Free profinite $\mathcal{R}$-trivial monoids[1]

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Résumé: Cet article est consacré à l'étude de la structure des semigroupes d'opérations implicites sur **R**, la pseudovariété des semigroupes $\mathcal{R}$-triviaux. Nous donnons deux descriptions complémentaires de ces semigroupes : d'abord à l'aide d'ordinaux étiquetés; puis en utilisant des arbres étiquetés infinis, de profondeur finie. Dans la première représentation, le produit d'opérations implicites admet une caractérisation élémentaire. Dans la seconde, c'est la topologie des opérations implicites qui est rendue le mieux visible. Une étude similaire est menée pour les opérations implicites sur **DRG**, l'analogue non apériodique de **R**. Cette dernière étude nous conduit à la description de la variété de langages associée à **DRG** et au calcul de certains sup de pseudovariétés.

Abstract: This article is concerned with the structure of semigroups of implicit operations on **R**, the pseudovariety of all $\mathcal{R}$-trivial semigroups. We give two complementary descriptions for these semigroups: first by means of labeled ordinals, and second by means of labeled infinite trees of finite depth. In the first representation, the product of implicit operations has a simple characterization. In the second one, the topology on implicit operations is best visible. A similar study is conducted for the implicit operations on **DRG**, the non-aperiodic analogue of **R**. This leads to a description of the corresponding variety of recognizable languages and to the computation of certain joins of pseudovarieties.

Monoids in which all regular elements lie in groups are not very far from the much better understood completely regular monoids, but, nevertheless, the step from one class of monoids to the other is rather nontrivial. Thus, as pointed out by Schützenberger [22], the pseudovariety **DS** of all finite monoids in which all regular elements lie in groups should play an important role in the theory of finite monoids.

The monoids of implicit operations on pseudovarieties of monoids have been shown to be worthy objects of study: the topological and algebraic structure of these compact monoids contains valuable information on the structure of the finite monoids in the pseudovariety, and on the associated variety of recognizable languages [1, 2, 5]. Many applications of this approach can be found, for instance, in the works of Almeida, Azevedo, Selmi, Weil, Zeitoun and others [5, 6, 14, 10, 11, 23, 25].

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The pseudovariety **DS** and its subpseudovarieties appear to be particularly amenable to a description of the structure of their semigroups of implicit operations. A factorization of implicit operations over the pseudovariety **J** of finite $\mathcal{J}$-trivial semigroups in terms of component projections and regular elements was obtained in [3]. Azevedo [12, 13, 14] showed that the same kind of factorization exists for implicit operations on **DS** and, therefore, on any subpseudovariety of **DS**. While a certain form of such a factorization is canonical for **J** [3], a fact which yields several nontrivial applications [5], no such form has yet been found for **DS**.

This paper is a contribution towards a complete understanding of the structure of implicit operations on **DS**. It deals with the special instance of monoids where the Green relations $\mathcal{R}$ and $\mathcal{H}$ coincide in each regular $\mathcal{D}$-class. The monoids with this property form a subpseudovariety of **DS**, denoted **DRG**. For any pseudovariety of groups **H**, we let **DRH** be the pseudovariety of monoids in **DRG**, all of whose subgroups lie in **H**. If **H** is the trivial pseudovariety, then **DRH** = **R**, the pseudovariety of $\mathcal{R}$-trivial monoids.

We give two descriptions of the monoids of implicit operations on **DRH** (and in particular on **DRG** and on **R**), in terms of labeled trees and in terms of labeled ordinals satisfying certain conditions. As a result, we are able to completely describe the algebraic and topological structure of these monoids. In particular, we can answer a question posed by the first author on the idempotents in $\mathcal{R}$-trivial monoids [5, Problem 15]. We then derive from our work a characterization of the variety of languages recognized by monoids in **DRH**, which extends Eilenberg's result on the languages recognized by $\mathcal{R}$-trivial monoids [16]. We also give a unique factorization statement for the implicit operations on **DRH**. This last result is particularly interesting: very few such unique factorization statements are known for pseudovarieties containing all $\mathcal{J}$-trivial monoids (see [10, 9]). Finally, our results are used to compute certain joins of pseudovarieties. Let **DLG** be the left right dual of **DRG**, that is, the pseudovariety of monoids where the Green relations $\mathcal{L}$ and $\mathcal{H}$ coincide in each regular $\mathcal{D}$-class, and let **L** be the pseudovariety of $\mathcal{L}$-trivial monoids. We compute the pseudovariety $\mathbf{DRG} \vee \mathbf{DLG}$:

$$\mathbf{DRG} \vee \mathbf{DLG} = [\![(xy)^\omega (yx)^\omega (xy)^\omega = (xy)^\omega\,,\ (xy)^\omega x^\omega (zx)^\omega = (xy)^\omega (zx)^\omega]\!],$$

and more generally, we compute $\mathbf{DRH} \vee \mathbf{DLH}$. It follows from these computations that if **H** is a decidable pseudovariety of groups, then $\mathbf{DRH} \vee \mathbf{DLH}$ is decidable. The aperiodic case, i.e. the join $\mathbf{R} \vee \mathbf{L}$ had been computed by Almeida and Azevedo [7]. We also give unique factorization statements for the implicit operations on these joins.

Formally, the paper is divided as follows. The first section summarizes the main definitions and basic results which we will need, concerning pseudovarieties, implicit operations and recognizable languages. In the second section, we present a first set of results on the monoids of implicit operations on the pseudovarieties of the form **DRH**. Sections 3 and 4 are devoted to the description of the algebraic and topological structure of the monoids of implicit operations on **R**, and sections 5 and 6 describe the structure of the monoids of implicit operations on **DRH** in general. Finally, section 7 applies the previous results to the computation of the joins of the form $\mathbf{DRH} \vee \mathbf{DLH}$.

1 Preliminaries

We first briefly review some definitions and notation concerning pseudovarieties. We then discuss the monoids of implicit operations $\overline{\Omega}_A\mathbf{V}$ associated with a pseudovariety of monoids **V**, and the definition of pseudovarieties by pseudoidentities. Next we focus on recognizable languages, and on the connection between these languages and the topology of $\overline{\Omega}_A\mathbf{V}$. Finally we summarize some results of Almeida and Azevedo regarding the structure of the monoids $\overline{\Omega}_A\mathbf{DS}$. For missing definitions and for the proofs omitted here, the reader is referred to the books of Eilenberg [16], Pin [20] and Almeida [5].

Throughout this paper, we use the word *alphabet* to mean a finite, non empty, set.

1.1 Pseudovarieties

A *pseudovariety of monoids* is a class of finite monoids closed under taking submonoids, homomorphic images and finite direct products. Pseudovarieties of semigroups and groups are defined in the same fashion. $\mathbf{M}$ denotes the pseudovariety of all finite monoids, and $\mathbf{G}$ denotes the pseudovariety of all finite groups. Particularly important in this paper are the pseudovarieties $\mathbf{R}$, $\mathbf{L}$, $\mathbf{J}$, $\mathbf{A}$, $\mathbf{J}_1$ and $\mathbf{DS}$. By definition, $\mathbf{R}$, $\mathbf{L}$, $\mathbf{J}$ and $\mathbf{A}$ are respectively the pseudovarieties of $\mathcal{R}$-trivial, $\mathcal{L}$-trivial, $\mathcal{J}$-trivial and $\mathcal{H}$-trivial monoids (where $\mathcal{R}$, $\mathcal{L}$, $\mathcal{J}$ and $\mathcal{H}$ are the Green relations). $\mathbf{J}_1$ denotes the pseudovariety of semilattices, that is, idempotent and commutative monoids. Finally $\mathbf{DS}$ denotes the pseudovariety of all monoids in which the regular elements all lie in groups. That is, $S \in \mathbf{DS}$ if and only if each regular $\mathcal{D}$-class of S is a semigroup, which is necessarily simple. We will give more background on $\mathbf{DS}$ and its subpseudovarieties in Section 1.4.

Let $\mathbf{V}$ and $\mathbf{W}$ be pseudovarieties. Then $\mathbf{V} \vee \mathbf{W}$ denotes the least pseudovariety containing both $\mathbf{V}$ and $\mathbf{W}$. A monoid M lies in $\mathbf{V} \vee \mathbf{W}$ if and only if it divides a direct product $V \times W$ such that $V \in \mathbf{V}$ and $W \in \mathbf{W}$.

1.2 Implicit operations and pseudoidentities

An important tool in the study of pseudovarieties of monoids is the investigation of the structure and properties of the corresponding relatively free profinite monoids. These monoids are also known as the monoids of implicit operations. Here we review some basic definitions and results concerning these monoids. For details and proofs, the reader is referred to the first author's book [5] and to the survey [9].

We say that a monoid is *profinite* if it is a projective limit of finite monoids. More generally, if $\mathbf{V}$ is a pseudovariety of monoids, we say that a monoid is *pro-*$\mathbf{V}$ if it is a projective limit of elements of $\mathbf{V}$. It is known that a monoid is profinite if and only if it is compact and 0-dimensional. If $\mathbf{V}$ is a pseudovariety, then a profinite monoid is pro-$\mathbf{V}$ if and only if all its finite continuous homomorphic images are in $\mathbf{V}$. Of course, all elements of $\mathbf{V}$ are pro-$\mathbf{V}$, and if $\mathbf{W}$ is a pseudovariety contained in $\mathbf{V}$, then all pro-$\mathbf{W}$ monoids are pro-$\mathbf{V}$. A profinite monoid S is said to be generated by an alphabet A, *A-generated* for short, if there exists a mapping $\sigma\colon A \to S$ such that the submonoid generated by $\sigma(A)$ is dense in S.

Let $\mathbf{V}$ be a pseudovariety and let A be an alphabet. We denote by $\overline{\Omega}_A\mathbf{V}$ the projective limit of the A-generated elements of $\mathbf{V}$. The topological monoids of the form $\overline{\Omega}_A\mathbf{V}$ have been widely studied [5, 9]. $\overline{\Omega}_A\mathbf{V}$ can be viewed as the completion of a certain uniform structure on the free monoid A^*, or as the monoid of $|A|$-ary implicit operations on $\mathbf{V}$. In fact, we often call the elements of $\overline{\Omega}_A\mathbf{V}$ (*$|A|$-ary*) *implicit operations* (*on* $\mathbf{V}$). Whenever we are specifically interested in the cardinality of the alphabet A, we write $\overline{\Omega}_k\mathbf{V}$ for $\overline{\Omega}_A\mathbf{V}$ if $|A| = k$.

If $\mathbf{V}$ admits a finite free object $F_A(\mathbf{V})$ over the alphabet A, then $\overline{\Omega}_A\mathbf{V} = F_A(\mathbf{V})$. This is the case in particular for the pseudovariety $\mathbf{J}_1$ of semilattices. In that case, $\overline{\Omega}_A\mathbf{J}_1$ is the monoid $\mathcal{P}(A)$

of subsets of A, under union. Relatively few monoids $\overline{\Omega}_A\mathbf{V}$ have been computed when $\mathbf{V}$ does not admit free objects over A. One of the more striking results in this direction is the description of the structure of $\overline{\Omega}_A\mathbf{J}$ (see [5]).

We summarize below the main properties of the monoids $\overline{\Omega}_A\mathbf{V}$, which will be used freely in the sequel.

Theorem 1.2.1 *Let A be an alphabet and let $\mathbf{V}$ be a pseudovariety of monoids.*

1. *$\overline{\Omega}_A\mathbf{V}$ is A-generated and pro-$\mathbf{V}$: in particular there exists a continuous mapping $\imath: A \to \overline{\Omega}_A\mathbf{V}$, which is one-to-one if $\mathbf{V}$ is non-trivial, such that $\imath(A)$ generates a dense submonoid of $\overline{\Omega}_A\mathbf{V}$.*

2. *$\overline{\Omega}_A\mathbf{V}$ is the free pro-$\mathbf{V}$ monoid over A: If S is pro-$\mathbf{V}$ and if $\sigma: A \to S$ is a continuous mapping, then σ induces a unique continuous morphism $\hat{\sigma}: \overline{\Omega}_A\mathbf{V} \to S$ such that $\hat{\sigma} \circ \imath = \sigma$.*

3. *A finite monoid is in $\mathbf{V}$ if and only if it is a continuous homomorphic image of $\overline{\Omega}_A\mathbf{V}$ for some alphabet A.*

Since the submonoid generated by $\imath(A)$ is dense in $\overline{\Omega}_A\mathbf{V}$, its elements are limits of sequences of $\imath$-images of words on the alphabet A. In this context, it is usually more convenient to omit $\imath$, even though it may not be one-to-one, and to consider that each word u on the alphabet A is also an element of $\overline{\Omega}_A\mathbf{V}$.

An important application of Theorem 1.2.1 is the following: If $\mathbf{W}$ is a subpseudovariety of $\mathbf{V}$, then the identity of A induces a continuous onto morphism $\pi: \overline{\Omega}_A\mathbf{V} \to \overline{\Omega}_A\mathbf{W}$, which is called the *natural projection of $\overline{\Omega}_A\mathbf{V}$ onto $\overline{\Omega}_A\mathbf{W}$*. If $x \in \overline{\Omega}_A\mathbf{V}$, we often call $\pi(x)$ the *restriction of x to $\mathbf{W}$*.

The notion of (alphabetic) content of a word can be extended to all the elements of $\overline{\Omega}_A\mathbf{V}$ whenever $\mathbf{V}$ contains $\mathbf{J}_1$. Recall that the *alphabetic content* of a word $u \in A^*$ is the set of letters of A which occur in u. Let $\mathbf{V}$ be a pseudovariety containing all semilattices. We define the *content morphism* $c: \overline{\Omega}_A\mathbf{V} \to \mathcal{P}(A)$ to be the natural projection from $\overline{\Omega}_A\mathbf{V}$ onto $\overline{\Omega}_A\mathbf{J}_1 = \mathcal{P}(A)$. It is immediately verified that $c(u)$ is the alphabetic content of u, for all $u \in A^*$.

The ω-power, which traditionally denotes the idempotent power of an element of a finite semigroup [16, 20], can also be extended to all the elements of $\overline{\Omega}_A\mathbf{V}$. For any $x \in \overline{\Omega}_A\mathbf{V}$, the sequence $(x^{n!})_n$ converges in $\overline{\Omega}_A\mathbf{V}$. Its limit, written x^ω, is idempotent and, if $\sigma: \overline{\Omega}_A\mathbf{V} \to S$ is a continuous morphism into a finite monoid, then $\sigma(x^\omega)$ is the unique idempotent power of $\sigma(x)$. We extend this notation by writing $x^{\omega+k}$ for $\lim_n x^{n!+k}$ for each integer k.

Theorem 1.2.1 also leads to the statement of Reiterman's theorem on the definition of pseudovarieties by pseudoidentities [21]. Let A be a finite set and let $x, y \in \overline{\Omega}_A\mathbf{M}$. We say that a profinite monoid S *satisfies the pseudoidentity* $x = y$ if $\sigma(x) = \sigma(y)$ for each continuous morphism $\sigma: \overline{\Omega}_A\mathbf{V} \to S$. We say that a class $\mathbf{W}$ of profinite monoids satisfies a given pseudoidentity if each element of $\mathbf{W}$ satisfies it. The class $\mathbf{W}$ satisfies a set Σ of pseudoidentities if each element of $\mathbf{W}$ satisfies each element of Σ. The class of all finite monoids which satisfy Σ is said to be *defined by* Σ and is denoted $[\![\Sigma]\!]$.

Pseudoidentities suffice to describe all pseudovarieties, as expressed by the following theorem, due to Reiterman [21].

Theorem 1.2.2 *Let $\mathbf{V}$ be a class of finite monoids. Then $\mathbf{V}$ is a pseudovariety if and only if there exists a set Σ of pseudoidentities such that $\mathbf{V} = [\![\Sigma]\!]$.*

Example 1.2.3 Here are definitions by pseudoidentities of some of the important pseudovarieties in this paper.

- $\mathbf{J}_1 = [\![xy = yx,\ x^2 = x]\!]$;
- $\mathbf{G} = [\![x^\omega = 1]\!]$;
- $\mathbf{A} = [\![x^{\omega+1} = x^\omega]\!]$;
- $\mathbf{R} = [\![(xy)^\omega x = (xy)^\omega]\!]$;
- $\mathbf{L} = [\![y(xy)^\omega = (xy)^\omega]\!]$;
- $\mathbf{J} = [\![(xy)^\omega x = (xy)^\omega = y(xy)^\omega]\!]$;
- $\mathbf{DS} = [\![((xy)^\omega(yx)^\omega(xy)^\omega)^\omega = (xy)^\omega]\!]$.

1.3 Recognizable languages

A language on the alphabet A, that is, a subset L of the free monoid A^*, is said to be *recognizable* if there exists a finite monoid S and a morphism $\sigma\colon A^* \to S$ such that $L = \sigma^{-1}(\sigma(L))$. In that case, we say that S recognizes L. If, in addition, S can be chosen in a given pseudovariety $\mathbf{V}$, we say that L is $\mathbf{V}$-*recognizable*. The *syntactic monoid* $S(L)$ of a language L is the quotient of A^* by the congruence $\sim_L$, given by

$$u \sim_L v \qquad \text{if and only if} \qquad xuy \in L \Longleftrightarrow xvy \in L \quad \text{for all } x, y \in A^*.$$

It is well known that L is recognizable if and only if $S(L)$ is finite. Moreover, L is $\mathbf{V}$-recognizable if and only if $S(L) \in \mathbf{V}$. We refer the reader to [16, 20] for more details on recognizable languages.

A $*$-*class of recognizable languages* is a correspondence $\mathcal{C}$ which associates with each finite alphabet A a set $A^*\mathcal{C}$ of recognizable languages of A^*. A $*$-*variety* is a $*$-class of recognizable languages $\mathcal{V}$ such that

(1) for every alphabet A, $A^*\mathcal{V}$ is closed under finite union, finite intersection and complement;

(2) for every morphism $\varphi\colon A^* \to B^*$, $L \in B^*\mathcal{V}$ implies $\varphi^{-1}(L) \in A^*\mathcal{V}$,

(3) If $X \in A^*\mathcal{V}$ and $a \in A$, then $a^{-1}L = \{u \in A^* \mid au \in L\}$ and $La^{-1} = \{u \in A^* \mid ua \in L\}$ are in $A^*\mathcal{V}$.

Pseudovarieties of monoids and $*$-varieties are closely related. To each pseudovariety $\mathbf{V}$, we associate the $*$-class $\mathcal{V}$ such that, for each alphabet A, $A^*\mathcal{V}$ is the set of $\mathbf{V}$-recognizable languages of A^*. One can show that $\mathcal{V}$ is a $*$-variety. More precisely, we have the following result, due to Eilenberg [16].

Theorem 1.3.1 *The correspondence $\mathbf{V} \to \mathcal{V}$ defines a bijective correspondence between the pseudovarieties of monoids and the $*$-varieties.*

There is a very interesting connection between the topology of $\overline{\Omega}_A\mathbf{V}$ and the $*$-variety $\mathcal{V}$ associated with $\mathbf{V}$. Let $\mathbf{V}$ be a non trivial pseudovariety. Recall that $\imath$ denotes the canonical mapping from A into $\overline{\Omega}_A\mathbf{V}$ such that $\imath(A)$ generates a dense submonoid of $\overline{\Omega}_A\mathbf{V}$. We also write $\imath$ for the morphism from A^* into $\overline{\Omega}_A\mathbf{V}$ extending this mapping. Observe that the morphism $\imath$ is one-to-one on A^* whenever $\mathbf{V}$ satisfies no non trivial word identity. This is the case in particular as soon as $\mathbf{V}$ contains $\mathbf{J}$ or $\mathbf{G}$.

One can prove the following characterization of $\mathbf{V}$-recognizable languages (Almeida [5]). The following are equivalent for each language L in A^*.

(1) L is $\mathbf{V}$-recognizable.

(2) $L = \imath^{-1}(K)$ for some clopen subset K of $\overline{\Omega}_A\mathbf{V}$.

(3) The topological closure $\overline{\imath(L)}$ of $\imath(L)$ in $\overline{\Omega}_A\mathbf{V}$ is clopen and $L = \imath^{-1}(\overline{\imath(L)})$.

Furthermore, the collection of all $\overline{\imath(L)}$ ($L \subseteq A^*$, $\mathbf{V}$-recognizable) is a basis of clopen sets for the topology of $\overline{\Omega}_A\mathbf{V}$.

Finally, we note two technical lemmas which will be used in the sequel.

Lemma 1.3.2 *Let K and L be $\mathbf{V}$-recognizable languages in A^*. Then $\overline{\imath(K)}\,\overline{\imath(L)} = \overline{\imath(KL)}$ and $\overline{\imath(K)} \cap \overline{\imath(L)} = \overline{\imath(K \cap L)}$ in $\overline{\Omega}_A\mathbf{V}$.*

Proof. Since $\overline{\Omega}_A\mathbf{V}$ is compact, closed subsets are compact as well. Therefore, if X and Y are subsets of $\overline{\Omega}_A\mathbf{V}$, then $\overline{X}\,\overline{Y} = \overline{XY}$ since multiplication is continuous. This proves the first statement of the lemma.

The containment $\overline{\imath(K \cap L)} \subseteq \overline{\imath(K)} \cap \overline{\imath(L)}$ is trivial. Conversely, let $x \in \overline{\imath(K)} \cap \overline{\imath(L)}$ and let $(u_n)_n$ be a sequence of words converging to x in $\overline{\Omega}_A\mathbf{V}$, that is, such that $x = \lim_n \imath(u_n)$. Then almost all the u_n lie in $K = \imath^{-1}(\overline{\imath(K)})$. Similarly, almost all the u_n lie in L, and hence almost all of them lie in $K \cap L$. Thus $x \in \overline{\imath(K \cap L)}$, which concludes the proof. $\square$

Lemma 1.3.3 *Let $\mathbf{V}$ and $\mathbf{W}$ be pseudovarieties such that $\mathbf{V} \subseteq \mathbf{W}$ and let $L \subseteq A^*$ be a $\mathbf{V}$-recognizable language. Let $\pi\colon \overline{\Omega}_A\mathbf{W} \to \overline{\Omega}_A\mathbf{V}$ be the canonical projection. Let L' and L'' be the topological closures of $\imath(L)$ in $\overline{\Omega}_A\mathbf{V}$ and $\overline{\Omega}_A\mathbf{W}$ respectively. Then $\pi^{-1}(L') = L''$.*

Proof. $\pi^{-1}(L')$ is closed by continuity of π, so $L'' \subseteq \pi^{-1}(L')$. Conversely, let $x \in \pi^{-1}(L')$ and let $(u_n)_n$ be a sequence of words converging to x in $\overline{\Omega}_A\mathbf{W}$. Then $\pi(x) \in L'$. But L' is clopen and $\pi(x)$ is the limit of the sequence $(u_n)_n$ in $\overline{\Omega}_A\mathbf{W}$. So almost all the u_n lie in $L = \imath^{-1}(\overline{\imath(L)})$, and hence $x \in L''$. $\square$

1.4 DS and its subpseudovarieties

The pseudovariety $\mathbf{DS}$ of monoids in which each regular $\mathcal{D}$-class is a subsemigroup has received particular attention. Observe that $\mathbf{J}_1$, $\mathbf{R}$, $\mathbf{L}$ and $\mathbf{J}$ are contained in $\mathbf{DS}$. We will study in detail the pseudovariety $\mathbf{DRG}$ of semigroups in which $\mathcal{R}$=$\mathcal{H}$ in each regular $\mathcal{D}$-class. That is, a semigroup lies in $\mathbf{DRG}$ if and only if each of its regular $\mathcal{D}$-classes is an $\mathcal{L}$-class, and hence a left group:

$$\mathbf{DRG} = [\![((xy)^\omega x)^\omega = (xy)^\omega]\!] = [\![(xy)^\omega (yx)^\omega = (xy)^\omega]\!].$$

$\mathbf{R}$ and $\mathbf{G}$ are contained in $\mathbf{DRG}$, and $\mathbf{DRG}$ is contained in $\mathbf{DS}$.

The left-right dual of $\mathbf{DRG}$ is $\mathbf{DLG}$, the pseudovariety of all semigroups in which $\mathcal{L}=\mathcal{H}$ in each regular $\mathcal{D}$-class: $\mathbf{DLG} = [\![(y(xy)^\omega)^\omega = (xy)^\omega]\!]$. If $\mathbf{H}$ is a pseudovariety of groups, we let $\mathbf{DRH}$ (resp. $\mathbf{DLH}$) be the pseudovariety of semigroups in $\mathbf{DRG}$ (resp. $\mathbf{DLG}$) all of whose subgroups lie in $\mathbf{H}$. If $\mathbf{H} = \mathbf{I}$, the trivial pseudovariety, then $\mathbf{DRH} = \mathbf{R}$ and $\mathbf{DLH} = \mathbf{L}$.

Both $\mathbf{DRG}$ and $\mathbf{DLG}$ are contained in $\mathbf{DO} = [\![(xy)^\omega(yx)^\omega(xy)^\omega = (xy)^\omega]\!]$, the subpseudovariety of $\mathbf{DS}$ consisting of all semigroups in which the regular $\mathcal{D}$-classes are orthodox subsemigroups, and hence rectangular groups (i.e. the direct product of a group and a rectangular band).

Almeida and Azevedo [8] gave a number of factorization and regularity results for the implicit operations on sub-pseudovarieties of $\mathbf{DS}$, which will prove very useful in this paper. These results are summarized in the following proposition, in the form in which we will use them.

Proposition 1.4.1 *Let* $\mathbf{M}$ *be the pseudovariety of all finite monoids and let* $x, y \in \overline{\Omega}_A\mathbf{M}$. *Let also* $\mathbf{V}$ *be a subpseudovariety of* $\mathbf{DS}$ *containing all* $\mathcal{J}$*-trivial monoids.*

1. *x can be written as a product of the form*

$$x = u_k x_k u_{k-1} \cdots x_0 u_0$$

where the u_i are words and the x_i are implicit operations whose restriction to $\mathbf{DS}$ *is regular.*

2. *The restriction of x to* $\mathbf{V}$ *is regular if and only if its restriction to* $\mathbf{DS}$ *is regular, if and only if* $\mathbf{DS}$ *satisfies $x^{\omega+1} = x$.*

3. *If the restrictions of x and y to* $\mathbf{DS}$ *are regular, then* $\mathbf{DS}$ *satisfies $x \mathrel{\mathcal{J}} y$ if and only if $c(x) = c(y)$. Conversely, if $c(x) = c(y)$, if $x \leq_{\mathcal{J}} y$ and if the restriction of y to* $\mathbf{DS}$ *is regular, then* $\mathbf{DS}$ *satisfies $x \mathrel{\mathcal{J}} y$.*

4. *If $x, y \in \overline{\Omega}_A\mathbf{M}$, $u \in A^+$, $c(u) \subseteq c(y)$ and $x = uy$ (resp. $x = yu$), then the restriction of x to* $\mathbf{DS}$ *is regular if and only if the restriction of y to* $\mathbf{DS}$ *is regular. In that case,* $\mathbf{DS}$ *satisfies $x \mathrel{\mathcal{L}} y$ (resp. $x \mathrel{\mathcal{R}} y$).*

5. *For each $S \in$* $\mathbf{DS}$, *there exists $k \geq 1$ such that S satisfies $u_1 \cdots u_k = (u_1 \cdots u_k)^{\omega+1}$ whenever $c(u_1) = \cdots = c(u_k)$ (that is, $u_1 \cdots u_k$ is regular in S).*

2 Infinite products and free pro-DRG monoids

We start by introducing (rightwards) infinite products in topological monoids and study some aspects of their convergence in monoids of implicit operations. A partial study of the languages recognized by monoids in $\mathbf{DRG}$ will then allow us to give a canonical factorization for the implicit operations on $\mathbf{DRG}$.

2.1 Convergence of infinite products

Let S be a topological monoid and let $(s_n)_{n\geq 1}$ be a sequence of elements of S. The associated *infinite product* is the sequence $(s_1 \cdots s_n)_{n\geq 1}$; the infinite product *converges* if the sequence that defines it converges, in which case its limit is called the *value* of the infinite product.

Lemma 2.1.1 *Let S be a compact monoid. Then any two accumulation points of every infinite product in S are $\mathcal{R}$-equivalent.*

Proof. Let $(s_n)_{n\geq 1}$ be a sequence of elements of S and let a and b be two accumulation points of the infinite product $(s_1 \cdots s_n)_{n\geq 1}$. Then there exist two sequences $(n_k)_{k\geq 1}$ and $(m_k)_{k\geq 1}$ of positive integers such that $n_1 < m_1 < n_2 < m_2 < \cdots$, $(s_1 \cdots s_{n_k})_k \to a$ and $(s_1 \cdots s_{m_k})_k \to b$. Let u and v be accumulation points of the sequences $(s_{n_k+1} \cdots s_{m_k})_{k\geq 1}$ and $(s_{m_k+1} \cdots s_{n_{k+1}})_{k\geq 1}$, respectively. Then, by continuity of the multiplication in S, we have $au = b$ and $bv = a$, so that a and b are $\mathcal{R}$-equivalent. □

In the case of semigroups of implicit operations, we may therefore relate convergence of every infinite product with the pseudovariety $\mathbf{R}$.

Proposition 2.1.2 *Let $\mathbf{V}$ be a pseudovariety of monoids. Then*

i) every infinite product of $\overline{\Omega}_1\mathbf{V}$ converges if and only if $\mathbf{V} \subseteq \mathbf{A}$;

ii) every infinite product of $\overline{\Omega}_2\mathbf{V}$ converges if and only if $\mathbf{V} \subseteq \mathbf{R}$;

iii) if $\mathbf{V} \subseteq \mathbf{R}$, then every infinite product of $\overline{\Omega}_n\mathbf{V}$ converges.

Proof. (i) If $\mathbf{V} \subseteq \mathbf{A}$, then every infinite product of $\overline{\Omega}_1\mathbf{V}$ converges to the unique minimal idempotent since this property holds in every finite aperiodic cyclic monoid. Conversely, if $\mathbf{V} \not\subseteq \mathbf{A}$, then, for the generator a of $\Omega_1\mathbf{V}$, we have $a^{\omega+1} \neq a^\omega$ and a^ω and $a^{\omega+1}$ are both accumulation points of the infinite product with constant factor a, showing that this infinite product does not converge.

(iii) Suppose that $\mathbf{V} \subseteq \mathbf{R}$ and let $(x_k)_{k\geq 1}$ be a sequence of elements of $\overline{\Omega}_n\mathbf{V}$. Then, since any two accumulation points of the infinite product $(x_1 \cdots x_k)_{k\geq 1}$ are $\mathcal{R}$-equivalent by Lemma 2.1.1, and $\overline{\Omega}_n\mathbf{V}$ is $\mathcal{R}$-trivial, it follows that $(x_1 \cdots x_k)_{k\geq 1}$ converges.

(ii) It remains to show that, if $\mathbf{V} \not\subseteq \mathbf{R}$, then $\overline{\Omega}_2\mathbf{V}$ has an infinite product which does not converge. Indeed, consider the two-letter alphabet $A = \{a, b\}$. Then $(ab)^\omega a \neq (ab)^\omega$ in $\overline{\Omega}_2\mathbf{V}$. But the two implicit operations $(ab)^\omega a$ and $(ab)^\omega$ are both accumulation points of the infinite product starting with a and alternating a and b. Hence $\overline{\Omega}_2\mathbf{V}$ has a non-convergent infinite product. □

The limit of a sequence $(s_1 \cdots s_k)_{k\geq 1}$ will be represented by $\prod_{k=1}^{\infty} s_k$ or, more simply, by $\prod_{k\geq 1} s_k$ or $\prod_k s_k$. The following proposition lists a number of elementary observations on infinite products.

Proposition 2.1.3 *Let $\mathbf{V}$ be a pseudovariety contained in $\mathbf{R}$ and let s and s_k $(k \geq 1)$ be elements of $\overline{\Omega}_A\mathbf{V}$. Then*

a) $\prod_{k\geq 1} s_k = s_1 \prod_{k\geq 2} s_k$;

b) for any $k_1 < k_2 < \cdots$, $\prod_{k\geq 1} s_k = \prod_{i\geq 1}(s_{k_{i-1}+1} \cdots s_{k_i})$ where $k_0 = 0$;

c) $\prod_k s = s^\omega$;

d) if every letter of $c(x)$ occurs in an infinite number of the s_k, then $(\prod_k s_k)x = \prod_k s_k$;

e) if every variable which occurs in any of the s_k occurs in an infinite number of factors, then $\prod_k s_k$ is an idempotent.

Proof. (a) follows by associativity and continuity of multiplication.

(b) In view of associativity, the infinite product on the right side is described by a subsequence of the infinite product on the left side.

(c) This is true on any finite aperiodic semigroup.

(d) Since $\mathbf{V} \subseteq \mathbf{R}$, it suffices to prove (e).

(e) follows from [5, Theorem 8.1.10]. □

For pseudovarieties comprised between $\mathbf{R}$ and $\mathbf{DS}$, we have the following analogue of Proposition 2.1.3 (e), which also derives from [5, Theorem 8.1.10].

Proposition 2.1.4 *Let* $\mathbf{V}$ *be a pseudovariety such that* $\mathbf{R} \subseteq \mathbf{V} \subseteq \mathbf{DS}$ *and let* $(s_1 \cdots s_n)_{n\geq 1}$ *be an infinite product in* $\overline{\Omega}_A\mathbf{V}$*. If every variable occurring in any* s_n *occurs in an infinite number of them, then the unique* $\mathcal{R}$*-class containing the accumulation points of* $(s_1 \cdots s_n)_{n\geq 1}$ *is regular.*

2.2 Languages recognized by monoids in DRG

Let A be a finite alphabet, let $L_0, L_1, \ldots, L_k$ be languages over A and let $a_1, \ldots, a_k \in A$. The product $L_0a_1L_1 \cdots a_kL_k$ is said to be *deterministic* if, for every word $w \in L_0a_1L_1 \cdots a_kL_k$ and for each $1 \leq i \leq k$, w admits a unique prefix in $L_0a_1 \cdots L_{i-1}a_i$.

We say that a pseudovariety of monoids $\mathbf{V}$ is *closed under deterministic product* if, for all deterministic products $L = L_0a_1L_1 \cdots a_kL_k$ such that the L_i are $\mathbf{V}$-recognizable, L is also $\mathbf{V}$-recognizable. The pseudovarieties of monoids closed under deterministic product were characterized algebraically by Pin [19]. Let $\mathbf{K}$ be the pseudovariety of semigroups given by $\mathbf{K} = [\![x^\omega y = x^\omega]\!]$, and let $\mathbf{K} \ⓜ \mathbf{V}$ be the pseudovariety of monoids generated by the monoids M such that there exists a morphism $\varphi: M \to N$ with $N \in \mathbf{V}$ and $\varphi^{-1}(e) \in \mathbf{K}$ for each idempotent e of N.

Proposition 2.2.1 *Let* $\mathbf{V}$ *be a pseudovariety of monoids. Then* $\mathbf{V}$ *is closed under deterministic product if and only if* $\mathbf{V} = \mathbf{K} \ⓜ \mathbf{V}$*.*

It is now easy to verify the following.

Proposition 2.2.2 *Let* $\mathbf{H}$ *be a pseudovariety of groups. Then* $\mathbf{DRH}$ *is closed under deterministic product.*

Proof. By Proposition 2.2.1, it suffices to verify that $\mathbf{DRH} = \mathbf{K} \ⓜ \mathbf{DRH}$. Let $\varphi: M \to N$ be a morphism with $N \in \mathbf{DRH}$ and $\varphi^{-1}(e) \in \mathbf{K}$ for each idempotent e of N. Let G be a group in M. Then $\varphi(G)$ is a group in N, so $\varphi(G) \in \mathbf{H}$. Moreover, if e is the unit of G, then $G \cap \varphi^{-1}(\varphi(e)) \in \mathbf{K}$ and $G \cap \varphi^{-1}(\varphi(e))$ is a subgroup of G, so $G \cap \varphi^{-1}(\varphi(e)) = \{e\}$, and hence G is isomorphic to $\varphi(G)$. Thus $G \in \mathbf{H}$.

Let now e and f be $\mathcal{D}$-equivalent idempotents of M. We only need to prove that $e \mathrel{\mathcal{L}} f$. Since $N \in \mathbf{DRG}$, we have $\varphi(e) \mathrel{\mathcal{L}} \varphi(f)$, so that $\varphi(ef) = \varphi(e)$. But $\varphi^{-1}(\varphi(e)) \in \mathbf{K}$, so $e \cdot ef = e$ and hence $e \leq_{\mathcal{L}} f$. By symmetry it follows that $e \mathrel{\mathcal{L}} f$. □

Let us finally exhibit an important family of $\mathbf{DRH}$-recognizable languages. A language is an *elementary* $\mathbf{DRH}$*-language* if it can be written as a product $L_0a_1L_1 \cdots a_kL_k$ where for each $0 \leq i \leq k$, L_i is an $\mathbf{H}$-recognizable language over some alphabet $B_i \subseteq A$ and where $a_{i+1} \in A \setminus B_i$.

Note that, according to this definition, some of the L_i may be reduced to the empty word, since $\{1\} = \emptyset^*$. For instance, if $\mathbf{H}$ is the trivial group variety, $\mathbf{DRH} = \mathbf{R}$ and the elementary $\mathbf{DRH}$-languages are the languages of the form $B_0^* a_1 B_1^* \cdots a_k B_k^*$ with $a_i \notin B_{i-1}$ for all $1 \le i \le k$.

Lemma 2.2.3 *Let $B \subseteq A$. If $L \subseteq B^*$ is an $\mathbf{H}$-recognizable language of B^*, then L is $(\mathbf{H} \vee \mathbf{J}_1)$-recognizable in A^*.*

If $L \subseteq A^*$ is $\mathbf{H}$-recognizable, then $L \cap B^*$ is $(\mathbf{H} \vee \mathbf{J}_1)$-recognizable in A^*, and it is $\mathbf{H}$-recognizable in B^*.

Proof. Let $\eta \colon B^* \to G$ be the syntactic morphism of L in B^*, and let $\varphi \colon A^* \to G \times \{0,1\}$ be defined by $\varphi(a) = (\eta(a), 1)$ if $a \in B$ and $\varphi(a) = (1, 0)$ if $a \notin B$. Then $L = \varphi^{-1}(\eta(L) \times \{1\})$, and hence L is $(\mathbf{H} \vee \mathbf{J}_1)$-recognizable in A^*.

Since B^* is $\mathbf{J}_1$-recognizable, $L \cap B^*$ is immediately $(\mathbf{H} \vee \mathbf{J}_1)$-recognizable. Let now $\eta' \colon A^* \to G'$ be the syntactic morphism of L in A^* and let φ' be the restriction of η' to B^*. Then $\varphi'^{-1}(\eta'(L)) = L \cap B^*$, so that $L \cap B^*$ is $\mathbf{H}$-recognizable in B^*. □

Proposition 2.2.4 *Let $\mathbf{H}$ be a pseudovariety of groups. Then the elementary $\mathbf{DRH}$-languages are $\mathbf{DRH}$-recognizable.*

Proof. Let $L = L_0 a_1 L_1 \cdots a_k L_k$ be an elementary $\mathbf{DRH}$-language with the a_i and L_i as above. By Lemma 2.2.3, each L_i is $\mathbf{DRH}$-recognizable. In addition, observe that L is given as a deterministic product. It now follows from Proposition 2.2.2 that L is $\mathbf{DRH}$-recognizable. □

2.3 Basic factorization of implicit operations on DRG

Let $\mathbf{H}$ be a pseudovariety of groups and let $x \in \overline{\Omega}_A\mathbf{DRH} \setminus \{1\}$. A *left basic factorization* of x is a factorization
$$x = yaz$$
with $a \in A$, $a \notin c(y)$ and $c(ya) = c(x)$.

Each non empty word admits a unique left basic factorization: it is obtained by locating the last occurrence for the first time of a letter when the word is read from left to right. More generally, every element of $\overline{\Omega}_A\mathbf{DRH} \setminus \{1\}$ admits a left basic factorization. Let indeed $(u_n)_{n>0}$ be a sequence of words of A^+ converging to x, and for each n, let $u_n = v_n a_n w_n$ be the left basic factorization of u_n. Using the finiteness of A and the compactness of $\overline{\Omega}_A\mathbf{DRH}$, and up to extracting a subsequence of $(u_n)_n$, we may assume that the a_n are all equal and that the sequences $(v_n)_n$ and $(w_n)_n$ converge. If a is the common value of the a_n, if $y = \lim_n v_n$ and $z = \lim_n w_n$, then $x = yaz$ is a left basic factorization of x.

We now prove the uniqueness of the left basic factorization. This result and its proof are adapted from the first author's [6, Prop. 3.4].

Proposition 2.3.1 *Let $\mathbf{H}$ be a pseudovariety of groups. Every element of $\overline{\Omega}_A\mathbf{DRH} \setminus \{1\}$ admits a unique left basic factorization.*

Proof. For each $B \subseteq A$, let $[B]$ be the set of all words with content exactly B. This language is $\mathbf{J}_1$-recognizable, and hence $\mathbf{DRH}$-recognizable. Let $x \in \overline{\Omega}_A\mathbf{DRH} \setminus \{1\}$ and let $x = yaz$ be a left basic factorization of x. Let $\mathcal{N}(y)$ (resp. $\mathcal{N}(z)$) be the class of all $\mathbf{DRH}$-recognizable languages $K \subseteq A^*$ such that $K \subseteq [c(y)]$ (resp. $[c(z)]$) and $y \in \overline{K}$ (resp. $z \in \overline{K}$). Then $(\overline{K})_{K\in\mathcal{N}(y)}$ (resp. $(\overline{K})_{K\in\mathcal{N}(z)}$) constitutes a basis of neighborhoods for y (resp. z). Let $Y \in \mathcal{N}(y)$, let $Z \in \mathcal{N}(z)$ and let $L = YaZ$. For every word $u \in L$, the basic factorization of u is of the form $u = vaw$ with $v \in Y$ and $w \in Z$. Moreover, the product YaZ is deterministic, so L is $\mathbf{DRH}$-recognizable. Finally, since $\overline{Y}a\overline{Z} \subseteq \overline{L}$, we have $x \in \overline{L}$.

Let now $x = y'a'z'$ be another left basic factorization of x, let $(v_n)_{n>0}$ and $(w_n)_{n>0}$ be sequences of words converging respectively to y' and z', and let $u_n = v_n a' w_n$ for all n. Without loss of generality, we may assume that $c(v_n) = c(y')$ and $c(w_n) = c(z')$ for all n. In particular $v_n a' w_n$ is the left basic factorization of u_n. Moreover we may assume that $u_n \in \overline{L} \cap A^* = L$ for all n. As observed above, it follows that $a = a'$, $v_n \in Y$ and $w_n \in Z$ for all n, and hence $y' \in \overline{Y}$ and $z' \in \overline{Z}$. Since this holds for all choices of $Y \in \mathcal{N}(y)$ and $Z \in \mathcal{N}(z)$, we have $y' = y$ and $z' = z$. This completes the proof. $\square$

3 Representation of the implicit operations on R

We first deal with the aperiodic case, i.e. the pseudovariety $\mathbf{R}$ of $\mathcal{R}$-trivial monoids. We give two connected representations of the implicit operations on $\mathbf{R}$, each of which renders more adequately one aspect, topological or algebraic, of the properties of the monoids $\overline{\Omega}_A\mathbf{R}$. Finally, we connect these representations with known results on implicit operations on $\mathbf{R}$.

3.1 Factorization of implicit operations

Our first result is based on the iteration of the left basic factorization, described in section 2.3. Let $x \in \overline{\Omega}_A\mathbf{R} \setminus \{1\}$. We let $y_0 = x$ and we define recursively x_k, a_k and y_k ($k \geq 1$) so that $x_k a_k y_k$ is the left basic factorization of y_{k-1}. By Proposition 2.3.1, the possibility of continuing such a construction goes on forever or until $y_k = 1$. In either case, we obtain a product (see Proposition 2.1.2) which we represent by

$$x' = \prod_{k\geq 1} x_k a_k. \tag{1}$$

By convention, if $x = 1$, we let $x' = 1$.

Lemma 3.1.1 *For every $x \in \overline{\Omega}_A\mathbf{R}$, we have $x = x'$.*

Proof. By the definition of x_k, a_k and y_k, $x = (\prod_{k=1}^{r} x_k a_k) y_r$ for every r such that the factors involved are defined. So, if $y_r = 1$, then $x = \prod_{k=1}^{r} x_k a_k = x'$.

On the other hand, if no y_r is 1, then $x = x'y$ for any accumulation point y of the sequence $(y_k)_{k\geq 1}$. We claim that every letter $a \in c(y)$ appears in the content of infinitely many factors $x_k a_k$. Indeed, by continuity of the content morphism, $a \in c(y_k)$ for infinitely many values of k and $c(y_k) = c(x_{k+1}a_{k+1})$. It now follows from Proposition 2.1.3(d) that $x = x'y = x'$. $\square$

The next factorization step consists in applying successively the factorization (1) to each of the factors x_k appearing in it. Such a process cannot go on forever.

Proposition 3.1.2 *Consider the partial operations on $\overline{\Omega}_A\mathbf{R}$ that associate to $x_0,\ldots, x_r \in \overline{\Omega}_A\mathbf{R}$ (resp. $x_0, x_1, \ldots \in \overline{\Omega}_A\mathbf{R}$) the product $x_0a_0\ldots x_ra_r$ (resp. $\prod_{k\geq 0} x_ka_k$) when, for each i, the a_i are letters such that $a_i \notin c(x_i)$ and x_ia_i has the same content as the product $x_ia_ix_{i+1}a_{i+1}\ldots$ Then every element of $\overline{\Omega}_A\mathbf{R}$ can be obtained from 1 by applying these operations in such a way that the nesting height is bounded by the cardinality of A.*

Proof. By Lemma 3.1.1, any $x \in \overline{\Omega}_A\mathbf{R} \setminus \{1\}$ has a factorization $x = \prod_{k\geq 0} x_ka_k$ involving either a finite or infinite product with each $a_k \notin c(x_k)$. In particular, $c(x_k)$ is strictly contained in $c(x)$ for every $k \geq 0$. The proposition follows immediately by induction on $|c(x)|$. $\square$

Let $u = a_0a_1\cdots$ be a word in $A^+ \cup A^\omega$, of finite or infinite length. We denote by op_u the corresponding operation, described in Proposition 3.1.2. The arity of op_u is the length of u.

Example 3.1.3 To illustrate Proposition 3.1.2, consider the following decompositions of implicit operations on $\mathbf{R}$:

$$\begin{aligned}
(x^\omega y)^\omega(y^\omega z)^\omega &= \mathrm{op}_{zz\ldots}((x^\omega y)^\omega y^\omega, y^\omega, y^\omega, \ldots)\\
(x^\omega y)^\omega y^\omega &= \mathrm{op}_{yy\ldots}(x^\omega, x^\omega, \ldots)\\
y^\omega &= \mathrm{op}_{yy\ldots}(1, 1, \ldots)\\
x^\omega &= \mathrm{op}_{xx\ldots}(1, 1, \ldots)
\end{aligned}$$

so that

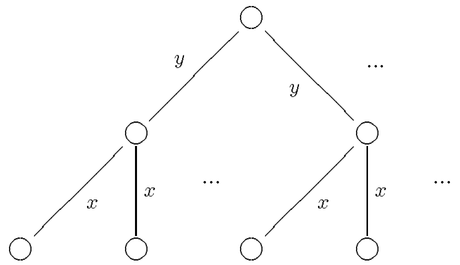
$$(x^\omega y)^\omega(y^\omega z)^\omega = ((x\cdot x\cdot\ldots)y\cdot(x\cdot x\cdot\ldots)y\ldots)\,z\cdot(y\cdot y\cdot\ldots)z\cdot(y\cdot y\cdot\ldots)z\cdot\ldots$$

3.2 Tree representation and uniqueness of the factorization

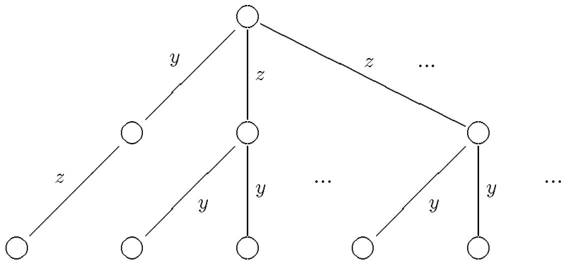
Let A be a finite alphabet. We define a set $\mathbf{T}(A)$ of possibly infinite oriented edge-labeled trees of finite height as follows. The set $\mathbf{T}(A)$ consists of all rooted trees such that

- the direct descendants of each node are well-ordered and their order type is $\leq \omega$;
- for each edge e, the label of e does not belong to the set of labels of the edges of the subtree attached below e (the label of e together with this set of labels constitutes the *content* of e);
- for each node v, the contents of the edges descending from v come in decreasing order.

Example 3.2.1 The following labeled trees lie in $\mathbf{T}(A)$.

t_{ex1} =


$t_{ex2} =$



Note. By definition, the depth of the elements of $\mathbf{T}(A)$ is at most $|A|$. Indeed, the contents of the edges along a descending path from the root form a strictly decreasing sequence.

We now proceed to define a mapping $\lambda\colon \mathbf{T}(A) \to \overline{\Omega}_A\mathbf{R}$. Let $t \in \mathbf{T}(A)$. Then $\lambda(t)$ is obtained by a bottom-up left-to-right traversal of t. More formally, $\lambda(t)$ is defined by induction on the depth of t. If t has depth 0, t consists of a single node, and we let $\lambda(t) = 1$. Let now t have non-zero depth, let $a_1, a_2, \ldots$ be the labels of the edges descending from the root of t, and let $t_1, t_2, \ldots$ be the subtrees attached below these edges. Then the t_i have strictly smaller depths than t, and we let

$$\lambda(t) = \prod_{k\geq 1} \lambda(t_k)a_k.$$

By Proposition 2.1.2, the mapping λ is well defined.

Example 3.2.2 The λ-images of the trees t_{ex1} and t_{ex2} in Example 3.2.1 are

$$\lambda(t_{ex1}) = (x^\omega y)^\omega \quad \text{and} \quad \lambda(t_{ex2}) = (zy^\omega)^\omega.$$

It is easily verified that $c(\lambda(t_k)a_k)$ is the content of the k-th edge descending from the root of t. In particular, $c(\lambda(t))$ is equal to the set of all letters appearing in t. Moreover, the factorization $\lambda(t) = \prod_{k\geq 1} \lambda(t_k)a_k$ is exactly the iteration of the left basic factorization of $\lambda(t)$.

Theorem 3.2.3 *The mapping* $\lambda\colon \mathbf{T}(A) \to \overline{\Omega}_A\mathbf{R}$ *is a bijection.*

Proof. The fact that λ is onto is simply a re-statement of Proposition 3.1.2. Formally, we consider $x \in \overline{\Omega}_A\mathbf{R}$, and we proceed by induction on $|c(x)|$. If $|c(x)| = 0$, then $x = 1$ and $x = \lambda(t)$ where t is the tree with a single node. If $|c(x)| \neq 0$, then $x = \mathrm{op}_u(x_0, x_1, \ldots)$ where $u = a_0a_1\cdots$ is a word in A^+ or A^ω. In particular, $c(x) = \bigcup_{k\geq 0} c(x_ka_k)$ and $a_k \notin c(x_k)$ for all k, so that $c(x_k)$ is strictly contained in $c(x)$ for each k. Let $t_0, t_1, \ldots$ be elements of $\mathbf{T}(A)$ such that $\lambda(t_k) = x_k$. Then $x = \lambda(t)$ where t is the tree with edges descending from the root labeled $a_0, a_1, \ldots$, and the trees $t_0, t_1, \ldots$ respectively attached at the other end.

Let now t and t' be distinct elements of $\mathbf{T}(A)$. Let $a_1, a_2, \ldots$ be the labels of the edges descending from the root of t and let $t_1, t_2, \ldots$ be the subtrees of t attached under these edges. Let $a'_1, a'_2, \ldots$ and $t'_1, t'_2, \ldots$ be defined similarly for t'. Let k be minimal such that either $t_k \neq t'_k$ or $a_k \neq a'_k$, and consider the factorizations yielded by the k-fold iteration of the left basic factorization of $\lambda(t)$ and $\lambda(t')$:

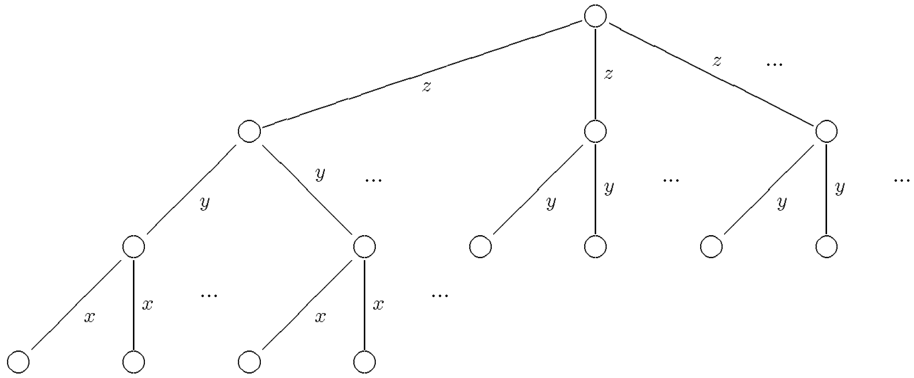
$$\lambda(t) = \lambda(t_1)a_1\cdots\lambda(t_k)a_ky \quad \text{and} \quad \lambda(t') = \lambda(t'_1)a'_1\cdots\lambda(t'_k)a'_ky'.$$

Proceeding by induction on the depth of t, we have either $\lambda(t_k) \neq \lambda(t'_k)$ or $a_k \neq a'_k$. But the left basic factorization is unique by Proposition 2.3.1. So this implies that $\lambda(t) \neq \lambda(t')$. $\square$

3.3 Representation by labeled ordinals

We will see in Section 4 that the representation by trees introduced above is well adapted to render the topological properties of $\overline{\Omega}_A\mathbf{R}$. However, it does not make it easy to understand the multiplication in $\overline{\Omega}_A\mathbf{R}$, as is illustrated in the following example.

Example 3.3.1 With reference to Example 3.2.1, the tree corresponding (via λ^{-1}) to the operation $\lambda(t_{ex_1})\lambda(t_{ex_2}) = (x^\omega y)^\omega(zy^\omega)^\omega$ is represented below.



We now introduce a new representation by means of labeled ordinals. In order to clarify our constructions, we briefly recall a few facts about ordinals (see [17]). An ordinal is a particular representative of an isomorphism class of well-ordered sets. More precisely, each such class contains a unique set α which satisfies the two following conditions:

- membership ($\in$) is a well-order on α;
- if $x \in \alpha$, then $x \subset \alpha$.

Such sets are called *ordinals*. If α and β are distinct ordinals, then exactly one of ($\alpha \in \beta$) and ($\beta \in \alpha$) holds. We write $\alpha < \beta$ if $\alpha \in \beta$. The smallest ordinal is the empty set, denoted 0. The successor of an ordinal α, denoted $\alpha + 1$, is $\alpha \cup \{\alpha\}$. The successor of 0 is $\{0\}$, denoted 1. Then come $2 = \{0, \{0\}\}$, $3 = \{0, \{0\}, \{0, \{0\}\}\}$, etc. We say that a non-zero ordinal α is a *limit ordinal* if it is not a successor. Then a non-zero ordinal α is a limit ordinal if and only if $\beta < \alpha$ implies $\beta + 1 < \alpha$, if and only if $\alpha = \sup_{\beta<\alpha}\beta = \bigcup_{\beta<\alpha}\beta$. The least limit ordinal is denoted ω. Finally, recall that if β is an ordinal and $(\alpha_\gamma)_{\gamma<\beta}$ is a family of ordinals, then $\sum_{\gamma<\beta}\alpha_\gamma$ is defined to be the ordinal corresponding to the disjoint union of the ordered sets α_γ where, in addition, each element of α_γ is less than each element of $\alpha_{\gamma'}$ whenever $\gamma < \gamma'$. If the $\alpha_\gamma = \alpha$ for all $\gamma < \beta$, we write $\alpha\beta$ for $\sum_{\gamma<\beta}\alpha$.

Let A be a finite alphabet. An *A-labeled ordinal* is a pair (α, ℓ) where α is an ordinal and ℓ is a mapping $\ell\colon \alpha \to A$. If α is finite, then (α, ℓ) is simply a (finite) word. It is sometimes useful to think of labeled ordinals as transfinite words. The *content* of (α, ℓ), denoted $c(\alpha, \ell)$, is the range of ℓ. If β is a limit ordinal such that $\beta \le \alpha$ we define the *cumulative content* of β (relative to (α, ℓ)) to be the set $\overline{c}(\beta)$ of all letters $a \in A$ such that there exists a sequence $(\gamma_n)_n$ of ordinals with $\bigcup_n \gamma_n = \beta$ and $\gamma_n < \beta$, $\ell(\gamma_n) = a$ for all n. Note that, for any limit ordinal $\beta \le \alpha$,

- $\overline{c}(\beta) = \bigcup\bigcap_k\{\ell(\gamma_k)\}$, where the union runs over all strictly increasing sequences $(\gamma_k)_k$ such that $\beta = \bigcup_k \gamma_k$;

- $\overleftarrow{c}(\beta)$ is not empty.

The last statement is an immediate consequence of the finiteness of the alphabet A. We say that the labeled ordinal (α, ℓ) is *reduced* if, for each limit ordinal $\beta < \alpha$, we have $\ell(\beta) \notin \overleftarrow{c}(\beta)$.

Lemma 3.3.2 *Let (α, ℓ) be an A-labeled ordinal. If (α, ℓ) is reduced, then for each ordinal β and for each $n \geq 1$ such that $\beta + \omega^n \leq \alpha$, we have $|\overleftarrow{c}(\beta + \omega^n)| \geq n$. In particular, $\alpha \leq \omega^{|A|}$.*

Proof. The proof is by induction on n. As observed above, for each limit ordinal $\beta < \alpha$, and in particular for each ordinal of the form $\beta + \omega$ such that $\beta + \omega \leq \alpha$, the cumulative content $\overleftarrow{c}(\beta + \omega)$ has at least one element. Suppose we proved that, for each ordinal β such that $\beta + \omega^n \leq \alpha$, we have $|\overleftarrow{c}(\beta + \omega^n)| \geq n$. We now consider β such that $\beta + \omega^{n+1} \leq \alpha$. Then $\beta + \omega^{n+1} = \bigcup_{k<\omega} \beta + \omega^n k$, and for each $k < \omega$, $|\overleftarrow{c}(\beta + \omega^n k)| \geq n$. Because A is finite, there exists a strictly increasing sequence of integers $(k_i)_{i<\omega}$ such that the $\overleftarrow{c}(\beta + \omega^n k_i)$ are all equal and such that the $\ell(\beta + \omega^n k_i)$ are all equal. Observe that $\ell(\beta + \omega^n k_i) \notin \overleftarrow{c}(\beta + \omega^n k_i)$ since (α, ℓ) is reduced.

For each $a \in \overleftarrow{c}(\beta + \omega^n k_1)$ and for each $1 \leq h < \omega$, there exists a strictly increasing sequence of ordinals $(\gamma_{h,j})_{j<\omega}$ such that $\bigcup_j \gamma_{h,j} = \beta + \omega^n k_h$ and $\beta + \omega^n k_{h-1} < \gamma_{h,j}$, $\ell(\gamma_{h,j}) = a$ for all j. Then the sequence $(\gamma_{h,h})_h$ is strictly increasing and for all h,

$$\beta + \omega^n k_{h-1} \leq \gamma_{h,h} \leq \beta + \omega^n k_h.$$

Thus $\bigcup_h \gamma_{h,h} = \beta + \omega^{n+1}$. In particular, $a \in \overleftarrow{c}(\beta + \omega^{n+1})$, that is, $\overleftarrow{c}(\beta + \omega^n k_1) \subseteq \overleftarrow{c}(\beta + \omega^{n+1})$. Let now $b = \ell(\beta + \omega^n k_h)$ for all $h \geq 1$. The sequence $(\beta + \omega^h k_h)_h$ also is strictly increasing and unions up to $\beta + \omega^{n+1}$. So $b \in \overleftarrow{c}(\beta + \omega^{n+1})$. It follows that $\overleftarrow{c}(\beta + \omega^{n+1})$ has at least $n + 1$ elements.

If $\alpha > \omega^{|A|}$, then $|\overleftarrow{c}(w^{|A|})| \geq |A|$, and hence $\overleftarrow{c}(\omega^{|A|}) = A$. This contradicts our assumption that (α, ℓ) is reduced, i.e. that $\ell(\omega^{|A|}) \notin \overleftarrow{c}(\omega^{|A|})$. So $\alpha \leq \omega^{|A|}$. □

Let $\mathbf{rLO}(A)$ be the set of reduced A-labeled ordinals. With each tree $t \in \mathbf{T}(A)$, we associate a reduced labeled ordinal $\mu(t)$ by ordering the set of edges of t from bottom to top and from left to right. More precisely, the set of edges of t, $E(t)$, is ordered as follows: for any edges e and e', let v_d be the common node ancestor of e and e' with largest depth. Among the edges descending from v_d, let e_d be the edge under which e lies (if e is adjacent to v_d, then $e_d = e$). The edge e'_d descending from v_d, is defined in the same fashion for e'. If $e_d < e'_d$ in the given order of the descendants of v_d, then we say that $e \leq e'$. If $e_d = e'_d$, note that one of e and e', say e', is equal to e_d. That is, e' is an edge ancestor of e or $e' = e$. Then we say that $e \leq e'$. It is easily verified that this defines a well-order on $E(t)$. We let α be the corresponding ordinal, we let $\ell\colon \alpha \to A$ be the corresponding labeling function (mapping each edge of t to its label), and we let $\mu(t) = (\alpha, \ell)$.

We now verify that $\mu(t) \in \mathbf{rLO}(A)$. Observe that an edge e of t corresponds to a limit ordinal $\beta < \alpha$ if and only if the order type of the direct descendants of its lower extremity is ω (otherwise, it is the successor of its last child). Thus, by definition of $\mathbf{T}(A)$, $\mu(t)$ is reduced. So we have defined a mapping $\mu\colon \mathbf{T}(A) \to \mathbf{rLO}(A)$.

Example 3.3.3 Let t_{ex1} and t_{ex2} be the trees given in Example 3.2.1. Then $\mu(t_{ex1}) = (\omega^2, \ell_1)$ and $\mu(t_{ex2}) = (\omega^2, \ell_2)$, with

$$\ell_1(0) \quad = \quad x,$$

$$\begin{aligned}
\ell_1(\omega k+n) &= x \quad \text{for all } 1\le n<\omega \text{ and } 0\le k<\omega,\\
\ell_1(\omega k) &= y \quad \text{for all } 1\le k<\omega, \text{ and}\\
\ell_2(\omega k) &= z \quad \text{for all } 0\le k<\omega,\\
\ell_2(\omega k+n) &= y \quad \text{for all } 1\le n<\omega \text{ and } 0\le k<\omega.
\end{aligned}$$

Lemma 3.3.4 *Let $t\in\mathbf{T}(A)$ be a tree of non-zero depth, let $a_1,a_2,\dots$ be the labels of the edges descending from the root of t, and let $t_1,t_2,\dots$ be the subtrees of t attached below these edges. If $\mu(t_n)=(\alpha_n,\ell_n)$ for all n, then $\mu(t)=(\alpha,\ell)$ with*

$$\begin{aligned}
\sum_{n\ge1}(\alpha_n+1) &= \alpha\\
\ell\left(\sum_{i=1}^{n-1}(\alpha_i+1)+\gamma\right) &= \ell_n(\gamma) \quad \textit{if } \gamma<\alpha_n\\
\ell(\sum_{i=1}^{n}(\alpha_i+1)) &= a_n.
\end{aligned}$$

Proof. The verification is immediate by induction on the depth of t. □

We want to verify that μ is a bijection between $\mathbf{T}(A)$ and $\mathbf{rLO}(A)$. For this purpose, in a process comparable to that described in Sections 2.3 and 3.1, we construct and iterate a left basic partition of a non-empty reduced labeled ordinal. Let $(\alpha,\ell)\in\mathbf{rLO}(A)$ with $\alpha\ne0$. For each letter $a\in A$, we consider the least ordinal γ_a such that $\gamma_a<\alpha$ and $\ell(\gamma_a)=a$, and we let $\alpha'=\max\{\gamma_a\mid a\in A\}$. Finally, we let α'' be the ordinal such that $\alpha=\alpha'+1+\alpha''$. This equality is called the *left basic partition* of (α,ℓ). Let us now iterate this process. Let $\beta_0=\alpha$ and let $\ell'_0=\ell$. For each n such that $\beta_n\ne0$, let $\beta_n=\alpha_{n+1}+1+\beta_{n+1}$ be the left basic partition of (β_n,ℓ'_n), let ℓ_{n+1} be the restriction of ℓ'_n to the initial segment α_{n+1} of β_n and let ℓ'_{n+1} be the labeling of β_{n+1} given by $\ell'_{n+1}(\gamma)=\ell'_n(\alpha_{n+1}+1+\gamma)$. Then for each $n\ge1$, we have

$$\begin{aligned}
\alpha &= \sum_{i=1}^{n}(\alpha_i+1)+\beta_n,\\
\ell_n(\gamma) &= \ell\left(\sum_{i=1}^{n-1}(\alpha_i+1)+\gamma\right) \quad \text{for all } \gamma<\alpha_n \text{ and}\\
\ell'_n(\gamma) &= \ell\left(\sum_{i=1}^{n}(\alpha_i+1)+\gamma\right) \quad \text{for all } \gamma<\beta_n.
\end{aligned}$$

If $\beta_r=0$ for some $r\ge1$, then $\alpha=\sum_{i=1}^{r}(\alpha_i+1)$. If $\beta_r\ne0$ for all r, we claim that $\alpha=\sum_{i\ge1}(\alpha_i+1)$. Suppose indeed that $\alpha'=\sum_{i\ge1}(\alpha_i+1)<\alpha$. Then, for each $n\ge1$, $\sum_{i=1}^{n}(\alpha_i+1)<\alpha'$, so $\ell(\alpha')\in c(\beta_n,\ell'_n)=c(\alpha_{n+1},\ell_{n+1})\cup\{\ell(\sum_{i=1}^{n+1}(\alpha_i+1))\}$. Therefore, $\ell(\alpha')\in\overleftarrow{c}(\alpha')$, which contradicts the reducedness of (α,ℓ). Thus $\alpha=\sum_{i\ge1}(\alpha_i+1)$ in all cases. This partition is called the *iterated left basic partition* of (α,ℓ).

Let now $\nu\colon\mathbf{rLO}(A)\to\mathbf{T}(A)$ be defined as follows. If $\alpha=0$, then $\nu(\alpha,\ell)$ is the tree consisting of a single node. Otherwise, $\alpha\ne0$, and we consider the iterated left basic partition of (α,ℓ), $\alpha=\sum_{i\ge1}(\alpha_i+1)$. By definition, $c(\alpha_i,\ell_i)$ is strictly contained in $c(\alpha,\ell)$ for each $i\ge1$. The tree $\nu(\alpha,\ell)$ is then defined by induction on $|c(\alpha,\ell)|$: If the sum $\alpha = \sum_{i\geq 1}(\alpha_i+1)$ consists of r (resp. infinitely many) terms, there are r (resp. ω) edges descending from the root of $\nu(\alpha,\ell)$, the corresponding labels are $\ell(\alpha_1+1)$, $\ell(\alpha_1+1+\alpha_2+1),\ldots$ and the subtrees attached below these edges are $\nu(\alpha_1,\ell_1)$, $\nu(\alpha_2,\ell_2),\ldots$ The reducedness of (α,ℓ) implies immediately that $\nu(\alpha,\ell)\in\mathbf{T}(A)$.

Theorem 3.3.5 *The mappings μ and ν are mutually reciprocal bijections.*

Proof. Let $t\in\mathbf{T}(A)$. If t has depth 0, then $\nu(\mu(t)) = t$ trivially. Let now t have non-zero depth and let $\mu(t) = (\alpha,\ell)$. Using the notations of Lemma 3.3.4, it is easily verified that the partition $\alpha = \sum_{n\geq 1}(\alpha_n+1)$ is exactly the iterated left basic partition of (α,ℓ), and hence, by Lemma 3.3.4, $\nu(\mu(t)) = t$.

Conversely, let $(\alpha,\ell)\in\mathbf{rLO}(A)$. If $\alpha = 0$, then $\mu(\nu(\alpha,\ell)) = (\alpha,\ell)$ by definition. Suppose now that $c(\alpha,\ell)\neq\emptyset$ and let $\alpha = \sum_{i\geq 1}(\alpha_i+1)$ be the iterated left basic partition of (α,ℓ). For each $i\geq 1$, let $\ell_i\colon\alpha_i\to A$ be given by $\ell_i(\gamma) = \ell(\sum_{j<i}(\alpha_j+1)+\gamma)$ for all $\gamma<\alpha_i$. By definition, the edges from the root of $\nu(\alpha,\ell)$ are labeled $\ell(\alpha_1+1)$, $\ell(\alpha_1+1+\alpha_2+1),\ldots$ and the subtrees attached below these edges are $\nu(\alpha_1,\ell_1)$, $\nu(\alpha_2,\ell_2),\ldots$ It now follows from Lemma 3.3.4 that $\mu(\nu(\alpha,\ell)) = (\alpha,\ell)$. $\square$

4 Algebra and topology on $\mathbf{T}(A)$ and $\mathbf{rLO}(A)$

We have constructed the following bijections,

$$\begin{tikzcd}
\mathbf{T}(A) \arrow[r, "\lambda"] \arrow[d, shift right, "\mu"'] & \overline{\Omega}_A\mathbf{R} \\
\mathbf{rLO}(A) \arrow[u, shift right, "\nu"'] \arrow[ur, "\psi"'] &
\end{tikzcd}
\qquad
\begin{aligned}
\mu\circ\nu &= \mathrm{id}_{\mathbf{rLO}(A)}\\
\nu\circ\mu &= \mathrm{id}_{\mathbf{T}(A)}\\
\psi &= \lambda\circ\nu
\end{aligned}$$

and we wish to use them to study the algebraic and topological properties of $\overline{\Omega}_A\mathbf{R}$.

4.1 Product of reduced labeled ordinals

Let us first define a product on $\mathbf{rLO}(A)$. Let $(\alpha,\ell),(\beta,m)\in\mathbf{rLO}(A)$. If α is not a limit ordinal, we let

$$(\alpha,\ell)(\beta,m) = (\alpha+\beta,\ell')$$

where $\ell'(\gamma) = \ell(\gamma)$ if $\gamma<\alpha$ and $\ell'(\alpha+\gamma) = m(\gamma)$ if $\gamma<\beta$. If we consider labeled ordinals as transfinite words, this is simply the concatenation of the two transfinite words (α,ℓ) and (β,m). If α is a limit ordinal, we partition β as $\beta = \beta_1+\beta_2$ where β_1 is minimal such that $m(\beta_1)\notin\overleftarrow{c}(\alpha)$ (if $c(\beta,m)\subseteq\overleftarrow{c}(\alpha)$, we let $\beta_1 = \beta$ and $\beta_2 = 0$). We then let

$$(\alpha,\ell)(\beta,m) = (\alpha+\beta_2,\ell')$$

where $\ell'(\gamma) = \ell(\gamma)$ if $\gamma<\alpha$ and $\ell'(\alpha+\gamma) = m(\beta_1+\gamma)$ if $\gamma<\beta_2$. In other words, $(\alpha,\ell)(\beta,m)$ is obtained by deleting the initial segment of (β,m) whose label is contained in $\overleftarrow{c}(\alpha)$, and then concatenating the resulting transfinite word to the right of (α,ℓ). In all cases, $(\alpha,\ell)(\beta,m)\in\mathbf{rLO}(A)$.

This product is then transferred to $\mathbf{T}(A)$ via $\nu = \mu^{-1}$: we let

$$tt' = \nu(\mu(t)\mu(t'))$$

for all $t, t' \in \mathbf{T}(A)$. In particular, μ and ν are isomorphisms.

Theorem 4.1.1 *The mapping* $\lambda\colon \mathbf{T}(A) \to \overline{\Omega}_A\mathbf{R}$ *is an isomorphism.*

We already know that λ is a bijection, so it suffices to prove that λ is multiplicative. We will not give the proof of this theorem at this point, since it is a special case of a more general result, namely Theorem 6.1.1 below.

Theorem 4.1.1 allows the effective computation of certain products in $\overline{\Omega}_A\mathbf{R}$ in a natural way.

Example 4.1.2 We prove that $\mathbf{R}$ satisfies $(x^\omega y)^\omega (y^\omega z)^\omega = (x^\omega y)^\omega (zy^\omega)^\omega$. It suffices to prove that this equality holds in $\overline{\Omega}_A\mathbf{R}$ when x, y, z are letters of A. The elements $(x^\omega y)^\omega$ and $(y^\omega z)^\omega$ are given "naturally" as

$$\begin{aligned}
(x^\omega y)^\omega = \lambda(\nu(\omega^2, \ell_1)) \quad &\text{and} \quad (y^\omega z)^\omega = \lambda(\nu(\omega^2, \ell_2)) \quad \text{with} \\
\ell_1(0) = x \quad &\text{and} \quad \ell_2(0) = y \\
\ell_1(\omega k + n) = x \quad &\text{and} \quad \ell_2(\omega k + n) = y \quad \text{for } 0 \leq k < \omega,\ 1 \leq n < \omega \\
\ell_1(\omega k) = y \quad &\text{and} \quad \ell_2(\omega k) = z \quad \text{for } 0 < k < \omega
\end{aligned}$$

With respect to (ω^2, ℓ_1), $\overleftarrow{c}(\omega^2) = \{x, y\}$. The least ordinal $\beta_1 < \omega^2$ such that $\ell_2(\beta_1) \notin \{x, y\}$ is $\beta_1 = \omega$. Now $\omega^2 = \omega + \omega^2$, so

$$(\omega^2, \ell_1)(\omega^2, \ell_2) = (\omega^2 2, \ell)$$

with $\ell(\gamma) = \ell_1(\gamma)$ and $\ell(\omega^2 + \gamma) = \ell_1(\omega + \gamma)$ for all $\gamma < \omega^2$. This proves immediately the announced equality.

We also derive easily the following characterization of the idempotents of $\overline{\Omega}_A\mathbf{R}$.

Corollary 4.1.3 *Let* $x \in \overline{\Omega}_A\mathbf{R}$ *and let* (α, ℓ) *be the corresponding reduced labeled ordinal, that is,* $(\alpha, \ell) = \nu(\lambda^{-1}(x))$. *Then* x *is idempotent if and only if* α *is a limit ordinal and* $c(\alpha, \ell) = \overleftarrow{c}(\alpha)$.

Proof. Since λ, μ and ν are isomorphisms, x is idempotent in $\overline{\Omega}_A\mathbf{R}$ if and only if (α, ℓ) is idempotent in $\mathbf{rLO}(A)$. The characterization of the idempotents of $\mathbf{rLO}(A)$ is immediate by definition of the product on $\mathbf{rLO}(A)$. $\square$

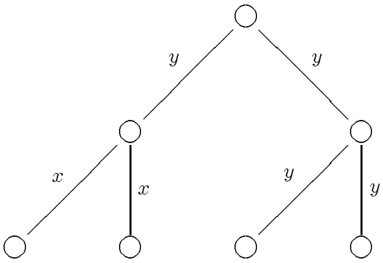
4.2 The topology of $\mathbf{T}(A)$

We now transport the topology of $\overline{\Omega}_A\mathbf{R}$ on $\mathbf{T}(A)$ via λ^{-1}. Since $\lambda\colon \mathbf{T}(A) \to \overline{\Omega}_A\mathbf{R}$ is an isomorphism, it follows that the product on $\mathbf{T}(A)$ is continuous, and λ is a homeomorphism.

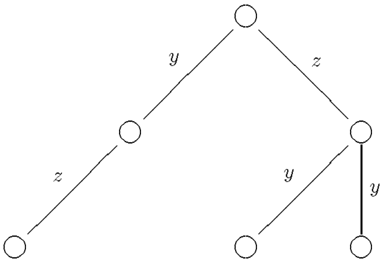
In addition, each element of $\mathbf{T}(A)$ admits a simple and graphical basis of neighborhoods. Let $t \in \mathbf{T}(A)$ and let $k \geq 1$. We define a finite tree $\mathrm{restr}(t, k)$ by dropping in t all the nodes that are the

n-th child of their parent if $n > k$, and all the subtrees attached under such nodes. For instance, if t_{ex1} and t_{ex2} are as in Example 3.2.1, then $\mathrm{restr}(t_{ex1}, 2)$ and $\mathrm{restr}(t_{ex2}, 2)$ are the following trees.

$\mathrm{restr}(t_{ex1}, 2) \quad =$



$\mathrm{restr}(t_{ex2}, 2) \quad =$



The following lemma is easily verified.

Lemma 4.2.1 *Let $t \in \mathbf{T}(A)$ be a tree of non-zero depth, let $e_1, e_2, \ldots$ be the edges descending from the root of t and let $t_1, t_2, \ldots$ be the subtrees of t attached below these edges. If the root of t has $r \leq k$ (resp. more than k) descendants, then the root of* $\mathrm{restr}(t, k)$ *has r (resp. k) descendants, namely $e_1, \ldots, e_r$ (resp. $e_1, \ldots, e_k$) and the subtree of* $\mathrm{restr}(t, k)$ *attached below each e_i is* $\mathrm{restr}(t_i, k)$.

Let $N(t, k)$ be the set of all elements $t' \in \mathbf{T}(A)$ such that $\mathrm{restr}(t', k) = \mathrm{restr}(t, k)$. We also define a family of languages $L(t, k)$ $(k \geq 1)$, by induction on the depth of t. If t consists of a single node, then $L(t, k) = \{1\}$. Let now t have non-zero depth. With the notation of Lemma 4.2.1, let $a_1, a_2, \ldots$ be the labels of $e_1, e_2, \ldots$ If the root of t has $r \leq k$ children, then we let

$$L(t, k) = L(t_1, k)a_1 L(t_2, k)a_2 \cdots L(t_r, k)a_r.$$

If the root of t has at least $k + 1$ children, and if B is the content of the $(k + 1)$-st edge descending from the root (and hence contains the content of all the subsequent children of the root), then we let

$$L(t, k) = L(t_1, k)a_1 L(t_2, k)a_2 \cdots L(t_k, k)a_k B^*.$$

Example 4.2.2 For the trees of Example 3.2.1, the languages $L(t, 2)$ are respectively

$$L(t_{ex1}, 2) = xxx^*yxxx^*y\{x, y\}^* \quad \text{and} \quad L(t_{ex2}, 2) = zyyyy^*z\{y, z\}^*.$$

Lemma 4.2.3 *Let $t \in \mathbf{T}(A)$ and let $k \geq 1$.*

1. *$L(t, k)$ is $\mathbf{R}$-elementary.*

2. *$\lambda(t) \in \overline{L(t, k)}$, and every element of $L(t, k)$ has content $c(\lambda(t))$.*

3. *Let $x \in \overline{\Omega}_A\mathbf{R}$. If the root of t has $r \leq k$ children, then $x \in \overline{L(t,k)}$ if and only if the left basic factorization of x can be iterated exactly r times, and it yields a factorization $x = x_1a_1 \cdots x_ra_r$ such that $x_i \in \overline{L(t_i,k)}$. If the root of t has at least $k+1$ children, then $x \in \overline{L(t,k)}$ if and only if the left basic factorization of x can be iterated at least k times, yielding a factorization $x = x_1a_1 \cdots x_ka_ky$ such that $x_i \in \overline{L(t_i,k)}$ and $y \in \overline{B^*}$.*

4. *There exists a finite tree u such that $\lambda(N(t,k)) = \overline{L(u,k)}$ and $L(t,k) \subseteq L(u,k)$.*

Proof. It is immediately verified that $L(t,k)$ is a product of letters and of languages of the form B^* ($B \subseteq A$). Moreover, the letter following a factor B^* does not belong to B, by definition of $\mathbf{T}(A)$. So $L(t,k)$ is $\mathbf{R}$-elementary.

The proof of (2) is immediate by induction on the depth of t. Let us now prove (3). The equality

$$\begin{array}{rcl} \overline{L(t_1,k)}a_1 \cdots \overline{L(t_r,k)}a_r &=& \overline{L(t_1,k)a_1L(t_2,k)a_2 \cdots L(t_r,k)a_r} \\ \text{(resp. } \overline{L(t_1,k)}a_1 \cdots \overline{L(t_k,k)}a_k\overline{B^*} &=& \overline{L(t_1,k)a_1L(t_2,k)a_2 \cdots L(t_k,k)a_kB^*}\text{)} \end{array}$$

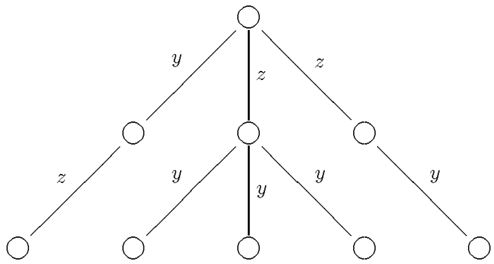
follows from the compactness of $\overline{\Omega}_A\mathbf{R}$. Let now $x \in \overline{L(t,k)}$. There exists a sequence $(u_n)_n$ of words of $L(t,k)$ such that $x = \lim_n u_n$. Using (2), the expected property of the iterated left basic factorization is easily verified for the words u_n. So each u_n can be factored as

$$u_n = u_{n,1}a_1 \cdots u_{n,r}a_r \quad \text{(resp. } u_n = u_{n,1}a_1 \cdots u_{n,k}a_kv_n \text{)}$$

with $u_{n,i} \in L(t_i,k)$ (and $v_n \in B^*$). By compactness, we may assume that each sequence $(u_{n,i})_n$ (and $(v_n)_n$) converges with, say, $\lim_n u_{n,i} = x_i$ (and $\lim_n v_n = y$). Then $x = x_1a_1 \cdots x_ra_r$ (resp. $x = x_1a_1 \cdots x_ka_ky$) is the r-fold (resp. k-fold) iteration of the left basic factorization of x, and $x_i \in \overline{L(t_i,k)}$ for each i (and $y \in \overline{B^*}$). This concludes the proof of (3).

Finally, let u be a tree obtained from the finite tree $\mathrm{restr}(t,k)$ by attaching under each node v of $\mathrm{restr}(t,k)$ with k children a new edge with a finite tree attached beneath it, and with the same content as the k-th edge attached under v. The resulting tree is finite. The verification that $\lambda(N(t,k)) = \overline{L(u,k)}$ and $L(t,k) \subseteq L(u,k)$ is immediate by induction on the depth of t. □

Example 4.2.4 For the tree t_{ex2} of Example 3.2.1, we have $\lambda(N(t_{ex2},2)) = \overline{L(u,2)}$ for the following finite tree u:



$$\lambda(u) = zy^4zyz$$
$$L(u,2) = zy^3y^*z\{y,z\}^*.$$

We can now describe convenient bases of neighborhoods for the elements of $\mathbf{T}(A)$.

Proposition 4.2.5 *Let $t \in \mathbf{T}(A)$. Then the $N(t,k)$ ($k \geq 1$) constitute a basis of clopen neighborhoods of t, and the $\overline{L(t,k)}$ constitute a basis of clopen neighborhoods of $\lambda(t)$.*

Proof. By Lemma 4.2.3 (1), each $L(t,k)$ is $\mathbf{R}$-elementary, and hence $\mathbf{R}$-recognizable (Proposition 2.2.4). Therefore, by Lemma 4.2.3 (2), $\overline{L(t,k)}$ is a clopen neighborhood of $\lambda(t)$, and $\lambda^{-1}(\overline{L(t,k)})$ is a clopen neighborhood of t. We also observe that Lemma 4.2.3 (4) implies that each $N(t,k)$ is a clopen subset of $\mathbf{T}(A)$.

We now prove that, for each $t \in \mathbf{T}(A)$ and $k \geq 1$, we have

$$\lambda(N(t,k+1)) \subseteq \overline{L(t,k)} \subseteq \lambda(N(t,k)).$$

It is easily verified, by induction on the depth of t, that $\mathrm{restr}(t,k+1)$ entirely determines $L(t,k)$. Therefore, if $t' \in N(t,k+1)$, then $L(t',k) = L(t,k)$, and hence $\lambda(t') \in \overline{L(t,k)}$. Thus $\lambda(N(t,k+1)) \subseteq \overline{L(t,k)}$. The containment $\overline{L(t,k)} \subseteq \lambda(N(t,k))$ follows from Lemma 4.2.3 (4): with the notation of the lemma, we have $\overline{L(t,k)} \subseteq \overline{L(u,k)} = \lambda(N(t,k))$.

It now suffices to verify that if $t' \neq t$, then $t' \notin N(t,k)$ for some k. We proceed by induction on the depth of t. Let $a_1, a_2, \ldots$ be the labels of the edges descending from the root of t and let $t_1, t_2, \ldots$ be the subtrees of t attached under these edges. Let $a'_1, a'_2, \ldots$ and $t'_1, t'_2, \ldots$ be defined similarly for t'. Four cases may occur:

1. there exists k such that $a_i = a'_i$ and $t_i = t'_i$ for $i < k$, and $t_k \neq t'_k$;
2. there exists k such that $a_i = a'_i$ for $i < k$, $t_i = t'_i$ for $i \leq k$, and $a_k \neq a'_k$;
3. the root of t has $k-1$ children and that of t' has at least k children;
4. the root of t' has $k-1$ children and that of t has at least k children.

In case (1), by induction hypothesis, there exists n such that $t'_k \notin N(t_k, n)$. Therefore, by Lemma 4.2.1, $t' \notin N(t, \max\{n,k\})$. In the other cases, it follows from Lemma 4.2.1 that $\mathrm{restr}(t,k) \neq \mathrm{restr}(t',k)$, so $t' \notin N(t,k)$. □

Corollary 4.2.6 *The closures of the* $\mathbf{R}$*-elementary languages form a basis of the topology of* $\overline{\Omega}_A\mathbf{R}$.

As a result, we get the following description of the $\mathbf{R}$-recognizable languages, which was first obtained by Eilenberg [16].

Corollary 4.2.7 *Let A be a finite alphabet. A language of A^* is* $\mathbf{R}$*-recognizable if and only if it is a finite disjoint union of languages of the form*

$$A_0^* a_1 A_1^* \cdots a_k A_k^*,$$

where $A_i \subseteq A$ $(0 \leq i \leq k)$ and $a_i \in A \setminus A_{i-1}$ $(1 \leq i \leq k)$.

Proof. Let $L \subseteq A^*$ be $\mathbf{R}$-recognizable. Then $\overline{L}$ is a clopen subset of $\overline{\Omega}_A\mathbf{R}$. By Proposition 4.2.5, for each $x \in \overline{L}$, there exist $t \in \mathbf{T}(A)$ and $k \geq 1$ such that $x \in \lambda(N(t,k)) \subseteq \overline{L}$. Each $\lambda(N(t,k))$ is open, and $\overline{L}$ is compact: thus $\overline{L}$ is a finite union of $\lambda(N(t,k))$, say

$$\lambda(N(t,k)) = \lambda(N(t_1,k_1)) \cup \ldots \cup \lambda(N(t_n,k_n)).$$

Let $k = \max(k_1, \ldots, k_n)$. For this fixed value of k, the $\lambda(N(t,k))$ are pairwise equal or disjoint, so each $\lambda(N(t_i,k_i))$ is a disjoint union of the $\lambda(N(t,k))$ which it contains. By compactness again, this union must be finite, so $\overline{L}$ is a finite disjoint union of $\lambda(N(t,k))$. Finally, by Lemma 4.2.3 (4), each $\lambda(N(t,k))$ is of the form $\overline{L(u,k)}$ for some finite tree u. Thus $L = \overline{L} \cap A^*$ is a finite disjoint union of $L(u,k)$, that is, of $\mathbf{R}$-elementary languages. □

4.3 Remarks on the representation of $\overline{\Omega}_A\mathbf{R}$

Our first remarks concern the two representations, by labeled trees and by labeled ordinals, presented in the above sections. It turns out that the first one is connected with a result of the first author on a hierarchy of subpseudovarieties of $\mathbf{R}$ [4], while the second one is connected with a more general result of the authors on pseudovarieties of the form $\mathbf{J}_1 * \mathbf{V}$ [11].

We also draw some consequences of our representations regarding the factorizations in letters and regular elements given by Almeida and Azevedo (Proposition 1.4.1 above).

(1) It is well known that $\mathbf{R} = \mathbf{J}_1 * \mathbf{J}$ [15]. In [11], denoting by $\Gamma_A\mathbf{V}$ the profinite Cayley graph of $\overline{\Omega}_A\mathbf{V}$, the authors gave the following description of the implicit operations on a pseudovariety of the form $\mathbf{J}_1 * \mathbf{V}$: they are in bijection with the pairs (X, x) where X is a profinitely connected, profinitely quasi-linear closed subgraph of $\Gamma_A\mathbf{V}$ such that 1 lies in the initial profinitely bonded component of X, and x lies in its final profinitely bonded component. For missing definitions, the reader is referred to [11].

In this context, understanding $\overline{\Omega}_A\mathbf{R}$ amounts to understanding the profinitely connected, profinitely quasi-linear closed subgraphs of $\Gamma_A\mathbf{J}$. This can be done by using the results of the first author in [3], which yield a unique factorization statement for the elements of $\overline{\Omega}_A\mathbf{J}$ as products of letters and of idempotents, and show that the profinitely bonded components of $\Gamma_A\mathbf{J}$ each have a single vertex. Then the profinitely connected, profinitely quasi-linear closed subgraphs of $\Gamma_A\mathbf{J}$ starting at 1 can be identified with certain transfinite paths in that graph, which are naturally encoded by elements of $\mathbf{rLO}(A)$.

(2) Another approach to $\mathbf{R}$ was used by the first author in [4]. It is known that $\mathbf{R}$ is the closure of the pseudovariety $\mathbf{J}_1$ under semidirect product [24]. In [4], Almeida characterized the identities which hold in each iterate $\mathbf{J}_1^{(k)}$ (where $\mathbf{J}_1^{(1)} = \mathbf{J}_1$ and $\mathbf{J}_1^{(k+1)} = \mathbf{J}_1 * \mathbf{J}_1^{(k)}$). Implicit in that paper (in particular in Lemma 3.3 and Theorem 3.4), is a representation of the elements of $\overline{\Omega}_A\mathbf{J}_1^{(k)}$ using labeled trees that are closely related to those described in Section 3.2, and based on properties also reminiscent of those reported in Section 4.2 (Proposition 4.2.5 in particular).

The difference between the two papers is the following. In [4], only finite semidirect iterates $\mathbf{J}_1^{(k)}$ are considered, using the fact that these pseudovarieties all admit finite free objects over finite alphabets. In contrast, the present paper considers directly the limit of the $\mathbf{J}_1^{(k)}$, namely $\mathbf{R}$, which imposes the consideration of a topological space of infinite labeled trees.

(3) By Proposition 1.4.1, each element of $\overline{\Omega}_A\mathbf{R}$ admits a factorization in letters and regular (and hence idempotent) elements. We want to provide a refinement of this statement in the form of a unique factorization theorem, and to connect this factorization with the representation of the elements of $\overline{\Omega}_A\mathbf{R}$ by reduced labeled ordinals.

Let $\psi = \lambda \circ \nu \colon \mathbf{rLO}(A) \to \overline{\Omega}_A\mathbf{R}$. Then ψ is a bijection. Let $x \in \overline{\Omega}_A\mathbf{R}$ and let $(\alpha, \ell) = \psi^{-1}(x)$. We define a finite set $L_{\alpha,\ell}$ of limit ordinals less than or equal to α as follows: If α is finite, then $L_{\alpha,\ell} = \emptyset$. Otherwise, α is infinite and we let $\gamma_0 = \alpha$. If γ_n is defined and is infinite, we let β_n be the maximal limit ordinal such that $\beta_n \leq \gamma_n$. Note that β_n is well-defined: if γ_n is not itself a limit ordinal, we may consider the decreasing sequence obtained by successively decrementing γ_n by 1. In a well-ordered set, all decreasing sequences are finite, so that the last value of this sequence is a limit ordinal, namely β_n. We then let

$$\gamma_{n+1} = \min\{\gamma < \beta_n \mid \forall \gamma \leq \beta < \beta_n,\ \ell(\beta) \in \overleftarrow{c}(\beta_n)\}.$$

We observe that β_{n+1} is defined if γ_{n+1} is not finite, and that in that case, $\beta_{n+1} < \beta_n$. Thus we have defined a decreasing sequence, which must then be finite (that is, for some n, γ_n is finite). We let $L_{\alpha,\ell}$ be the set of the β_n. For each $0 < i \leq k$, we let u_i be the word

$$u_i = \ell(\beta_i)\ell(\beta_i + 1)\cdots\ell(\gamma_i).$$

We also define u_0 and u_{k+1} by

$$\begin{aligned} u_{k+1} &= \ell(0)\ell(1)\cdots\ell(\gamma_{k+1}), \\ u_0 &= \begin{cases} 1 & \text{if } \gamma_0 = \beta_0, \\ \ell(\beta_0)\ell(\beta_0+1)\cdots\ell(\alpha-1) & \text{otherwise.} \end{cases} \end{aligned}$$

Observe that $|u_i| \geq 1$ if $i \neq 0$, that the first letter of u_i $(0 \leq i \leq k)$ does not lie in $\overleftarrow{c}(\beta_i)$, that the last letter of u_i $(0 < i \leq k+1)$ lies in $\overleftarrow{c}(\beta_{i-1})$ but that its penultimate letter (if it exists) does not lie in $\overleftarrow{c}(\beta_{i-1})$.

For each $0 \leq i \leq k$, let δ_i be the ordinal such that $\gamma_i + \delta_i = \beta_{i-1}$. Let also $\ell_i \colon \delta_i \to A$ be defined by $\ell_i(\gamma) = \ell(\gamma_i + \gamma)$. Then, by Corollary 4.1.3, $\psi(\delta_i, \ell_i)$ is an idempotent with content $\overleftarrow{c}(\beta_i)$ and we have

$$x = \psi(\alpha, \ell) = u_{k+1}\psi(\delta_k, \ell_k)u_k \cdots \psi(\delta_0, \ell_0)u_0.$$

This factorization is called the *canonical factorization* of $x = \psi(\alpha, \ell)$.

Theorem 4.3.1 *Let $\pi\colon \overline{\Omega}_A\mathbf{M} \to \overline{\Omega}_A\mathbf{R}$ be the canonical projection. Then each element x of $\overline{\Omega}_A\mathbf{M}$ can be written as a product $x = u_{k+1}x_ku_k \cdots x_0u_0$ where*

- *$u_0 \in A^*$, $u_i \in A^+$ if $i \neq 0$, and $\pi(x_i)$ is idempotent for all i;*
- *the first letter of u_i $(0 \leq i \leq k)$ does not lie in $c(x_i)$;*
- *the last letter of u_i $(0 < i \leq k+1)$ lies in $c(x_{i-1})$;*
- *if $|u_i| \geq 2$, then the penultimate letter of u_i does not lie in $c(x_{i-1})$.*

Moreover $u_{k+1}(\pi(x_k))u_k \cdots (\pi(x_0))u_0$ is the canonical factorization of $\pi(x)$. In particular, if $x = u_{k+1}x_ku_k \cdots x_0u_0$ and $y = v_{n+1}y_nv_n \cdots y_0v_0$ are factorizations of this type, and if $\mathbf{R}$ satisfies $x = y$, then $k = n$, $u_i = v_i$ and $\mathbf{R}$ satisfies $x_i = y_i$ for all i.

Proof. Let $x \in \overline{\Omega}_A\mathbf{M}$ and let $x = u_{k+1}x_ku_k \cdots x_0u_0$ be a factorization of x as a product of words u_i and of elements x_i such that $\pi(x_i)$ is idempotent. The existence of such a factorization is ensured by Proposition 1.4.1. For each $0 \leq i \leq k$, let v_i be the longest prefix of u_i consisting only of letters of $c(x_i)$. Then $u_i = v_iu'_i$ and $\pi(x_iv_i) = \pi(x_i)$ (by Proposition 2.1.3 (d)). So, replacing x_i by x_iv_i and u_i by u'_i, we may assume that no u_i $(0 \leq i \leq k)$ starts with a letter of $c(x_i)$. If u_i has a non empty suffix in $c(x_{i-1})^+$, let us factor u_i as $u_i = u'_ia_iv_i$ with $c(a_iv_i) \subseteq c(x_{i-1})$ and $u'_i \notin A^*c(x_{i-1})$. Observe that $\pi(v_ix_{i-1})\ \mathcal{L}\ \pi(x_{i-1})$, so $\pi(v_ix_{i-1})$ is idempotent (by Proposition 1.4.1): we then replace x_i by v_ix_{i-1} and u_i by u'_i in the factorization of x. If $u_i \neq 1$ but its last letter does not lie in $c(x_{i-1})$, let us consider the first letter of x_{i-1}, say a_i. Then $x_{i-1} = a_ix'_{i-1}$ and $\pi(x_{i-1})\ \mathcal{L}\ \pi(x'_{i-1})$, so $\pi(x'_{i-1})$ is idempotent. We then modify the factorization of x by replacing u_i by u_ia_i and x_{i-1} by x'_{i-1}. Finally, let us consider the case where $u_i = 1$. If $c(x_i)$ and $c(x_{i-1})$ are inclusion-comparable, then, using again Proposition 1.4.1, it follows that $\pi(x_ix_{i-1})$ is idempotent, so grouping x_i and

x_{i-1} provides us with a shorter factorization. If $c(x_i)$ and $c(x_{i-1})$ are not inclusion-comparable, we consider the first letter of x_{i-1} not in $c(x_i)$, say a_i. Then $x_{i-1} = x'_{i-1}a_i x''_{i-1}$ with $c(x'_{i-1}) \subseteq c(x_i)$ and $\pi(x_{i-1}) \mathrel{\mathcal{L}} \pi(x''_{i-1})$, so $\pi(x''_{i-1})$ is idempotent. We again modify the factorization of x by replacing $u_i = 1$ by a_i and x_{i-1} by x''_{i-1}. In this fashion, we construct a factorization of x with the announced properties.

For each $0 \le i \le k$, let $(\delta_i, \ell_i) = \psi^{-1}(\pi(x_i)) \in \mathbf{rLO}(A)$. For $0 \le i \le k+1$, let also $(\varepsilon_i, \ell'_i) = \psi^{-1}(\pi(u_i))$: ε_i is finite and equal to the length of u_i. Then we have

$$\begin{aligned}
\pi(x) = \psi(\alpha, \ell) &= u_{k+1}\psi(\delta_k, \ell_k)u_k \cdots \psi(\delta_0, \ell_0)u_0 \\
&= \psi(\varepsilon_{k+1}, \ell'_{k+1})\psi(\delta_k, \ell_k)\psi(\varepsilon_k, \ell'_k) \cdots \psi(\delta_0, \ell_0)\psi(\varepsilon_0, \ell'_0) \\
&= \psi\left((\varepsilon_{k+1}, \ell'_{k+1})(\delta_k, \ell_k)(\varepsilon_k, \ell'_k) \cdots (\delta_0, \ell_0)(\varepsilon_0, \ell'_0)\right)
\end{aligned}$$

Since the ε_i are finite and the first letter of each u_i does not lie en $c(x_i)$ (that is, the range of ℓ_i), it follows that

$$\begin{aligned}
\alpha &= \varepsilon_{k+1} + \delta_k + \varepsilon_k + \cdots + \delta_0 + \varepsilon_0 \\
\ell(\gamma) &= \ell'_{k+1}(\gamma) \quad \text{if } \gamma < \varepsilon_{k+1} \\
\ell(\varepsilon_{k+1} + \gamma) &= \ell_k(\gamma) \quad \text{if } \gamma < \delta_k \\
\ell(\varepsilon_{k+1} + \delta_k + \cdots + \delta_i + \gamma) &= \ell'_i(\gamma) \quad \text{if } \gamma < \varepsilon_i \text{ and } 0 \le i \le k \\
\ell(\varepsilon_{k+1} + \delta_k + \cdots + \varepsilon_{i+1} + \gamma) &= \ell_i(\gamma) \quad \text{if } \gamma < \delta_i \text{ and } 1 \le i < k
\end{aligned}$$

Let us now calculate $L_{\alpha,\ell}$: We first let $\gamma_0 = \alpha$. Since $\varepsilon_0 < \omega$, we have $\beta_0 = \varepsilon_{k+1}+\delta_k+\varepsilon_k+\cdots+\delta_0$. It is not difficult to verify that $\overline{c}(\beta_0) = c(\delta_0, \ell_0)$ and that the subsequent values of γ_i and β_i are

$$\begin{aligned}
\gamma_i &= \varepsilon_{k+1} + \delta_k + \varepsilon_k + \cdots + \delta_i + \varepsilon_i \ \ (i \le k), \\
\beta_i &= \varepsilon_{k+1} + \delta_k + \varepsilon_k + \cdots + \delta_i \ \ (i \le k), \\
\gamma_{k+1} &= \varepsilon_{k+1}.
\end{aligned}$$

That is, $\psi(\alpha, \ell) = u_{k+1}\psi(\delta_k, \ell_k)u_k \cdots \psi(\delta_0, \ell_0)u_0$ is the canonical factorization of $x = \psi(\alpha, \ell)$, which concludes the proof. $\square$

5 Implicit operations on DRG

We now turn our attention to the non-aperiodic case. In the rest of the paper, $\mathbf{H}$ is a fixed pseudovariety of groups. Recall that $\mathbf{DRH}$ is the pseudovariety of all finite semigroups in which each regular $\mathcal{D}$-class is an $\mathcal{L}$-class and each subgroup is in $\mathbf{H}$, or equivalently, in which each regular $\mathcal{R}$-class is a group in $\mathbf{H}$. Then $\mathbf{DRH}$ trivially contains $\mathbf{R}$ and $\mathbf{H}$, and we let $\pi\colon \overline{\Omega}_A\mathbf{DRH} \to \overline{\Omega}_A\mathbf{R}$ and $\sigma\colon \overline{\Omega}_A\mathbf{DRH} \to \overline{\Omega}_A\mathbf{H}$ be the canonical projections.

Our aim in this section is to adapt the results obtained in the aperiodic case to this more general setting. Before we proceed, recall that, by Proposition 2.1.4, if $(s_1 \cdots s_n)_n$ is an infinite product in $\overline{\Omega}_A\mathbf{DRH}$ such that all the s_n have equal contents, then the accumulation points of this infinite product all lie in the same regular $\mathcal{R}$-class, and therefore in the same $\mathcal{H}$-class, which is a group. If e is the idempotent of this group, we say that the infinite product $(s_1 \cdots s_n)_n$ *designates* e (even though the infinite product may not converge, or may converge to another element of the $\mathcal{H}$-class of e).

5.1 The subgroups of $\overline{\Omega}_A\mathbf{DRH}$

Azevedo proved the following property of **DO** [13].

Proposition 5.1.1 *Let* $\mathbf{V}$ *be a pseudovariety contained in* $\mathbf{DO}$*. Let* $x, y, e \in \overline{\Omega}_A\mathbf{V}$ *be such that* $e = e^2$ *and* $c(x) \cup c(y) \subseteq c(e)$*. Then* $exye = exeye$*.*

We deduce from this result the following important structural property of the free pro-**DRH** monoids, and more generally of the free pro-$\mathbf{V}$ monoids for all $\mathbf{V}$ such that $\mathbf{H} \subseteq \mathbf{V} \subseteq \mathbf{DO} \cap \overline{\mathbf{H}}$.

Proposition 5.1.2 *Let* $\mathbf{H}$ *be a pseudovariety of groups, let* $\mathbf{V}$ *be a pseudovariety such that* $\mathbf{H} \subseteq \mathbf{V} \subseteq \mathbf{DO} \cap \overline{\mathbf{H}}$ *and let* $\sigma\colon \overline{\Omega}_A\mathbf{V} \to \overline{\Omega}_A\mathbf{H}$ *be the canonical projection. Then the regular* $\mathcal{H}$*-classes of* $\overline{\Omega}_A\mathbf{V}$ *are free pro-*$\mathbf{H}$ *groups on their content. More precisely, if* e *is an idempotent of* $\overline{\Omega}_A\mathbf{V}$ *and if* H_e *is its* $\mathcal{H}$*-class, then letting* $\varphi_e(a) = eae$ *for each* $a \in c(e)$ *defines a unique homeomorphism* $\varphi_e\colon \overline{\Omega}_{c(e)}\mathbf{H} \to H_e$ *whose inverse is the restriction of* σ *to* H_e*.*

Proof. We first observe that H_e is pro-$\mathbf{V}$, and hence pro-$\mathbf{H}$. Moreover, the subsemigroup generated by $\varphi_e(c(e))$ is dense in H_e. Indeed, each element x of H_e is the limit of a sequence of words on the alphabet $c(x) = c(e)$. If $u = a_1 \cdots a_r \in c(e)^+$, then $eue = e(a_1 \cdots a_r)e = (ea_1e) \cdots (ea_re)$ by Proposition 5.1.1. But $x = exe$, so x is the limit of a sequence of elements of the form eue ($u \in c(e)^+$), and hence of elements of the subsemigroup generated by $\varphi_e(c(e))$.

$$\begin{tikzcd}
c(e) \arrow[r, "\varphi_e"] \arrow[d, hook] & H_e \arrow[dr, hook] & \\
\overline{\Omega}_{c(e)}\mathbf{H} \arrow[ur, "\varphi_e"'] \arrow[r, hook] & \overline{\Omega}_A\mathbf{H} & \overline{\Omega}_A\mathbf{V} \arrow[l, "\sigma"']
\end{tikzcd}$$

Therefore, φ_e admits a unique extension to a continuous onto morphism (still denoted by φ_e) from $\overline{\Omega}_{c(e)}\mathbf{H}$ onto H_e. But, for each $a \in c(e)$, we have $\sigma(\varphi_e(a)) = \sigma(eae) = a$. Thus $\sigma \circ \varphi_e$ coincides with the identity of $\overline{\Omega}_{c(e)}\mathbf{H}$ on $c(e)$. By continuity, and using the fact that $\overline{\Omega}_{c(e)}\mathbf{H}$ is $c(e)$-generated, it follows that $\sigma \circ \varphi_e = \mathrm{id}_{\overline{\Omega}_{c(e)}\mathbf{H}}$, which concludes the proof. □

5.2 Representation by decorated trees

By Proposition 2.3.1, each element of $\overline{\Omega}_A\mathbf{DRH} \setminus \{1\}$ admits a unique left basic factorization. Proceeding as in Section 3.1, we may iterate this factorization: we let $y_0 = x$, and for each $k \geq 0$ such that $y_k \neq 1$, we let $y_k = x_{k+1}a_{k+1}y_{k+1}$ be the left basic factorization of y_k. For each such k, we have $x = x_1a_1 \cdots x_{k+1}a_{k+1}y_{k+1}$.

Let us assume that this factorization process goes on forever, that is, $y_k \neq 1$ for each $k \geq 0$. By definition of left basic factorizations, the x_ka_k have decreasing contents. Since A is finite, this sequence of contents is ultimately constant, and we consider the least integer m such that $c(x_na_n) = c(x_ma_m)$ for all $n \geq m$. Then $x = x_1a_1 \cdots x_{m-1}a_{m-1}y_{m-1}$. Let e be the idempotent designated by the infinite product $(x_ma_mx_{m+1}a_{m+1} \cdots x_na_n)_{n \geq m}$, and let H_e be the $\mathcal{H}$-class of e. Since y_{m-1} has the same content as any of the products $x_ma_mx_{m+1}a_{m+1} \cdots x_na_n$ ($n \geq m$) and since

$$y_{m-1} = x_ma_mx_{m+1}a_{m+1} \cdots x_na_ny_n \leq_{\mathcal{R}} x_ma_mx_{m+1}a_{m+1} \cdots x_na_n,$$

it follows from Proposition 1.4.1 that y_{m-1} is regular and $y_{m-1} \in H_e$. Moreover $\varphi_e(\sigma(y_{m-1})) = y_{m-1}$ by Proposition 5.1.2.

Thus, if the left basic factorization of x can be iterated an unbounded number of times, then $x = x_1a_1 \cdots x_{m-1}a_{m-1}\varphi_e(\sigma(y_{m-1}))$ for e and m chosen as above. Naturally, if the left basic factorization of x can be iterated only finitely many times, say we have $y_k = 1$, then $x = x_1a_1 \cdots x_ka_k$.

An $\mathbf{H}$-*decorated tree* (or *decorated tree*, since $\mathbf{H}$ is fixed) is a pair $t = (\underline{t}, \delta)$ where $\underline{t} \in \mathbf{T}(A)$ and δ is a partial labeling function from the set of nodes of t into $\overline{\Omega}_A\mathbf{H}$. We will talk, abusively, of the edges and the nodes of t, instead of $\underline{t}$. If v is a node of t with infinite arity, the sequence of contents of the edges descending from the root of t is ultimately constant. We call the ultimate value of this sequence the *cumulative content* of v, denoted $\vec{c}(v)$. This is coherent with the use of this phrase for reduced labeled ordinals, in view of the definition of the bijection μ (Section 3.3). Let $V_\omega(t)$ be the set of nodes of t of infinite arity. Let also τ be the projection from the set of decorated trees into $\mathbf{T}(A)$ defined by $\tau(t) = \underline{t}$ if $t = (\underline{t}, \delta)$.

Let $\mathbf{T_H}(A)$ be the set of decorated trees $t = (\tau(t), \delta)$ such that

- the domain of δ is $V_\omega(t)$, and
- $\delta(v) \in \overline{\Omega}_{\vec{c}(v)}\mathbf{H}$ for each $v \in V_\omega(t)$.

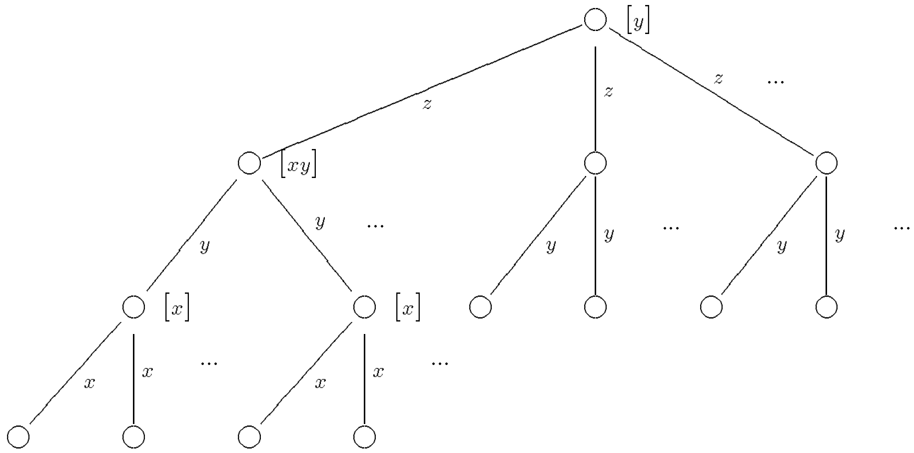
We now define $\lambda_\mathbf{H}(t)$ for each $t \in \mathbf{T_H}(A)$, by induction on the depth of t. If t has depth 0, then $\lambda_\mathbf{H}(t) = 1$. If t has non-zero depth, let $a_1, a_2, \ldots$ be the labels of the edges descending from the root r of t, and let $t_1, t_2, \ldots$ be the subtrees attached under these edges. These subtrees are viewed as elements of $\mathbf{T_H}(A)$ by taking the appropriate restriction of the node-labeling function δ: we then talk of *decorated subtrees*. If r has finitely many children, say n, then

$$\lambda_\mathbf{H}(t) = \lambda_\mathbf{H}(t_1)a_1 \cdots \lambda_\mathbf{H}(t_n)a_n.$$

Otherwise, r has infinite arity. Let m be the least integer such that the content of the m-th edge descending from r is equal to $\vec{c}(r)$. Let e be the idempotent designated by the infinite product $(\lambda_\mathbf{H}(t_m)a_m \cdots \lambda_\mathbf{H}(t_n)a_n)_{n\geq m}$. Then

$$\lambda_\mathbf{H}(t) = \lambda_\mathbf{H}(t_1)a_1 \cdots \lambda_\mathbf{H}(t_{m-1})a_{m-1}\varphi_e(\delta(r)).$$

Example 5.2.1 Let t_{ex3} be the following element of $\mathbf{T_H}(A)$. (The labels in $\overline{\Omega}_A\mathbf{H}$ of the nodes are between square brackets. We omit the label when it is equal to 1.)



Then $\lambda_{\mathbf{H}}(t_{ex3}) = (x^{\omega+1}y)^\omega xyz(y^\omega z)^\omega y$.

Lemma 5.2.2 *The following diagram is commutative.*

$$\begin{tikzcd}
\mathbf{T_H}(A) \arrow[r, "\lambda_{\mathbf{H}}"] \arrow[d, "\tau"'] & \overline{\Omega}_A\mathbf{DRH} \arrow[d, "\pi"] \\
\mathbf{T}(A) \arrow[r, "\lambda"] & \overline{\Omega}_A\mathbf{R}
\end{tikzcd}$$

Proof. The proof that $\pi(\lambda_{\mathbf{H}}(t)) = \lambda(\tau(t))$ is by induction on the depth of t. In view of the definitions of λ and $\lambda_{\mathbf{H}}$, we only need to verify that $\pi(\varphi_e(\delta(r))) = \prod_{n\geq m}\lambda(t_n)a_n$ (with the above notations). Recall indeed that each infinite product converges in $\overline{\Omega}_A\mathbf{R}$. But $\varphi_e(\delta(r))$ is $\mathcal{R}$-equivalent to each accumulation point of the sequence $(\lambda_{\mathbf{H}}(t_m)a_m\cdots\lambda_{\mathbf{H}}(t_n)a_n)_{n\geq m}$, so $\pi(\varphi_e(\delta(r)))$ is $\mathcal{R}$-equivalent to $\prod_{n\geq m}\lambda(t_n)a_n$, and hence they are equal. $\square$

In particular, it follows that the content of $\lambda_{\mathbf{H}}(t)$ is that of $\lambda(\tau(t))$, that is, the content of the first edge descending from the root of t, or equivalently, the set of all labels of edges of t. We now prove the expected result.

Theorem 5.2.3 *The mapping* $\lambda_{\mathbf{H}}\colon \mathbf{T_H}(A) \to \overline{\Omega}_A\mathbf{DRH}$ *is a bijection.*

Proof. Let $x \in \overline{\Omega}_A\mathbf{DRH}$. The existence of a decorated tree $t \in \mathbf{T_H}(A)$ such that $\lambda_{\mathbf{H}}(t) = x$ is proved by induction on $|c(x)|$, using the discussion at the beginning of this section.

Let now $t = (\tau(t), \delta)$ and $t' = (\tau(t'), \delta')$ be distinct elements of $\mathbf{T_H}(A)$. If $\tau(t) \neq \tau(t')$, then by Theorem 3.2.3 and Lemma 5.2.2, we have $\lambda_{\mathbf{H}}(t) \neq \lambda_{\mathbf{H}}(t')$. Let us now assume that $\tau(t) = \tau(t')$, let $a_1, a_2, \ldots$ be the labels of the edges descending from the root of t (and t'), let $t_1, t_2, \ldots$ be the decorated subtrees of t attached under these edges, and let $t'_1, t'_2, \ldots$ be the decorated subtrees of t' attached under the same edges. In particular, $\tau(t_i) = \tau(t'_i)$ for each i. Suppose first that $t_i = t'_i$ for each i. Since $t \neq t'$, the root r of t and t' has infinite arity and $\delta(r) \neq \delta'(r)$. Let m be the least integer such that the content of the m-th edge descending from r is $\overleftarrow{c}(r)$. Then the $(m-1)$-st iterations of the left basic factorizations of $\lambda_{\mathbf{H}}(t)$ and $\lambda_{\mathbf{H}}(t')$ are

$$\begin{aligned}
\lambda_{\mathbf{H}}(t) &= \lambda_{\mathbf{H}}(t_1)a_1\cdots\lambda_{\mathbf{H}}(t_{m-1})a_{m-1}\varphi_e(\delta(r))\\
\lambda_{\mathbf{H}}(t') &= \lambda_{\mathbf{H}}(t_1)a_1\cdots\lambda_{\mathbf{H}}(t_{m-1})a_{m-1}\varphi_e(\delta'(r))
\end{aligned}$$

(since $t_i = t'_i$ for each i). Since φ_e is a bijection, we have $\varphi_e(\delta(r)) \neq \varphi_e(\delta'(r))$ and hence, by uniqueness of the left basic factorization (Proposition 2.3.1), it follows that $\lambda_{\mathbf{H}}(t) \neq \lambda_{\mathbf{H}}(t')$.

We must now consider the case where $t_i \neq t'_i$ for some i. Let k be minimal such that $t_k \neq t'_k$. Proceeding by induction on the depth of t, we have $\lambda_{\mathbf{H}}(t_k) \neq \lambda_{\mathbf{H}}(t'_k)$. We claim that $\lambda_{\mathbf{H}}(t_k)$ and $\lambda_{\mathbf{H}}(t'_k)$ appear in the k-fold iteration of the left basic factorizations of $\lambda_{\mathbf{H}}(t)$ and $\lambda_{\mathbf{H}}(t')$, thus proving that these factorizations are distinct. By Proposition 2.3.1, it follows that $\lambda_{\mathbf{H}}(t) \neq \lambda_{\mathbf{H}}(t')$. The claim is clear if r has finite arity. If r has infinite arity and m is defined as above, the claim is clear again if $k < m$. Finally, if $m \leq k$, then $\varphi_e(\delta(r))\ \mathcal{R}\ e \leq_{\mathcal{R}} \lambda_{\mathbf{H}}(t_m)a_m\cdots\lambda_{\mathbf{H}}(t_k)a_k$, so $\varphi_e(\delta(r)) = \lambda_{\mathbf{H}}(t_m)a_m\cdots\lambda_{\mathbf{H}}(t_k)a_k y$ for some $y \in \overleftarrow{c}(r)^*$. Moreover,

$$\lambda_{\mathbf{H}}(t) = \lambda_{\mathbf{H}}(t_1)a_1\cdots\lambda_{\mathbf{H}}(t_k)a_k y$$

is the k-fold iteration of the left basic factorizations of $\lambda_{\mathbf{H}}(t)$, thus establishing the claim, and completing the proof. □

5.3 Representation by decorated ordinals

For each ordinal α, let L_α be the set of limit ordinals $\beta \leq \alpha$. A *decorated reduced labeled ordinal* is a triple (α, ℓ, g) such that $(\alpha, \ell) \in \mathbf{rLO}(A)$ and g is a mapping $g\colon L_\alpha \to \overline{\Omega}_A\mathbf{H}$ verifying

$$g(\beta) \in \overline{\Omega}_{\overline{c}(\beta)}\mathbf{H} \quad \text{for each } \beta \in L_\alpha.$$

Let $\mathbf{rLO_H}(A)$ be the set of all decorated reduced labeled ordinals and let ρ be the projection from $\mathbf{rLO_H}(A)$ onto $\mathbf{rLO}(A)$ given by $\rho(\alpha, \ell, g) = (\alpha, \ell)$.

As in the aperiodic case, there exists a natural mapping $\mu_{\mathbf{H}}\colon \mathbf{T_H}(A) \to \mathbf{rLO_H}(A)$. Let $t = (\tau(t), \delta) \in \mathbf{T_H}(A)$ and let $(\alpha, \ell) = \mu(\tau(t))$. Let $\gamma \mapsto e_\gamma$ denote the order-preserving bijection from α onto $E(\tau(t))$. Recall that the limit ordinals $\beta < \alpha$ are exactly the ordinals such that the lower extremity of e_β has infinite arity. Similarly, α itself is a limit ordinal if and only if the root of t has infinite arity. So it makes sense to define $\mu_{\mathbf{H}}(t) = (\alpha, \ell, g)$ by letting, for each $\beta \in L_\alpha$

$$g(\beta) = \begin{cases} \delta(v) & \text{if } \beta < \alpha \text{ and } v \text{ is the lower extremity of } e_\beta, \\ \delta(r) & \text{is } \beta = \alpha \text{ and } r \text{ is the root of } t. \end{cases}$$

It is immediate, from the definition, that $\mu_{\mathbf{H}}(t) \in \mathbf{rLO_H}(A)$ and that the following diagram is commutative.

$$\begin{tikzcd} \mathbf{T_H}(A) \arrow[r, "\mu_{\mathbf{H}}"] \arrow[d, "\tau"'] & \mathbf{rLO_H}(A) \arrow[d, "\rho"] \\ \mathbf{T}(A) \arrow[r, "\mu"] & \mathbf{rLO}(A) \end{tikzcd}$$

Example 5.3.1 If t_{ex3} is the decorated tree in Example 5.2.1, then we have $\mu_{\mathbf{H}}(t_{ex3}) = (\omega^2 2, \ell, g)$ where ℓ is described in Example 4.1.2 and g is given by

$$g(\omega k) = x,\ g(\omega^2) = xy,\ g(\omega^2 + \omega k) = 1 \text{ and } g(\omega^2 2) = zy \text{ for all } k \geq 1.$$

For each $(\alpha, \ell, g) \in \mathbf{rLO_H}(A)$, we also define $\nu_{\mathbf{H}}(\alpha, \ell, g) = (\nu(\alpha, \ell), \delta)$ by letting

$$\delta(v) = \begin{cases} g(\beta) & \text{if } v \text{ is the lower extremity of } e_\beta \text{ with } \beta \text{ a limit ordinal,} \\ g(\alpha) & \text{if } v \text{ is the root of } t \text{ and } v \text{ has infinite arity (i.e. } \alpha \text{ is a limit).} \end{cases}$$

Proposition 5.3.2 *The mappings*

$$\mu_{\mathbf{H}}\colon \mathbf{T_H}(A) \to \mathbf{rLO_H}(A) \quad \textit{and} \quad \nu_{\mathbf{H}}\colon \mathbf{rLO_H}(A) \to \mathbf{T_H}(A)$$

are mutually reciprocal bijections.

Proof. This is immediate since μ and ν are mutually reciprocal bijections (Theorem 3.3.5). □

6 Algebra and topology on $\mathbf{T_H}(A)$ and $\mathbf{rLO_H}(A)$

As in Section 4, we define a product for decorated labeled ordinals, and we transport it to trees so that $\lambda_{\mathbf{H}}$, $\mu_{\mathbf{H}}$ and $\nu_{\mathbf{H}}$ are isomorphisms. We also transport the topology of $\overline{\Omega}_A\mathbf{DRH}$ onto $\mathbf{T_H}(A)$ via $\lambda_{\mathbf{H}}^{-1}$, so $\lambda_{\mathbf{H}}$ is a homeomorphism.

6.1 Product of decorated reduced labeled ordinals

If $(\alpha, \ell, g), (\beta, m, h) \in \mathbf{rLO_H}(A)$, we define their product by letting

$$(\alpha, \ell, g)(\beta, m, h) = (\alpha + \beta_2, \ell', g'), \quad \text{where}$$

- The partition $\beta = \beta_1 + \beta_2$ satisfies the following property: if α is not a limit ordinal, then $\beta_1 = 0$. If α is a limit ordinal and $c(\beta, m) \subseteq \overrightarrow{c}(\alpha)$, then $\beta_1 = \beta$. Otherwise, β_1 is the least ordinal such that $m(\beta_1) \notin \overrightarrow{c}(\alpha)$.
- $(\alpha, \ell)(\beta, m) = (\alpha + \beta_2, \ell')$.
- $g'(\gamma) = g(\gamma)$ if $\gamma \in L_\alpha$. If $\gamma \in L_{\beta_2}$, then $g'(\alpha + \gamma) = h(\beta_1 + \gamma)$. Finally, if α is a limit ordinal, then $g'(\alpha) = g(\alpha)\sigma(\lambda_{\mathbf{H}}(\nu_{\mathbf{H}}(\beta_1, m_{|\beta_1}, h_{|L_{\beta_1}})))$.

We next transfer this product on $\mathbf{T_H}(A)$ by letting

$$t\ t' = \nu_{\mathbf{H}}(\mu_{\mathbf{H}}(t)\mu_{\mathbf{H}}(t'))$$

for all $t, t' \in \mathbf{T_H}(A)$. It is immediate from the definition that $\mu_{\mathbf{H}}$ and $\nu_{\mathbf{H}}$ are isomorphisms between $\mathbf{rLO_H}(A)$ and $\mathbf{T_H}(A)$, and that the projection $\rho\colon \mathbf{rLO_H}(A) \to \mathbf{rLO}(A)$ preserves the product of decorated labeled ordinals, that is, ρ is a morphism. Therefore, $\tau\colon \mathbf{T_H}(A) \to \mathbf{T}(A)$ is a morphism as well. We now prove that the bijection $\lambda_{\mathbf{H}}\colon \mathbf{T_H}(A) \to \overline{\Omega}_A\mathbf{DRH}$ is an isomorphism. The special case where $\mathbf{H}$ is the trivial pseudovariety of groups corresponds to Theorem 4.1.1 above.

Theorem 6.1.1 *The bijection $\lambda_{\mathbf{H}}\colon \mathbf{T_H}(A) \to \overline{\Omega}_A\mathbf{DRH}$ is an isomorphism.*

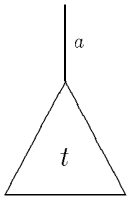
Before we prove this result, let us summarize in a diagram the different morphisms which will play a role in the proof (with e an idempotent of $\overline{\Omega}_A\mathbf{DRH}$).

$$\begin{tikzcd}
\mathbf{rLO_H}(A) \arrow[d, "\rho"'] \arrow[r, shift right, "\nu_{\mathbf{H}}"'] & \mathbf{T_H}(A) \arrow[l, shift right, "\mu_{\mathbf{H}}"'] \arrow[l, phantom, "\simeq"] \arrow[r, "\lambda_{\mathbf{H}}"] \arrow[d, "\tau"'] & \overline{\Omega}_A\mathbf{DRH} \arrow[r, "\sigma"] \arrow[d, "\pi"'] & \overline{\Omega}_A\mathbf{H} & \\
\mathbf{rLO}(A) \arrow[r, shift right, "\nu"'] & \mathbf{T}(A) \arrow[l, shift right, "\mu"'] \arrow[l, phantom, "\simeq"] \arrow[r, "\lambda"] & \overline{\Omega}_A\mathbf{R} & H_e \arrow[ul, hook] & \overline{\Omega}_{c(e)}\mathbf{H} \arrow[l, "\varphi_e", "\simeq"'] \arrow[ul, hook]
\end{tikzcd}$$

The proof of Theorem 6.1.1 relies on three lemmas, each of which proves the multiplicativity of $\lambda_{\mathbf{H}}$ in a particular case.

Lemma 6.1.2 *Let $t \in \mathbf{T_H}(A)$ and let a be a letter not in the content of t. Then $\lambda_{\mathbf{H}}(ta) = \lambda_{\mathbf{H}}(t)a$.*

Proof. Let u be the decorated tree with exactly one edge, say e, descending from the root, such that e is labeled a and the decorated subtree attached below e is t.

$u =$ 

By definition of $\lambda_{\mathbf{H}}$, we have $\lambda_{\mathbf{H}}(u) = \lambda_{\mathbf{H}}(t)a$. If $(\alpha, \ell, g) = \mu_{\mathbf{H}}(t)$, then, by definition of $\mu_{\mathbf{H}}$, we have $\mu_{\mathbf{H}}(u) = (\alpha + 1, \ell', g)$, with

$$\ell'(\gamma) = \begin{cases} \ell(\gamma) & \text{if } \gamma < \alpha, \\ a & \text{if } \gamma = \alpha. \end{cases}$$

Then $\mu_{\mathbf{H}}(u) = (\alpha, \ell, g)\mu_{\mathbf{H}}(a) = \mu_{\mathbf{H}}(t)\mu_{\mathbf{H}}(a)$, and hence $u = ta$, concluding the proof. $\square$

Lemma 6.1.3 *Let $t, t' \in \mathbf{T}_{\mathbf{H}}(A)$. If the root r of t has infinitely many children and if the content of t' is contained in the cumulative content of r, $c(t') \subseteq \overline{c}(r)$, then $\lambda_{\mathbf{H}}(tt') = \lambda_{\mathbf{H}}(t)\lambda_{\mathbf{H}}(t')$.*

Proof. Let $(\alpha, \ell, g) = \mu_{\mathbf{H}}(t)$ and $(\alpha', \ell', g') = \mu_{\mathbf{H}}(t')$. Then α is a limit ordinal and $c(\alpha', \ell') \subseteq \overline{c}(\alpha)$, so that $(\alpha, \ell, g)(\alpha', \ell', g') = (\alpha, \ell, h)$, with $h(\gamma) = g(\gamma)$ if $\gamma < \alpha$ and $h(\alpha) = g(\alpha)\sigma(\lambda_{\mathbf{H}}(t'))$.

Now $tt' = \nu_{\mathbf{H}}(\alpha, \ell, h)$. That is, tt' is obtained from t by substituting $g(\alpha)\sigma(\lambda_{\mathbf{H}}(t'))$ for $g(\alpha)$ as the decoration of the root. Let $e_1, e_2, \ldots$ be the edges of t descending from the root r, let $a_1, a_2, \ldots$ be their labels, and let $t_1, t_2, \ldots$ be the decorated subtrees of t (and of tt') attached below these edges. Let also m be the least integer such that the content of e_m is equal to $\overline{c}(r)$. By definition of $\lambda_{\mathbf{H}}$, if e is the idempotent designated by the infinite product $(\lambda_{\mathbf{H}}(t_m)a_m \cdots \lambda_{\mathbf{H}}(t_n)a_n)_{n \geq m}$, then we have

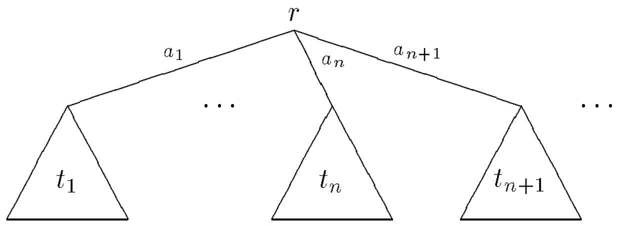
$$\begin{aligned} \lambda_{\mathbf{H}}(t) &= \lambda_{\mathbf{H}}(t_1)a_1 \cdots \lambda_{\mathbf{H}}(t_{m-1})a_{m-1}\varphi_e(g(\alpha)) \\ \lambda_{\mathbf{H}}(tt') &= \lambda_{\mathbf{H}}(t_1)a_1 \cdots \lambda_{\mathbf{H}}(t_{m-1})a_{m-1}\varphi_e(h(\alpha)). \end{aligned}$$

Therefore $\lambda_{\mathbf{H}}(tt') = \lambda_{\mathbf{H}}(t)\varphi_e(\sigma(\lambda_{\mathbf{H}}(t')))$.

By definition of σ, $\sigma(\lambda_{\mathbf{H}}(t')) = \sigma(e\lambda_{\mathbf{H}}(t'))$, and since $c(\lambda_{\mathbf{H}}(t')) \subseteq c(e) = \overline{c}(r)$, $e\lambda_{\mathbf{H}}(t')$ lies in H_e, the $\mathcal{H}$-class of e. By Proposition 5.1.2, it follows that $\varphi_e(\sigma(\lambda_{\mathbf{H}}(t'))) = e\lambda_{\mathbf{H}}(t')$, and hence $\lambda_{\mathbf{H}}(tt') = \lambda_{\mathbf{H}}(t)e\lambda_{\mathbf{H}}(t')$. But $\varphi_e(g(\alpha)) = \varphi_e(g(\alpha))e$, so $\lambda_{\mathbf{H}}(t) = \lambda_{\mathbf{H}}(t)e$, and the lemma is proved. $\square$

The third lemma deals with the following situation. Let $u \in \mathbf{T}_H(A)$ and let n be an integer such that the root of u has at least $n+1$ children. Let $e_1, e_2, \ldots$ be the edges of u descending from the root of u, let $a_1, a_2, \ldots$ be their labels, and let $u_1, u_2, \ldots$ be the decorated subtrees of u attached below these edges. Let t be the decorated tree whose root has n chilren, with corresponding edge labels $a_1, \ldots, a_n$ and corresponding attached decorated subtrees $t_1, \ldots, t_n$. Finally, let t' be the decorated tree with edges descending from the root $e_{n+1}, \ldots$, corresponding edge labels $a_{n+1}, \ldots$ and corresponding attached decorated subtrees $u_{n+1}, \ldots$ In order to completely specify t', we need to give the label of its root in case it has infinite arity. In that case, the root of u also has infinite arity. Let $g \in \overline{\Omega}_A\mathbf{H}$ be its label, and let m be the least integer such that the content of e_m is equal to $\overline{c}(r)$.

The decoration of the root of t' is then defined to be g if $m > n$, and $\sigma(\lambda_{\mathbf{H}}(t_m)a_m \cdots \lambda_{\mathbf{H}}(t_n)a_n)^{-1}g$ if $m \leq n$.



In this situation, we have:

Lemma 6.1.4 *$u = tt'$ and $\lambda_{\mathbf{H}}(u) = \lambda_{\mathbf{H}}(t)\lambda_{\mathbf{H}}(t')$.*

Proof. Let $\mu_{\mathbf{H}}(u) = (\alpha, \ell, g)$. Then, by Lemma 3.3.4, $\alpha = \sum_{k\geq 1} \alpha_k$ where the α_k are successor ordinals and

- $t = \nu_{\mathbf{H}}(\alpha_1 + \cdots + \alpha_n, \ell_1, g_1)$, where ℓ_1 is the restriction of ℓ to the initial segment $\alpha_1 + \cdots + \alpha_n$ of α and g_1 is the restriction of g to $L_{\alpha_1+\cdots+\alpha_n}$,
- $t' = \nu_{\mathbf{H}}(\sum_{k>n} \alpha_k, \ell_2, g_2)$, where $\ell_2(\gamma) = \ell(\alpha_1 + \cdots + \alpha_n + \gamma)$, and $g_2(\gamma) = g(\alpha_1 + \cdots + \alpha_n + \gamma)$ if $\gamma \in L_{\alpha_1+\cdots+\alpha_n}$ and $\gamma < \sum_{k>n} \alpha_k$. In addition, if α (and hence $\sum_{k>n} \alpha_k$) is a limit ordinal, then
$$g_2(\sum_{k>n} \alpha_k) = \begin{cases} g(\alpha) & \text{if } m > n, \\ \sigma(\lambda_{\mathbf{H}}(t_m)a_m \cdots \lambda_{\mathbf{H}}(t_n)a_n)^{-1}g(\alpha) & \text{if } m \leq n. \end{cases}$$

Since each α_k is a successor ordinal, $\sum_{k>n} \alpha_k$ is not a limit, and it is immediately verified that $tt' = u$.

We now verify that $\lambda_{\mathbf{H}}(u) = \lambda_{\mathbf{H}}(t)\lambda_{\mathbf{H}}(t')$. We first consider the case where the root r of u has finitely many, say q, children. Then, by definition of $\lambda_{\mathbf{H}}$, we have

$$\lambda_{\mathbf{H}}(u) = \lambda_{\mathbf{H}}(t_1)a_1 \cdots \lambda_{\mathbf{H}}(t_n)a_n \cdot \lambda_{\mathbf{H}}(t_{n+1})a_{n+1} \cdots \lambda_{\mathbf{H}}(t_q)a_q = \lambda_{\mathbf{H}}(t)\,\lambda_{\mathbf{H}}(t').$$

If r has infinite arity, then

$$\lambda_{\mathbf{H}}(u) = \lambda_{\mathbf{H}}(t_1)a_1 \cdots \lambda_{\mathbf{H}}(t_{m-1})a_{m-1}\varphi_e(g(\alpha))$$

where e is the idempotent designated by $(\lambda_{\mathbf{H}}(t_m)a_m \cdots \lambda_{\mathbf{H}}(t_k)a_k)_{k\geq m}$. If $m > n$, this gives immediately $\lambda_{\mathbf{H}}(u) = \lambda_{\mathbf{H}}(t)\lambda_{\mathbf{H}}(t')$.

If $m \leq n$, then $\lambda_{\mathbf{H}}(t') = \varphi_{e'}(g_2(\sum_{k>n} \alpha_k))$, where e' is the idempotent designated by the infinite product $(\lambda_{\mathbf{H}}(t_{n+1})a_{n+1} \cdots \lambda_{\mathbf{H}}(t_k)a_k)_{k\geq n+1}$. Note that the $\lambda_{\mathbf{H}}(t_k)a_k$ ($k \geq m$) have the same content as e and e', so $e\ \mathcal{L}\ e'$ and the left translate $\lambda_{\mathbf{H}}(t_m)a_m \cdots \lambda_{\mathbf{H}}(t_n)a_n e'$ is $\mathcal{H}$-equivalent to the idempotent designated by the infinite product $(\lambda_{\mathbf{H}}(t_m)a_m \cdots \lambda_{\mathbf{H}}(t_k)a_k)_{k\geq m}$, namely e. Therefore

$$\begin{aligned} \lambda_{\mathbf{H}}(t_m)a_m \cdots \lambda_{\mathbf{H}}(t_n)a_n e' &= e\lambda_{\mathbf{H}}(t_m)a_m \cdots \lambda_{\mathbf{H}}(t_n) \\ &= \varphi_e(\sigma(\lambda_{\mathbf{H}}(t_m)a_m \cdots \lambda_{\mathbf{H}}(t_n))). \end{aligned}$$

Moreover, if $x \in \overline{\Omega}_A\mathbf{H}$, then $\sigma(e'\varphi_e(x)) = x = \sigma(\varphi_{e'}(x))$ and $e'\varphi_e(x)\ \mathcal{H}\ e'$, so $e'\varphi_e(x) = \varphi_{e'}(x)$ by Proposition 5.1.2. It follows that

$$
\begin{aligned}
\lambda_{\mathbf{H}}(u) &= \lambda_{\mathbf{H}}(t_1)a_1 \cdots \lambda_{\mathbf{H}}(t_{m-1})a_{m-1}\varphi_e(g(\alpha)) \\
&= \lambda_{\mathbf{H}}(t_1)a_1 \cdots \lambda_{\mathbf{H}}(t_{m-1})a_{m-1} \cdot e\lambda_{\mathbf{H}}(t_m)a_m \cdots \lambda_{\mathbf{H}}(t_n)a_n \\
&\qquad\qquad \varphi_e(\sigma(\lambda_{\mathbf{H}}(t_m)a_m \cdots \lambda_{\mathbf{H}}(t_n)a_n)^{-1}g(\alpha)) \\
&= \lambda_{\mathbf{H}}(t_1)a_1 \cdots \lambda_{\mathbf{H}}(t_{m-1})a_{m-1}\lambda_{\mathbf{H}}(t_m)a_m \cdots \lambda_{\mathbf{H}}(t_n)a_n\, e' \\
&\qquad\qquad \varphi_e(\sigma(\lambda_{\mathbf{H}}(t_m)a_m \cdots \lambda_{\mathbf{H}}(t_n)a_n)^{-1}g(\alpha)) \\
&= \lambda_{\mathbf{H}}(t)e'\varphi_e(g_2(\sum_{k>n}\alpha_k)) \\
&= \lambda_{\mathbf{H}}(t)\varphi_{e'}(g_2(\sum_{k>n}\alpha_k)) \\
&= \lambda_{\mathbf{H}}(t)\lambda_{\mathbf{H}}(t').
\end{aligned}
$$

This concludes the proof of the lemma. □

We are now ready to give the proof of Theorem 6.1.1.

Proof of Theorem 6.1.1. Let $t, t' \in \mathbf{T_H}(A)$ and let $\mu_{\mathbf{H}}(t) = (\alpha, \ell, g)$ and $\mu_{\mathbf{H}}(t') = (\beta, m, h)$. Then $tt' = (\alpha + \beta_2, \ell', g')$ with $\beta = \beta_1 + \beta_2$ and

$$
\begin{aligned}
\beta_1 &= \begin{cases} 0 & \text{if } \alpha \text{ is not a limit} \\ \beta & \text{if } \alpha \text{ is a limit and } c(\beta, m) \subseteq \overline{c}(\alpha) \\ \min\{\gamma \mid m(\gamma) \notin \overline{c}(\alpha)\} & \text{otherwise} \end{cases} \\
\ell'(\gamma) &= \begin{cases} \ell(\gamma) & \text{if } \gamma < \alpha \\ m(\beta_1 + \gamma') & \text{if } \gamma = \alpha + \gamma' \end{cases} \\
g'(\gamma) &= \begin{cases} g(\gamma) & \text{if } \gamma \in L_\alpha \setminus \{\alpha\} \\ h(\beta_1 + \gamma') & \text{if } \gamma = \alpha + \gamma' \text{ with } \gamma' \in L_{\beta_2} \\ g(\alpha)\sigma(\lambda_{\mathbf{H}}(\nu_{\mathbf{H}}(\beta_1, m_{|\beta_1}, h_{|L_{\beta_1}}))) & \text{if } \gamma = \alpha \text{ and } \alpha \text{ is a limit.} \end{cases}
\end{aligned}
$$

Let $\gamma \mapsto e_\gamma$ denote the bijection from $\alpha + \beta_2$ onto the set of edges of tt' and consider the edges $e_{\gamma_1}, e_{\gamma_2}, \ldots$ descending from the root of tt'. We distinguish several cases depending on the position of the ordinals $\gamma_1, \gamma_2, \ldots$ with respect to α.

Suppose first that $\gamma_i < \alpha$ for all i. Since each edge of tt' lies below one of the e_{γ_i}, and hence each element of α is bounded above by one of the γ_i, it follows that $\beta_2 = 0$, that is, α is a limit and $c(t')$ is contained in the cumulative content of the root of t. Then, by Lemma 6.1.3, we have $\lambda_{\mathbf{H}}(tt') = \lambda_{\mathbf{H}}(t)\lambda_{\mathbf{H}}(t')$.

We may now assume that $\gamma_j \geq \alpha$ for some j, and we let i be the least such index. If $i > 1$, then writing $\alpha = \gamma_{i-1} + \widetilde{\alpha}$, we get $t = t_1 t_2$ where

$$
\begin{aligned}
t_1 &= \nu_{\mathbf{H}}(\gamma_{i-1}, \ell_{|\gamma_{i-1}}, g_{|L_{\gamma_{i-1}}}) \\
t_2 &= \nu_{\mathbf{H}}(\widetilde{\alpha}, \widetilde{\ell}, \widetilde{g}) \\
\widetilde{\ell}(\gamma) &= \ell(\gamma_{i-1} + \gamma) \quad \text{if } \gamma < \widetilde{\alpha} \\
\widetilde{g}(\gamma) &= g(\gamma_{i-1} + \gamma) \quad \text{if } \gamma \in L_{\widetilde{\alpha}}.
\end{aligned}
$$

Then, by Lemma 6.1.4, we have $\lambda_{\mathbf{H}}(tt') = \lambda_{\mathbf{H}}(t_1(t_2t')) = \lambda_{\mathbf{H}}(t_1)\lambda_{\mathbf{H}}(t_2t')$. This reduces the proof to the case $i = 1$, that is, $\gamma_1 \geq \alpha$.

In that last case, all the letters of $c(t)$ appear in the decorated subtree attached below e_{γ_1}, and $a_1 \notin c(t)$. Thus we have $t' = t'_1a_1t'_2$ with $a_1 \notin c(t'_1) \cup c(t)$ and so

$$\begin{aligned}
\lambda_{\mathbf{H}}(tt') &= \lambda_{\mathbf{H}}(tt'_1a_1t'_2) \\
&= \lambda_{\mathbf{H}}(tt'_1a_1)\lambda_{\mathbf{H}}(t'_2) \quad \text{by Lemma 6.1.4} \\
&= \lambda_{\mathbf{H}}(tt'_1)a_1\lambda_{\mathbf{H}}(t'_2) \quad \text{by Lemma 6.1.2} \\
&= \lambda_{\mathbf{H}}(t)\lambda_{\mathbf{H}}(t'_1)a_1\lambda_{\mathbf{H}}(t'_2) \quad \text{by induction on the content} \\
&= \lambda_{\mathbf{H}}(t)\lambda_{\mathbf{H}}(t'_1a_1t'_2) \quad \text{by the same reasoning as above} \\
&= \lambda_{\mathbf{H}}(t)\lambda_{\mathbf{H}}(t').
\end{aligned}$$

This concludes the proof of the theorem. □

For instance, we can derive from Theorem 6.1.1 a characterization of the regular elements and of the idempotents of $\overline{\Omega}_A\mathbf{DRH}$.

Corollary 6.1.5 *Let $x \in \overline{\Omega}_A\mathbf{DRH}$ and let $(\alpha, \ell, g) = \nu_{\mathbf{H}}(\lambda_{\mathbf{H}}^{-1}(x))$ be the corresponding decorated reduced labeled ordinal. Then x is regular (and hence a group element) if and only if α is a limit ordinal and $c(\alpha, \ell) = \overline{c}(\alpha)$. Moreover, x is idempotent if and only if x is regular and $g(\alpha) = 1$.*

6.2 The topology of $\mathbf{T}_{\mathbf{H}}(A)$

As in the aperiodic case, we transport the topology of $\overline{\Omega}_A\mathbf{DRH}$ onto $\mathbf{T}_{\mathbf{H}}(A)$ via $\lambda_{\mathbf{H}}^{-1}$. In particular, the product in $\mathbf{T}_{\mathbf{H}}(A)$ is continuous, and $\lambda_{\mathbf{H}}$ is a homeomorphism.

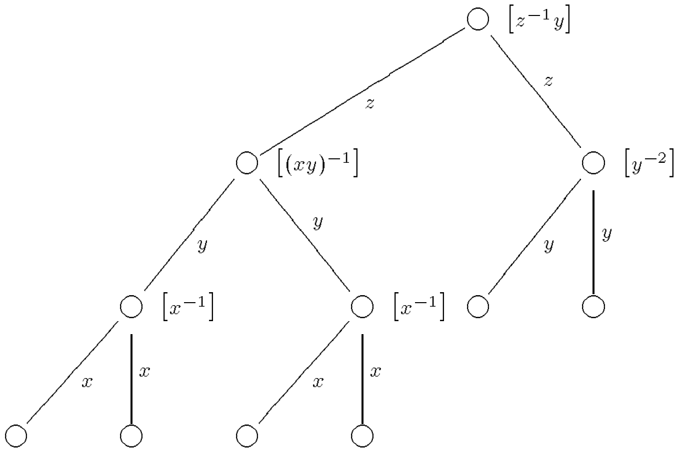
In order to characterize the neighborhoods of an element $t \in \mathbf{T}_{\mathbf{H}}(A)$, we again define restrictions $\mathrm{restr}(t, k)$ $(k \geq 1)$ which are finite decorated trees. However, in the following definition, $\mathrm{restr}(t, k)$ is not an element of $\mathbf{T}_{\mathbf{H}}(A)$ because all of its nodes (none of which has infinite arity) receive a decoration in $\overline{\Omega}_A\mathbf{H}$.

More precisely, let $t = (\tau(t), \delta) \in \mathbf{T}_{\mathbf{H}}(A)$ and let $k \geq 1$. We let $\mathrm{restr}(t, k)$ be a decoration of the finite tree $\mathrm{restr}(\tau(t), k)$, that is, we let $\mathrm{restr}(t, k) = (\underline{t}, \delta_k)$ where $\underline{t} = \mathrm{restr}(\tau(t), k)$ and where δ_k is defined as follows. If v is a node of $\mathrm{restr}(\tau(t), k)$ and v has at most $k - 1$ children, then $\delta_k(v) = 1$. In particular, the leaves of $\mathrm{restr}(\tau(t), k)$ receive label 1. If v has k children, let t_v be the decorated subtree of t attached under v, let $a_1, \ldots, a_k$ be the labels of the k first edges descending from v in t, and let $t_1, \ldots, t_k$ be the corresponding decorated subtrees of t. Then

$$\delta_k(v) = (\sigma(\lambda_{\mathbf{H}}(t_1)a_1 \cdots \lambda_{\mathbf{H}}(t_k)a_k))^{-1}\,\sigma(\lambda_{\mathbf{H}}(t_v)).$$

If v has exactly k children in t, then $\delta_k(v) = 1$. Otherwise, let B be the content of the $(k+1)$-st edge descending from v in t. Then it is not difficult to verify that $\delta_k(v) \in \overline{\Omega}_B\mathbf{H}$.

Example 6.2.1 Let t_{ex3} be the decorated tree of Example 5.2.1. Then $\mathrm{restr}(t_{ex3}, 2)$ is the following tree.



We now want to exhibit a simple basis of neighborhoods for the elements of $\mathbf{T_H}(A)$. Let us first consider a family $(L_q)_{q\geq 1}$ of $\mathbf{H}$-recognizable languages of A^* such that

- the topological closures L'_q of the L_q in $\overline{\Omega}_A\mathbf{H}$ ($q \geq 1$) form a basis of clopen neighborhoods of 1;
- for each q, L'_q is a normal subgroup of $\overline{\Omega}_A\mathbf{H}$.

Such a family certainly exists: it suffices to consider all the onto morphisms $\varphi_q: A^* \to G_q$ onto groups of $\mathbf{H}$, and to let $L_q = \varphi_q^{-1}(1)$.

For each $x \in \overline{\Omega}_A\mathbf{H}$, the xL'_q ($q \geq 1$) form a basis of clopen neighborhoods of x. We denote by $L_q(x)$ the $\mathbf{H}$-recognizable language $xL'_q \cap A^*$. In particular, the closure of $L_q(x)$ in $\overline{\Omega}_A\mathbf{H}$ is xL'_q.

Lemma 6.2.2 *For each $q \geq 1$, the xL'_q ($x \in \overline{\Omega}_A\mathbf{H}$) form a finite clopen partition of $\overline{\Omega}_A\mathbf{H}$, and the $\overline{L_q(x)}$ form a finite clopen partition of $\overline{\Omega}_A\mathbf{DRH}$. Moreover, if $x, y \in \overline{\Omega}_A\mathbf{H}$ and $a \in A$, then $L_q(x)L_q(y) \subseteq L_q(xy)$ and $L_q(x)aL_q(y) \subseteq L_q(xay)$.*

Proof. By definition, the left cosets xL'_q form a clopen partition of $\overline{\Omega}_A\mathbf{H}$. By compactness, this partition must be finite. Then, by Lemma 1.3.3, $\overline{L_q(x)} = \sigma^{-1}(xL'_q)$, so the $\overline{L_q(x)}$ form a finite clopen partition of $\overline{\Omega}_A\mathbf{DRH}$. Now, we have $L_q(x)aL_q(y) = (xL'_q \cap A^*)a(yL'_q \cap A^*) \subseteq (xL'_q ayL'_q) \cap A^*$. But L'_q is a normal subgroup of $\overline{\Omega}_A\mathbf{H}$, so $L_q(x)aL_q(y) \subseteq (xay)L'_q \cap A^* = L_q(xay)$. The statement concerning $L_q(x)L_q(y)$ is proved in the same fashion. □

Let $t \in \mathbf{T_H}(A)$, let $k \geq 1$, and let $\mathrm{restr}(t,k) = (\mathrm{restr}(\tau(t),k), \delta_k)$. For each $q \geq 1$, we define $N(t,k,q)$ to be the set of all $t' \in \mathbf{T_H}(A)$ such that $\mathrm{restr}(t',k) = (\mathrm{restr}(\tau(t'),k), \delta'_k)$ satisfies $\mathrm{restr}(\tau(t'),k) = \mathrm{restr}(\tau(t),k)$ and $\delta'_k(v) \in \delta_k(v)L'_q$ for each node v of $\mathrm{restr}(\tau(t),k)$.

Lemma 6.2.3 *Let $t, t' \in \mathbf{T_H}(A)$ and let $k, q \geq 1$. If $N(t,k,q) \cap N(t',k,q) \neq \emptyset$, then $N(t,k,q) = N(t',k,q)$.*

Proof. Let $x \in N(t,k,q) \cap N(t',k,q)$. Then $\tau(x) \in N(\tau(t),k) \cap N(\tau(t'),k)$. We already know that this implies $N(\tau(t),k) = N(\tau(t'),k)$, that is,

$$\mathrm{restr}(\tau(t),k) = \mathrm{restr}(\tau(t'),k) = \mathrm{restr}(\tau(x),k).$$

Let δ_k, δ'_k and ε_k be the decorating functions of $\mathrm{restr}(t,k)$, $\mathrm{restr}(t',k)$ and $\mathrm{restr}(x,k)$: all of them are defined on the same domain, namely the set of nodes of $\mathrm{restr}(\tau(t),k)$. For each node v in this set, we have $\varepsilon_k(v) \in \delta_k(v)L'_q \cap \delta'_k(v)L'_q$. But the cosets of L'_q form a partition of $\overline{\Omega}_A\mathbf{H}$, so it follows $\delta_k(v)L'_q = \delta'_k(v)L'_q$ for each v, and hence $N(t,k,q) = N(t',k,q)$. ☐

We also define a family of languages $L(t,k,q)$ by induction on the depth of t. If t has depth 0, then $L(t,k,q) = \{1\}$. If t has non-zero depth, let r be its root, let $a_1, a_2, \ldots$ be the labels of the edges descending from r and let $t_1, t_2, \ldots$ be the corresponding decorated subtrees of t. If r has $n < k$ children, then we let

$$L(t,k,q) = L(t_1,k,q)a_1 \cdots L(t_n,k,q)a_n.$$

If r has at least k children and if B is the content of the k-th edge descending from r in t, then we let

$$L(t,k,q) = L(t_1,k,q)a_1 \cdots L(t_k,k,q)a_k(L_q(\delta_k(r)) \cap B^*).$$

Lemma 6.2.4 *Let* $t \in \mathbf{T}_{\mathbf{H}}(A)$*, let* $k \geq 1$ *and let* $\mathrm{restr}(t,k) = (\mathrm{restr}(\tau(t)), \delta_k)$.

1. $L(t,k,q) \subseteq L(\tau(t),k)$.
2. $L(t,k,q)$ *is* $\mathbf{DRH}$*-elementary.*
3. $L(t,k,q) \subseteq L_q(\sigma(\lambda_{\mathbf{H}}(t)))$.
4. *Let us now assume that* t *has non-zero depth and let* $u \in \overline{\Omega}_A\mathbf{DRH}$*. With reference to the above notations, if the root* r *of* t *has* $n < k$ *children, then* $u \in \overline{L(t,k,q)}$ *if and only if the left basic factorization of* u *can be iterated exactly* n *times, yielding a factorization of* u *of the form*
$$u = u_1a_1 \cdots u_na_n$$
with each u_i *in* $\overline{L(t_i,k,q)}$*. If* r *has at least* k *children, then* $u \in \overline{L(t,k,q)}$ *if and only if the left basic factorization of* u *can be iterated at least* k *times, yielding a factorization of* u *of the form*
$$u = u_1a_1 \cdots u_ka_kv$$
with each u_i *in* $\overline{L(t_i,k,q)}$ *and with* $v \in \overline{L_q(\delta_k(r)) \cap B^*}$ *and* $c(v) = B$.

Proof. Statement (1) is immediate by induction on the depth of t. Statement (2) follows from Lemma 2.2.3 and Statement (3) follows from Lemma 6.2.2. Finally, as in the proof of Lemma 4.2.3, Statement (4) is easily verified (using (1)) if u is a word, and it is extended to the general case by compactness. ☐

We also have the following analogue of Lemma 4.2.3 (4).

Lemma 6.2.5 *Let $t \in \mathbf{T_H}(A)$ and let $k \geq 1$. Then there exists $u \in \mathbf{T_H}(A)$ such that, for each $q \geq 1$, we have $\lambda_{\mathbf{H}}(N(t,k,q)) = \overline{L(u,k,q)}$ and $L(t,k,q) \subseteq L(u,k,q)$.*

Proof. Let δ_k be the decoration function of $\mathrm{restr}(t,k)$, that is, $\mathrm{restr}(t,k) = (\mathrm{restr}(\tau(t),k),\delta_k)$.

The decorated tree $u = (\tau(u),\varepsilon)$ is defined as follows. The tree $\tau(u)$ is obtained from $\mathrm{restr}(\tau(t),k)$ by adding to each node v with k children, an infinite number of new children, each having the same content as the k-th child of v, and each carrying a finite subtree representing a linear word with that content. In particular, $\mathrm{restr}(\tau(u),k) = \mathrm{restr}(\tau(t),k)$. For each node v of $\mathrm{restr}(\tau(t),k)$, we let t_v and u_v be the decorated subtrees, respectively of t and u, attached under v.

Let us now define the decoration function ε of u. The only nodes of u with an infinite number of children are those that originated from nodes of $\mathrm{restr}(\tau(t),k)$ with k children. For such a node v, let $v_1,\ldots,v_k$ be the k children of v in $\mathrm{restr}(\tau(t),k)$, and let $a_1,\ldots,a_k$ be the labels of the corresponding edges. Let m be the least index such that the content of the m-th child of v is equal to the content of its k-th child. Then we let

$$\varepsilon(v) = (\sigma(\lambda_{\mathbf{H}}(u_{v_1})a_1 \cdots \lambda_{\mathbf{H}}(u_{v_{m-1}})a_{m-1}))^{-1}\sigma(\lambda_{\mathbf{H}}(t_v)).$$

This defines ε by bottom-up induction on the set of nodes of infinite arity of u. It follows from the definition of $\lambda_{\mathbf{H}}$ that $\sigma(\lambda_{\mathbf{H}}(u_v)) = \sigma(\lambda_{\mathbf{H}}(t_v))$ for each node v of $\mathrm{restr}(\tau(t),k)$. Observe also that the construction of u from t is inductive: if v is a node of $\mathrm{restr}(\tau(t),k)$, then u_v is the decorated tree constructed from t_v.

Let ε_k be the decorating function of $\mathrm{restr}(u,k)$ (defined on the nodes of $\mathrm{restr}(\tau(u),k) = \mathrm{restr}(\tau(t),k)$). We claim that $\varepsilon_k(v) = \delta_k(v)$ for each node v. Indeed, if v has less than k children, then $\delta_k(v) = \varepsilon_k(v) = 1$. If v has k children, with $v_1,\ldots,v_k$ and $a_1,\ldots,a_k$ as above, we have by definition

$$\begin{aligned}
\delta_k(v) &= (\sigma(\lambda_{\mathbf{H}}(t_{v_1})a_1 \cdots \lambda_{\mathbf{H}}(t_{v_k})a_k))^{-1}\sigma(\lambda_{\mathbf{H}}(t_v)) \\
\varepsilon_k(v) &= (\sigma(\lambda_{\mathbf{H}}(u_{v_1})a_1 \cdots \lambda_{\mathbf{H}}(u_{v_k})a_k))^{-1}\sigma(\lambda_{\mathbf{H}}(u_v)).
\end{aligned}$$

But we already observed that $\sigma(\lambda_{\mathbf{H}}(t_v)) = \sigma(\lambda_{\mathbf{H}}(u_v))$ for each v, so $\varepsilon_k(v) = \delta_k(v)$.

It is now easy to verify that $\lambda_{\mathbf{H}}(N(t,k,q)) = \overline{L(u,k,q)}$ by induction on the depth of t, using Lemma 6.2.4 (4). The containment $L(t,k,q) \subseteq L(u,k,q)$ is also immediate by induction on the depth of t. $\square$

Proposition 6.2.6 *Let $t \in \mathbf{T_H}(A)$. Then the $N(t,k,q)$ $(k,q \geq 1)$ constitute a basis of clopen neighborhoods of t, and the $\overline{L(t,k,q)}$ constitute a basis of clopen neighborhoods of $\lambda_{\mathbf{H}}(t)$.*

Proof. By Lemma 6.2.4 (2), each $L(t,k,q)$ is **DRH**-elementary, and hence **DRH**-recognizable (Proposition 2.2.4), so that $\overline{L(t,k,q)}$ is clopen. Moreover, by Lemma 6.2.5, $N(t,k,q)$ is clopen as well.

Let us verify that

$$\lambda_{\mathbf{H}}(N(t,k+1,q)) \subseteq \overline{L(t,k,q)} \subseteq \lambda_{\mathbf{H}}(N(t,k,q)).$$

The second containment follows easily from Lemma 6.2.5: with the notations of the lemma, we have $\overline{L(t,k,q)} \subseteq \overline{L(u,k,q)} = \lambda_{\mathbf{H}}(N(t,k,q))$. Let us now prove the first containment by induction on the depth of t. The result is elementary if t has depth 0, so we assume that t has non-zero depth and we consider $t' \in N(t, k+1, q)$. If the root r of t has $n \le k$ children, then the same holds for t'. With the usual notations, we have $\lambda_{\mathbf{H}}(t') = \lambda_{\mathbf{H}}(t'_1)a_1 \cdots \lambda_{\mathbf{H}}(t'_n)a_n$, and it follows, by induction hypothesis, that

$$\lambda_{\mathbf{H}}(t') \in \overline{L(t_1, k, q)}a_1 \cdots \overline{L(t_n, k, q)}a_n = \overline{L(t, k, q)}.$$

If r has at least $k+1$ children, recall that the $(k+1)$-st edges descending from the roots of t and t' have the same content (that of the $(k+1)$-st edge attached to the root of $\mathrm{restr}(\tau(t), k+1) = \mathrm{restr}(\tau(t'), k+1)$), namely B. Now $\lambda_{\mathbf{H}}(t') = \lambda_{\mathbf{H}}(t'_1)a_1 \cdots \lambda_{\mathbf{H}}(t'_k)a_k\lambda_{\mathbf{H}}(t'_{k+1})a_{k+1}y$, with $\sigma(y) = \delta_{k+1}(r)$ and $\lambda_{\mathbf{H}}(t'_{k+1})a_{k+1}y \in \overline{B^*}$. By induction hypothesis, $\lambda_{\mathbf{H}}(t'_i) \in \overline{L(t_i, k, q)}$ for $1 \le i \le k+1$. In particular, by Lemma 6.2.4, $\lambda_{\mathbf{H}}(t'_{k+1}) \in \overline{L_q(\sigma(\lambda_{\mathbf{H}}(t_{k+1})))}$, and by Lemma 6.2.2, $\lambda_{\mathbf{H}}(t'_{k+1})a_{k+1}y \in \overline{L_q(\sigma(\lambda_{\mathbf{H}}(t_{k+1})a_{k+1}y))} = \overline{L(\delta_k(r))}$. Thus

$$\lambda_{\mathbf{H}}(t') \in \overline{L(t_1, k, q)}a_1 \cdots \overline{L(t_k, k, q)}a_k(\overline{L_q(\delta(r)) \cap B^*}) = \overline{L(t, k, q)}.$$

In order to conclude, we need to prove that if $t \ne t' \in \mathbf{T_H}(A)$, then there exist $k, q \ge 1$ such that $t' \notin N(t, k, q)$. By Proposition 4.2.5, we know that if $\tau(t') \ne \tau(t)$, then there exists $k \ge 1$ such that $\tau(t') \notin N(\tau(t), k)$, and hence $t' \notin N(t, k, q)$ for any q. Let us now assume that $\tau(t) = \tau(t')$. In particular, the roots of t and t' have the same number of edges attached under them, with the same labels $a_1, a_2, \ldots$, and the corresponding decorated subtrees, $t_1, t_2, \ldots$ in t and $t'_1, t'_2, \ldots$ in t', satisfy $\tau(t_i) = \tau(t'_i)$ for each i.

We first consider the case where $t_k \ne t'_k$ for some k. By induction hypothesis, there exist $n, q \ge 1$ such that $t'_k \notin N(t_k, n, q)$, and hence $t'_k \notin N(t_k, \max(n, k), q)$ (the containment $N(t_k, \max(n, k), q) \subseteq N(t_k, n, q)$ follows from the inclusions established above. Now, the tree $\mathrm{restr}(t'_k, \max(n, k))$ is one of the decorated subtrees attached under the root of $\mathrm{restr}(t', \max(n, k))$, so $t' \notin N(t, \max(n, k), q)$.

The remaining case is that where $t_i = t'_i$ for all i. Since $t \ne t'$, this implies that the root r of t and t' has infinite arity, and that δ and δ' coincide on all nodes except r. Then there exists q such that $\delta'(r) \notin \delta(r)L'_q$. Let m be the least integer such that the content of the m-th edge descending from r is $\overline{c}(r)$ and let e be the idempotent designated by $(\lambda_{\mathbf{H}}(t_m)a_m \cdots \lambda_{\mathbf{H}}(t_n)a_n)_{n \ge m}$. Then

$$\lambda_{\mathbf{H}}(t) = \lambda_{\mathbf{H}}(t_1)a_1 \cdots \lambda_{\mathbf{H}}(t_{m-1})a_{m-1}\varphi_e(\delta(r)),$$

so the label of the root of $\mathrm{restr}(t, m)$ is

$$\begin{aligned}
\sigma(\lambda_{\mathbf{H}}(t_1)a_1 \cdots \lambda_{\mathbf{H}}(t_m)a_m)^{-1}\sigma(\lambda_{\mathbf{H}}(t)) &= \sigma(\lambda_{\mathbf{H}}(t_m)a_m)^{-1}\sigma(\varphi_e(\delta(r))) \\
&= \sigma(\lambda_{\mathbf{H}}(t_m)a_m)^{-1}\delta(r)
\end{aligned}$$

Similarly, the label of the root of $\mathrm{restr}(t', m)$ is $\sigma(\lambda_{\mathbf{H}}(t'_m)a_m)^{-1}\delta'(r) = \sigma(\lambda_{\mathbf{H}}(t_m)a_m)^{-1}\delta'(r)$. By assumption, $\delta'(r) \notin \delta(r)L'_q$, so

$$\sigma(\lambda_{\mathbf{H}}(t_m)a_m)^{-1}\delta'(r) \notin \sigma(\lambda_{\mathbf{H}}(t_m)a_m)^{-1}\delta(r)L'_q,$$

and hence $t' \notin N(t, m, q)$. $\square$

Corollary 6.2.7 *The closures of the* **DRH**-*elementary languages form a basis of the topology of* $\overline{\Omega}_A\mathbf{DRH}$.

This implies the following description of the **DRH**-languages, with the same proof as that of Corollary 4.2.7.

Corollary 6.2.8 *Let A be a finite alphabet. A language of A^* is* **DRH**-*recognizable if and only if it is a finite disjoint union of languages of the form*

$$(L_0 \cap A_0^*)a_1(L_1 \cap A_1^*)\cdots a_k(L_k \cap A_k^*),$$

where $A_i \subseteq A$ $(0 \leq i \leq k)$, $a_i \in A \setminus A_i$ $(1 \leq i \leq k)$ and L_i $(0 \leq i \leq k)$ is **H**-*recognizable.*

Corollary 6.2.9 $\mathbf{DRH} = \mathbf{K} ⓜ (\mathbf{H} \vee \mathbf{J}_1)$.

Proof. Since **H** and $\mathbf{J}_1$ are contained in **DRH**, and since **DRH** is closed under deterministic products (Proposition 2.2.2), $\mathbf{K} ⓜ (\mathbf{H} \vee \mathbf{J}_1)$ is contained in **DRH**. But Corollary 6.2.8 above states that the **DRH**-recognizable languages are unions of deterministic products of $(\mathbf{H} \vee \mathbf{J}_1)$-languages, so $\mathbf{DRH} \subseteq \mathbf{K} ⓜ (\mathbf{H} \vee \mathbf{J}_1)$, thus establishing the announced equality. □

6.3 Factorization in products of letters and regular elements

We can also generalize Theorem 4.3.1 for all pseudovarieties of the form **DRH**.

Theorem 6.3.1 *Let* **H** *be a pseudovariety of groups and let $\pi: \overline{\Omega}_A\mathbf{M} \to \overline{\Omega}_A\mathbf{DRH}$ be the canonical projection. Then each element x of $\overline{\Omega}_A\mathbf{M}$ can be written as a product $x = u_{k+1}x_k u_k \cdots x_0 u_0$ where*

- *$u_0 \in A^*$, $u_i \in A^+$ if $i \neq 0$, and $\pi(x_i)$ is regular for all i;*
- *the first letter of u_i $(0 \leq i \leq k)$ does not lie in $c(x_i)$;*
- *the last letter of u_i $(0 < i \leq k+1)$ lies in $c(x_{i-1})$;*
- *if $|u_i| \geq 2$, then the penultimate letter of u_i does not lie in $c(x_{i-1})$.*

Moreover, if $x = u_{k+1}x_k u_k \cdots x_0 u_0$ and $y = v_{n+1}y_n v_n \cdots y_0 v_0$ are factorizations of this type, and if **DRH** *satisfies $x = y$, then $k = n$, $u_i = v_i$ and* **DRH** *satisfies $x_i = y_i$ for all i.*

Proof. By Theorem 4.3.1, each element of $\overline{\Omega}_A\mathbf{M}$ admits a factorization of the required form. Let us now assume that $x = u_{k+1}x_k u_k \cdots x_0 u_0$ and $y = v_{n+1}y_n v_n \cdots y_0 v_0$ are factorizations of this form, and that **DRH** satisfies $x = y$. We proceed as in the proof of Theorem 4.3.1. Let $\psi_{\mathbf{H}} = \lambda_{\mathbf{H}} \circ \nu_{\mathbf{H}}: \mathbf{rLO}_{\mathbf{H}}(A) \to \overline{\Omega}_A\mathbf{DRH}$. Then $\psi_{\mathbf{H}}$ is a bijection. Let $(\alpha, \ell, g) = \psi_{\mathbf{H}}(\pi(x))$, let $(\delta_i, \ell_i, g_i) = \psi_{\mathbf{H}}(\pi(x_i))$ $(0 \leq i \leq k)$ and let $(\varepsilon_i, \ell'_i, g'_i) = \psi_{\mathbf{H}}(u_i)$ $(0 \leq i \leq k+1)$. By Theorem 4.3.1, the integer k, the words u_i, the ordinals δ_i and the labeling function ℓ_i are entirely determined by $\pi(x)$. Moreover, we saw in the proof of Theorem 4.3.1 that $\alpha = \varepsilon_{k+1} + \delta_k + \cdots + \delta_0 + \varepsilon_0$, so L_α is the union of the $\varepsilon_{k+1} + \delta_k + \cdots + \varepsilon_{i+1} + L_{\delta_i}$ $(0 \leq i \leq k)$. In addition, this implies that

$$g(\varepsilon_{k+1} + \delta_k + \cdots + \varepsilon_{i+1} + \gamma) = g_i(\gamma)$$

for each $\gamma \in L_{\delta_i}, 0 \leq i \leq k$. In other words, the g_i too are entirely determined by $\pi(x)$. This completes the proof. □

7 Application to the computation of certain joins

In this section, we generalize the result of Almeida and Azevedo [7] on the computation of the join $\mathbf{R} \vee \mathbf{L}$, and we compute $\mathbf{DRH} \vee \mathbf{DLH}$ for each pseudovariety of groups $\mathbf{H}$. Let $\mathbf{V}$ be the pseudovariety

$$\mathbf{V} = [\![(xy)^\omega x^\omega (zx)^\omega = (xy)^\omega (zx)^\omega]\!] \cap \mathbf{DO}.$$

Let $\mathbf{H}$ be a pseudovariety of groups and let $\mathbf{V}(\mathbf{H})$ be the pseudovariety consisting of all the elements of $\mathbf{V}$ all of whose subgroups are in $\mathbf{H}$.

Lemma 7.1 *$\mathbf{V}$ is the pseudovariety defined by the pseudoidentities of the form $yx^\omega z = yz$ where $c(x) \subseteq c(y) \cap c(z)$ and the restrictions of y and z are regular in $\mathbf{DS}$.*

Proof. Let $\mathbf{W}$ be the pseudovariety in the statement of the lemma, and let $x, y, z \in \overline{\Omega}_A \mathbf{M}$ be such that $c(x) \subseteq c(y) \cap c(z)$ and the restrictions of y and z are regular in $\mathbf{DS}$. Since $\mathbf{V}$ is contained in $\mathbf{DS}$, it follows from Proposition 1.4.1 that $\mathbf{V}$ satisfies $y^\omega = (y^\omega x y^\omega)^\omega = y^\omega (xy^\omega)^\omega$, and hence $y = yy^\omega = y(xy^\omega)^\omega$. By symmetry, $\mathbf{V}$ satisfies $z = (z^\omega x)^\omega z$. Therefore, $\mathbf{V}$ satisfies

$$\begin{aligned} yz &= y(xy^\omega)^\omega (z^\omega x)^\omega z \\ &= y(xy^\omega)^\omega x^\omega (z^\omega x)^\omega z \\ &= yx^\omega z. \end{aligned}$$

Thus, $\mathbf{V}$ is contained in $\mathbf{W}$.

Conversely, it is immediately clear from the definition that $\mathbf{W}$ satisfies $(xy)^\omega (yx)^\omega (xy)^\omega = (xy)^\omega (xy)^\omega = (xy)^\omega$ and $(xy)^\omega x^\omega (yz)^\omega = (xy)^\omega (yz)^\omega$, so $\mathbf{W}$ is contained in $\mathbf{V}$. □

For each $k \geq 1$, let $\mathbf{V}_k$ be the pseudovariety

$$[\![u_1 \cdots u_k v^\omega w_k \cdots w_1 = u_1 \cdots u_k w_k \cdots w_1 \mid \begin{array}{l} c(v) \subseteq c(u_k) \subseteq \cdots \subseteq c(u_1), \\ c(v) \subseteq c(w_k) \subseteq \cdots \subseteq c(w_1) \end{array}]\!].$$

Lemma 7.2 $\mathbf{V} = \bigcup_{k \geq 1} \mathbf{V}_k$.

Proof. It is trivial that $\mathbf{V}_k \subseteq \mathbf{V}$ for each k. Let now $S \in \mathbf{V}$. By Proposition 1.4.1, there exists an integer k such that S satisfies $u_1 \cdots u_k = (u_1 \cdots u_k)^{\omega+1}$ whenever the u_i all have the same content. Then we verify that $S \in \mathbf{V}_{k|A|}$. Let indeed $u_1, \ldots, u_{k|A|}, v, w_1, \ldots, w_{k|A|}$ be such that $c(v) \subseteq c(u_{k|A|}) \subseteq \cdots \subseteq c(u_1)$ and $c(v) \subseteq c(w_{k|A|}) \subseteq \cdots \subseteq c(w_1)$. Then there exist indices $i, j \leq k(|A|-1)$ such that $c(u_{i+1}) = \cdots = c(u_{i+k})$ and $c(w_{j+1}) = \cdots = c(w_{j+k})$. Then S satisfies $u_{i+1} \cdots u_{i+k} = (u_{i+1} \cdots u_{i+k})^{\omega+1}$, and hence $u_{i+1} \cdots u_{k|A|} = (u_{i+1} \cdots u_{k|A|})^{\omega+1}$ (since $c(u_h) \subseteq c(u_{i+k})$ for all $h > i+k$). Similarly, S satisfies $w_{k|A|} \cdots w_{j+1} = (w_{k|A|} \cdots w_{j+1})^{\omega+1}$. Therefore, S satisfies

$$\begin{aligned} u_1 \cdots u_k w_k \cdots w_1 &= u_1 \cdots u_i (u_{i+1} \cdots u_{k|A|})^{\omega+1} (w_{k|A|} \cdots w_{j+1})^{\omega+1} w_j \cdots w_1 \\ &= u_1 \cdots u_i (u_{i+1} \cdots u_{k|A|})^{\omega+1} v^\omega (w_{k|A|} \cdots w_{j+1})^{\omega+1} w_j \cdots w_1 \\ &= u_1 \cdots u_k v^\omega w_k \cdots w_1. \end{aligned}$$

□

If we let $\mathbf{V}_k(\mathbf{H})$ be the pseudovariety of all the elements of $\mathbf{V}_k$ all of whose subgroups are in $\mathbf{H}$, it follows that $\mathbf{V}(\mathbf{H}) = \bigcup_{k\geq 1} \mathbf{V}_k(\mathbf{H})$.

We now prove the following technical property of $\mathbf{V}(\mathbf{H})$.

Lemma 7.3 *If $c(u) = c(v) \subseteq c(y)$ and if* **DRH** *satisfies $u = v$, then* $\mathbf{V}(\mathbf{H})$ *satisfies $uy^\omega = vy^\omega$.*

Proof. The proof is by induction on $|c(u)| = |c(v)|$. If $c(u) = \emptyset$, then $u = v = 1$ and the result is trivial. So we assume that $c(u) \neq \emptyset$ and that the result holds for smaller contents.

By Proposition 2.3.1, the left basic factorizations of u and v can be iterated the same number of times, and the resulting factorizations coincide in **DRH**. If they can be iterated only finitely many times, say n times, then we have

$$u = u_1a_1 \cdots u_na_n \quad \text{and} \quad v = v_1a_1 \cdots v_na_n$$

with $c(u_i) = c(v_i) \subset c(u)$ and $u_i = v_i$ in **DRH** for all i. But in **DS**, and hence in $\mathbf{V}(\mathbf{H})$, a_ny^ω is regular, so by induction hypothesis, $\mathbf{V}(\mathbf{H})$ satisfies $uy^\omega = u_1a_1 \cdots u_{n-1}a_{n-1}u_n(a_ny^\omega) = u_1a_1 \cdots u_{n-1}a_{n-1}v_n(a_ny^\omega)$. Iterating this reasoning, it follows that $\mathbf{V}(\mathbf{H})$ satisfies the pseudoidentity $uy^\omega = v_1a_1 \cdots v_na_ny^\omega = vy^\omega$.

Let us now assume that the left basic factorizations of u and v can be iterated indefinitely: for each n, we have

$$u = u_1a_1 \cdots u_na_nr_n \quad \text{and} \quad v = v_1a_1 \cdots v_na_ns_n$$

with $c(u_n) = c(v_n) \subset c(u)$, $u_n = v_n$ and $r_n = s_n$ in **DRH** for all n. If for some n, $c(r_n) = c(s_n)$ is strictly contained in $c(u)$, then by the same method as above, $\mathbf{V}(\mathbf{H})$ satisfies

$$uy^\omega = u_1a_1 \cdots u_na_nr_ny^\omega = v_1a_1 \cdots v_na_ns_ny^\omega = vy^\omega.$$

The last case is that where $c(r_n) = c(u)$ for all n, that is, where u and v are regular in **DRH**, and hence in **DS** and in $\mathbf{V}(\mathbf{H})$. Then $\mathbf{V}(\mathbf{H})$ satisfies $uy^\omega = uv^\omega y^\omega$, so it suffices to prove that $\mathbf{V}(\mathbf{H})$ satisfies $uv^\omega = v$. By induction hypothesis, for each $k \geq 1$, $\mathbf{V}_k(\mathbf{H})$ satisfies

$$\begin{aligned} uv^\omega &= u_1a_1 \cdots u_{2k}a_{2k}r_{2k}v^\omega \\ &= v_1a_1 \cdots v_{2k}a_{2k}r_{2k}v^\omega. \end{aligned}$$

But in $\mathbf{V}_k(\mathbf{H})$, $v_1a_1 \cdots v_{2k}a_{2k}$ is equal to

$$v_1a_1 \cdots v_ka_k(v_{k+1}a_{k+1} \cdots v_{2k}a_{2k}v_1a_1 \cdots v_ka_k)^\omega v_{k+1}a_{k+1} \cdots v_{2k}a_{2k},$$

and hence to $(v_1a_1 \cdots v_{2k}a_{2k})^{\omega+1}$. That is, $v_1a_1 \cdots v_{2k}a_{2k}$ is regular in $\mathbf{V}_k(\mathbf{H})$. But it has the same content as v, so in $\mathbf{V}_k$, $v_1a_1 \cdots v_{2k}a_{2k}$ is $\mathcal{R}$-equivalent to v, and hence uv^ω is $\mathcal{H}$-equivalent to $vv^\omega = v$. In addition, since **DRH** satisfies $u = v$, the pseudovariety **H** also satisfies $u = v$. Since $\mathbf{V}_k(\mathbf{H})$ is contained in **DO**, this implies that $\mathbf{V}_k(\mathbf{H})$ satisfies $uv^\omega = v$, by Proposition 5.1.2.

Finally, since $\mathbf{V}_k(\mathbf{H})$ satisfies $uv^\omega = v$ for all k, $\mathbf{V}(\mathbf{H})$ itself satisfies $uv^\omega = v$, which concludes the proof. □

We can now prove the expected theorem.

Theorem 7.4 $\mathbf{DRH} \vee \mathbf{DLH} = \mathbf{V}(\mathbf{H})$.

Proof. By Reiterman's theorem, it suffices to prove that if $x, y \in \overline{\Omega}_A\mathbf{M}$ are such that **DRH** and **DLH** satisfy $x = y$, then $\mathbf{V}(\mathbf{H})$ satisfies $x = y$.

By Theorem 6.3.1, we can factor x as

$$x = u_{k+1}x_k u_k \cdots x_0 u_0,$$

where

- $u_0 \in A^*$, $u_i \in A^+$ if $i \neq 0$, and x_i is regular in **DS** for all i;
- the first letter of u_i $(0 \leq i \leq k)$ does not lie in $c(x_i)$;
- the last letter of u_i $(0 < i \leq k+1)$ lies in $c(x_{i-1})$;
- if $|u_i| \geq 2$, then the penultimate letter of u_i does not lie in $c(x_{i-1})$.

In addition, for each $1 \leq i \leq k$, there exists $a_i \in c(x_i)$ and $x_i' \in \overline{\Omega}_A\mathbf{M}$ such that $x_i = a_i x_i'$ and $x_i \mathrel{\mathcal{L}} x_i'$ in **DS**. Let us indeed consider a sequence of words $(u_n)_n$ converging to x_i. Since A is finite, we may assume that all the u_n are of the form $u_n = au_n'$ for some letter a. By compactness, we may assume that the sequence $(u_n')_n$ converges, say its limit is x_i'. Then $ax_i' = x_i$. It now suffices to apply Proposition 1.4.1.

So we now have a factorization of x of the form

$$x = u_{k+1}a_k x_k b_k u_k a_{k-1} x_{k-1} \cdots a_0 x_0 b_0 u_0,$$

where

- $u_i \in A^*$ $(0 \leq i \leq k+1)$, $a_i, b_i \in c(x_i)$ and x_i is regular in **DS** $(0 \leq i \leq k)$;
- if $u_i \neq 1$, its first letter does not lie in $c(x_i)$ $(0 \leq i \leq k)$;
- if $u_i \neq 1$, its last letter does not lie in $c(x_{i-1})$ $(1 \leq i \leq k+1)$;
- if $u_i = 1$, then $b_i \notin c(x_{i-1})$ and $a_{i-1} \notin c(x_i)$ $(1 \leq i \leq k)$.

Let $y = v_{\ell+1}a_\ell' y_\ell b_\ell' v_\ell a_{\ell-1}' y_{\ell-1} \cdots a_0' y_0 b_0' v_0$ be a factorization of y of the same form. By Theorem 6.3.1, the hypothesis that **DRH** satisfies $x = y$ implies that $k = \ell$, $b_i = b_i'$ $(0 \leq i \leq k)$, $u_i = v_i$ $(0 \leq i \leq k+1)$ and **DRH** satisfies $a_i' y_i = a_i x_i$ $(0 \leq i \leq k)$. Repeated applications of Proposition 2.3.1 now imply that $a_{i+1} = a_{i+1}'$ and that **DRH** satisfies $x_i = y_i$ $(0 \leq i \leq k)$. Dually, **DLH** also satisfies $x_i = y_i$ $(0 \leq i \leq k)$.

Now Lemma 7.3 implies that $\mathbf{V}(\mathbf{H})$ satisfies $x_i^\omega y_i^\omega = y_i^\omega y_i^\omega = y_i^\omega$. By left-right duality, $\mathbf{V}(\mathbf{H})$ also satisfies $x_i^\omega y_i^\omega = y_i^\omega$, so $\mathbf{V}(\mathbf{H})$ satisfies $x_i^\omega = y_i^\omega$, and hence $x_i = x_i x_i^\omega = x_i y_i^\omega = y_i y_i^\omega = y_i$ in $\mathbf{V}(\mathbf{H})$ (using Lemma 7.3 again). In particular, $\mathbf{V}(\mathbf{H})$ satisfies $x = y$, which concludes the proof. □

The above proof also implies the following unique factorization result for $\mathbf{V}(\mathbf{H})$.

Corollary 7.5 *Let $x \in \overline{\Omega}_A\mathbf{M}$. If $x \neq 1$, then x admits a factorization of the form*

$$x = u_{k+1}a_k x_k b_k u_k a_{k-1} x_{k-1} \cdots a_0 x_0 b_0 u_0$$

where

- *$u_i \in A^*$ $(0 \le i \le k+1)$, $a_i, b_i \in c(x_i)$ and x_i is regular in* **DS** *$(0 \le i \le k)$;*
- *the second letter of $b_i u_i a_{i-1}$ does not lie in $c(x_i)$ $(1 \le i \le k)$, and if $u_0 \ne 1$, then the second letter of $a_0 u_0$ does not lie in $c(x_0)$;*
- *the penultimate letter of $b_i u_i a_{i-1}$ does not lie in $c(x_{i-1})$ $(1 \le i \le k)$, and if $u_{k+1} \ne 1$ then the penultimate letter of $u_{k+1} a_k$ does not lie in $c(x_k)$.*

Moreover, if $y = v_{n+1} a'_n y_n b'_n v_n a'_{n-1} y_{n-1} \cdots a'_0 y_0 b'_0 v_0$ is another factorization of this type, and if $\mathbf{V}(\mathbf{H})$ *satisfies $x = y$, then $k = n$, $a_i = a'_i$, $b_i = b'_i$, $u_i = v_i$ and* $\mathbf{V}(\mathbf{H})$ *satisfies $x_i = y_i$ for all i.*

Finally, the following decidability result is immediately derived from Theorem 7.4.

Corollary 7.6 *If* **H** *is a decidable pseudovariety of groups, then* $\mathbf{DRH} \vee \mathbf{DLH}$ *is decidable.*

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