}$ of X^* is a sequence of languages $\mathcal{I} = (I_n)_{n\geq 0}$ such that

$$X^* = I_0 \supseteq I_1 \supseteq \cdots \supseteq I_n \supseteq \cdots$$

and such that $\bigcap_{n\geq 0} I_n = \emptyset$. We say that $\mathcal{I}$ is in a class of languages $\mathcal{L}$ if each I_n is in $\mathcal{L}$. We will be interested in particular in rational filtrations (each I_n is a rational language) and in ideal filtrations ($I_n = X^* I_n X^*$ for each $n \geq 0$).

Let $\overline{X^*}$ be the monoid $\overline{X^*} = X^* \cup \{\infty\}$ consisting of X^* and a new element ∞ such that

$$x\,(\infty) = (\infty)\,x = \infty, \qquad \text{for all } x \in X^*.$$

Note that ∞ is a zero of $\overline{X^*}$ and that X^* is a submonoid of $\overline{X^*}$. It is classical to consider the topology $\mathrm{Top}_{\mathcal{I}}$ on $\overline{X^*}$ defined by a filtration $\mathcal{I} = (I_n)_n$ on X^*. For this topology, a basis for the open sets is

$$\big\{\{w\} \mid w \in X^*\big\} \cup \big\{I_n \cup \{\infty\} \mid n \geq 0\big\}.$$

For example, let $\overline{\mathbf{N}} = \{1, 2, \ldots, n, \ldots\} \cup \{\infty\}$ be the one-point compactification of the positive integers: a basis for the open sets of $\overline{\mathbf{N}}$ is

$$\big\{\{n\} \mid n \geq 1\big\} \cup \big\{F_n \cup \{\infty\} \mid n \geq 1\big\}$$

where $F_n = \{i \in \mathbf{N} \mid i > n\}$.

If $\mathcal{I} = (I_n)_n$ is a filtration on X^*, we define a function $r_{\mathcal{I}}\colon \overline{X^*} \to \overline{\mathbf{N}}$ by $r_{\mathcal{I}}(\infty) = \infty$ and, for $w \in X^*$,

$$r_{\mathcal{I}}(w) = \min\{n \geq 0 \mid w \notin I_n\}.$$

In other words, if $w \in X^*$, then

$$r_{\mathcal{I}}(w) = n \quad \text{if and only if} \quad w \in I_k \text{ for all } k < n, \text{ and } w \notin I_k \text{ for all } k \geq n.$$

Therefore $r_{\mathcal{I}}^{-1}(F_n \cup \{\infty\}) = I_n \cup \{\infty\}$. In particular, $\mathrm{Top}_{\mathcal{I}}$ is the least topology on $\overline{X^*}$ that makes $r_{\mathcal{I}}$ continuous. It is easy to verify that a sequence $(w_n)_n$ of $\overline{X^*}$ converges to w if and only if it is eventually stationary at w, or

$$w = \infty \qquad \text{and} \qquad \lim_n r_{\mathcal{I}}(w) = \infty$$

(i.e., the sequence $(w_n)_n$ "falls further and further down in $\mathcal{I}$").

If $\mathcal{I}$ is an ideal filtration, then one can verify that $\overline{X^*}$ is a topological monoid (the multiplication in $\overline{X^*}$ is a continuous function). Furthermore, for each $k \geq 0$, $\overline{X^*}/(I_k \cup \{\infty\})$ is a monoid, and if this monoid is endowed with the discrete topology, then the projection $\pi_k\colon \overline{X^*} \to \overline{X^*}/(I_k \cup \{\infty\})$ is a continuous morphism.

For any filtration $\mathcal{I}$ on X^*, one can define a ultrametric distance function $d_{\mathcal{I}}$ on $\overline{X^*}$ by letting

$$\begin{aligned}
d_{\mathcal{I}}(w, w) &= 0 \qquad \text{for all } w \in \overline{X^*}, \\
d_{\mathcal{I}}(w, \infty) &= \frac{1}{r_{\mathcal{I}}(w)} \qquad \text{for all } w \in X^*, \text{ and} \\
d_{\mathcal{I}}(w_1, w_2) &= \max\left(d_{\mathcal{I}}(w_1, \infty), d_{\mathcal{I}}(w_2, \infty)\right) \qquad \text{for all distinct } w_1, w_2 \text{ in } X^*.
\end{aligned}$$

Note that, for all $n \geq 1$ and $w_1 \neq w_2$ in X^*, we have

$$d(w_1, w_2) < 1/n \qquad \Longleftrightarrow \qquad w_1, w_2 \in I_n.$$

For $w \in \overline{X^*}$ and $\varepsilon > 0$, let $B(w, \varepsilon)$ be the open ball with center w and radius ε. Then we have

$$B(\infty, \varepsilon) = I_n \cup \{\infty\} \qquad \text{where } n = \min\{k \mid \tfrac{1}{k} < \varepsilon\}$$

and for $w \in X^*$

$$B(w, \varepsilon) = \begin{cases} \{w\} & \text{if } \varepsilon \leq \frac{1}{r_{\mathcal{I}}(w)} \\ B(\infty, \varepsilon) = I_n \cup \{\infty\} & \text{where } n = \min\{k \mid \frac{1}{k} < \varepsilon\}, \text{ if } \varepsilon > \frac{1}{r_{\mathcal{I}}(w)}. \end{cases}$$

Therefore, the topology induced by $d_{\mathcal{I}}$ on $\overline{X^*}$ is exactly $\mathrm{Top}_{\mathcal{I}}$.

3.2 Sequences of languages

We will be interested in languages of X^* which are limits in a certain sense of a sequence of languages with respect to a given filtration $\mathcal{I}$ on X^*. More precisely, the metric $d_{\mathcal{I}}$ induces a Hausdorff metric on the set of closed subsets of $\overline{X^*}$: if L and L' are closed subsets of $\overline{X^*}$, then $D_{\mathcal{I}}(L, L')$ is defined by

$$D_{\mathcal{I}}(L, L') = \min\{\varepsilon > 0 \mid \forall w \in L,\ \exists w' \in L',\ d_{\mathcal{I}}(w, w') < \varepsilon \text{ and } \forall w' \in L',\ \exists w \in L,\ d_{\mathcal{I}}(w, w') < \varepsilon\}.$$

Note that a set $L \subseteq \overline{X^*}$ is closed if and only if $\infty \in L$ or $L \subseteq X^* \setminus I_n$ for some $n \geq 0$. In particular, if $L \subseteq X^*$, then $L \cup \{\infty\}$ is always closed, and the closure of L is either L or $L \cup \{\infty\}$.

If $(L_n)_n$ is a sequence of languages of X^* and $L \subseteq X^*$, we will write $L = \lim_{\mathcal{I}} L_n$ if $L \cup \{\infty\}$ is the limit of the sequence of closed spaces $(L_n \cup \{\infty\})_n$, that is, if

$$\lim_n D_{\mathcal{I}}(L_n \cup \{\infty\}, L \cup \{\infty\}) = 0.$$

One can verify the following proposition.

Proposition 3.1 *Let $L, L_n \subseteq X^*$ and let $\mathcal{I} = (I_n)_n$ be a filtration on X^*. Then*

$$L = \lim_{\mathcal{I}} L_n \quad \text{if and only if} \quad \forall n,\ \exists k \geq 0,\ \forall m \geq k, \quad L_m \setminus I_n = L \setminus I_n.$$

Example Let $L \subseteq X^*$ and let I_n ($n \geq 0$) be the set of all words of length greater than or equal to n. Let $L_n = L \setminus I_n$ for each $n \geq 0$, and let $\mathcal{I} = (I_n)_n$. Then $\mathcal{I}$ is a cofinite ideal filtration, each L_n is finite and $L = \lim_{\mathcal{I}} L_n$.

Note the following easy property of this topology on the set of languages.

Proposition 3.2 *Let $\mathcal{I}$ be a filtration on X^* and let $(L_n)_n$ be a Cauchy sequence of languages over X (meaning that the sequence of closed subsets of X^* $(L_n \cup \{\infty\})_n$ is Cauchy for $D_{\mathcal{I}}$). Then $(L_n)_n$ is a convergent sequence.*

Proof. Our hypothesis on $(L_n)_n$ is the following:

$$\forall n \geq 1,\ \exists k \geq 1,\ \forall m \geq k, \quad L_m \setminus I_n = L_k \setminus I_n.$$

We may then choose an increasing sequence of integers $(k_n)_n$ such that for all $m \geq k_n$, we have $L_m \setminus I_n = L_{k_n} \setminus I_n$. In particular, $L_{k_n} \setminus I_n \subseteq L_{k_{n+1}} \setminus I_{n+1}$. Let now $L = \bigcup_n L_{k_n} \setminus I_n$. It is easy to verify that $L \setminus I_n = L_{k_n} \setminus I_n$ for all n and hence that $L = \lim_{\mathcal{I}} L_n$. $\square$

In the sequel we will use the following notion. We say that a filtration $\mathcal{I} = (I_n)_n$ is *fast* if

$$\forall w \in X^*,\ \exists k \geq 0,\ \forall n \geq 1, \quad w^n \notin I_k.$$

Example Let $X = \{a, b\}$, let $I_0 = X^*$, and let $I_n = X^* a^n b X^*$ for all $n \geq 1$. Then $\mathcal{I} = (I_n)_n$ is a fast rational ideal filtration. It is easy to verify that $\mathcal{I}$ is a filtration. Suppose now that $w^n \in I_k$ for some $w \in X^*$, $n \geq 3$ and $k \geq 1$. Then $a^k b$ is a factor of w^n, and hence $a^k b$ is a factor of w^2. Since $\bigcap_n I_n = \emptyset$, there exists k such that w and w^2 do not lie in I_k. Then, for all $n \geq 1$, $w^n \notin I_k$.

3.3 Sequences of automata

We now consider automata (with unspecified set of final states) over X. The reader can consult Appendix 1 for a simpler version, stated for congruences instead of automata.

Recall that we assume all automata to be accessible, and that $\mathcal{A}(X)$ is the class of all (accessible) automata over X (see Section 1.1). To an automaton $A = (Q, X, q_0, \lambda) \in \mathcal{A}(X)$, we associate the equivalence relation

$$C_A = \{(u, v) \in X^* \times X^* \mid q_0 \cdot u = q_0 \cdot v\}.$$

Note that, for all $u, v, w \in X^*$, if $(u, v) \in C_A$, then $(uw, vw) \in C_A$. That is, C_A is a right congruence of X^*.

A *morphism* φ from $A = (Q, X, q_0, \lambda) \in \mathcal{A}(X)$ into $A' = (Q', X, q'_0, \lambda') \in \mathcal{A}(X)$ is a mapping $\varphi: Q \to Q'$ such that, for all $u \in X^*$, $\big(\lambda(q_0, u)\big)\varphi = \lambda'(q'_0, u)$. Thus, there exists a morphism from A to A' if and only if the equivalence relation C_A refines $C_{A'}$. In that case, the morphism from A to A' is unique. Furthermore, since the automata we consider are accessible, the mapping φ is necessarily onto.

If $A = (Q, X, q_0, \lambda)$ and $A' = (Q', X, q'_0, \lambda')$ are in $\mathcal{A}(X)$, we define $D_{\mathcal{I}}(A, A')$ as follows (where $\triangle$ denotes the symmetric difference).

$$\begin{aligned} D_{\mathcal{I}}(A, A') &= \max\{d(u, v) \mid (u, v) \in C_A \triangle C_{A'}\} \\ &= \max\{d(u, v) \mid (q_0 \cdot u = q_0 \cdot v \text{ and } q'_0 \cdot u \neq q'_0 \cdot v) \text{ or} \\ &\qquad\qquad (q_0 \cdot u \neq q_0 \cdot v \text{ and } q'_0 \cdot u = q'_0 \cdot v)\}. \end{aligned}$$

Let us remark that $D_{\mathcal{I}}(A, A') = 0$ if and only if $C_A = C_{A'}$, that is, if and only if, for all $u, v \in X^*$, $q_0 \cdot u = q_0 \cdot v$ if and only if $q_0' \cdot u = q_0' \cdot v$. Therefore, $D_{\mathcal{I}}(A, A') = 0$ if and only if A and A' are isomorphic. In the next few lines we will verify that $D_{\mathcal{I}}$ is an ultrametric distance function on the set of isomorphism classes of automata over X. It would be equivalent to define $D_{\mathcal{I}}$ directly on the set of right congruences of X^*.

It suffices to verify that for all A, B and C in $\mathcal{A}(X)$, we have $D_{\mathcal{I}}(A, C) \leq \max\big(D_{\mathcal{I}}(A, B), D_{\mathcal{I}}(B, C)\big)$, which is an easy consequence of the containment

$$C_A \bigtriangleup C_C \subseteq (C_A \bigtriangleup C_B) \cup (C_B \bigtriangleup C_C).$$

Let $(A_n)_n$ and let A be automata over X. We say that the sequence $(A_n)_n$ *converges to* A (*modulo* $\mathcal{I}$), and we write $A = \lim_{\mathcal{I}} A_n$, if $\lim_n D_{\mathcal{I}}(A_n, A) = 0$. From the above discussion, it follows that the limit of a convergent sequence of automata is unique up to isomorphism.

Proposition 3.3 *Let $A = (Q, X, q_0, \lambda)$ and $A_n = (Q_n, X, q_0^n, \lambda_n)$ $(n \geq 0)$ be automata and let $\mathcal{I} = (I_n)_n$ be a filtration on X^*. Then*

$$A = \lim_{\mathcal{I}} A_n \text{ if and only if } \begin{cases} \forall n \geq 0, \quad \exists k \geq 0, \quad \forall m \geq k, \\ \forall u, v \in X^* \setminus I_n, \quad q_0^m \cdot u = q_0^m \cdot v \Longleftrightarrow q_0 \cdot u = q_0 \cdot v. \end{cases}$$

Proof. This is immediate since we noticed that, for all $n \geq 1$ and $u \neq v \in X^*$, $d(u, v) < 1/n$ if and only if both $u, v \in I_n$. □

Let $\big(A_n = (Q_n, X, q_0^n, \lambda_n)\big)_n$ be a sequence of automata. We say as usual that $(A_n)_n$ is a *Cauchy sequence* (*modulo* $\mathcal{I}$) if

$$\forall \varepsilon > 0, \quad \exists k \geq 0, \quad \forall n, m \geq k, \qquad D_{\mathcal{I}}(A_n, A_m) < \varepsilon.$$

Equivalently, $(A_n)_n$ is a Cauchy sequence if and only if

$$\begin{aligned} &\forall n \geq 0, \quad \exists k_n \geq 0, \quad \forall m \geq k_n, \\ &\forall u, v \in X^* \setminus I_n, \quad q_0^{k_n} \cdot u = q_0^{k_n} \cdot v \Longleftrightarrow q_0^m \cdot u = q_0^m \cdot v. \end{aligned}$$

We have the following result, which is the analogue of Proposition 3.2.

Proposition 3.4 *Let $\left(A_n = (Q_n, X, q_0^n, \lambda_n)\right)_n$ be a sequence of automata over X and let $\mathcal{I}$ be an ideal filtration on X^*. If $(A_n)_n$ is a Cauchy sequence modulo $\mathcal{I}$, then $(A_n)_n$ converges modulo $\mathcal{I}$.*

Proof. For each $n \geq 0$, there exists $k_n \geq 0$ such that

$$\forall m \geq k_n, \quad \forall u, v \in X^* \setminus I_n, \quad q_0^{k_n} \cdot u = q_0^{k_n} \cdot v \Longleftrightarrow q_0^m \cdot u = q_0^m \cdot v.$$

We may assume the sequence $(k_n)_n$ to be non-decreasing. For $u, v \in X^*$, let us define $u \sim v$ by

$$u \sim v \quad \text{if and only if} \quad \text{for all } n \text{ such that } u, v \notin I_n,\ q_0^{k_n} \cdot u = q_0^{k_n} \cdot v.$$

This is equivalent to

$$u \sim v \quad \text{if and only if} \quad \text{there exists } n \text{ such that } u, v \notin I_n \text{ and } q_0^{k_n} \cdot u = q_0^{k_n} \cdot v.$$

We now verify that $\sim$ is a right congruence. Let indeed $u, v, w \in X^*$ and let us assume that $u \sim v$. For each n such that $uw, vw \notin I_n$, we have $u, v \notin I_n$ (since I_n is an ideal), so $q_0^{k_n} \cdot u = q_0^{k_n} \cdot v$ and hence $q_0^{k_n} \cdot uw = q_0^{k_n} \cdot vw$.

We then define Q as the quotient set $X^*/\sim$, we let q_0 be the $\sim$-class of the empty word, and we define $\lambda\colon Q \times X \to Q$ by $\lambda([u]_\sim, x) = [ux]_\sim$ for all $u \in X^*$ and $x \in X$. It is easy to verify that λ is well-defined, that $A = (Q, X, q_0, \lambda) \in \mathcal{A}(X)$, and that $A = \lim_{\mathcal{I}} A_n$. $\square$

Finally we will need the following notion. Let $A = (Q, X, q_0, \lambda)$ and $A_n = (Q_n, X, q_0^n, \lambda_n)$ $(n \geq 0)$ be automata, and let $\mathcal{I} = (I_n)_n$ be a filtration on X^*. We say that the sequence $(A_n)_n$ *approximates* A (*modulo* $\mathcal{I}$) if

$$\begin{aligned} &\forall n \geq 0, \quad \exists k_n \geq 0, \quad \forall m \geq k_n, \\ &\forall u, v \in X^* \setminus I_n, \quad q_0^m \cdot u = q_0^m \cdot v \Longrightarrow q_0 \cdot u = q_0^m \cdot v. \end{aligned}$$

Quite clearly, if $(A_n)_n$ converges to A, then $(A_n)_n$ approximates A. However a given sequence $(A_n)_n$ may approximate several automata.

Proposition 3.5 *Let A, B and A_n $(n \geq 0)$ be automata over X, and let $\mathcal{I} = (I_n)_n$ be a filtration on X^* such that $A = \lim_{\mathcal{I}} A_n$. Then $(A_n)_n$ approximates B modulo $\mathcal{I}$ if and only if there exists a morphism $\varphi\colon A \to B$.*

Proof. Let us assume that the sequence $(A_n = (Q_n, X, q_0^n, \lambda_n))_n$ approximates $B = (Q', X, q_0', \lambda')$. If $A = (Q, X, q_0, \lambda)$, we need to verify that $q_0 \cdot u = q_0 \cdot v$ implies $q_0' \cdot u = q_0' \cdot v$ for all $u, v \in X^*$. Let $n \geq 0$ be such that $u, v \notin I_n$. There exist integers k and k' such that, for each $m \geq k$ (resp. $m \geq k'$) and for each $x, y \in X^* \setminus I_n$, the equality $q_0^m \cdot x = q_0^m \cdot y$ is equivalent to $q_0 \cdot x = q_0 \cdot y$ (resp. implies $q_0' \cdot x = q_0' \cdot y$). By considering an integer $m \geq \max(k, k')$, we find that $q_0 \cdot u = q_0 \cdot v$ implies $q_0' \cdot x = q_0' \cdot y$.

The converse implication is trivial. □

In the sequel, we will want to further qualify approximations. Let $A = (Q, X, q_0, \lambda)$ and $A' = (Q', X, q_0', \lambda')$ be automata, and let φ be a morphism from A to A'. We say that φ is *torsion* (resp. *aperiodic*) if, for each $u \in X^*$ such that $q' \cdot u = q' \cdot u^2$ for all $q' \in Q'$, there exists an integer k such that $q \cdot u^k = q \cdot u^{2k}$ for all $q \in Q$. Similarly, for a fixed integer k, we say that φ is *k-bounded torsion* (resp. *k-bounded aperiodic*) if, for each $u \in X^*$ such that $q' \cdot u = q' \cdot u^2$ for all $q' \in Q'$, we have $q \cdot u^k = q \cdot u^{2k}$ (resp. $q \cdot u^k = q \cdot u^{k+1}$) for all $q \in Q$. Finally, φ is *bounded torsion* (resp. *bounded aperiodic*) if it is k-bounded torsion (resp. k-bounded aperiodic) for some k.

Let now A, B and A_n ($n \geq 0$) be automata over X, let $\mathcal{I}$ be a filtration on X^*, and let us assume that $(A_n)_n$ converges to A and that $(A_n)_n$ approximates B modulo $\mathcal{I}$. By Proposition 3.5, there exists a morphism φ from A to B. We say that the approximation is *torsion* (resp. *aperiodic, bounded torsion, bounded aperiodic*) if the morphism φ is torsion (resp. aperiodic, bounded torsion, bounded aperiodic.

4 Main results

We will now state the main theorems of this paper. These results are generalizations of Kleene's, Schützenberger's and Straubing's theorems and they characterize torsion, aperiodic, bounded torsion and bounded aperiodic languages. The first set of results deals with $\mathcal{L}_{\text{btor}}$ and $\mathcal{L}_{\text{bap}}$, and the second one deals with $\mathcal{L}_{\text{tor}}$ and $\mathcal{L}_{\text{ap}}$. The proofs of these theorems will be given in Section 5.

4.1 Bounded torsion and bounded aperiodic languages

Before we proceed with the results, let us mention a few important closure properties for a class of languages $\mathcal{L}$. We say that

- $\mathcal{L}$ satisfies (BT) if, for each $L_1, L_2 \in \mathcal{L}$, we have $BT(L_1, L_2) \subseteq \mathcal{L}$;
- $\mathcal{L}$ satisfies (H^{-1}) if, for each morphism $\varphi\colon X_1^* \to X_2^*$ between finitely generated free monoids and for each $L \subseteq X_2^*$ in $\mathcal{L}$, we have $L\varphi^{-1} \in \mathcal{L}$;
- $\mathcal{L}$ satisfies (Pr) if, for each $L, L' \subseteq X^*$ in $\mathcal{L}$, we have $LL' \in \mathcal{L}$;
- $\mathcal{L}$ satisfies (St) if, for each $L \in \mathcal{L}$, we have $L^* \in \mathcal{L}$.
- $\mathcal{L}$ satisfies (Lpq) if, for each prime p, each integer $0 \leq q < p$ and each language $L \in \mathcal{L}$, we have $\langle L, p, q\rangle \in \mathcal{L}$.

Let $(L_n)_n$ be a sequence of languages over X and let $\mathcal{I} = (I_n)_n$ be a filtration. Suppose that $L = \lim_{\mathcal{I}} L_n$. We say that *the limit is bounded* if there exists $k \geq 0$ such that $L_n \in \mathcal{L}^{(k)}_{\mathrm{btor}}$ for each n. It is *bounded rational* (resp. *bounded rational subgroup-cyclic*) if furthermore the L_n are rational (resp. rational subgroup-cyclic) for all n. We say that a class of languages $\mathcal{L}$ is *closed under bounded limits* (resp. bounded rational limits, bounded rational subgroup-cyclic limits) if, whenever $L = \lim_{\mathcal{I}} L_n$ and the limit is bounded (resp. bounded rational, bounded rational subgroup-cyclic) and $L_n \in \mathcal{L}$ for each n, then $L \in \mathcal{L}$.

Theorem 4.1 *For each integer k, $\mathcal{L}^{(k)}_{\mathrm{btor}}$ satisfies (BT) and (H^{-1}) and is closed under limits. $\mathcal{L}_{\mathrm{btor}}$ satisfies (BT), (H^{-1}), (Pr) and (Lpq), and it is closed under bounded limits.*

Theorem 4.2 *For each integer k, $\mathcal{L}^{[k]}_{\mathrm{bap}}$ satisfies (BT) and (H^{-1}) and is closed under limits. $\mathcal{L}_{\mathrm{bap}}$ satisfies (BT), (H^{-1}) and (Pr), and it is closed under bounded limits.*

Caution It is important to note that, for any fixed k, $\mathcal{L}^{(k)}_{\mathrm{btor}}$ does not satisfy (Pr) and (Lpq), and that $\mathcal{L}^{[k]}_{\mathrm{bap}}$ does not satisfy (Pr). For instance, the product of two k-bounded torsion (resp. aperiodic) languages is bounded torsion (resp. aperiodic), but its torsion bound may be greater than k.

Theorem 4.3 *Let $k \geq 1$, let $k' = \max(k, 4)$ and $L \subseteq X^*$. If $L \in \mathcal{L}^{(k)}_{\mathrm{btor}}$ (resp. $L \in \mathcal{L}^{[k]}_{\mathrm{bap}}$), then there exists a sequence $(L_n)_n$ and a fast ideal filtration $\mathcal{I} = (I_n)_n$ such that $L = \lim_{\mathcal{I}} L_n$ and L_n and I_n are rational (resp. star-free) and in $\mathcal{L}^{(k')}_{\mathrm{btor}}$ (resp. $\mathcal{L}^{[k']}_{\mathrm{bap}}$).*

The above results yield immediately the following descriptions of $\mathcal{L}_{\mathrm{btor}}$ and $\mathcal{L}_{\mathrm{bap}}$, which generalize the theorems of Kleene and Schützenberger.

Theorem 4.4 **(Bounded generalization of Kleene's theorem)** *$\mathcal{L}_{\mathrm{btor}}$ is the least family of languages containing the rational languages and closed under bounded limits.*

Theorem 4.5 **(Bounded generalization of Schützenberger's theorem)** *$\mathcal{L}_{\mathrm{bap}}$ is the least family of languages containing the star-free languages and closed under bounded limits.*

Equivalently, $\mathcal{L}_{\mathrm{bap}}$ is the least family of languages containing the finite languages and closed under Boolean operations, product and bounded limits.

Remark Theorems 4.1, 4.2 and 4.3 allow in fact more precise descriptions: $\mathcal{L}_{\mathrm{btor}}$ (resp. $\mathcal{L}_{\mathrm{bap}}$) is the least family of languages containing the rational (resp. star-free) languages and closed under bounded rational limits with respect to fast ideal bounded rational filtrations.

The following result, which is connected to Straubing's theorem on rational subgroup-solvable languages, is more surprising.

Theorem 4.6 *Let $k \geq 1$, let $k' = \max(k, 4)$ and $L \subseteq X^*$. If $L \in \mathcal{L}^{(k)}_{\mathrm{btor}}$, then there exists a sequence $(L_n)_n$ and a fast ideal filtration $\mathcal{I} = (I_n)_n$ such that $L = \lim_{\mathcal{I}} L_n$ and L_n and I_n are rational subgroup-cyclic and in $\mathcal{L}^{(k')}_{\mathrm{btor}}$.*

Theorem 4.7 **(Bounded generalization of Straubing's theorem)** *$\mathcal{L}_{\mathrm{btor}}$ is the least family of languages containing the finite languages, satisfying (Lpq) and closed under Boolean operations, product and bounded limits.*

Remark As above, Theorem 4.6 implies in fact a more precise statement: $\mathcal{L}_{\mathrm{btor}}$ is the least family of languages containing the rational subgroup-cyclic languages and closed under bounded rational subgroup-cyclic limits with respect to fast ideal bounded rational subgroup-cyclic filtrations.

4.2 Torsion and aperiodic languages

In order to characterize the torsion and aperiodic languages in general, it is not sufficient to consider only limits of sequences of languages. In fact, as we noticed in Section 3.2, every language is the limit of a sequence of finite languages modulo a cofinite filtration.

A class of languages $\mathcal{L}$ is *closed under approximation* (resp. *under fast approximation*) if, for each filtration (resp. fast filtration) $\mathcal{I}$, whenever an automaton A is approximated modulo $\mathcal{I}$ by a sequence of automata $(A_n)_n$ such that $\mathcal{L}(A_n) \subseteq \mathcal{L}$ for all n, then $\mathcal{L}(A) \subseteq \mathcal{L}$.

Theorem 4.8 *$\mathcal{L}_{\mathrm{tor}}$ satisfies (BT), (H^{-1}), (Pr), (St) and (Lpq), and is closed under fast approximation.*

Theorem 4.9 *$\mathcal{L}_{\mathrm{ap}}$ satisfies (BT), (H^{-1}) and (Pr), and is closed under fast approximation.*

Theorem 4.10 *Let A be an automaton over X such that $\mathcal{L}(A) \subseteq \mathcal{L}_{\mathrm{tor}}$ (resp. $\mathcal{L}(A) \subseteq \mathcal{L}_{\mathrm{ap}}$). Then there exists a fast rational (resp. star-free) ideal filtration $\mathcal{I}$ and a sequence of finite automata $(A_n)_n$ (resp. such that $\mathcal{L}(A_n) \subseteq \mathcal{L}_{\mathrm{ap}}$ for all n), such that $(A_n)_n$ converges modulo $\mathcal{I}$, $(A_n)_n$ approximates A modulo $\mathcal{I}$, and the approximation is bounded aperiodic.*

As in Section 4.1, these results yield descriptions of $\mathcal{L}_{\mathrm{tor}}$ and $\mathcal{L}_{\mathrm{ap}}$ which generalize Kleene's and Schützenberger's theorems.

Theorem 4.11 (Unbounded generalization of Kleene's theorem) *$\mathcal{L}_{\mathrm{tor}}$ is the least family of languages containing the rational languages and closed under fast approximation.*

Equivalently, $\mathcal{L}_{\mathrm{tor}}$ is the least family of languages containing the finite languages and closed under union, product, star and fast approximation.

Theorem 4.12 (Unbounded generalization of Schützenberger's theorem) *$\mathcal{L}_{\mathrm{ap}}$ is the least family of languages containing the star-free languages and closed under fast approximation.*

Equivalently, $\mathcal{L}_{\mathrm{ap}}$ is the least family of languages containing the finite languages and closed under Boolean operations, product and fast approximation.

Remark As above, we can in fact make these statements more precise: $\mathcal{L}_{\mathrm{tor}}$ (resp. $\mathcal{L}_{\mathrm{ap}}$) is the least family of languages containing the rational (resp. star-free) languages and closed under fast bounded aperiodic approximation with respect to a fast rational (resp. star-free) ideal filtration.

Analogous to Theorem 4.6, we also have

Theorem 4.13 *Let A be an automaton over X such that $\mathcal{L}(A) \subseteq \mathcal{L}_{\mathrm{tor}}$ (resp. $\mathcal{L}(A) \subseteq \mathcal{L}_{\mathrm{ap}}$). Then there exists a fast rational subgroup-cyclic ideal filtration $\mathcal{I}$ and a sequence of finite automata $(A_n)_n$ such that the languages recognized by the A_n are rational cyclic, $(A_n)_n$ converges modulo $\mathcal{I}$, $(A_n)_n$ approximates A modulo $\mathcal{I}$, and the approximation is aperiodic.*

Theorem 4.14 (Unbounded generalization of Straubing's theorem) *$\mathcal{L}_{\text{tor}}$ is the least family of languages containing the rational subgroup-cyclic languages and closed under fast approximation.*

Equivalently, $\mathcal{L}_{\text{tor}}$ is the least family of languages containing the finite languages, satisfying (Lpq) and closed under Boolean operations, product and fast approximation.

4.3 Arbitrary languages

The important Theorems 4.10 and 4.13 above are "fast" versions of results that hold for arbitrary automata, namely Theorems 4.15 and 4.16 below. Note that the latter differ from Theorems 4.10 and 4.13 only by the fact that the filtrations they involve need not be fast. As we will see in Section 5.2, the proofs of Theorems 4.10, 4.13, 4.15 and 4.16 are very similar.

Theorem 4.15 (Arbitrary language theorem) *Let A be an automaton over X. Then there exists a rational ideal filtration $\mathcal{I}$ and a sequence of finite automata $(A_n)_n$ such that $(A_n)_n$ converges modulo $\mathcal{I}$, $(A_n)_n$ approximates A modulo $\mathcal{I}$, and the approximation is bounded aperiodic.*

Theorem 4.16 (Cyclic arbitrary language theorem) *Let A be an automaton over X. Then there exists a rational subgroup-cyclic ideal filtration $\mathcal{I}$ and a sequence of finite automata $(A_n)_n$ such that the languages recognized by the A_n are rational subgroup-cyclic, $(A_n)_n$ converges modulo $\mathcal{I}$, $(A_n)_n$ approximates A modulo $\mathcal{I}$, and the approximation is aperiodic.*

5 Proofs of the main results

The proofs of Theorems 4.1, 4.2, 4.8 and 4.9 can be obtained using more or less classical methods, and we shall give these proofs in Section 5.1. As for Theorems 4.3, 4.6, 4.10, 4.13, 4.15 and 4.16, their proof is more delicate and requires some deep results of semigroup theory. These proofs will be discussed in Section 5.2. Finally, we note that given these theorems, the bounded and unbounded generalizations of Kleene's, Schützenberger's and Straubing's theorems (Theorems 4.4, 4.5, 4.7, 4.11, 4.12 and 4.14) are immediate.

5.1 Proof of Theorems 4.1, 4.2, 4.8 and 4.9

First we observe that rational languages are trivially in $\mathcal{L}_{\rm btor}$ and that aperiodic (star-free) rational languages are in $\mathcal{L}_{\rm bap}$.

The fact that the families $\mathcal{L}^{(k)}_{\rm btor}$, $\mathcal{L}^{[k]}_{\rm bap}$, $\mathcal{L}_{\rm btor}$, $\mathcal{L}_{\rm bap}$, $\mathcal{L}_{\rm tor}$ and $\mathcal{L}_{\rm ap}$ satisfy (BT) is a direct consequence of Proposition 1.3, together with Lemma 1.2 and Proposition 2.3. The same Lemma 1.2 and Proposition 2.3 show that these classes of languages satisfy (H^{-1}).

Let us now show that they all satisfy (Pr), and that $\mathcal{L}_{\rm tor}$ and $\mathcal{L}_{\rm btor}$ satisfy (Lpq). The proof uses the properties of the Schützenberger product and of its 'mod p'-variant. For more details on these products, see in particular [15,22,23].

Proposition 5.1 *$\mathcal{L}_{\rm btor}$, $\mathcal{L}_{\rm bap}$, $\mathcal{L}_{\rm tor}$ and $\mathcal{L}_{\rm ap}$ satisfy (Pr).*

Proof. Let L and L' be languages in X^*, let $S(L)$ and $S(L')$ be their syntactic monoids, and let η_L and $\eta_{L'}$ be their syntactic morphisms. Let η be the morphism from X^* into the multiplicative monoid of (2,2)-matrices with entries in the semiring $\big(\mathcal{P}(S(L)\times S(L')),\cup,\cdot\big)$ defined by

$$x\eta = \begin{pmatrix} \{(x\eta_L,1)\} & \{(1,1)\} \\ \emptyset & \{(1,x\eta_{L'})\} \end{pmatrix} \quad \text{for all } x\in X.$$

One can check that, for all $w\in X^*$, we have

$$w\eta = \begin{pmatrix} \{(w\eta_L,1)\} & P_w \\ \emptyset & \{(1,w\eta_{L'})\} \end{pmatrix}$$

where $P_w = \{(u\eta_L, v\eta_{L'}) \mid w = uv\}$. Then, $LL' = K\eta^{-1}$, where K is the set of all matrices in the form

$$\begin{pmatrix} \{(s,1)\} & P \\ \emptyset & \{(1,s')\} \end{pmatrix}$$

with $P\cap(L\eta_L\times L'\eta_{L'})\neq\emptyset$. The range of η is denoted $X^*\eta = \Diamond_2(S(L),S(L'))$ and is called the Schützenberger product of $S(L)$ and $S(L')$. Let now π be the morphism from $\Diamond_2(S(L),S(L'))$ into $S(L)\times S(L')$ defined by $w\eta\pi = (w\eta_L, w\eta_{L'})$. Let e and e' be idempotents, respectively in $S(L)$ and $S(L')$. Then the inverse image $(e,e')\pi^{-1}$ satisfies the identity $x^3 = x^4$. Let indeed $m\in(e,e')\pi^{-1}$, say,

$$m = \begin{pmatrix} \{(e,1)\} & P \\ \emptyset & \{(1,e')\} \end{pmatrix}$$

Then, for each $n \geq 1$, m^n is in the form

$$m^n = \begin{pmatrix} \{(e,1)\} & P_n \\ \emptyset & \{(1,e')\} \end{pmatrix}$$

where $P_1 = P$ and $P_{n+1} = (e,1)P \cup P_n(1,e')$. In particular

$$\begin{aligned} P_2 &= (e,1)P \cup P(1,e') \\ P_3 &= (e,1)P \cup (e,1)P(1,e') \cup P(1,e') \\ P_n &= P_3 \quad \text{for all } n \geq 3. \end{aligned}$$

Therefore π is a bounded aperiodic morphism and hence (see Proposition 2.2), if $S(L)$ and $S(L')$ are in **Tor**, **bTor**, **Ap** or **bAp**, then so are $\diamondsuit_2(S(L), S(L'))$ and $S(LL')$. $\square$

Proposition 5.2 *$\mathcal{L}_{\mathrm{btor}}$ and $\mathcal{L}_{\mathrm{tor}}$ satisfy (Lpq).*

Proof. The proof is quite similar to the proof of Proposition 5.1. Let p be a prime number, let L be a language in X^*, and let $\eta_L: X^* \to S(L)$ be its syntactic morphism. Let η be the morphism from X^* into the multiplicative monoid of (2,2)-matrices with entries in the semiring $\mathbf{Z}_p[S(L)]$ defined by

$$x\eta = \begin{pmatrix} x\eta_L & 1 \\ \emptyset & 1 \end{pmatrix} \quad \text{for all } x \in X.$$

$\mathbf{Z}_p[S(L)]$ is the semiring of formal linear combinations of elements of $S(L)$ with coefficients in the cyclic group of order p, $\mathbf{Z}_p$. (Note that the semiring $(\mathcal{P}(S), \cup, \cdot)$ can be identified with $\mathbf{B}[S]$ where $\mathbf{B}$ is the Boolean semiring.) One can check that, for all $w \in X^*$, we have

$$w\eta = \begin{pmatrix} w\eta_L & P_w \\ 0 & 1 \end{pmatrix}$$

with $P_w = \sum u\eta_L$ where the sum runs over all prefixes u of w. Then η recognizes $\langle L, p, q\rangle$. Indeed, if K is the set of matrices of the form

$$\begin{pmatrix} s & P \\ 0 & 1 \end{pmatrix}$$

where $P = \sum_{s \in S(L)} c_s\, s$ with $\sum_{s \in L\eta_L} c_s \equiv q \pmod{p}$, then $\langle L, p, q\rangle = K\eta^{-1}$. The range of η is denoted $X^*\eta = \mathbf{Z}_p\diamondsuit_2(S(L), 1)$. Let now π be the morphism from $\mathbf{Z}_p\diamondsuit_2(S(L),1)$ into $S(L)$ defined by $w\eta\pi = w\eta_L$. Let e be an idempotent of $S(L)$ and let $m \in e\pi^{-1}$, that is,

$$m = \begin{pmatrix} e & P \\ 0 & 1 \end{pmatrix}$$

Then, one verifies by induction that for each $n \geq 1$,

$$m^n = \begin{pmatrix} e & (n-1)P + P \\ 0 & 1 \end{pmatrix}$$

Therefore $m^{p+1} = m$ and hence π is a bounded torsion morphism. By Proposition 2.2, if $S(L)$ is torsion (resp. bounded torsion), then so are $\mathbf{Z}_p\diamondsuit_2(S(L),1)$ and $S(\langle L,p,q\rangle)$. □

Remark Let $L_0, L_1, \ldots, L_k \subseteq X^*$, $a_1, \ldots, a_k \in X$ and $r \geq 0$, $n \geq 2$. We define the product with counter $(L_0a_1L_1 \cdots a_kL_k)_{r,n}$ to be the set of all words $w \in X^*$ for which the number of factorizations in the form

$$w = w_0a_1w_1 \cdots a_kw_k \qquad \text{with } w_i \in L_i \text{ for } 0 \leq i \leq k$$

is congruent to r modulo n. This product was studied in detail in [22,23], and the above proof shows in fact that $\mathcal{L}_{\rm btor}$ and $\mathcal{L}_{\rm tor}$ are closed under such products.

Our next task will be to prove the following result. Our proof extends ideas contained in the proof of a less general result, due to Straubing [21].

Proposition 5.3 *$\mathcal{L}_{\rm tor}$ satisfies (St).*

Proof. Let $L \subseteq X^*$ be a torsion language and let $\eta_L: X^* \to S(L)$ be its syntactic morphism. Let $w \in X^*$. Then there exists $n \geq 1$ such that $w^n\eta_L = w^{2n}\eta_L$. Let $x = w^n$. Then $x\eta_L = x^2\eta_L$. It will be sufficient to prove that there exist k and $k' \geq 1$ such that $x^k\eta_{L^*} = x^{k+k'}\eta_{L^*}$.

Let $k_1 = 2 + 4|x|$ and $k_2 = (k_1-2)!$. We will first prove that, for all $u, v \in X^*$,

$$\begin{aligned} &ux^{k_1}v \in L^* \Longrightarrow ux^{k_1+k_2}v \in L^*, \\ &ux^{k_1}v \in L^* \Longrightarrow \exists\ 1 \leq t \leq k_1 - 2,\ ux^{k_1-t}v \in L^*. \end{aligned}$$

Indeed, if $ux^{k_1}v \in L^*$, then $ux^{k_1}v = x_1 \cdots x_p$ for some $x_1, \cdots, x_p \in L$. For each $0 \leq r \leq 1 + 2|x| = \frac{k_1}{2}$, let

$$i_r = \min\{1 \leq i \leq p \mid ux^{2r} \text{ is a prefix of } x_1 \cdots x_i\}.$$

Then $r \mapsto i_r$ is a non-decreasing function from $\{0, \ldots, 1 + 2|x|\}$ into $\{1, \ldots, p\}$. If the function $r \mapsto i_r$ is not injective, then there exists $0 \leq r \leq 2|x|$ such that $i_r = i_{r+1}$, that is, $ux^{2r} = x_1 \cdots x_{i_r - 1}u'$ and $x_1 \cdots x_{i_r} = ux^{2(r+1)}v'$ for some non-empty prefix u' of x_{i_r} and for some $v' \in X^*$. In particular $x_{i_r} = u'x^2v'$ and $v'x_{i_r+1} \cdots x_p = x^{k_1 - 2(r+1)}v$. Since $x\eta_L = x^2\eta_L = x^t\eta_L$ for all $t \geq 1$, we have $u'xv' \in L$ and $u'x^{2+t}v' \in L$ for all $t \geq 1$. Therefore

$$ux^{k_1-1}v = ux^{2r}xx^{k_1-2(r+1)}v = x_1 \cdots x_{i_r-1}(u'xv')x_{i_r+1} \cdots x_p \in L^*$$

and, for $t \geq 1$,

$$ux^{k_1+t}v = ux^{2r}x^{2+t}x^{k_1-2(r+1)}v = x_1 \cdots x_{i_r-1}(u'x^{2+t}v')x_{i_r+1} \cdots x_p \in L^*.$$

Now, if the function $r \mapsto i_r$ is injective, then for each $0 \leq r \leq 2|x|$, we have

$$\begin{aligned} x_1 \cdots x_{i_r} &= ux^{2r}w_1(r) \\ x_{i_r+1} \cdots x_p &= w_2(r)x^{k_1-2(r+1)}v \end{aligned}$$

for some words $w_1(r)$ and $w_2(r)$ such that $w_1(r)w_2(r) = x^2$ and $w_2(r) \neq 1$. Since r can assume $2|x| + 1$ values, there exist $0 \leq r < r' \leq 2|x|$ such that

$$w_1(r) = w_1(r') \text{ and } w_2(r) = w_2(r').$$

Therefore

$$ux^{k_1-2(r'-r)}v = ux^{2r}w_1(r)w_2(r')x^{k_1-2(r+1)}v = x_1 \cdots x_{i_r}x_{i_{r'}+1} \cdots x_p \in L^*,$$

and, since $x_{i_r+1} \cdots x_{i_{r'}} = w_2(r)x^{2(r'-r-1)}w_1(r')$, for all $t \geq 1$ we have

$$\begin{aligned} ux^{k_1+2t(r'-r)}v &= ux^{2r}w_1(r)\big(w_2(r)x^{2(r'-r-1)}w_1(r')\big)^t w_2(r')x^{k_1-2(r+1)}v \\ &= x_1 \cdots x_{i_r}(x_{i_r+1} \cdots x_{i_{r'}})^t x_{i_{r'}+1} \cdots x_p \in L^*. \end{aligned}$$

Note that $2(r'-r) \leq 2r' \leq 4|x| \leq k_1-2$. Therefore, in all cases and if $k_2 = (k_1-2)!$,

$$\begin{aligned} &ux^{k_1}v \in L^* \Longrightarrow ux^{k_1+k_2}v \in L^*, \\ &ux^{k_1}v \in L^* \Longrightarrow ux^{k_1-t}v \in L^* \text{ for some } 1 \leq t \leq k_1 - 2. \end{aligned}$$

Let now $y = x^{k_2}$ and $k_3 = 2 + 4|y|$. Since $y\eta_L = y^2\eta_L$, the above computation shows that

$$uy^{k_3}v \in L^* \Longrightarrow uy^{k_3-t}v \in L^* \text{ for some } 1 \leq t \leq k_3 - 2,$$

that is,

$$ux^{k_2k_3}v \in L^* \Longrightarrow ux^{(k_3-t)k_2}v \in L^* \text{ for some } 1 \leq t \leq k_3 - 2.$$

Now, $(k_3 - t)k_2 \geq 2k_2 \geq k_1$, so that

$$ux^{(k_3-t)k_2}v \in L^* \Longrightarrow ux^{(k_3-t)k_2+sk_2}v \in L^* \text{ for all } s \geq 1$$

and hence

$$ux^{k_2k_3}v \in L^* \Longrightarrow ux^{k_2k_3-k_2}v \in L^*.$$

Conversely, since $k_2k_3 - k_2 \geq k_1$,

$$ux^{k_2k_3-k_2}v \in L^* \Longrightarrow ux^{k_2k_3}v \in L^*.$$

Therefore $x^{k_2k_3-k_2}\eta_{L^*} = x^{k_2k_3}\eta_{L^*}$, which concludes the proof. □

The following proposition will help us show that $\mathcal{L}^{(k)}_{\text{btor}}$ and $\mathcal{L}^{[k]}_{\text{bap}}$ are closed under limits.

Proposition 5.4 *Let $\mathcal{I} = (I_n)_n$ be a filtration of X^* and let $L \subseteq X^*$ be the limit $L = \lim_{\mathcal{I}} L_n$ for some sequence of languages $(L_n)_n$. Then $S(L) < \prod_n S(L_n)$.*

Proof. Let $\eta_n: X^* \to S(L_n)$ be the syntactic morphism of L_n and let $\eta: X^* \to \prod_n S(L_n)$ be defined by $u\eta = (u\eta_n)_n$ for all $u \in X^*$. By Lemma 1.2, it suffices to show that η recognizes L. Let us assume that $u, v \in X^*$ are such that $u\eta = v\eta$ and $u \in L$. Since $\bigcap_n I_n = \emptyset$, there exists $n_0 \geq 1$ such that u and v are not in I_{n_0}. Since $L = \lim_{\mathcal{I}} L_n$, there exists n_1 such that $L \setminus I_{n_0} = L_{n_1} \setminus I_{n_0}$. In particular, $u \in L_{n_1}$. But $u\eta = v\eta$ implies $u\eta_{n_1} = v\eta_{n_1}$, so $v \in L_{n_1} \setminus I_{n_0}$ and hence $v \in L$. Thus η recognizes L. □

Corollary 5.5 *For all k, the classes $\mathcal{L}^{(k)}_{\text{btor}}$ and $\mathcal{L}^{[k]}_{\text{bap}}$ are closed under limits. In particular, $\mathcal{L}_{\text{btor}}$ and $\mathcal{L}_{\text{bap}}$ are closed under bounded limits.*

Proof. We know that the classes $\mathbf{bTor}_k$ and $\mathbf{bAp}_k$ are closed under arbitrary direct product and taking finitely generated divisors. In particular, if $\mathcal{I}$ is a filtration of X^* and if $L = \lim_{\mathcal{I}} L_n$ where the L_n are all in $\mathcal{L}^{(k)}_{\text{btor}}$ (resp. $\mathcal{L}^{[k]}_{\text{bap}}$), then so is L. This shows immediately that the classes $\mathcal{L}_{\text{btor}}$ and $\mathcal{L}_{\text{bap}}$ satisfy the required closure property. □

To complete this section we prove the following important proposition.

Proposition 5.6 *$\mathcal{L}_{\text{tor}}$ and $\mathcal{L}_{\text{ap}}$ are closed under fast approximation.*

Proof. Let $\mathcal{I} = (I_n)_n$ be a fast filtration, and let $A = (Q, X, q_0, \lambda)$ and $A_n = (Q_n, X, q_0^n, \lambda_n)$ $(n \geq 0)$ be automata such that $\mathcal{L}(A_n) \subseteq \mathcal{L}_{\text{tor}}$ for all n and such that the sequence $(A_n)_n$ approximates A modulo $\mathcal{I}$. By Proposition 2.5, we know that $S(A_n) \in \mathbf{Tor}$ for each n, and we need to show that $S(A) \in \mathbf{Tor}$.

Let $u \in X^*$. Since $\mathcal{I}$ is a fast filtration, there exists an integer n such that $u^* \cap I_n = \emptyset$. Let now k_n be the integer associated to n in the definition of approximation. Since $S(A_n) \in \mathbf{Tor}$, there exists $r \geq 1$ such that $q \cdot u^r = q \cdot u^{2r}$ for all $q \in Q_{k_n}$. This is equivalent to $q_0^{k_n} \cdot vu^r = q_0^{k_n} \cdot vu^{2r}$ for all $v \in X^*$. Because of the way we chose k_n, this implies $q_0 \cdot vu^r = q_0 \cdot vu^{2r}$ for all $v \in X^*$, and hence $q \cdot u^r = q \cdot u^{2r}$ for all $q \in Q$. Thus $S(A)$ is torsion and hence $\mathcal{L}(A) \subseteq \mathcal{L}_{\mathrm{tor}}$.

The proof of the statement concerning $\mathcal{L}_{\mathrm{ap}}$ is similar. $\square$

5.2 Proof of Theorems 4.3, 4.6, 4.10, 4.13, 4.15 and 4.16

These proofs require three deep results from semigroup theory. The first of these results (Theorem 5.7 below) is a property of finitely generated Burnside monoids, defined by identities of the form $x^k = x^{k+\ell}$. This result was first proved by McCammond [11] for $k \geq 6$, and extended to $k \geq 4$ by Pereira do Lago [14] using different methods. The two other results are properties of the semigroup expansions $S \mapsto \widehat{S}^{(3)}$ and $S \mapsto \overline{S}$ (cut-down to generators) considered by Rhodes and others. A complete study of these expansions, including the proof of the results stated here, can be found in the works of Birget, Henckell, Lazarus and Rhodes [1,17,18,6]. See also Appendix 2 below.

Theorem 5.7 *(See [11,14]) Let X be a finite alphabet, and let k and ℓ be integers with $k \geq 4$ and $\ell \geq 1$. Let $B_X(k, \ell)$ be the monoid generated by X and defined by the identity $x^k = x^{k+\ell}$.*

(1) The maximal subgroups of $B_X(k, \ell)$ are finite cyclic groups.

(2) There exists a sequence $(J_n)_n$ of ideals of $B_X(k, \ell)$ such that

$$B_X(k, \ell) = J_0 \supseteq J_1 \supseteq \cdots \supseteq J_n \supseteq \cdots$$

$\bigcap_n J_n = \emptyset$, and $B_X(k, \ell)/J_n$ is a finite monoid.

Theorem 5.8 *(See [1]) Let S be a finitely generated semigroup, and let $\sigma: X^* \to S$ be an onto morphism, with X a finite alphabet. There exists a semigroup $\widehat{S}^{(3)}$, an onto morphism $\tau: X^* \to \widehat{S}^{(3)}$ and a morphism $\pi: \widehat{S}^{(3)} \to S$ which satisfy the following property*

(1) $\tau\pi = \sigma$.

(2) π is a 5-bounded-aperiodic morphism.

(3) The subgroups of $\widehat{S}^{(3)}$ are isomorphic to the finite subgroups of S.
(4) There exists a sequence $(J_n)_n$ of ideals of $\widehat{S}^{(3)}$ such that

$$\widehat{S}^{(3)} = J_0 \supseteq J_1 \supseteq \cdots \supseteq J_n \supseteq \cdots$$

$\bigcap_n J_n = \emptyset$, *and $\widehat{S}^{(3)}/J_n$ is a finite monoid.*

Theorem 5.9 *(See [17,18]) Let S be a finitely generated semigroup, and let $\sigma: X^* \to S$ be an onto morphism, with X a finite alphabet. There exists a semigroup $\overline{S}$, an onto morphism $\varphi: X^* \to \overline{S}$ and a morphism $\psi: \overline{S} \to S$ which satisfy the following property*
(1) $\varphi\psi = \sigma$.
(2) ψ is an aperiodic morphism.
(3) The maximal subgroups of $\overline{S}$ are finite cyclic groups.

We are now ready to give the remaining proofs.

Proof of Theorems 4.3 and 4.6. Let $L \subseteq X^*$ and let $\sigma: X^* \to S$ be its syntactic morphism. If $L \in \mathcal{L}_{\rm btor}$, then $S \in \mathbf{bTor}$, and hence $S \in \mathbf{bTor}_k$ for some integer k. Since $k \leq k'$, S belongs to $\mathbf{bTor}_{k'}$ as well. Let μ be the canonical morphism from X^* onto $B_X(k', k')$. Then there exists an onto morphism $\varphi: B_X(k', k') \to S$ such that $\mu\varphi = \sigma$. Let $(J_n)_n$ be given by Theorem 5.7 and let $I_n = J_n\mu^{-1}$ for each $n \geq 0$.

$\mathcal{I} = (I_n)_n$ is trivially an ideal filtration on X^*. Furthermore, each I_n is recognized by the morphism $\mu_n: X^* \to B_X(k', k')/J_n$ obtained by composing μ with the canonical projection of $B_X(k', k')$ onto $B_X(k', k')/J_n$. This proves that I_n is rational subgroup-cyclic and k-bounded-torsion. The fact that $B_X(k', k')$ is torsion also implies that $\mathcal{I}$ is a fast filtration. Indeed for each $w \in X^*$, the set $w^*\mu$ is finite, so there exists $n \geq 1$ such that $w^*\mu \cap J_n = \emptyset$ and hence $w^* \cap I_n = \emptyset$.

Let now $L_n = L \backslash I_n$ for each $n \geq 0$. Since L is recognized by σ, it is recognized by μ too, and hence L_n is recognized by the morphism μ_n, so that L_n is rational subgroup-cyclic and k'-bounded-torsion for all n. It is then easy to verify that $L = \lim_{\mathcal{I}} L_n$ and the limit is bounded rational subgroup-cyclic.

In the case where $L \in \mathcal{L}_{\rm bap}^{[k]}$, we have $S \in \mathbf{bAp}_k$, and we consider $B_X(k', 1)$ instead of $B_X(k', k')$. The corresponding statement of Theorem 4.3 follows since $B_X(k', 1) \in \mathbf{bAp}_{k'}$ by Theorem 5.7. □

Proof of Theorem 4.15. Let $A = (Q, X, q_0, \lambda)$ be an automaton and let $\sigma: X^* \to S = S(A)$ be its transition morphism. Let also $\tau: X^* \to \widehat{S}^{(3)}$, $\pi: \widehat{S}^{(3)} \to S$

and $(J_n)_n$ be given by Theorem 5.8 and let $I_n = J_n\tau^{-1}$ for each $n \geq 0$. As in the above proof, one verifies that, for each n, I_n is recognized by the onto morphism $\tau_n: X^* \to \widehat{S}^{(3)}/J_n$, obtained by composing τ with the projection of $\widehat{S}^{(3)}$ onto $\widehat{S}^{(3)}/J_n$. Note that $\widehat{S}^{(3)}/J_n$ is finite. So $\mathcal{I} = (I_n)_n$ is a rational ideal filtration. For each n, let $A_n = A(\widehat{S}^{(3)}/J_n)$ be the automaton associated with the morphism τ_n. It is not difficult to verify that the sequence $(A_n)_n$ converges to $A(\widehat{S}^{(3)})$ modulo $\mathcal{I}$. Furthermore

$$\begin{aligned} \varphi: \widehat{S}^{(3)} &\longrightarrow Q \\ u\tau &\longmapsto q_0 \cdot u \end{aligned}$$

is a well-defined morphism from $A(\widehat{S}^{(3)})$ to A, so that the sequence $(A_n)_n$ approximates A modulo $\mathcal{I}$ (Proposition 3.5). There remains to verify that the approximation is bounded aperiodic. Let $u \in X^*$ be such that $q \cdot u = q \cdot u^2$ for all $q \in Q$. Then $u\sigma = u^2\sigma$ and hence $u^5\tau = u^6\tau$ since $\pi: \widehat{S}^{(3)} \to S$ is 5-bounded aperiodic. Therefore, if s is any state of $A(\widehat{S}^{(3)})$, then $s \cdot u^5 = s(u^5\tau) = s(u^6\tau) = s \cdot u^6$. This concludes the proof. □

Proof of Theorem 4.10. The proof of Theorem 4.15 given above can be copied *verbatim*. We only need to verify that the filtration $\mathcal{I}$ is fast, and that the I_n and the languages of the A_n are star-free if $\mathcal{L}(A) \subseteq \mathcal{L}_{\mathrm{ap}}$. Since π is 5-bounded-aperiodic, $\widehat{S}^{(3)}$ is torsion (resp. aperiodic) if S is, that is, if $\mathcal{L}(A)$ consists only of torsion (resp. aperiodic) languages. In particular, if $\mathcal{L}(A) \subseteq \mathcal{L}_{\mathrm{ap}}$, then the monoids $\widehat{S}^{(3)}/J_n$ are finite and aperiodic, so that I_n and the languages of A_n are star-free. The proof that $\mathcal{I}$ is fast is as in the proof of Theorem 4.3. □

Proof of Theorem 4.16. This proof is quite similar to the proof of Theorem 4.15. Let $A = (Q, X, q_0, \lambda)$ be an automaton and let $\sigma: X^* \to S = S(A)$ be its transition morphism. Let $\varphi: X^* \to \overline{S}$ and $\psi: \overline{S} \to S$ be given by Theorem 5.9. We now apply Theorem 5.8 to the morphism φ. Let $T = \overline{S}^{\frown(3)}$. We obtain onto morphisms $\tau: X^* \to T$, $\pi: T \to \overline{S}$ and a sequence $(J_n)_n$ of ideals of T. Let now $\mathcal{I} = (I_n)_n$ with $I_n = J_n\tau^{-1}$ for all $n \geq 0$. Reasoning as in the proof of Theorem 4.15, we verify that each I_n is recognized by a morphism $\tau_n: X^* \to T/J_n$, which proves that I_n is rational subgroup-cyclic. Again, we verify that, if $A_n = A(T/J_n)$, the sequence $(A_n)_n$ converges to $A(T)$ modulo $\mathcal{I}$, and there exists a torsion morphism from $A(T)$ to A. Thus $(A_n)_n$ approximates A modulo $\mathcal{I}$ and the approximation is aperiodic. □

Proof of Theorem 4.13. We only need to complete the proof of Theorem 4.16

with the consequences of the fact that $\mathcal{L}(A) \subseteq \mathcal{L}_{\rm tor}$, that is, $S \in \mathbf{Tor}$, and this is done as in the proof of Theorem 4.10. $\square$

Appendix 1 Sequences of morphisms

Instead of considering sequences of automata as we did in Section 3.3, we could have considered sequences of X-generated monoids, or more precisely, of onto morphisms defined on X^*. To such a morphism $\sigma: X^* \to S$ we associate the congruence

$$C_\sigma = \{(u, v) \in X^* \times X^* \mid u\sigma = v\sigma\}.$$

Given a filtration $\mathcal{I}$ on X^*, we then define a distance function $D_{\mathcal{I}}$ by

$$D_{\mathcal{I}}(\sigma, \tau) = \max \{d(u, v) \mid (u, v) \in C_\sigma \bigtriangleup C_\tau\}.$$

As in Section 3.3, D is in fact a distance function on the set of isomorphism classes of onto morphisms defined on X^*, which is ultrametric. Results analogous to Propositions 3.3 to 3.5 hold for this metric too. In particular, one can define the concept of a sequence of onto morphisms approximating an onto morphism modulo a filtration, and the concept of a class of monoids being closed under (fast) approximation.

Within this framework, statements similar to those of Sections 4.2 and 4.3 hold, and they are obtained with proofs quite similar to those reported for Proposition 5.6 and in Section 5.2.

Appendix 2 Rational filtration associated to an arbitrary congruence

Let $\equiv$ be an arbitrary congruence on X^* and let $\sigma: X^* \to S = X^*/\equiv$ be the canonical projection morphism. In this section we will show how a rational ideal filtration is naturally associated with $\equiv$. Let us emphasize that $\equiv$ is arbitrary but that the (simple) objects which we construct from $\equiv$-computations are rational!

Recall the definition of Green's relation $\mathcal{J}$. If s and t are elements of a monoid S, we say that $s \leq_{\mathcal{J}} t$ if $SsS \subseteq StS$ and that $s\,\mathcal{J}\,t$ if $SsS = StS$. Then $\mathcal{J}$ is an equivalence relation and $\leq_{\mathcal{J}}$ is a quasi-order. If $s \leq_{\mathcal{J}} t$ and $not(s\,\mathcal{J}\,t)$, we write $s <_{\mathcal{J}} t$. We say that a word u is a factor of a word v if $v = xuy$ for some $x, y \in X^*$. This is equivalent to saying that $v \leq_{\mathcal{J}} u$ in the $\mathcal{J}$-order of X^*. Let $\mathcal{I}_{\equiv} = (I_n)_n$ be the sequence of subsets of X^* defined for each $n \geq 0$ by

$$I_n = \{w \in X^* \mid |\{u\sigma \mid u \text{ is a factor of } w\}| \geq n\}.$$

Theorem A.2.1 *Let $\equiv$ be an arbitrary congruence on X^*. Then $\mathcal{I}_{\equiv}$ is a rational ideal filtration.*

The proof of this theorem is actually an immediate consequence of the proof of Theorem 5.8 applied to $\sigma: X^* \to S$. We will indicate the structure of the proof of this theorem, as it is needed to prove Theorem A.2.1. Details can be found in [1,6].

Sketch of the proof of Theorem 5.8. Let $\mathcal{P}_f(S^3)$ be the set of finite subsets of $S^3 = S \times S \times S$. For (s_1, s_2, s_3) and (s_4, s_5, s_6) in S^3 we define

$$(s_1, s_2, s_3) * (s_4, s_5, s_6) = \{(\prod_{i=1}^{n_1} s_i, \prod_{i=n_1+1}^{n_2} s_i, \prod_{i=n_2+1}^{6} s_i) \mid 0 \leq n_1 \leq n_2 \leq 6\}$$

with empty products equal to 1 by convention. This operation is extended to $\mathcal{P}_f(S^3)$ by

$$A * B = \{a * b \mid a \in A,\ b \in B\}$$

for all $A, B \in \mathcal{P}_f(S^3)$, which makes $\mathcal{P}_f(S^3)$ a monoid. Moreover, letting

$$w\tau = \{(w_1\sigma, w_2\sigma, w_3\sigma) \mid w_1, w_2, w_3 \in X^*,\ w = w_1w_2w_3\}$$

for each $w \in X^*$ defines a morphism from X^* into $\mathcal{P}_f(S^3)$. Let $\widehat{S}^{(3)} = X^*\tau$. For each (s_1, s_2, s_3) in $\mathcal{P}_f(S^3)$, let $(s_1, s_2, s_3)\pi = s_1s_2s_3$. Then $\sigma = \tau\pi$.

Next one verifies that if $w \in X^*$ and $w\sigma$ is an idempotent of S, then $w^5\tau = w^6\tau$. The argument is standard: if w^6 is factored as $w^6 = w_1w_2w_3$, then one of w_1, w_2 and w_3 contains w^2 as as factor. This shows that π is bounded aperiodic.

For each $x \in \widehat{S}^{(3)}$, let $\mathrm{set}(x)$ and $F(x)$ be the sets

$$\mathrm{set}(x) = \{s \in S \mid \exists s_1, s_2 \in S,\ (s_1, s, s_2) \in x\} \quad \text{and} \quad F(x) = \{y \in \widehat{S}^{(3)} \mid x \leq_{\mathcal{J}} y\}.$$

Then $\mathrm{set}(x)$ is a finite set, containing $x\pi$, and $x \leq_{\mathcal{J}} y$ implies $\mathrm{set}(y) \subseteq \mathrm{set}(x)$. Therefore, for each fixed x, the set $\{\mathrm{set}(y) \mid y \in F(x)\}$ is finite, and hence $F(x)$ is finite. By elementary facts of semigroup theory, it follows that each strictly ascending chain in the $<_{\mathcal{J}}$-order is finite, each $\mathcal{J}$-class of $\widehat{S}^{(3)}$ is finite, and so is each of its subgroups.

For each $n \geq 1$ let us now define

$$J_n = \{x \in \widehat{S}^{(3)} \mid |\mathrm{set}(x)| \geq n\}.$$

Since $x \leq_{\mathcal{J}} y$ implies $\mathrm{set}(y) \subseteq \mathrm{set}(x)$, J_n is an ideal, and $\bigcap_n J_n = \emptyset$. We now want to prove that $\widehat{S}^{(3)} \setminus J_n$ is finite for all n.

Since $J_0 = \widehat{S}^{(3)}$ it will be sufficient to show that

$$J_n \setminus J_{n+1} = \{x \in \widehat{S}^{(3)} \mid |\mathrm{set}(x)| = n\}$$

is finite for all n. Note that J_0 is a finitely generated semigroup. We will prove by induction on n that $J_n \setminus J_{n+1}$ is finite and J_{n+1} is finitely generated (as a semigroup) for all $n \geq 0$. If J_n is generated by a finite set $X_n \subseteq J_n$ and if $x = x_1 \cdots x_k$ ($x_i \in X_n$), then we have $\mathrm{set}(x_i) \subseteq \mathrm{set}(x)$ for all i. Thus $|\mathrm{set}(x)| = n$ implies $\mathrm{set}(x) = \mathrm{set}(x_i)$ for all $1 \leq i \leq k$. Therefore $J_n \setminus J_{n+1}$ is contained in the union of the sets $\mathrm{set}^{-1}(\mathrm{set}(x))$ when $x \in X_n$ and hence is finite. Let now

$$Y_n = \{rx \mid r \in J_n \setminus J_{n+1} \cup \{1\},\ x \in X_n \text{ and } rx \in J_{n+1}\}$$

and $X_{n+1} = Y_n \cup Y_n(J_n \setminus J_{n+1})$. Then X_{n+1} is finite and, since J_{n+1} is an ideal, X_{n+1} is contained in J_{n+1}. Let now $y \in J_{n+1}$. Then $y = x_1 \cdots x_k$ for some $x_1, \cdots, x_k \in X_n$. Let $k_1 \geq 0$ be minimal such that $x_1 \cdots x_{k_1+1} \in J_{n+1}$. Then $r_1 = x_1 \cdots x_{k_1} \in J_n \setminus J_{n+1}$ and $r_1 x_{k_1+1} \in Y_n$. Iterating this reasoning we obtain a factorization

$$y = x_1 \cdots x_k = (r_1 x_{k_1+1})(r_2 x_{k_2+1}) \cdots (r_t x_{k_t+1}) r_{t+1}$$

with $r_1 x_{k_1+1} \in Y_n, \ldots, r_t x_{k_t+1}$ in Y_n and $r_{t+1} \in J_n \setminus J_{n+1} \cup \{1\}$. Thus X_{n+1} is a finite system of generators for J_{n+1}. □

Remark Using the notations of the above proof, let

$$K_n = \{x \in \widehat{S}^{(3)} \mid h_{\mathcal{J}}(x) \geq n\}.$$

We could prove in a similar fashion that $(K_n)_n$ is another sequence of ideals of $\widehat{S}^{(3)}$ satisfying condition (4) of Theorem 5.8.

Proof of Theorem A.2.1. We use the notations of the above proof. Let $\tau_n \colon X^* \to \widehat{S}^{(3)}/J_n$ be the composition of the morphism $\tau \colon X^* \to \widehat{S}^{(3)}$ with the projection of $\widehat{S}^{(3)}$ onto $\widehat{S}^{(3)}/J_n$. For $w \in X^*$, note that

$$\mathrm{set}(w\tau) = \{u\sigma \mid u \text{ is a factor of } w\}.$$

Therefore $I_n = J_n \tau^{-1}$, so that I_n is an ideal and I_n is recognized by the morphism τ_n. Since $\widehat{S}^{(3)}/J_n$ is finite, I_n is rational. □

Appendix 3 Generalization of Eilenberg's variety theorem

In this section we give a generalization of Eilenberg's variety (or stream) theorem (see [4,16]) which encompasses the classes of (bounded) torsion and aperiodic languages. Recall that for us, a language is always a subset of some free monoid X^* where the alphabet X is finite. We say that a monoid S is *syntactic* if it is the syntactic monoid of some language. In particular S is necessarily finitely generated. If P is a subset of a monoid S, we say that P is *disjunctive* if for all $s, t \in S$

$$\big(\forall u, v \in S, \quad usv \in P \Longleftrightarrow utv \in P\big) \quad \Longleftrightarrow \quad s = t.$$

(In other words, the syntactic congruence of P in S is trivial.) The following result is well-known.

Lemma A.3.1 *Let S be a finitely generated monoid. Then S is syntactic if and only if S contains a disjunctive subset.*

This lemma allows the description of a large class of syntactic monoids.

Proposition A.3.2 *Let S be a finitely generated semigroup. Let us assume that S admits a sequence of ideals $(J_n)_n$ such that*

$$S = J_0 \supseteq J_1 \supseteq \cdots \supseteq J_n \supseteq \cdots$$

$\bigcap_n J_n = \emptyset$, and, for all $n \geq 0$ and for all pairs of distinct elements of S, $s \neq t$, there exists $u, v \in S$ such that

$$usv \neq utv \qquad \text{and} \qquad usv, utv \in J_n.$$

Then S is a syntactic monoid.

Proof. For each $x \in S$, let $r(x) = \min\{n \geq 0 \mid x \notin J_n\}$. Notice that S is countable since it is finitely generated. Let

$$(s_0, t_0), (s_1, t_1), \ldots, (s_n, t_n), \ldots$$

be an enumeration of the pairs of distinct elements of S. We now construct by induction two sequences $(u_n)_n$ and $(v_n)_n$ of elements of S such that, for each n,

$$(*) \qquad \begin{array}{c} u_n s_n v_n \neq u_n t_n v_n \qquad \text{and} \\ \max\big\{r(u_n s_n v_n), r(u_n t_n v_n)\big\} > \max\big\{r(u_i s_i v_i), r(u_i t_i v_i) \bigm| 0 \leq i < n\big\}. \end{array}$$

By hypothesis, there exist u_0 and v_0 such that $u_0s_0v_0 \neq u_0t_0v_0$. Let us now assume that $u_0, \ldots, u_m$ and $v_0, \ldots, v_m$ have been chosen which satisfy $(*)$ for all $n \leq m$. Let $r_m = \max\{r(u_is_iv_i), r(u_it_iv_i) \mid 0 \leq i \leq m\}$ By hypothesis, there exists u_{m+1} and v_{m+1} such that $u_{m+1}s_{m+1}v_{m+1} \neq u_{m+1}t_{m+1}v_{m+1}$ and $u_{m+1}s_{m+1}v_{m+1}, u_{m+1}t_{m+1}v_{m+1} \in J_r$, that is, $r(u_{m+1}s_{m+1}v_{m+1}) > r_m$ and $r(u_{m+1}t_{m+1}v_{m+1}) > r_m$. For these values of u_{m+1} and v_{m+1}, $(*)$ is again satisfied.

Let now $P = \{u_ns_nv_n \mid n \geq 0\}$. We claim that P is a disjunctive subset of S, and hence that S is syntactic by Lemma A.3.1. Indeed, if $s \neq t \in S$, then $s = s_n$ and $t = t_n$ for some n. Then $u_nsv_n = u_ns_nv_n \in P$. Furthermore, $u_ntv_n = u_nt_nv_n$ is not equal to u_nsv_n, and for all $i < n < j$, we have

$$r(u_is_iv_i) < r(u_ntv_n) < r(u_js_jv_j),$$

so that $u_ntv_n \notin P$. $\square$

Corollary A.3.3 *If X is a finite alphabet, then X^* is syntactic.*

Proof. It is easy to verify that X^* satisfies the hypothesis of Proposition A.3.2 for $J_n = \{w \in X^* \mid |w| \geq n\}$ $(n \geq 0)$. $\square$

Let $\mathbf{V}$ be a class of finitely generated monoids. We say that $\mathbf{V}$ is a *variety of finitely generated monoids*, or *fg-variety*, if:

(1) $\mathbf{V}$ is not empty.
(2) If $S_1, S_2 \in \mathbf{V}$, then $S_1 \times S_2 \in \mathbf{V}$.
(3) If $S \in \mathbf{V}$, if T is finitely generated and if T divides S, then $T \in \mathbf{V}$.
(4) For each $S \in \mathbf{V}$, there exists a finite collection $S_1, \ldots, S_n$ of monoids in $\mathbf{V}$ which are syntactic and such that $S < S_1 \times \cdots \times S_n$.

Remark We could also define an fg-variety by properties (1)–(3), and then restrict ourselves to fg-varieties which are generated by a class of syntactic monoids.

Example Let **FG** be the class of all finitely generated monoids. Then **FG** is an fg-variety. Indeed, by Corollary A.3.3, each finitely generated monoid is a quotient of a syntactic monoid. The usual varieties of finite monoids, or **M**-varieties are exactly the fg-varieties consisting only of finite monoids.

Let $\mathcal{L}$ be a class of languages. We say that $\mathcal{L}$ is a *variety of languages* if:

(1) $\mathcal{L}$ is not empty.
(2) $\mathcal{L}$ satisfies (H^{-1}).
(3) $\mathcal{L}$ satisfies (BT).

Example The class $\mathcal{L}_{\text{all}}$ of all languages is a variety. The results of Section 4 show that, for $k \geq 1$, $\mathcal{L}^{(k)}_{\text{btor}}$, $\mathcal{L}^{[k]}_{\text{bap}}$, $\mathcal{L}_{\text{btor}}$, $\mathcal{L}_{\text{bap}}$, $\mathcal{L}_{\text{tor}}$ and $\mathcal{L}_{\text{ap}}$ are varieties of languages. Also, the usual varieties of rational languages are exactly the varieties of languages consisting only of rational languages.

Let $\mathbf{V}$ be an fg-variety. We define $\mathcal{L}(\mathbf{V})$ to be the class of all languages whose syntactic monoid is in $\mathbf{V}$, or equivalently, the class of all languages that are recognized by some monoid in $\mathbf{V}$. Also, if $\mathcal{L}$ is a class of languages, we let $\mathbf{V}(\mathcal{L})$ be the fg-variety generated by the syntactic monoids of the languages of $\mathcal{L}$. With these notations, we can state the following generalization of Eilenberg's variety theorem.

Theorem A.3.4 *The correspondence $\mathbf{V} \mapsto \mathcal{L}(\mathbf{V})$ is one-to-one and onto from the class of all fg-varieties onto the class of all varieties of languages. Furthermore the reciprocal correspondence is given by $\mathcal{L} \mapsto \mathbf{V}(\mathcal{L})$.*

Proof. The proof is very similar to that of Eilenberg's variety theorem. Let $\mathbf{V}$ be an fg-variety. We verify that $\mathcal{L}(\mathbf{V})$ is a variety of languages by an immediate application of Lemma 1.2 and Proposition 1.3.

It is clear that, if $\mathbf{V}$ and $\mathbf{W}$ are fg-varieties and $\mathbf{V} \subseteq \mathbf{W}$, then $\mathcal{L}(\mathbf{V}) \subseteq \mathcal{L}(\mathbf{W})$. Let us prove that the converse holds, that is, that $\mathcal{L}(\mathbf{V}) \subseteq \mathcal{L}(\mathbf{W})$ implies $\mathbf{V} \subseteq \mathbf{W}$. Note that this will prove that the correspondence $\mathbf{V} \mapsto \mathcal{L}(\mathbf{V})$ is one-to-one. Let $S \in \mathbf{V}$. By definition of an fg-variety, we have $S < S_1 \times \cdots \times S_n$ where $S_i \in \mathbf{V}$ and S_i is the syntactic monoid of some language L_i. Then each L_i is in $\mathcal{L}(\mathbf{V})$ and hence in $\mathcal{L}(\mathbf{W})$. Therefore $S_i \in \mathbf{W}$ for each i and hence $S \in \mathbf{W}$.

Finally let $\mathcal{L}$ be a variety of languages. We will show that $\mathcal{L}\big(\mathbf{V}(\mathcal{L})\big) = \mathcal{L}$, thus showing that the correspondences $\mathbf{V} \mapsto \mathcal{L}(\mathbf{V})$ and $\mathcal{L} \mapsto \mathcal{L}(\mathbf{V})$ are mutually reciprocal. The inclusion $\mathcal{L} \subseteq \mathcal{L}\big(\mathbf{V}(\mathcal{L})\big)$ is trivial. To prove the converse we consider $L \subseteq X^*$ with $L \in \mathcal{L}\big(\mathbf{V}(\mathcal{L})\big)$. We know that the syntactic monoid of L, $S = S(L)$, is in $\mathbf{V}(\mathcal{L})$ and hence that there exist finitely many languages $L_1 \subseteq X_1^*, \ldots, L_n \subseteq X_n^*$ such that $L_1, \ldots, L_n \in \mathcal{L}$ and S divides $S_1 \times \cdots \times S_n$ (where S_i is the syntactic monoid of L_i). Let $\eta_1, \ldots, \eta_n$ be the syntactic morphisms of $L_1, \ldots, L_n$ and let e be a symbol not in $\bigcup_{i=1}^n X_i$. For each $1 \leq i \leq n$ we define a morphism $\sigma_i \colon (X_i \cup \{e\})^* \to X_i^*$ by $x\sigma_i = x$ for all $x \in X_i$ and $e\sigma_i = 1$. Let

$$Y = \big(X_1 \cup \{e\}\big) \times \cdots \times \big(X_n \cup \{e\}\big).$$

Then $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a morphism from Y^* onto $X_1^* \times \cdots \times X_n^*$. Finally let $\eta = (\eta_1, \ldots, \eta_n)$. Then η is an onto morphism from $\prod_{i=1}^n X_i^*$ onto $\prod_{i=1}^n S_i$. Now

S divides $S_1 \times \cdots \times S_n$, so L is recognized by $S_1 \times \cdots \times S_n$. That is, there exists a morphism $\varphi\colon X^* \to S_1 \times \cdots \times S_n$ and a subset P such that $L = P\varphi^{-1}$. Since $\sigma\eta$ is onto there exists a morphism $\pi\colon X^* \to Y^*$ such that $\varphi = \pi(\sigma\eta)$. Therefore $L = P(\sigma\eta)^{-1}\,\pi^{-1}$ and it suffices to show that $P(\sigma\eta)^{-1} \in \mathcal{L}$. The situation is summarized by the following commutative diagram where π_i and π'_i are the i-th projections.

$$\begin{tikzcd}
Y^* \arrow[r, "\sigma"] & X_1^* \times \cdots \times X_n^* \arrow[r, "\pi_i"] \arrow[d, "\eta"] & X_i^* \arrow[d, "\eta_i"] \\
X^* \arrow[u, "\pi"] \arrow[r, "\sigma"] & S_1 \times \cdots \times S_n \arrow[r, "\pi'_i"] & S_i
\end{tikzcd}$$

In particular $\pi_i\eta_i = \eta\pi'_i$ for all $1 \le i \le n$. Let $P_i = L_i\eta_i$ and let

$$\begin{aligned}
L'_i &= (S_1 \times \cdots S_{i-1} \times P_i \times S_{i+1} \times \cdots \times S_n)\eta^{-1}\,\sigma^{-1} \\
&= (X_1^* \times \cdots X_{i-1}^* \times L_i \times X_{i+1}^* \times \cdots \times X_n^*)\sigma^{-1} \\
&= L_i\pi_i^{-1}\,\sigma^{-1}.
\end{aligned}$$

Note that $L'_i = P_i(\sigma\eta\pi'_i)^{-1}$ so that $\sigma\eta\pi'_i$ recognizes L_i. We show that $\sigma\eta\pi'_i = \sigma\pi_i\eta_i$ is in fact the syntactic morphism of L'_i. Let $u, v \in Y^*$ be syntactically equivalent (for L'_i). Then for all $x, y \in Y^*$, $(xuy)\sigma\eta\pi'_i = (xuy)\sigma\pi_i\eta_i \in P_i$ if and only if $(xvy)\sigma\eta\pi'_i = (xvy)\sigma\pi_i\eta_i \in P_i$. Therefore $(xuy)\sigma\pi_i \in L_i$ if and only if $(xvy)\sigma\pi_i \in L_i$, that is, $u\,\sigma\pi_i\eta_i = v\,\sigma\pi_i\eta_i$ and hence $u\,\sigma\eta\pi'_i = v\,\sigma\pi_i\eta_i$.

By Proposition 1.3, $P(\sigma\eta)^{-1} \in BT(L'_1, \ldots, L'_n)$. But $L'_i = L_i\pi_i'^{-1}$, so $L'_i \in \mathcal{L}$ and hence $P(\sigma\eta)^{-1} \in \mathcal{L}$. □

Example We already remarked that the correspondences between **M**-varieties and varieties of rational languages (see Pin [16]) are instances of the correspondence described in Theorem A.3.4. Other examples are given in the following theorem.

Theorem A.3.5 *Let k be an integer with $k \ge 6$. The classes* $\mathbf{FG}$, $\mathbf{FG} \cap \mathbf{bTor}_k$, $\mathbf{FG} \cap \mathbf{bAp}_k$, $\mathbf{FG} \cap \mathbf{bTor}$, $\mathbf{FG} \cap \mathbf{bAp}$, $\mathbf{FG} \cap \mathbf{Tor}$ *and* $\mathbf{FG} \cap \mathbf{Ap}$ *are fg-varieties. We have the following correspondences:*

$$\begin{array}{ccc}
\mathbf{bTor}_k \longmapsto \mathcal{L}_{\mathrm{btor}}^{(k)} & \mathbf{bAp}_k \longmapsto \mathcal{L}_{\mathrm{bap}}^{[k]} & \\
\mathbf{bTor} \longmapsto \mathcal{L}_{\mathrm{btor}} & \mathbf{bAp} \longmapsto \mathcal{L}_{\mathrm{bap}} & \mathbf{FG} \longmapsto \mathcal{L}_{\mathrm{all}}. \\
\mathbf{Tor} \longmapsto \mathcal{L}_{\mathrm{tor}} & \mathbf{Ap} \longmapsto \mathcal{L}_{\mathrm{ap}} &
\end{array}$$

Proof. After the results of Section 4, the only part of the statement that remains to be established is that $\mathbf{FG}$, $\mathbf{FG} \cap \mathbf{bTor}_k$, $\mathbf{FG} \cap \mathbf{bAp}_k$, $\mathbf{FG} \cap \mathbf{bTor}$, $\mathbf{FG} \cap \mathbf{bAp}$, $\mathbf{FG} \cap \mathbf{Tor}$ and $\mathbf{FG} \cap \mathbf{Ap}$ are fg-varieties, that is, that these classes are generated by their syntactic members. This was noted earlier for $\mathbf{FG}$.

Each monoid in $\mathbf{FG} \cap \mathbf{bTor}_k$ (resp. $\mathbf{FG} \cap \mathbf{bAp}_k$) divides a monoid of the form $B_X(k,k)$ (resp. $B_X(k,1)$) for some finite alphabet X. Note that $B_X(k,k) \in \mathbf{FG} \cap \mathbf{bTor}_k$ and $B_X(k,1) \in \mathbf{FG} \cap \mathbf{bAp}_k$. In fact, each monoid in $\mathbf{FG} \cap \mathbf{bTor}$ (resp. $\mathbf{FG} \cap \mathbf{bAp}$) divides a monoid of the form $B_X(k,k)$ (resp. $B_X(k,1)$) for some finite alphabet X and some integer $k \geq 6$. So it suffices to show that, for $k \geq 6$, $\ell \geq 1$ and X finite, the monoid $B_X(k,\ell)$ is syntactic.

We will not prove this fact. Let us just say that readers familiar with McCammond [11] can use the notion of rank to prove that $B_X(k,\ell)$ satisfies the hypothesis of Proposition A.3.2, and hence is syntactic.

Similarly, in order to prove that $\mathbf{FG} \cap \mathbf{Tor}$ and $\mathbf{FG} \cap \mathbf{Ap}$ are fg-varieties, it suffices to show that, for all $S \in \mathbf{FG} \cap \mathbf{Tor}$ (resp. $S \in \mathbf{FG} \cap \mathbf{Ap}$), then $\bar{S}$ is syntactic. Note that $\bar{S} \in \mathbf{FG} \cap \mathbf{Tor}$ (resp. $\mathbf{FG} \cap \mathbf{Ap}$) since the morphism ψ of Theorem 5.9 is aperiodic. Again, we leave it to readers familiar with Rhodes [17] to prove that $\bar{S}$ satisfies the hypothesis of Proposition A.3.2. This can be done using the natural filtration given by the length of the infinite iteration matrix semigroup (IIMS) description of $\bar{S}$. $\square$

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