The lattice of pseudovarieties of idempotent semigroups and a non-regular analogue[1]

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Abstract: We use classical results on the lattice $\mathcal{L}(\mathcal{B})$ of varieties of band (idempotent) semigroups to obtain information on the structure of the lattice $Ps(\mathbf{DA})$ of subpseudovarieties of $\mathbf{DA}$, — where $\mathbf{DA}$ is the largest pseudovariety of finite semigroups in which all regular semigroups are band semigroups. We bring forward a lattice congruence on $Ps(\mathbf{DA})$, whose quotient is isomorphic to $\mathcal{L}(\mathcal{B})$, and whose classes are intervals with effectively computable least and greatest members. Also we characterize the pro-identities satisfied by the members of an important family of subpseudovarieties of $\mathbf{DA}$. Finally, letting $\mathbf{V}_k$ be the pseudovariety generated by the k-generated elements of $\mathbf{DA}$ ($k \geq 1$), we use all our results to compute the position of the congruence class of $\mathbf{V}_k$ in $\mathcal{L}(\mathcal{B})$.

Introduction

The lattice of pseudovarieties of finite semigroups has been the object of much attention over the past few decades, with motivations drawn not only from universal algebra, but also from theoretical computer science. The link between theoretical computer science and semigroup theory goes back to Schützenberger's work in the late fifties, and the role played there by pseudovarieties of finite semigroups was detailed by Eilenberg in the seventies [7]. For more recent developments, the reader is refered to the books by Pin [20] and Almeida [1].

Independently of this research, the lattice $\mathcal{L}(\mathcal{CR})$ of varieties of completely regular semigroups also received considerable attention, through the work of Polak [24, 25, 26], Reilly [27], Pastijn and Trotter [17], and others. Until recently however, the interaction between the techniques and results developed in the study of these two lattices was only minimal. This situation has changed in the last couple years, for instance in Reilly [28], Auinger, Hall, Reilly and Zhang [5], and Reilly and Zhang [29].

This paper is a contribution to the effort of using results on varieties of completely regular semigroups to study pseudovarieties of finite semigroups. Here we choose a restricted framework, namely that provided by the lattice of band varieties, $\mathcal{L}(\mathcal{B})$. This lattice, the first significant sublattice of $\mathcal{L}(\mathcal{CR})$ to be elucidated (Birjukov [6], Fennemore [8], Gerhard [10]), provided the model for the description of the (much more complex) lattice $\mathcal{L}(\mathcal{CR})$.

It is well known that $\mathcal{L}(\mathcal{B})$ is isomorphic to the lattice $Ps(\mathbf{B})$ of subpseudovarieties of the pseudovariety $\mathbf{B}$ of all finite band semigroups. Our aim is to lift the description of $Ps(\mathbf{B})$ to a non regular analogue.

Let $\mathbf{DA}$ be the pseudovariety of finite semigroups in which all regular elements are idempotents. Equivalently, a finite semigroup is in $\mathbf{DA}$ if and only if each of its regular $\mathcal{D}$-classes is a rectangular band. This pseudovariety is the largest in which every regular

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semigroup is a band. The pseudovariety $\mathbf{DA}$ arises naturally in the context of the study of operations on languages, and its importance was emphasized by Schützenberger [31], Fich and Brzozowski [9], Pin, Straubing and Thérien [21], Jones and Trotter [15], Almeida [2], etc.

The general idea in this paper is to use the mapping from the lattice $Ps(\mathbf{DA})$ of subpseudovarieties of $\mathbf{DA}$ onto $Ps(\mathbf{B})$, mapping $\mathbf{V}$ to $\mathbf{V}\cap\mathbf{B}$, to obtain information on the structure of $Ps(\mathbf{DA})$. Note that $Ps(\mathbf{B})$ is countable and has a simple structure as a lattice. In contrast, $Ps(\mathbf{DA})$ is not countable. It contains the lattice of subpseudovarieties of the pseudovariety $\mathbf{N}$ of finite nilpotent semigroups, and it is known that this lattice contains intervals which are isomorphic to the dual of the partition lattice of a countable set [14].

In spite of the intrinsic complexity of the lattice $Ps(\mathbf{DA})$, certain important properties of the mapping $\mathbf{V}\mapsto\mathbf{V}\cap\mathbf{B}$ contribute to making our undertaking meaningful.

Firstly this mapping is a complete lattice morphism, and the induced congruence classes in $Ps(\mathbf{DA})$ are intervals. This result was proved very recently, and independently from our work, by Reilly and Zhang [29]. We give another proof of this result in this paper, but we prove in fact a much more precise result: for each pseudovariety of bands $\mathbf{Z}$ (given, say, by a defining band identity), we give an explicit formula for the subpseudovariety $\mathbf{Z}^\dagger$ of $\mathbf{DA}$ such that, for each subpseudovariety $\mathbf{X}$ of $\mathbf{DA}$, we have

$$\mathbf{X}\cap\mathbf{B}=\mathbf{Z}\quad\text{if and only if}\quad\mathbf{X}\in[\mathbf{Z},\mathbf{Z}^\dagger].$$

As a result, for each given pseudovariety of bands $\mathbf{Z}$, membership in $\mathbf{Z}^\dagger$ is decidable. The subpseudovarieties of $\mathbf{DA}$ of the form $\mathbf{Z}^\dagger$ ($\mathbf{Z}\in Ps(\mathbf{B})$) evidently play a natural role with respect to the lattice structure of $Ps(\mathbf{DA})$, since the intervals $[\mathbf{Z},\mathbf{Z}^\dagger]$ form a partition of $Ps(\mathbf{DA})$ such that

$$\mathbf{X}\in[\mathbf{Z},\mathbf{Z}^\dagger],\ \mathbf{X}'\in[\mathbf{Z}',\mathbf{Z}'^\dagger]\quad\Longrightarrow\quad\begin{cases}\mathbf{X}\cap\mathbf{X}'\in[\mathbf{Z}\cap\mathbf{Z}',(\mathbf{Z}\cap\mathbf{Z}')^\dagger]\\ \mathbf{X}\vee\mathbf{X}'\in[\mathbf{Z}\vee\mathbf{Z}',(\mathbf{Z}\vee\mathbf{Z}')^\dagger]\end{cases}$$

(Note that Reilly and Zhang's result is wider in scope, since they consider the mapping $\mathbf{V}\mapsto\mathbf{V}\cap\mathbf{B}$ on the lattice of all pseudovarieties of finite semigroups; but it is less precise, since they only prove the existence of the pseudovarieties $\mathbf{Z}^\dagger$.)

Secondly, the close interplay between the mapping $\mathbf{V}\mapsto\mathbf{V}\cap\mathbf{B}$ and certain Mal'cev product related operators in the lattices $Ps(\mathbf{B})$ and $Ps(\mathbf{DA})$ allows us to take full advantage of the techniques of profinite semigroup theory. The role of the operators $\mathbf{V}\mapsto\mathbf{LZ}Ⓜ\mathbf{V}$ and $\mathbf{V}\mapsto\mathbf{RZ}Ⓜ\mathbf{V}$ ($\mathbf{LZ}$ and $\mathbf{RZ}$ are respectively the pseudovarieties of left zero and right zero semigroups, and Ⓜ denotes the Mal'cev product) in the lattice $Ps(\mathbf{B})$ was indirectly studied by Gerhard and Petrich [11]. The role of the analogous operators $\mathbf{V}\mapsto\mathbf{K}Ⓜ\mathbf{V}$, $\mathbf{V}\mapsto\mathbf{D}Ⓜ\mathbf{V}$ and $\mathbf{V}\mapsto L\mathbf{I}Ⓜ\mathbf{V}$ in the lattice of pseudovarieties of finite semigroups was emphasized by Auinger, Hall, Reilly and Zhang [5] and Hall and Weil [13] ($\mathbf{K}$, $\mathbf{D}$ and $L\mathbf{I}$ are respectively the pseudovarieties of all finite semigroups S such that $eS=e$, $Se=e$ and $eSe=e$ for each idempotent e). Using recent results of profinite semigroup theory, we are able to characterize the pro-identities satisfied by $\mathbf{DA}$ and by the pseudovarieties of the form $\mathbf{K}Ⓜ\mathbf{V}$, $\mathbf{D}Ⓜ\mathbf{V}$ and $L\mathbf{I}Ⓜ\mathbf{V}$ when $\mathbf{V}$ contains semilattices and is contained in $\mathbf{DS}$ ($\mathbf{DS}$ is the pseudovariety of all semigroups in which each regular $\mathcal{D}$-class is a subsemigroup). This characterization, which is analogous to a solution of the identity problem for these pseudovarieties, is then used to effectively compute the above mentioned operation $\mathbf{Z}\mapsto\mathbf{Z}^\dagger$ ($\mathbf{Z}\in Ps(\mathbf{B})$).

The use of relatively free profinite semigroups to study pseudovarieties of finite semigroups goes back to Reiterman's theorem [30], in the early eighties. It has since been highly developed by Almeida and other researchers [1, 3].

In the last part of the paper, we illustrate our results by computing the pseudovarieties of the form $\mathbf{V}_k \cap \mathbf{B}$, where $\mathbf{V}_k$ is the pseudovariety generated by the k-generated semigroups in $\mathbf{DA}$. Let $\mathbf{B}'_k$ be the pseudovariety generated by the k-generated bands. The position of $\mathbf{B}'_k$ in $Ps(\mathbf{B})$ is well known, and it is clear that $\mathbf{B}'_k$ is contained in $\mathbf{V}_k$. We prove that $\mathbf{V}_k \cap \mathbf{B} = \mathbf{B}'_k$ for $k \geq 3$. Surprisingly, $\mathbf{B}'_2$ is strictly contained in $\mathbf{V}_2 \cap \mathbf{B}$, whose exact value is also computed.

The formal organization of the paper is the following. Section 1 contains the necessary preliminaries on pseudovarieties, on profinite semigroups and pro-identities, and on the specific properties of the subpseudovarieties of $\mathbf{DS}$. In Section 2, some properties of the lattice $Ps(\mathbf{B})$ are reviewed, and the first properties of the mapping $\mathbf{V} \mapsto \mathbf{V} \cap \mathbf{B}$ are established. Section 3 is devoted to the syntactical study of the pseudovarieties of the form $\mathbf{K} \circledm \mathbf{V}$, $\mathbf{D} \circledm \mathbf{V}$ and $L\mathbf{I} \circledm \mathbf{V}$, when $\mathbf{V}$ is a subpseudovariety of $\mathbf{DS}$, and of the pro-identities they satisfy. In Section 4 we apply the previous results to furthering the study of the mapping $\mathbf{V} \mapsto \mathbf{V} \cap \mathbf{B}$ on $Ps(\mathbf{DA})$, and especially to proving that the equivalence classes it induces on $Ps(\mathbf{DA})$ are intervals with effectively computable least and greatest members. Finally, Section 5 illustrates our results and methods with the study of the pseudovariety $\mathbf{V}_k$ ($k \geq 1$) generated by the k-generated elements of $\mathbf{DA}$.

1 Preliminaries

A *pseudovariety of semigroups* is a class of finite semigroups which is closed under taking subsemigroups, homomorphic images and finite direct products. Pseudovarieties of finite monoids are defined in the same fashion. In this paper, we call any non empty finite set an *alphabet*.

Pseudovarieties which will play a central role in this paper are $\mathbf{SL}$, the class of all (finite) semilattices (i.e. idempotent and commutative semigroups); $\mathbf{B}$, the class of all idempotent semigroups, or *bands*; $\mathbf{DA}$, the class of all semigroups S in which each regular $\mathcal{D}$-class is a subsemigroup of S, which is a band; $\mathbf{DS}$, the class of all semigroups in which each regular $\mathcal{D}$-class is a subsemigroup. We have the following containments:

$$\mathbf{SL} \subsetneq \mathbf{B} \subsetneq \mathbf{DA} \subsetneq \mathbf{DS}.$$

We will also encounter $\mathbf{J}$, $\mathbf{R}$ and $\mathbf{L}$, respectively the classes of $\mathcal{J}$-trivial, $\mathcal{R}$-trivial and $\mathcal{L}$-trivial semigroups, which also sit between $\mathbf{SL}$ and $\mathbf{DA}$. For more details about pseudovarieties, and in particular for a proof that the above classes are pseudovarieties, see [20]. We denote by $\mathbf{S}$ (resp. $\mathbf{M}$, $\mathbf{G}$) the class of all finite semigroups (resp. monoids, groups). For any pseudovariety of monoids $\mathbf{V}$, the class $L\mathbf{V}$ of all finite semigroups S such that $eSe \in \mathbf{V}$ for each idempotent e of S, is a pseudovariety of semigroups. The trivial pseudovariety of monoids, consisting only of the 1-element monoid, is written $\mathbf{I}$. As well, the following subpseudovarieties of $L\mathbf{I}$ will be used: $\mathbf{K}$ (resp. $\mathbf{D}$) is the class of finite semigroups S such that $eS = e$ (resp. $Se = e$) for each idempotent e of S, and $\mathbf{N} = \mathbf{K} \cap \mathbf{D}$ ($\mathbf{N}$ is the pseudovariety of nilpotent semigroups).

1.1 Relatively free profinite semigroups and identities

We first review some basic definitions and results concerning relatively free profinite semigroups. For details and proofs, the reader is referred to Almeida's book [1] and to the survey [3].

We say that a semigroup is *profinite* if it is a projective limit of finite semigroups. More generally, if $\mathbf{V}$ is a pseudovariety of semigroups, we say that a semigroup is *pro*-$\mathbf{V}$

if it is a projective limit of elements of $\mathbf{V}$. It is known that a semigroup is profinite if and only if it is compact and 0-dimensional. If $\mathbf{V}$ is a pseudovariety, then a profinite semigroup is pro-$\mathbf{V}$ if and only if all its finite continuous homomorphic images are in $\mathbf{V}$. Of course, all elements of $\mathbf{V}$ are pro-$\mathbf{V}$, and if $\mathbf{W}$ is a pseudovariety contained in $\mathbf{V}$, then all pro-$\mathbf{W}$ semigroups are pro-$\mathbf{V}$. A profinite semigroup S is said to be generated by an alphabet A, *A-generated* for short, if there exists a mapping $\sigma: A \to S$ such that the subsemigroup generated by $\sigma(A)$ is dense in S.

Let $\mathbf{V}$ be a pseudovariety and let A be an alphabet. We denote by $\hat{F}_A(\mathbf{V})$ the projective limit of the A-generated elements of $\mathbf{V}$. The topological semigroup $\hat{F}_A(\mathbf{V})$ can be viewed as the completion of a certain uniform structure on the free semigroup A^+, or as the semigroup (resp. monoid) of $|A|$-ary implicit operations on $\mathbf{V}$. Whenever we are specifically interested in the cardinality of the alphabet A, as will be the case in Section 5 below, we write $\hat{F}_k(\mathbf{V})$ for $\hat{F}_A(\mathbf{V})$ if $|A| = k$.

In the particular case where $\mathbf{V}$ admits a finite free object $F_A(\mathbf{V})$ over the alphabet A, then $\hat{F}_A(\mathbf{V}) = F_A(\mathbf{V})$. This is the case in particular for the pseudovarieties of bands. For instance, if $\mathbf{V} = \mathbf{SL}$, the pseudovariety of semilattices, then $\hat{F}_A(\mathbf{SL}) = F_A(\mathbf{SL})$ is equal to the semigroup $\mathcal{P}(A)$ of non empty subsets of A under union.

We summarise below the main properties of the $\hat{F}_A(\mathbf{V})$ which will be used freely in the sequel.

Theorem 1.1 *Let A be an alphabet and let $\mathbf{V}$ be a pseudovariety of semigroups.*

1. *$\hat{F}_A(\mathbf{V})$ is A-generated and pro-$\mathbf{V}$: in particular there exists a mapping $\imath: A \to \hat{F}_A(\mathbf{V})$, which is one-to-one if $\mathbf{V}$ is non-trivial, such that $\imath(A)$ generates a dense subsemigroup of $\hat{F}_A(\mathbf{V})$.*

2. *$\hat{F}_A(\mathbf{V})$ is the free pro-$\mathbf{V}$ semigroup over A: If S is pro-$\mathbf{V}$ and if $\sigma: A \to S$ is a mapping, then σ induces a unique continuous morphism $\hat{\sigma}: \hat{F}_A(\mathbf{V}) \to S$ such that $\hat{\sigma} \circ \imath = \sigma$.*

3. *A finite semigroup is in $\mathbf{V}$ if and only if it is a continuous homomorphic image of $\hat{F}_A(\mathbf{V})$ for some alphabet A.*

An important application of Theorem 1.1 is the following: If $\mathbf{W}$ is a subpseudovariety of $\mathbf{V}$, then the identity map on A induces a continuous onto morphism $\pi: \hat{F}_A(\mathbf{V}) \to \hat{F}_A(\mathbf{W})$, which is called the *natural projection of $\hat{F}_A(\mathbf{V})$ onto $\hat{F}_A(\mathbf{W})$*. For instance, if $\mathbf{W} = \mathbf{SL} \subseteq \mathbf{V}$, the natural projection $c: \hat{F}_A(\mathbf{V}) \to \hat{F}_A(\mathbf{SL}) = \mathcal{P}(A)$ is called the *content morphism* on $\hat{F}_A(\mathbf{V})$. By definition, if $a \in A$ is a letter, then $c(a) = c(a\imath) = \{a\}$. More generally, if u is a word on A, then $c(u)$ is the set of letters occurring in u, i.e. the usual alphabetic content of u.

The ω-power, which traditionally denotes the idempotent power of an element of a finite semigroup [7, 20], can be defined on all semigroups of the form $\hat{F}_A(\mathbf{V})$. For any $x \in \hat{F}_A(\mathbf{V})$, the sequence $(x^{n!})_n$ converges in $\hat{F}_A(\mathbf{V})$. Its limit, written x^ω, is idempotent and, if $\sigma: \hat{F}_A(\mathbf{V}) \to S$ is a continuous morphism into a finite semigroup, then $\sigma(x^\omega)$ is the unique idempotent power of $\sigma(x)$.

1.2 Pro-identities

Theorem 1.1 leads to the statement of Reiterman's theorem on the definition of pseudovarieties by pro-identities [30]. Let A be an alphabet, let $\mathbf{V}$ be a pseudovariety, and let $x, y \in \hat{F}_A(\mathbf{V})$. We say that a pro-$\mathbf{V}$ semigroup S *satisfies the pro-$\mathbf{V}$-identity $x = y$*

if $\sigma(x) = \sigma(y)$ for each continuous morphism $\sigma: \hat{F}_A(\mathbf{V}) \to S$. If the pseudovariety $\mathbf{V}$ is understood, or if it is the pseudovariety $\mathbf{S}$ of all finite semigroups, we talk of *pro-identities*. Pro-identities are also called pseudoidentities [1]. If $|A| = k$, we say that $x = y$ is a k-variable pro-$\mathbf{V}$-identity. We say that a class $\mathbf{W}$ of pro-$\mathbf{V}$ semigroups satisfies a set Σ of pro-$\mathbf{V}$-identities if each element of $\mathbf{W}$ satisfies each element of Σ. The class of all finite semigroups which satisfy Σ is said to be *defined by* Σ and is denoted $[\![\Sigma]\!]_{\mathbf{V}}$ ($[\![\Sigma]\!]$ if $\mathbf{V} = \mathbf{S}$). It is important to remark the following [22, Proposition 3.2].

Proposition 1.2 *Let $\mathbf{W} \subseteq \mathbf{V}$ be pseudovarieties and let A be an alphabet. Let $\pi: \hat{F}_A(\mathbf{V}) \to \hat{F}_A(\mathbf{W})$ be the natural projection and let $x, y \in \hat{F}_A(\mathbf{V})$. Then $\mathbf{W}$ satisfies $x = y$ if and only if $\pi(x) = \pi(y)$.*

Pro-$\mathbf{V}$-identities suffice to describe all subpseudovarieties of $\mathbf{V}$, as expressed by the following theorem, due to Reiterman [30].

Theorem 1.3 *Let $\mathbf{V}$ be a pseudovariety and let $\mathbf{W}$ be a class of semigroups in $\mathbf{V}$. Then $\mathbf{W}$ is a pseudovariety if and only if there exists a set Σ of pro-$\mathbf{V}$-identities such that $\mathbf{W} = [\![\Sigma]\!]_{\mathbf{V}}$.*

Since the elements of $\hat{F}_A(\mathbf{V})$ are limits of sequences of words, a pro-$\mathbf{V}$-identity can be seen as the limit of a sequence of word identities. In this sense, Reiterman's theorem above is a topological version of Eilenberg and Schützenberger's theorem which states that pseudovarieties of finite semigroups are ultimately defined by sequences of (word) identities [7, 20].

Example. Here are descriptions by pro-identities of the pseudovarieties mentioned above.

- $\mathbf{SL} = [\![xy = yx,\ x^2 = x]\!]$ and $\mathbf{B} = [\![x^2 = x]\!]$;
- $\mathbf{J} = [\![(yx)^\omega = (xy)^\omega\,;\ x^\omega x = x^\omega]\!]$, $\mathbf{R} = [\![(xy)^\omega x = (xy)^\omega]\!]$ and $\mathbf{L} = [\![y(xy)^\omega = (xy)^\omega]\!]$;
- $\mathbf{DA} = [\![(xy)^\omega(yx)^\omega(xy)^\omega = (xy)^\omega\,;\ x^\omega x = x^\omega]\!]$;
- $\mathbf{DS} = [\![((xy)^\omega(yx)^\omega(xy)^\omega)^\omega = (xy)^\omega]\!]$;
- $L\mathbf{I} = [\![x^\omega y x^\omega = x^\omega]\!]$, $\mathbf{K} = [\![x^\omega y = x^\omega]\!]$, $\mathbf{D} = [\![yx^\omega = x^\omega]\!]$ and $\mathbf{N} = [\![x^\omega y = yx^\omega = x^\omega]\!]$.

1.3 Monoidal pseudovarieties

In the above sections, we have dealt only with pseudovarieties of semigroups. However, the same definitions can be given for pseudovarieties of monoids (up to the obvious changes), and analogous results hold. In this section, we want to clarify some of the interplay between pseudovarieties of semigroups and pseudovarieties of monoids.

Let $\mathbf{M}$ be the class of all finite monoids. If $\mathbf{V}$ is a pseudovariety of semigroups, then $\mathbf{V} \cap \mathbf{M}$ is a pseudovariety of monoids. We say that $\mathbf{V}$ is *supermonoidal* if, for each finite semigroup $S \in \mathbf{V}$, the monoid $S \cup \{1\}$ with 1 as a new identity element also lies in $\mathbf{V}$. It is *monoidal* if, for each semigroup S *which is not a monoid*, $S \in \mathbf{V}$ implies $S \cup \{1\} \in \mathbf{V}$. In [7, Prop. V.1.2], it is proved that a pseudovariety is monoidal if and only if it is generated by the monoids it contains. Of course, every supermonoidal pseudovariety is monoidal.

Examples. **SL** and **B** are supermonoidal, as well as **J**, **R**, **DA**, **DS**. It follows from the definition that each supermonoidal pseudovariety contains the semilattice $\{0,1\}$, and hence contains all of **SL**. The subpseudovarieties of $L\mathbf{G}$ are never supermonoidal, and among them, only the pseudovarieties consisting solely of groups are monoidal.

It is easily verified that the intersection of monoidal (resp. supermonoidal) pseudovarieties is again monoidal (resp. supermonoidal). The following will be useful in the sequel.

Proposition 1.4 *A pseudovariety of semigroups* $\mathbf{V}$ *is supermonoidal if and only if, for each alphabet* A, $\hat{F}_A(\mathbf{V})$ *is not a monoid and* $\hat{F}_A(\mathbf{V} \cap \mathbf{M}) = \hat{F}_A(\mathbf{V}) \cup \{1\}$, *where* 1 *is a new identity element and* $\{1\}$ *forms a clopen subset.*

Proof. Let us first assume that $\mathbf{V}$ is supermonoidal. Since $\mathbf{SL} \subseteq \mathbf{V}$, we may consider the content morphism from $\hat{F}_A(\mathbf{V})$ onto the semilattice of non empty subsets of A under union. It follows from the existence of this morphism that $\hat{F}_A(\mathbf{V})$ is not a monoid. Now the topological monoid $\hat{F}_A(\mathbf{V}) \cup \{1\}$ obtained by adding a new identity element in such a way that $\{1\}$ forms a clopen subset, is an A-generated profinite monoid. Its finite continuous homomorphic images are exactly the A-generated elements of $\mathbf{V}$ which are monoids, and the A-generated elements of $\mathbf{V}$ with a new identity adjoined; that is, exactly all the A-generated monoids in $\mathbf{V} \cap \mathbf{M}$. Thus $\hat{F}_A(\mathbf{V}) \cup \{1\} = \hat{F}_A(\mathbf{V} \cap \mathbf{M})$.

Conversely, let us assume that $\hat{F}_A(\mathbf{V} \cap \mathbf{M}) = \hat{F}_A(\mathbf{V}) \cup \{1\}$ with $\{1\}$ clopen for each alphabet A, and let $S \in \mathbf{V}$. There exists an onto continuous morphism $\sigma \colon \hat{F}_A(\mathbf{V}) \to S$. The extension of σ to a mapping from $\hat{F}_A(\mathbf{V} \cap \mathbf{M})$ onto $S \cup \{1\}$ given by $\sigma(1) = 1$ is a morphism, and it is continuous since $\{1\}$ is clopen in $\hat{F}_A(\mathbf{V} \cap \mathbf{M})$, so the monoid $S \cup \{1\}$ lies in $\mathbf{V} \cap \mathbf{M}$, and hence in $\mathbf{V}$. $\square$

1.4 Localized pseudovarieties

In the reverse direction, given a pseudovariety of monoids $\mathbf{V}$, one may consider the pseudovariety of semigroups $L\mathbf{V}$ consisting of all semigroups S such that $eSe \in \mathbf{V}$ for each idempotent e of S: $L\mathbf{V}$ is the largest pseudovariety of semigroups whose intersection with $\mathbf{M}$ is $\mathbf{V}$. We now describe a set of pro-identities defining $L\mathbf{V}$.

Let $\mathbf{W}$ be a supermonoidal pseudovariety. Let A be an alphabet, let z be a symbol not in A and let $A_z = A \cup \{z\}$. Finally, let $\varphi_z \colon \hat{F}_A(\mathbf{W}) \to \hat{F}_{A_z}(\mathbf{W})$ be the continuous morphism defined by $\varphi_z(a) = z^\omega a z^\omega$ for each $a \in A$. We then extend φ_z to $\hat{F}_A(\mathbf{W} \cap \mathbf{M}) = \hat{F}_A(\mathbf{W}) \cup \{1\}$ by letting $\varphi_z(1) = z^\omega$. The resulting morphism is again continuous since $\{1\}$ is clopen.

Proposition 1.5 *Let* $\mathbf{W}$ *be a supermonoidal pseudovariety, let* $\mathbf{V}$ *be a pseudovariety of monoids contained in* $\mathbf{W} \cap \mathbf{M}$, *and let* Σ *be a set of pro-*$(\mathbf{W} \cap \mathbf{M})$*-identities defining* $\mathbf{V}$. *Then* $L\mathbf{V} \cap \mathbf{W}$ *is defined by the pro-*$\mathbf{W}$*-identities of the form* $\varphi_z(u) = \varphi_z(v)$, *where* $(u = v) \in \Sigma$, A *is an alphabet such that* $u, v \in \hat{F}_A(\mathbf{W} \cap \mathbf{M})$, *and* z *is a symbol not in* A.

Proof. Suppose that $S \in \mathbf{W}$ satisfies the pro-$\mathbf{W}$-identities described above, and that S is a monoid. Let A be an alphabet, let $(u = v) \in \Sigma$ with $u, v \in \hat{F}_A(\mathbf{W} \cap \mathbf{M})$, and let $\sigma \colon \hat{F}_A(\mathbf{W} \cap \mathbf{M}) \to S$ be a continuous (monoid) morphism. Let z be a symbol not in A and let $\sigma' \colon \hat{F}_{A_z}(\mathbf{W}) \to S$ be the continuous morphism defined by $\sigma'(z) = 1$ and $\sigma'(a) = \sigma(a)$ for each $a \in A$. Then $\sigma'(\varphi_z(a)) = \sigma(a)$ for each $a \in A$, so $\sigma' \circ \varphi_z = \sigma$. By hypothesis, $\sigma'(\varphi_z(u)) = \sigma'(\varphi_z(v))$, so $\sigma(u) = \sigma(v)$. This proves that $S \in \mathbf{V}$, and in general that the pro-identities described in the statement of the proposition define a subpseudovariety of $L\mathbf{V} \cap \mathbf{W}$.

Conversely, suppose that $S \in L\mathbf{V} \cap \mathbf{W}$, let $(u = v) \in \Sigma$ with $u, v \in \hat{F}_A(\mathbf{W})$ for some alphabet A, let z be a symbol not in A and let $\sigma: \hat{F}_{A_z}(\mathbf{W}) \to S$ be a continuous morphism. Let $e = \sigma(z^\omega)$. For each $a \in A$, $\sigma(\varphi_z(a)) \in eSe$, so $\sigma \circ \varphi_z$ is a continuous monoid morphism from $\hat{F}_A(\mathbf{W} \cap \mathbf{M})$ into eSe. Therefore, our hypothesis implies that $\sigma \circ \varphi_z(u) = \sigma \circ \varphi_z(v)$. This proves that S satisfies the pro-identity $\varphi_z(u) = \varphi_z(v)$, thus concluding the proof of the proposition. $\square$

1.5 Mal'cev products

Let $\mathbf{V}$ be a pseudovariety of semigroups (resp. monoids) and let $\mathbf{W}$ be a pseudovariety of semigroups. The *Mal'cev product* $\mathbf{W} ⓜ \mathbf{V}$ is the pseudovariety of semigroups (resp. monoids) consisting of all homomorphic images of semigroups (resp. monoids) T such that there exists a morphism $\varphi: T \to V$ with $V \in \mathbf{V}$ and $\varphi^{-1}(e) \in \mathbf{W}$ for each idempotent e of V.

Pin and Weil gave a description of defining sets of pro-identities for $\mathbf{W} ⓜ \mathbf{V}$ in terms of pro-identities for $\mathbf{V}$ and $\mathbf{W}$. From that work, we will use the following result [23, Thm. 4.1].

Proposition 1.6 *Let* $\mathbf{W}$ *be a pseudovariety of semigroups, and let* $\mathbf{V}$ *be a pseudovariety of semigroups (resp. monoids). Let* A *be an alphabet and let* $\pi: \hat{F}_A(\mathbf{W} ⓜ \mathbf{V}) \to \hat{F}_A(\mathbf{V})$ *be the canonical projection. Then, for each idempotent* e *of* $\hat{F}_A(\mathbf{V})$*, the semigroup* $\pi^{-1}(e)$ *is pro-*$\mathbf{W}$*, that is, it satisfies all the pro-identities of* $\mathbf{W}$*.*

We also note the following.

Proposition 1.7 *Let* $\mathbf{V}$ *and* $\mathbf{W}$ *be pseudovarieties of semigroups. If* $\mathbf{V}$ *is supermonoidal, then so is* $\mathbf{W} ⓜ \mathbf{V}$*.*

Proof. By definition, if $S \in \mathbf{W} ⓜ \mathbf{V}$, then there exist onto morphisms $S \xleftarrow{\alpha} T \xrightarrow{\beta} V$ such that $V \in \mathbf{V}$ and $\beta^{-1}(e) \in \mathbf{W}$ for each idempotent e of V. Let α' and β' be the mappings $\alpha': T \cup \{1\} \to S \cup \{1\}$ and $\beta': T \cup \{1\} \to V \cup \{1\}$ defined by $\alpha'(1) = 1$, $\beta'(1) = 1$, and $\alpha'(t) = \alpha(t)$ and $\beta'(t) = \beta(t)$ for each $t \in T$. Then α' and β' are onto morphisms and, for each $v \in V$, $\beta'^{-1}(v) = \beta^{-1}(v)$. Moreover, $\beta'^{-1}(1) = \{1\}$. But $V \cup \{1\} \in \mathbf{V}$, so $S \cup \{1\} \in \mathbf{W} ⓜ \mathbf{V}$. $\square$

The cases where $\mathbf{W}$ is one of $\mathbf{N}$, $\mathbf{K}$, $\mathbf{D}$ and $L\mathbf{I}$ have received particular attention (see Pin [20], Pin, Straubing and Thérien [21], Almeida [1], Auinger, Hall, Reilly and Zhang [5] and Hall and Weil [13]). In view of the special role played by the operations $\mathbf{V} \mapsto \mathbf{W} ⓜ \mathbf{V}$ for these values of $\mathbf{W}$ in the study of the lattice of pseudovarieties, and in this paper in particular, we sometimes use the more compact notation $\mathbf{V}^{\mathbf{N}}$, $\mathbf{V}^{\mathbf{K}}$, $\mathbf{V}^{\mathbf{D}}$ and $\mathbf{V}^{L\mathbf{I}}$ for $\mathbf{N} ⓜ \mathbf{V}$, $\mathbf{K} ⓜ \mathbf{V}$, $\mathbf{D} ⓜ \mathbf{V}$ and $L\mathbf{I} ⓜ \mathbf{V}$. One can easily verify that $\mathbf{W} ⓜ \mathbf{W} = \mathbf{W}$ for these values of $\mathbf{W}$. It follows that the operations $\mathbf{V} \mapsto \mathbf{V}^{\mathbf{W}}$ are idempotent (say by [32, Lemma 1.4]). It is shown in [5] that for each pseudovariety $\mathbf{V}$, $\mathbf{V}^{L\mathbf{I}}$ is the union of all $\mathbf{V}^{(\mathbf{KD})^n}$, and also the union of all $\mathbf{V}^{(\mathbf{DK})^n}$. It is well known that $\mathbf{SL}^{\mathbf{K}} = \mathbf{R}$, $\mathbf{SL}^{\mathbf{D}} = \mathbf{L}$ and $\mathbf{SL}^{L\mathbf{I}} = \mathbf{DA}$ [20]. Therefore the sequences $\mathbf{SL}$, $\mathbf{SL}^{\mathbf{K}}$, $\mathbf{SL}^{\mathbf{KD}}$, $\mathbf{SL}^{\mathbf{KDK}}$, ... and $\mathbf{SL}$, $\mathbf{SL}^{\mathbf{D}}$, $\mathbf{SL}^{\mathbf{DK}}$, $\mathbf{SL}^{\mathbf{DKD}}$, ... provide strict hierarchies inside $\mathbf{DA}$ [5]. It is also shown in [5] that each finite semigroup S admits a unique greatest congruence over $\mathbf{N}$ (that is, such that the idempotent classes lie in $\mathbf{N}$), say ρ, that for each pseudovariety $\mathbf{V}$, $S \in \mathbf{V}^{\mathbf{N}}$ if and only if $S/\rho \in \mathbf{V}$ and that the equation $\mathbf{X}^{\mathbf{N}} = \mathbf{V}^{\mathbf{N}}$ (in the lattice of pseudovarieties) admits a least solution, denoted $\mathbf{V}_{\mathbf{N}}$. In particular, $\mathbf{V}^{\mathbf{N}} = \mathbf{N} ⓜ \mathbf{V}$ is the class of all semigroups T such that there exists a morphism $\varphi: T \to V$ with $V \in \mathbf{V}$ and $\varphi^{-1}(e) \in \mathbf{N}$ for each idempotent e of V. The same holds when we replace $\mathbf{N}$ by $\mathbf{K}$, $\mathbf{D}$ or $L\mathbf{I}$, and the pseudovarieties $\mathbf{V_K}$, $\mathbf{V_D}$ and $\mathbf{V}_{L\mathbf{I}}$ are defined similarly. In addition, $\mathbf{V_N}$, $\mathbf{V_K}$, $\mathbf{V_D}$ and $\mathbf{V}_{L\mathbf{I}}$ are generated by the quotients of the elements of $\mathbf{V}$ by their greatest congruence over $\mathbf{N}$, $\mathbf{K}$, $\mathbf{D}$ and $L\mathbf{I}$ respectively [5]. In particular, if $\mathbf{V}$ is monoidal, then so are $\mathbf{V_N}$, $\mathbf{V_K}$, $\mathbf{V_D}$ and $\mathbf{V}_{L\mathbf{I}}$.

1.6 Subpseudovarieties of DS

We will use freely the following important properties of the semigroups $\hat{F}_A(\mathbf{DS})$, which are due to Almeida and Azevedo. For a proof, see [1, Chap. 8].

Proposition 1.8 *Let A be an alphabet and let $x, y \in \hat{F}_A(\mathbf{DS})$.*

1. *If $c(x) = c(y)$, $x \leq_{\mathcal{J}} y$ and y is regular, then $x \,\mathcal{J}\, y$, and in particular, x is regular. The same result holds if we replace $\mathcal{J}$ by $\mathcal{R}$ or $\mathcal{L}$.*
2. *Let $(x_n)_{n>0}$ be a sequence of elements of $\hat{F}_A(\mathbf{DS})$ converging to x. Suppose that the x_n all have the same content, and for each n, x_n can be written as a product of at least n factors, each of which has content $c(x_n)$. Then x is regular.*
3. *Let $\mathbf{V}$ be a pseudovariety in the interval $[\mathbf{J}, \mathbf{DS}]$ and let $\pi: \hat{F}_A(\mathbf{DS}) \to \hat{F}_A(\mathbf{V})$ be the canonical projection. For each $x \in \hat{F}_A(\mathbf{DS})$, then x is regular if and only if $\pi(x)$ is regular.*

2 The mapping $\mathbf{V} \mapsto \mathbf{V} \cap \mathbf{B}$

We consider the mapping $\mathbf{V} \mapsto \mathbf{V} \cap \mathbf{B}$, defined on the lattice $Ps(\mathbf{DA})$ of subpseudovarieties of $\mathbf{DA}$ into the lattice $Ps(\mathbf{B})$ of band pseudovarieties. Our intention is to use this mapping, and the partition of $Ps(\mathbf{DA})$ it induces, to "chart" the lattice $Ps(\mathbf{DA})$. The properties of this mapping reported in this section, justify this approach since the mapping preserves a number of important operations on $Ps(\mathbf{DA})$. In particular, Reilly and Zhang recently proved that it is a complete lattice morphism, and consequently that each of the induced classes in $Ps(\mathbf{DA})$ is an interval. We will prove more properties of the mapping in Section 4. In particular we will give explicit formulæ for the greatest elements of these intervals and obtain a new proof of Reilly and Zhang's result.

Before considering the mapping itself, we summarize important results on $Ps(\mathbf{B})$ which are well known and will be used in Section 4.

2.1 The word problem in the free band

First let us review the solution of the word problem in the free band. Let A be an alphabet and let $w \in A^+$. The longest prefix of w with content strictly smaller than $c(w)$ is denoted by $0(w)$. The first letter in w after $0(w)$ is denoted by $\bar{0}(w)$. By definition, $0(w)\bar{0}(w)$ is the least prefix of w with content $c(w)$. Dually, we denote by $1(w)$ the longest suffix of w with content strictly less than $c(w)$, and by $\bar{1}(w)$ the letter immediately preceding $1(w)$ in w.

It is well known that the pseudovariety $\mathbf{B}$ of all bands admits a finite free object $F_A(\mathbf{B})$ over each alphabet A. In particular $\hat{F}_A(\mathbf{B}) = F_A(\mathbf{B})$. More precisely, if $\beta: A^+ \to F_A(\mathbf{B})$ is the natural projection, we have the following result [12].

Proposition 2.1 *Let $w, w' \in A^+$. Then $\beta(w) = \beta(w')$ if and only if*

$$\begin{aligned} \beta(0(w)) &= \beta(0(w')) & \qquad \beta(1(w)) &= \beta(1(w')) \\ \overline{0}(w) &= \overline{0}(w') & \overline{1}(w) &= \overline{1}(w'). \end{aligned}$$

Since $0(w)$ and $1(w)$ have strictly smaller contents than w, the solution of the word problem in $F_A(\mathbf{B})$ follows by induction.

2.2 The structure of the lattice of pseudovarieties of bands

The structure of the lattice of varieties of band semigroups was elucidated by Birjukov [6], Fennemore [8] and Gerhard [10]. The result can be transfered immediately to the lattice of pseudovarieties of band semigroups, since the free band over any finite alphabet is finite, so there is a one-to-one correspondance between varieties and pseudovarieties of band semigroups. This lattice is represented in Figure 1.

The lower part of the lattice consists of the band pseudovarieties contained in $L\mathbf{I}$, namely the trivial pseudovariety $\mathbf{I}$ and the pseudovarieties $\mathbf{LZ} = [\![xy = x]\!] = \mathbf{K} \cap \mathbf{B}$, $\mathbf{RZ} = [\![xy = y]\!] = \mathbf{D} \cap \mathbf{B}$ and $\mathbf{ReB} = [\![xyx = x]\!]_{\mathbf{B}} = L\mathbf{I} \cap \mathbf{B}$.

Of the rest of the lattice, the least element is $\mathbf{SL}$, followed by $\mathbf{LNB} = [\![xyz = xzy]\!]_{\mathbf{B}}$ and $\mathbf{RNB} = [\![xyz = yxz]\!]_{\mathbf{B}}$. Except for $\mathbf{B}$ itself, all elements of this sublattice can be obtained from $\mathbf{LNB}$ and $\mathbf{RNB}$ by using intersections and the two following operations: if $\mathbf{V}$ is a band pseudovariety, then $\mathbf{V}^{\kappa_\ell} = \mathbf{LZ} \circledm \mathbf{V} = \mathbf{V}^{\mathbf{K}} \cap \mathbf{B}$ and $\mathbf{V}^{\kappa_r} = \mathbf{RZ} \circledm \mathbf{V} = \mathbf{V}^{\mathbf{D}} \cap \mathbf{B}$. In Figure 1, for each pseudovariety $\mathbf{V}$ containing $\mathbf{SL}$, the pseudovariety $\mathbf{V}^{\kappa_\ell}$ (resp. $\mathbf{V}^{\kappa_r}$) is at the upper extremity of the line through $\mathbf{V}$ oriented N.-W. (resp. N.-E). In particular, the pseudovarieties of bands of the form $\mathbf{V}^{\kappa_\ell}$ (resp. $\mathbf{V}^{\kappa_r}$) are linearly ordered. See the references [6, 8, 10] mentioned above, and also Gerhard and Petrich [11].

By analogy, we denote by $\mathbf{V}_{\kappa_\ell}$ and $\mathbf{V}_{\kappa_r}$ the pseudovarieties at the lower extremity of the line through $\mathbf{V}$ oriented S.-E. and S.-W. respectively.

The following formulæ are immediate consequences of the structure of the lattice. If $\mathbf{V}$ and $\mathbf{W}$ are pseudovarieties of bands, then:

- $\mathbf{V} = \mathbf{V}^{\kappa_\ell} \cap \mathbf{V}^{\kappa_r} = \mathbf{V}_{\kappa_\ell} \vee \mathbf{V}_{\kappa_r}$.
- $\mathbf{V} \vee \mathbf{W} = \max(\mathbf{V}^{\kappa_\ell}, \mathbf{W}^{\kappa_\ell}) \cap \max(\mathbf{V}^{\kappa_r}, \mathbf{W}^{\kappa_r})$.

Wismath [33] showed that the lattice of pseudovarieties of band monoids consists exactly of the trivial pseudovariety, $\mathbf{B} \cap \mathbf{M}$, $\mathbf{SL} \cap \mathbf{M}$ and the intersections with $\mathbf{M}$ of the pseudovarieties obtained from $\mathbf{SL}$ using the operators κ_ℓ, κ_r and intersection. So it corresponds to the solid line portion of Figure 1 (excluding any intersection with the non solid lines). From Section 1.3 and Proposition 1.7, it follows that the pseudovarieties in that portion of Figure 1 are supermonoidal.

If, for each monoidal pseudovariety of bands $\mathbf{V}$ strictly contained in $\mathbf{B}$, we let $\mathbf{V}'$ be the largest pseudovariety containing $\mathbf{V}$ and containing no larger monoidal pseudovarieties, then $\mathbf{V}' = L\mathbf{V} \cap \mathbf{B}$. For instance, $\mathbf{RNB}^{\kappa_\ell} = L(\mathbf{SL}^{\kappa_\ell}) \cap \mathbf{B}$. On the operator $\mathbf{V} \mapsto \mathbf{V}'$ in $Ps(\mathbf{B})$ and on related operators, see [34].

In [11], defining identities are given for each (pseudo) variety of bands, along with word problem solutions for the relatively free bands. Let us briefly review these identities. First we have

$$\begin{aligned} \mathbf{SL} &= [\![x_1x_2 = x_2x_1]\!]_{\mathbf{B}} \\ \mathbf{LNB} &= [\![x_1x_2x_3 = x_1x_3x_2]\!]_{\mathbf{B}} \\ \mathbf{RNB} &= [\![x_1x_2x_3 = x_2x_1x_3]\!]_{\mathbf{B}} \end{aligned}$$

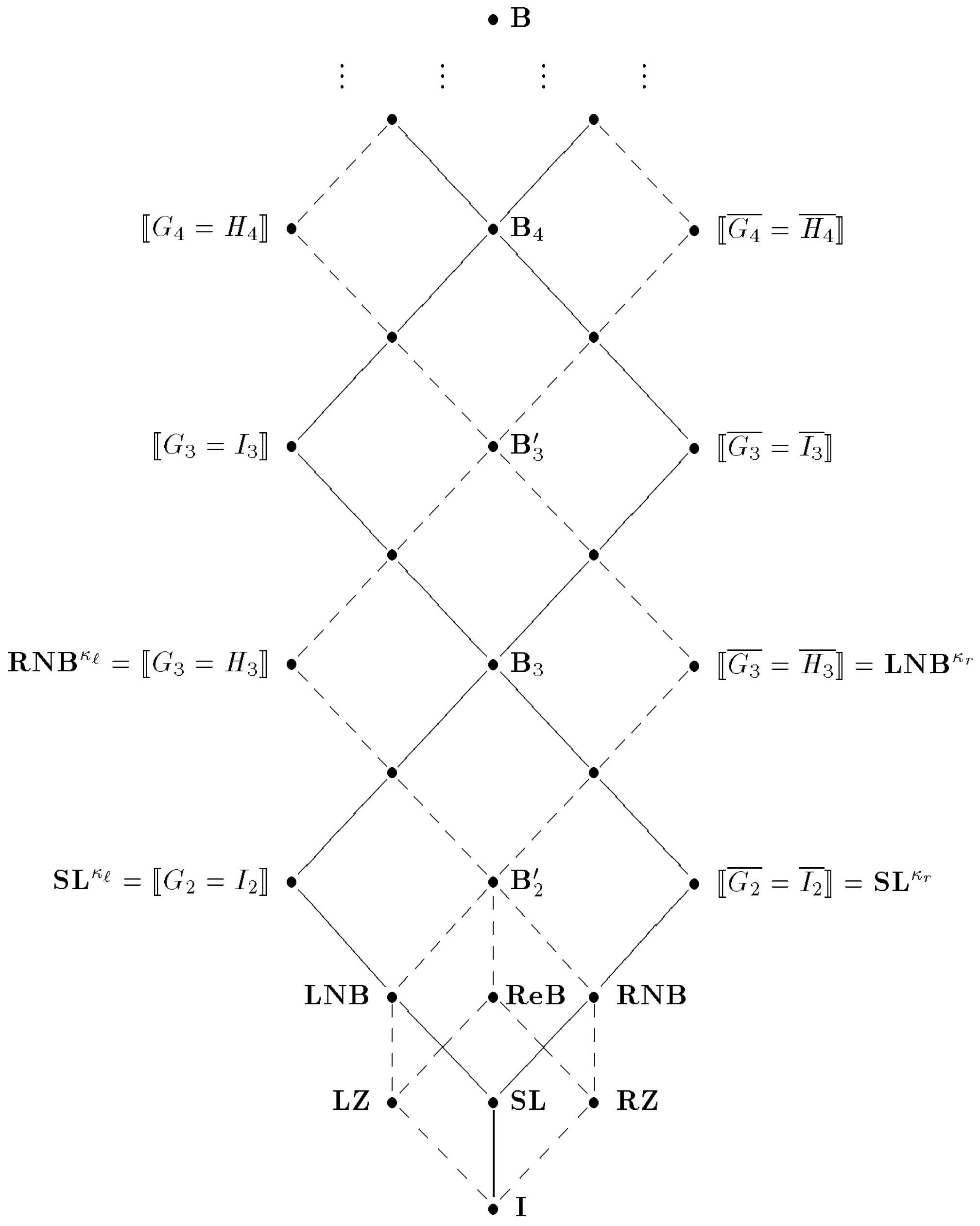

Figure 1: The lattice $Ps(\mathbf{B})$

Now, if $w = a_1a_2\cdots a_k$, we denote by $\overline{w}$ the *mirror image* of w, namely the word $\overline{w} = a_k\cdots a_2a_1$. Let $x_1, x_2, \ldots$ be distinct variables. We define

$$G_2 = x_2x_1, \qquad H_2 = x_2, \qquad I_2 = x_2x_1x_2,$$

and for $n > 2$

$$G_n = x_n\overline{G_{n-1}}, \qquad H_n = G_nx_n\overline{H_{n-1}}, \qquad I_n = G_nx_n\overline{I_{n-1}}.$$

Then, for $n \geq 0$,

$$\begin{array}{ll}
\mathbf{SL}^{(\kappa_\ell\kappa_r)^n\kappa_\ell} = [\![G_{2n+2} = I_{2n+2}]\!]_{\mathbf{B}} & \mathbf{SL}^{(\kappa_r\kappa_\ell)^{n+1}} = [\![G_{2n+3} = I_{2n+3}]\!]_{\mathbf{B}} \\
\mathbf{SL}^{(\kappa_r\kappa_\ell)^n\kappa_r} = [\![\overline{G_{2n+2}} = \overline{I_{2n+2}}]\!]_{\mathbf{B}} & \mathbf{SL}^{(\kappa_\ell\kappa_r)^{n+1}} = [\![\overline{G_{2n+3}} = \overline{I_{2n+3}}]\!]_{\mathbf{B}} \\
\mathbf{RNB}^{(\kappa_\ell\kappa_r)^n\kappa_\ell} = [\![G_{2n+3} = H_{2n+3}]\!]_{\mathbf{B}} & \mathbf{RNB}^{(\kappa_\ell\kappa_r)^{n+1}} = [\![\overline{G_{2n+4}} = \overline{H_{2n+4}}]\!]_{\mathbf{B}} \\
\mathbf{LNB}^{(\kappa_r\kappa_\ell)^n\kappa_r} = [\![\overline{G_{2n+3}} = \overline{H_{2n+3}}]\!]_{\mathbf{B}} & \mathbf{LNB}^{(\kappa_r\kappa_\ell)^{n+1}} = [\![G_{2n+4} = H_{2n+4}]\!]_{\mathbf{B}}
\end{array}$$

(see Fig. 1). As for the solution of the word problems in the corresponding relatively free bands, let us note the following, which is immediately derived from [12]. Let $\mathbf{V}$ be a pseudovariety of bands containing $\mathbf{SL}$, let $\gamma\colon A^+ \to F_A(\mathbf{V})$, $\delta\colon A^+ \to F_A(\mathbf{V}^{\kappa_\ell})$ and $\delta'\colon A^+ \to F_A(\mathbf{V}^{\kappa_r})$ be the natural projections. Then, for all $w, w' \in A^+$,

$$\begin{array}{rll}
\delta(w) = \delta(w') & \text{if and only if} & \gamma(w) = \gamma(w'),\ \delta(0(w)) = \delta(0(w')) \\
& & \text{and } \bar{0}(w) = \bar{0}(w'), \\
\delta'(w) = \delta'(w') & \text{if and only if} & \gamma(w) = \gamma(w'),\ \delta'(1(w)) = \delta'(1(w')) \\
& & \text{and } \bar{1}(w) = \bar{1}(w').
\end{array}$$

In the sequel, and in particular in Section 5, we will use the following notation: we let $\mathbf{B}_2 = \mathbf{SL}$ and, for each $k \geq 2$, $\mathbf{B}_{k+1} = \mathbf{B}_k^{\kappa_\ell\kappa_r} \cap \mathbf{B}_k^{\kappa_r\kappa_\ell}$. We also let $\mathbf{B}'_k = L\mathbf{B}_k \cap \mathbf{B}$ for each $k \geq 2$. From [11, Thm. 9.1], it is easy to check that $\mathbf{B}'_k$ is the smallest pseudovariety of bands that contains the free band on k generators, for $k \geq 2$. In other words, the free band on k generators generates the pseudovariety $\mathbf{B}'_k$.

2.3 Properties of the mapping $\mathbf{V} \mapsto \mathbf{V} \cap \mathbf{B}$

Let us first observe that the mapping $\mathbf{V} \mapsto \mathbf{V} \cap \mathbf{B}$ can be defined on $Ps(\mathbf{DA})$ as well as on $Ps_{\mathbf{M}}(\mathbf{DA})$, the lattice of pseudovarieties of monoids contained in $\mathbf{DA}$. Most of the results which follow hold for both mappings, even though they will often be stated in the framework of pseudovarieties of semigroups only.

It is immediate that this mapping preserves intersections. Reilly and Zhang proved that it preserves more [29].

Theorem 2.2 *The mapping $\mathbf{V} \mapsto \mathbf{V} \cap \mathbf{B}$ is a complete lattice morphism, from $Ps(\mathbf{DA})$ onto $Ps(\mathbf{B})$, and hence each class of the partition it induces on $Ps(\mathbf{DA})$, is an interval.*

In view of the role played by the operations $\mathbf{X} \mapsto \mathbf{X}^{\kappa_\ell}$, $\mathbf{X} \mapsto \mathbf{X}^{\kappa_r}$ and $\mathbf{X} \mapsto L\mathbf{X} \cap \mathbf{B}$ in the lattice $Ps(\mathbf{B})$, it is interesting to note the following results.

Proposition 2.3 *Let $\mathbf{V}$ be a pseudovariety of monoids. Then $L\mathbf{V} \cap \mathbf{B} = L(\mathbf{V} \cap \mathbf{B}) \cap \mathbf{B}$.*

Proof. The inclusion $L(\mathbf{V} \cap \mathbf{B}) \cap \mathbf{B} \subseteq L\mathbf{V} \cap \mathbf{B}$ is trivial. Conversely, if $S \in L\mathbf{V} \cap \mathbf{B}$, then for each idempotent e of S, the monoid eSe is in $\mathbf{V}$ and it is a band, so $eSe \in \mathbf{V} \cap \mathbf{B}$. $\square$

Proposition 2.4 *If* $\mathbf{V}$ *is a pseudovariety of semigroups or monoids, then* $\mathbf{V}^{\mathbf{K}} \cap \mathbf{B} = (\mathbf{V} \cap \mathbf{B})^{\kappa_\ell}$ *and* $\mathbf{V}^{\mathbf{D}} \cap \mathbf{B} = (\mathbf{V} \cap \mathbf{B})^{\kappa_r}$.

Proof. We know that $(\mathbf{V} \cap \mathbf{B})^{\kappa_\ell}$ is equal to $(\mathbf{V} \cap \mathbf{B})^{\mathbf{K}} \cap \mathbf{B}$, which is clearly contained in $\mathbf{V}^{\mathbf{K}} \cap \mathbf{B}$. Conversely, if $S \in \mathbf{V}^{\mathbf{K}} \cap \mathbf{B}$, then there exists a morphism φ from S onto an element V of $\mathbf{V}$ such that $\varphi^{-1}(e) \in \mathbf{K}$ for each idempotent e of V. But S is a band, so $V \in \mathbf{V} \cap \mathbf{B}$ and each $\varphi^{-1}(e)$ lies in fact in $\mathbf{K} \cap \mathbf{B} = \mathbf{LZ}$. That is, $S \in \mathbf{LZ} ⓜ (\mathbf{V} \cap \mathbf{B}) = (\mathbf{V} \cap \mathbf{B})^{\kappa_\ell}$. The proof of the equality $\mathbf{V}^{\mathbf{D}} \cap \mathbf{B} = (\mathbf{V} \cap \mathbf{B})^{\kappa_r}$ is by left-right duality. $\square$

Corollary 2.5 *Let* $\mathbf{V}$ *be a pseudovariety of bands. Then* $\mathbf{V}_{\mathbf{K}} = \mathbf{V}_{\kappa_\ell}$ *and* $\mathbf{V}_{\mathbf{D}} = \mathbf{V}_{\kappa_r}$.

Proof. By definition, $\mathbf{V}_{\kappa_\ell}$ is the least pseudovariety of bands $\mathbf{X}$ such that $\mathbf{X}^{\kappa_\ell} = \mathbf{V}^{\kappa_\ell}$. By Proposition 2.4, $\mathbf{X}^{\kappa_\ell} = \mathbf{X}^{\mathbf{K}} \cap \mathbf{B}$ and $\mathbf{V}^{\kappa_\ell} = \mathbf{V}^{\mathbf{K}} \cap \mathbf{B}$. Since ${\mathbf{V}_{\mathbf{K}}}^{\mathbf{K}} = \mathbf{V}^{\mathbf{K}}$, it follows that $\mathbf{V}_{\kappa_\ell} \subseteq \mathbf{V}_{\mathbf{K}}$.

Conversely, suppose that $\mathbf{X}^{\kappa_\ell} = \mathbf{V}^{\kappa_\ell}$. Let S be a band and let ρ be the greatest congruence on S with idempotent classes in $\mathbf{K}$. If $S \in \mathbf{V}$, then $S \in \mathbf{V}^{\kappa_\ell} = \mathbf{X}^{\kappa_\ell} \subseteq \mathbf{X}^{\mathbf{K}}$, so $S/\rho \in \mathbf{X}$. Since $\mathbf{V}_{\mathbf{K}}$ is generated by the S/ρ $(S \in \mathbf{V})$, it follows that $\mathbf{V}_{\mathbf{K}} \subseteq \mathbf{X}$, and hence $\mathbf{V}_{\mathbf{K}} \subseteq \mathbf{V}_{\kappa_\ell}$. $\square$

We will see however in Section 5.4 an example of a subpseudovariety of $\mathbf{DA}$ such that $\mathbf{V}_{\mathbf{K}} \cap \mathbf{B} \neq (\mathbf{V} \cap \mathbf{B})_{\kappa_\ell}$ and $\mathbf{V}_{\mathbf{D}} \cap \mathbf{B} \neq (\mathbf{V} \cap \mathbf{B})_{\kappa_r}$.

3 Pseudovarieties of the form $\mathbf{V}^{\mathbf{K}}$, $\mathbf{V}^{\mathbf{D}}$ and $\mathbf{V}^{L\mathbf{I}}$

In this section, we want to give more information on the subpseudovarieties of $\mathbf{DS}$ of the form $\mathbf{V}^{\mathbf{K}}$, $\mathbf{V}^{\mathbf{D}}$ and $\mathbf{V}^{L\mathbf{I}}$. It is known that $\mathbf{DA} = \mathbf{SL}^{L\mathbf{I}}$ is the union of a strictly ascending chain of pseudovarieties of the form $\mathbf{V}^{\mathbf{K}}$ and $\mathbf{V}^{\mathbf{D}}$ starting from $\mathbf{SL}$. In Section 3.3 we will give a description of the pro-identities satisfied by the pseudovarieties of the form $\mathbf{V}^{\mathbf{K}}$, $\mathbf{V}^{\mathbf{D}}$ and $\mathbf{V}^{L\mathbf{I}}$ when $\mathbf{SL} \subseteq \mathbf{V} \subseteq \mathbf{DS}$, in terms of the pro-identities satisfied by $\mathbf{V}$. These results constitute an extension of the solution of the word problem in relatively free bands reported in Sec 2.2.

3.1 Left, right and central basic factorizations

Let $\mathbf{V}$ be a pseudovariety containing $\mathbf{SL}$ and let $x \in \hat{F}_A(\mathbf{V})$. A *left basic factorization* of x is a factorization $x = par$ with $a \in A$, $p, r \in \hat{F}_A(\mathbf{V}) \cup \{1\}$, $a \notin c(p)$ and $c(pa) = c(x)$. (By convention, $c(1) = \emptyset$.) Similarly, a *right basic factorization* of x is a factorization $x = rbq$ with $b \in A$, $q, r \in \hat{F}_A(\mathbf{V}) \cup \{1\}$, $b \notin c(q)$ and $c(bq) = c(x)$. Finally, a *central basic factorization* of x is a factorization of one of the following forms:

standard form: $x = parbq$ with $a, b \in A$, $p, q, r \in \hat{F}_A(\mathbf{V}) \cup \{1\}$, $a \notin c(p)$, $b \notin c(q)$ and $c(pa) = c(bq) = c(x)$;

overlapping form: $x = pbraq$ with $a, b \in A$, $p, q, r \in \hat{F}_A(\mathbf{V}) \cup \{1\}$, $a \notin c(pbr)$, $b \notin c(raq)$ (in particular, $a, b \notin c(r)$) and $c(pbra) = c(braq) = c(x)$;

degenerate form: $x = paq$ with $a \in A$, $p, q \in \hat{F}_A(\mathbf{V}) \cup \{1\}$, $a \notin c(p)$, $a \notin c(q)$ and $c(pa) = c(aq) = c(x)$.

Each word from A^+ admits unique left, right and central basic factorizations. Using the notation of Section 2.1, the left and right basic factorizations of the word w are given by $w = 0(w)\bar{0}(w)w'$ and $w = w''\bar{1}(w)1(w)$. The central basic factorization of w simply expresses the degree of overlapping between these two factorizations. More generally, every element x of $\hat{F}_A(\mathbf{V})$ admits a left basic factorization. Let indeed $(u_n)_{n>0}$ be a sequence of words of A^+ converging to x, and for each n, let $u_n = p_n a_n r_n$ be the left basic factorization of u_n. Using the finiteness of A and the compactness of $\hat{F}_A(\mathbf{V})$, and up to extracting a subsequence of $(u_n)_n$, we may assume that the a_n are all equal and that the sequences $(p_n)_n$ and $(r_n)_n$ converge. If a is the common value of the a_n, if $p = \lim_n p_n$ and $r = \lim_n r_n$, then $x = par$ is a left basic factorization of x. Similarly, one can verify that each element x of $\hat{F}_A(\mathbf{V})$ admits a right basic factorization and a central basic factorization. In contrast with the case of words, x may admit several distinct such factorizations. For instance, if $\mathbf{V} = \mathbf{J}$, then $(ab)^\omega = (ba)^\omega$, so $(ab)^\omega = a\, b\, (ab)^\omega$ and $(ab)^\omega = b\, a\, (ab)^\omega$ are distinct left basic factorizations of the same element of $\hat{F}_A(\mathbf{J})$. However, Almeida proved that for certain pseudovarieties $\mathbf{V}$, the left, right and central basic factorizations of the elements of $\hat{F}_A(\mathbf{V})$ are unique [2, Prop. 3.4]. For such pseudovarieties, the left and right basic factorizations can be viewed as generalizations of the factorizations exhibiting the prefix $0(w)$ and the suffix $1(w)$ of a word w.

Proposition 3.1 *Let* $\mathbf{V}$ *be a pseudovariety containing* $\mathbf{SL}$*.*

1. *If* $\mathbf{V} = \mathbf{V}^{\mathbf{K}}$*, then every element of* $\hat{F}_A(\mathbf{V})$ *admits a unique left basic factorization.*
2. *If* $\mathbf{V} = \mathbf{V}^{\mathbf{D}}$*, then every element of* $\hat{F}_A(\mathbf{V})$ *admits a unique right basic factorization.*
3. *If* $\mathbf{V} = \mathbf{V}^{\mathbf{LI}}$*, then every element of* $\hat{F}_A(\mathbf{V})$ *admits a unique central basic factorization.*

In fact, [2, Prop. 3.4] only states and proves Statement (3) of Proposition 3.1, and uses the term "basic boundary factorization" for what we call central basic factorization. The other two statements can easily be derived using the same proof technique (involving classical results of rational language theory). A full proof of Statement (1) can be found in [4, Prop. 2.9]. We will also need the following result, which can be proved using the same technique, or which can be derived inductively from Proposition 3.1.

Proposition 3.2 *Let* $\mathbf{V}$ *be a pseudovariety containing* $\mathbf{SL}$ *and such that* $\mathbf{V} = \mathbf{V}^{\mathbf{K}}$ *(resp.* $\mathbf{V} = \mathbf{V}^{\mathbf{D}}$*). Let* A *be a finite set, let* $x \in \hat{F}_A(\mathbf{V})$*, and let* B *be a subset of* A *such that* $c(x) \not\subseteq B$*. Then* x *admits a unique longest prefix (resp. suffix) with content in* B*. More precisely, there exists a unique triple* (x_1, b, x_2) *(resp.* (x_2, b, x_1)*) such that* $x_1, x_2 \in \hat{F}_A(\mathbf{V}) \cup \{1\}$*,* $c(x_1) \subseteq B$*,* $b \in A \setminus B$ *and* $x = x_1 b x_2$ *(resp.* $x = x_2 b x_1$*).*

Note. Observe that $\mathbf{V} = \mathbf{V}^{\mathbf{LI}}$ implies $\mathbf{V}^{\mathbf{K}} = \mathbf{V} = \mathbf{V}^{\mathbf{D}}$. So if $\mathbf{V} = \mathbf{V}^{\mathbf{LI}}$, then each element x of $\hat{F}_A(\mathbf{V})$ admits a unique central, left and right basic factorization. In fact, if the central basic factorization of x is standard, $x = parbq$, then the left (resp. right) basic factorization of x is $x = p\, a\, (rbq)$ (resp. $x = (par)\, b\, q$). If the central basic factorization of x is overlapping, $x = pbraq$, then the left (resp. right) basic factorization of x is $x = (pbr)\, a\, q$ (resp. $x = p\, b\, (raq)$). Finally, if the central basic factorization of x is degenerate, $x = paq$, then the left (resp. right) basic factorization of x is $x = p\, a\, q$ (resp. $x = p\, a\, q$).

3.2 Iterating the basic factorizations

Let $\mathbf{V}$ be a pseudovariety containing $\mathbf{SL}$ and such that $\mathbf{V} = \mathbf{V}^{\mathbf{K}}$. Let $x = p_1a_1r_1$ be the left basic factorization of $x \in \hat{F}_A(\mathbf{V})$. If $c(r_1) = c(x)$, we may consider the left basic factorization $r_1 = p_2a_2r_2$ of r_1, thus yielding a new factorization of x, $x = p_1a_1p_2a_2r_2$. We iterate this process as long as $c(r_k) = c(x)$: in that case, we let $r_k = p_{k+1}a_{k+1}r_{k+1}$ be the left basic factorization of r_k, so $x = p_1a_1 \cdots p_{k+1}a_{k+1}r_{k+1}$: this is the $(k+1)$-st iteration of the left basic factorization of x, and r_{k+1} is the $(k+1)$-st remainder in this factorization. The values of the p_i, a_i and r_i are uniquely determined in $\hat{F}_A(\mathbf{V}) \cup \{1\}$ by Proposition 3.1. If $c(r_n) \neq c(x)$ for some $n \geq 1$, we say that $x = p_1a_1 \cdots p_na_nr_n$ is the *iterated left basic factorization* of x, that r_n is the *remainder* in this factorization, and that this factorization has length n. If $c(r_n) = c(x)$ for each n, we say that x has an *infinite iterated left basic factorization.*

If $\mathbf{V}$ is a pseudovariety containing $\mathbf{SL}$ and such that $\mathbf{V} = \mathbf{V}^{\mathbf{D}}$, then the iterated right basic factorizations of the elements of $\hat{F}_A(\mathbf{V})$ are defined dually.

Now let $\mathbf{V}$ be a pseudovariety containing $\mathbf{SL}$, such that $\mathbf{V} = \mathbf{V}^{L\mathbf{I}}$, and let $x \in \hat{F}_A(\mathbf{V})$. Let $r_0 = x$. If $c(r_k) = c(x)$, we consider the central basic factorization of r_k. If this factorization is standard, $r_k = p_{k+1}a_{k+1}r_{k+1}b_{k+1}q_{k+1}$, then the $(k+1)$-st iteration of the central basic factorization of x is $x = p_1a_1 \cdots p_{k+1}a_{k+1}r_{k+1}b_{k+1}q_{k+1} \cdots b_1q_1$, and r_{k+1} is the $(k+1)$-st remainder in the central basic factorization of x. We iterate this process as long as $c(r_k) = c(x)$ and r_k admits a standard central basic factorization. If $c(r_n) \neq c(x)$ for some $n \geq 1$, we say that $x = p_1a_1 \cdots p_na_nr_nb_nq_n \cdots b_1q_1$ is the *iterated central basic factorization* of x, that this factorization is *standard*, and that it has length n. If $c(r_{n-1}) = c(x)$ and r_{n-1} admits an overlapping central basic factorization ($n \geq 1$), $r_{n-1} = p_nb_nr_na_nq_n$, we say that $x = p_1a_1 \cdots p_{n-1}a_{n-1}p_nb_nr_na_nq_nb_{n-1}q_{n-1} \cdots b_1q_1$ is the *iterated central basic factorization* of x, that this factorization is *overlapping*, and that it has length n. If $c(r_{n-1}) = c(x)$ and r_{n-1} admits a degenerate central basic factorization ($n \geq 1$), $r_{n-1} = p_na_nq_n$, we say that $x = p_1a_1 \cdots p_{n-1}a_{n-1}p_na_nq_nb_{n-1}q_{n-1} \cdots b_1q_1$ is the *iterated central basic factorization* of x, that this factorization is *degenerate*, and that it has length n. The *remainder* in the iterated central basic factorization of x is r_n (1 in the degenerate case). Finally, if $c(r_n) = c(x)$ for all n, we say that x *admits an infinite iterated central basic factorization.*

Note that the factors occurring in the iterated left (resp. right, central) basic factorization of x, that is the p_i (resp. the q_i, the p_i and q_i) and the remainders (in the case of finite-length iterated factorizations) all have strictly smaller contents than x.

Proposition 3.3 *Let* $\mathbf{V}$ *be a pseudovariety containing* $\mathbf{SL}$ *and contained in* $\mathbf{DS}$. *If* $\mathbf{V} = \mathbf{V}^{\mathbf{K}}$ *(resp.* $\mathbf{V} = \mathbf{V}^{\mathbf{D}}$, $\mathbf{V} = \mathbf{V}^{L\mathbf{I}}$*), then* $x \in \hat{F}_A(\mathbf{V})$ *admits an infinite iterated left (resp. right, central) basic factorization if and only if* x *is regular.*

Proof. If x admits an infinite iterated left (resp. right, central) basic factorization, then x is regular by Proposition 1.8. Conversely, let us assume that $x \in \hat{F}_A(\mathbf{V})$ is regular. If $\mathbf{V} = \mathbf{V}^{\mathbf{K}}$ and $x = p_1a_1 \cdots p_na_nr_n$ is the n-th iteration of the left basic factorization of x ($n \geq 1$),then $x = xx^{\omega} = p_1a_1 \cdots p_na_n(r_nx^{\omega})$, so by Proposition 3.1, $r_n = r_nx^{\omega}$, and hence $c(r_n) = c(x)$. Thus x has an infinite iterated left basic factorization.

The statement concerning right basic factorization is proved in the same fashion. Let us now assume that $\mathbf{V} = \mathbf{V}^{L\mathbf{I}}$. Since x is regular, the central basic factorization of x must be standard: indeed, let $x = par$ and $x = sbq$ be the left and right basic factorizations of x. Then $x = xx^{\omega-1}x$, so $x = pa(rx^{\omega-1}s)bq$ is the central basic factorization of x and it is of the standard form. Moreover, the central factor, $rx^{\omega-1}s$, has the same content as x and is regular as well. It follows that x admits an infinite iterated central basic factorization. □

We will use the following technical lemma in the sequel.

Lemma 3.4 *Let* $\mathbf{V}$ *be a pseudovariety containing* $\mathbf{SL}$. *Let* $r, q_1, \ldots, q_n \in \hat{F}_A(\mathbf{V}) \cup \{1\}$ *and let* $b_1, \ldots, b_n \in A$ *be such that* $b_i \notin c(q_i)$ *and* $c(b_i q_i) = c(b_1 q_1)$ *for each* i, *and such that* $c(r)$ *is strictly contained in* $c(b_1 q_1)$. *If* $\mathbf{V} = \mathbf{V}^{\mathbf{K}}$, *then the iterated left basic factorization of* $r b_n q_n \cdots b_1 q_1$ *has length exactly* n.

Proof. Let $r b_n q_n = zct$ be the left basic factorization of $r b_n q_n$. We first verify that $c(t)$ is strictly contained in $c(b_1 q_1) = c(b_n q_n)$. The result is clear if r and q_n are words: in that case, zc is the shortest prefix of $r b_n q_n$ with content $c(b_1 q_1)$, and that is at least as long as $r b_n$. In general, r and q_n are limits of sequences of words, say $(v_k)_k$ and $(u_k)_k$ with contents respectively $c(r)$ and $c(q_n)$. The left basic factorization of $v_k b_n u_k$, say $v_k b_n u_k = z_k c_k t_k$, is such that $c(t_k)$ is strictly contained in $c(b_1 q_1)$. Up to extracting a subsequence, we may assume that $(c_k)_k$ is a constant sequence, with value c, and that the sequences $(z_k)_k$ and $(t_k)_k$ converge to, say, z and t. Then the iterated left basic factorization of $r b_n q_n$ is $r b_n q_n = zct$, and $c(t)$ is strictly contained in $c(b_1 q_1)$.

So the announced result holds for $n = 1$. If $n > 1$, observe that the left basic factorization of $r b_n q_n \cdots b_1 q_1$ is $r b_n q_n \cdots b_1 q_1 = zc(t b_{n-1} \cdots b_1 q_1)$. So, by induction, the iterated left basic factorization of $r b_n q_n \cdots b_1 q_1$ has length n. □

Corollary 3.5 *Let* $\mathbf{V}$ *be a pseudovariety such that* $\mathbf{V} = \mathbf{V}^{L\mathbf{I}}$ *and let* $x \in \hat{F}_A(\mathbf{V})$. *The iterated right and left factorizations of* x *have the same length.*

3.3 Pro-identities satisfied by $\mathbf{V}^{\mathbf{K}}$, $\mathbf{V}^{\mathbf{D}}$ and $\mathbf{V}^{L\mathbf{I}}$

Here we give a characterization of the pro-identities $u = v$ satisfied by $\mathbf{V}^{\mathbf{K}}$ (resp. $\mathbf{V}^{\mathbf{D}}$, $\mathbf{V}^{L\mathbf{I}}$) in terms of the iterated basic factorizations of u and v. As will appear, these characterizations are strongly reminiscent of the solution of the word problem for the free band, but we cannot describe our results as an algorithmic solution of the "pro-identity" problem for $\mathbf{V}^{\mathbf{K}}$, $\mathbf{V}^{\mathbf{D}}$ and $\mathbf{V}^{L\mathbf{I}}$, since the input of the problem, namely the pair (u, v), cannot in general be given in finite time.

Theorem 3.6 *Let* $\mathbf{V}$ *be a pseudovariety containing* $\mathbf{SL}$ *and contained in* $\mathbf{DS}$, *let* $\mathbf{W}$ *be a pseudovariety containing* $\mathbf{V}$ *and such that* $\mathbf{W} = \mathbf{W}^{\mathbf{K}}$, *and let* $x, y \in \hat{F}_A(\mathbf{W})$. $\mathbf{V}^{\mathbf{K}}$ *satisfies* $x = y$ *if and only if*

1. $\mathbf{V}$ *satisfies* $x = y$;
2. x *and* y *have iterated left basic factorizations of equal length;*
3. *the sequences of letters from* A *(finite or infinite) arising from the iterated left basic factorizations of* x *and* y *are equal;*
4. *the factors occurring in the iterated left basic factorizations of* x *and* y, *and in the case of finite-length factorizations, the remainders, are equal in* $\mathbf{V}^{\mathbf{K}}$.

Theorem 3.7 *Let* $\mathbf{V}$ *be a pseudovariety containing* $\mathbf{SL}$ *and contained in* $\mathbf{DS}$, *let* $\mathbf{W}$ *be a pseudovariety containing* $\mathbf{V}$ *and such that* $\mathbf{W} = \mathbf{W}^{\mathbf{D}}$, *and let* $x, y \in \hat{F}_A(\mathbf{W})$. $\mathbf{V}^{\mathbf{D}}$ *satisfies* $x = y$ *if and only if*

1. $\mathbf{V}$ *satisfies* $x = y$;
2. x *and* y *have iterated right basic factorizations of equal length;*

3. *the sequences of letters from A (finite or infinite) arising from the iterated right basic factorizations of x and y are equal;*

4. *the factors occurring in the iterated right basic factorizations of x and y, and in the case of finite-length factorizations, the remainders, are equal in $\mathbf{V}^{\mathbf{D}}$.*

Theorem 3.8 *Let $\mathbf{V}$ be a pseudovariety containing $\mathbf{SL}$ and contained in $\mathbf{DS}$, let $\mathbf{W}$ be a pseudovariety containing $\mathbf{V}$ and such that $\mathbf{W} = \mathbf{W}^{L\mathbf{I}}$, and let $x, y \in \hat{F}_A(\mathbf{W})$. $\mathbf{V}^{L\mathbf{I}}$ satisfies $x = y$ if and only if*

1. *$\mathbf{V}$ satisfies $x = y$;*

2. *x and y have iterated central basic factorizations of equal length. Moreover, if these iterated factorizations are finite, then they are of the same form (standard, overlapping or degenerate);*

3. *the pairs of sequences of letters from A (finite or infinite) arising from the iterated central basic factorizations of x and y are equal;*

4. *the factors occurring in the iterated central basic factorizations of x and y, and in the case of finite-length factorizations, the remainders, are equal in $\mathbf{V}^{L\mathbf{I}}$.*

Proof of Theorem 3.6. If $\mathbf{V}^{\mathbf{K}}$ satisfies $x = y$, then conditions (1)-(4) hold by Proposition 3.1. So we need to establish the converse implication. In the case where x and y have finite length iterated left basic factorizations, the result is trivial (in fact, in that case, condition (1) plays no role). Let us now assume that x and y admit infinite iterated left basic factorizations, say $x = x_1a_1 \cdots x_na_nr_n$ and $y = y_1a_1 \cdots y_na_ns_n$ for each $n \geq 1$, and let $\pi\colon \hat{F}_A(\mathbf{W}) \to \hat{F}_A(\mathbf{V}^{\mathbf{K}})$ and $\rho\colon \hat{F}_A(\mathbf{V}^{\mathbf{K}}) \to \hat{F}_A(\mathbf{V})$ be the canonical projections: by Proposition 3.3, it follows that $\pi(x)$ and $\pi(y)$ are regular. Condition (4) states that $\pi(x_n) = \pi(y_n)$ for each $n \geq 1$. Let z be a limit of the sequence $(x_1a_1 \cdots x_na_n)_n$ and let r be a limit of the sequence $(r_n)_n$: then $x = zr$, so $x \leq_{\mathcal{R}} z$. In addition, by Proposition 1.8(2), $\pi(z)$ is regular. Now, by Proposition 1.8(1), it follows that $\pi(x)\ \mathcal{R}\ \pi(z)$. Similarly, we verify that $\pi(y)\ \mathcal{R}\ \pi(z)$, so $\pi(x)\ \mathcal{R}\ \pi(y)$.

Condition (1) states that $\rho\pi(x) = \rho\pi(y)$, and hence also $\rho\pi(x^\omega) = \rho\pi(y^\omega)$. Then, by Proposition 1.6, it follows that $\pi(x^\omega)\pi(y^\omega) = \pi(x^\omega)$ and $\pi(y^\omega)\pi(x^\omega) = \pi(y^\omega)$, that is, $\pi(x^\omega)\ \mathcal{L}\ \pi(y^\omega)$. But $\pi(x)$ and $\pi(y)$ are regular, so they are $\mathcal{H}$-equivalent, respectively, to $\pi(x^\omega)$ and $\pi(y^\omega)$. Therefore $\pi(x)\ \mathcal{L}\ \pi(y)$, and hence $\pi(x)\ \mathcal{H}\ \pi(y)$ and $\pi(x^\omega) = \pi(y^\omega)$. Finally $\rho\pi(xy^{\omega-1}) = \rho\pi(x^\omega)$, so $\pi(x^\omega) = \pi(x^\omega xy^{\omega-1}) = \pi(xy^{\omega-1})$. Now, $\pi(y) = \pi(y^\omega y) = \pi(xy^{\omega-1}y) = \pi(xy^\omega) = \pi(xx^\omega) = \pi(x)$. □

Proof of Theorem 3.7. The proof is dual of that of Theorem 3.6. □

Proof of Theorem 3.8. The proof follows the same lines as that of Theorem 3.6. Let π and ρ denote the projections $\pi\colon \hat{F}_A(\mathbf{W}) \to \hat{F}_A(\mathbf{V}^{L\mathbf{I}})$ and $\rho\colon \hat{F}_A(\mathbf{V}^{L\mathbf{I}}) \to \hat{F}_A(\mathbf{V})$. The non-regular case is dealt with as above. Let us now assume that x and y are regular. It is proved as above that $\pi(x)\ \mathcal{R}\ \pi(y)$, and dually $\pi(x)\ \mathcal{L}\ \pi(y)$, so $\pi(x)\ \mathcal{H}\ \pi(y)$, and hence $\pi(x^\omega) = \pi(y^\omega)$. Also, $\rho\pi(x) = \rho\pi(y)$ implies that $\rho\pi(xy^{\omega-1}) = \rho\pi(x^\omega)$. By Proposition 1.6, it follows that

$$\pi(x^\omega) = \pi(x^\omega xy^{\omega-1}x^\omega) = \pi(xy^{\omega-1}y^\omega) = \pi(xy^{\omega-1}).$$

Again, this implies $\pi(x) = \pi(y)$. □

Of particular interest in this paper is the case where $\mathbf{V} = \mathbf{SL}$. In that case, condition (1) in Thms. 3.6, 3.7 and 3.8 vanishes, as it is implied by the other conditions. Recall that $\mathbf{SL}^{L\mathbf{I}} = \mathbf{DA}$, $\mathbf{SL}^{\mathbf{K}} = \mathbf{R}$ and $\mathbf{SL}^{\mathbf{D}} = \mathbf{L}$.

Corollary 3.9 *Every element of $\hat{F}_A(\mathbf{DA})$ admits a unique iterated central basic factorization. If x is regular, then this iterated factorization is infinite, and x is entirely determined by the pair of sequences of letters and of factors arising in the factorization. If x is not regular, then the iterated central basic factorization of x has finite length.*

We will therefore use the following notation for the elements of $\hat{F}_A(\mathbf{DA})$: if x is not regular, then it will be represented by its (finite) iterated central basic factorization; if it is regular and the n-th iteration of its central basic factorization is $x = p_1a_1\cdots p_na_nr_nb_nq_n\cdots b_1q_1$, we will write

$$x = p_1a_1p_2a_2 \quad \cdots \quad b_2q_2b_1q_1.$$

Remark. Similar (one-sided) statements can be formulated for $\hat{F}_A(\mathbf{R})$ and $\hat{F}_A(\mathbf{L})$. For a complete study of these semigroups, see Almeida and Weil [4].

The above characterizations of the pro-identities satisfied by $\mathbf{V}^{\mathbf{K}}$ and $\mathbf{V}^{\mathbf{D}}$ are formally similar to the solution of the word problems in the free object of $\mathbf{V}^{\kappa_\ell}$ and $\mathbf{V}^{\kappa_r}$ reported in Section 2.2 when $\mathbf{V}$ is a pseudovariety of bands. The difference is that it does not suffice to look at the left (right) basic factorization: we must take into account every factor of the iterated left (right) basic factorization. Similarly, the characterization of the elements of $\hat{F}_A(\mathbf{DA})$ given in Corollary 3.9 is a direct analogue of the characterization of the elements of the free band over A.

Let us observe however that these characterizations give a transfinite inductive method for verifying whether a pro-identity $x = y$ holds in a pseudovariety of the form $\mathbf{V}^{\mathbf{K}}$, $\mathbf{V}^{\mathbf{D}}$ or $\mathbf{V}^{L\mathbf{I}}$, when $\mathbf{V}$ is a subpseudovariety of $\mathbf{DS}$ containing $\mathbf{SL}$. Indeed, the satisfaction of $x = y$ in such a variety is reduced to the satisfaction of pro-identities between elements of strictly smaller content than x and y. In the case of regular elements, the number of lesser-content pro-identities to satisfy may be infinite. Nevertheless, it may be said that the above results reduce the pro-identity problem of $\mathbf{V}^{\mathbf{K}}$ (resp. $\mathbf{V}^{\mathbf{D}}$, $\mathbf{V}^{L\mathbf{I}}$) to the analogous problem in $\mathbf{V}$.

4 Charting $Ps(\mathbf{DA})$

We now consider the partition of $Ps(\mathbf{DA})$ induced by the mapping $\mathbf{V} \mapsto \mathbf{V} \cap \mathbf{B}$; two subpseudovarieties of $\mathbf{DA}$, say $\mathbf{V}$ and $\mathbf{W}$, lie in the same class if $\mathbf{V} \cap \mathbf{B} = \mathbf{W} \cap \mathbf{B}$. The fact that the mapping $\mathbf{V} \mapsto \mathbf{V} \cap \mathbf{B}$ is a lattice morphism on $Ps(\mathbf{DA})$ and the knowledge we have of the structure of the lattice $Ps(\mathbf{B})$ make this partition a "meaningful" chart of the lattice $Ps(\mathbf{DA})$.

It is clear that this partition is in bijection with $Ps(\mathbf{B})$: each class contains exactly one pseudovariety of bands, which is the smallest pseudovariety in the class. In this section, we give an effective method to compute, for each pseudovariety of bands $\mathbf{Z}$, the greatest element of the class of $\mathbf{Z}$, namely the subpseudovariety $\mathbf{Z}^\dagger$ of $\mathbf{DA}$ such that, for each $\mathbf{X} \in Ps(\mathbf{DA})$,

$$\mathbf{X} \cap \mathbf{B} = \mathbf{Z} \text{ if and only if } \mathbf{X} \in [\mathbf{Z}, \mathbf{Z}^\dagger].$$

In particular, this provides a new proof of Theorem 2.2.

4.1 Computing $\mathbf{Z}^\dagger$ for small values of $\mathbf{Z}$

In this section we use the detailed information on the structure of the lattice $Ps(\mathbf{B})$ given in Section 2. The existence of $\mathbf{Z}^\dagger$ for $\mathbf{Z} = \mathbf{B}$ and for the smaller values of $\mathbf{Z}$ is easily verified.

Proposition 4.1 *Let $\mathbf{X}$ be a subpseudovariety of $\mathbf{DA}$.*

1. $\mathbf{B}^\dagger = \mathbf{DA}$, *that is,* $\mathbf{X} \cap \mathbf{B} = \mathbf{B}$ *if and only if* $\mathbf{X} \in [\mathbf{B}, \mathbf{DA}]$.
2. $\mathbf{I}^\dagger = \mathbf{N}$, *that is,* $\mathbf{X} \cap \mathbf{B} = \mathbf{I}$ *if and only if* $\mathbf{X} \in [\mathbf{I}, \mathbf{N}]$.
3. $\mathbf{LZ}^\dagger = \mathbf{K}$, *that is,* $\mathbf{X} \cap \mathbf{B} = \mathbf{LZ}$ *if and only if* $\mathbf{X} \in [\mathbf{LZ}, \mathbf{K}]$.
4. $\mathbf{RZ}^\dagger = \mathbf{D}$, *that is,* $\mathbf{X} \cap \mathbf{B} = \mathbf{RZ}$ *if and only if* $\mathbf{X} \in [\mathbf{RZ}, \mathbf{D}]$.
5. $\mathbf{ReB}^\dagger = L\mathbf{I}$, *that is,* $\mathbf{X} \cap \mathbf{B} = \mathbf{ReB}$ *if and only if* $\mathbf{X} \in [\mathbf{ReB}, L\mathbf{I}]$.
6. $\mathbf{SL}^\dagger = \mathbf{J}$, *that is,* $\mathbf{X} \cap \mathbf{B} = \mathbf{SL}$ *if and only if* $\mathbf{X} \in [\mathbf{SL}, \mathbf{J}]$.
7. $(\mathbf{SL}^{\kappa_\ell})^\dagger = \mathbf{J}^{\mathbf{K}} = \mathbf{R}$, *that is,* $\mathbf{X} \cap \mathbf{B} = \mathbf{SL}^{\kappa_\ell}$ *if and only if* $\mathbf{X} \in [\mathbf{SL}^{\kappa_\ell}, \mathbf{J}^{\mathbf{K}}]$.
8. $(\mathbf{SL}^{\kappa_r})^\dagger = \mathbf{J}^{\mathbf{D}} = \mathbf{L}$, *that is,* $\mathbf{X} \cap \mathbf{B} = \mathbf{SL}^{\kappa_r}$ *if and only if* $\mathbf{X} \in [\mathbf{SL}^{\kappa_r}, \mathbf{J}^{\mathbf{D}}]$.

Proof. Statements (1)–(6) are immediate. Statement (8) is the left-right dual of Statement (7). By Proposition 2.4, we know that $\mathbf{J}^{\mathbf{K}} \cap \mathbf{B} = (\mathbf{J} \cap \mathbf{B})^{\kappa_\ell} = \mathbf{SL}^{\kappa_\ell}$. Conversely, if $\mathbf{X}$ is contained in $\mathbf{DA}$ but not in $\mathbf{J}^{\mathbf{K}} = \mathbf{R}$, then $\mathbf{X}$ contains a semigroup with 2 distinct $\mathcal{R}$-related idempotents, so $\mathbf{RZ} \subseteq \mathbf{X}$. But $\mathbf{RZ}$ is not contained in $\mathbf{SL}^{\kappa_\ell}$, so $\mathbf{X} \subseteq \mathbf{R}$. $\square$

4.2 Some properties of $\mathbf{Z}^\dagger$

If $\mathbf{Z}$ is a pseudovariety of band monoids, we denote in the same fashion by $\mathbf{Z}^\dagger$ the maximum subpseudovariety of $\mathbf{DA} \cap \mathbf{M}$ such that $\mathbf{Z}^\dagger \cap \mathbf{B} = \mathbf{Z}$, if it exists. The next proposition shows that it suffices to compute $\mathbf{Z}^\dagger$ for pseudovarieties of band semigroups.

If $\mathbf{V}$ is a class of finite monoids, we denote by $\langle \mathbf{V} \rangle$ the (monoidal) pseudovariety of semigroups generated by the class $\mathbf{V}$.

Proposition 4.2 *Let $\mathbf{Z}$ be a monoidal pseudovariety of bands. If $\mathbf{Z}^\dagger$ exists, then $(\mathbf{Z} \cap \mathbf{M})^\dagger$ exists and $(\mathbf{Z} \cap \mathbf{M})^\dagger = \mathbf{Z}^\dagger \cap \mathbf{M}$.*

Proof. By definition of monoidal pseudovarieties, if $\mathbf{X}$ is monoidal, then so is $\mathbf{X} \cap \mathbf{B}$, and hence $\mathbf{X} = \langle \mathbf{X} \cap \mathbf{M} \rangle$ and $\mathbf{X} \cap \mathbf{B} = \langle \mathbf{X} \cap \mathbf{B} \cap \mathbf{M} \rangle$.

We have immediately $(\mathbf{Z}^\dagger \cap \mathbf{M}) \cap \mathbf{B} = \mathbf{Z} \cap \mathbf{M}$. If $\mathbf{X}$ is a pseudovariety of monoids contained in $\mathbf{DA}$ and $\mathbf{X} \cap \mathbf{B} = \mathbf{Z} \cap \mathbf{M}$, then $\langle \mathbf{X} \rangle \cap \mathbf{B} = \langle \mathbf{X} \cap \mathbf{B} \rangle = \langle \mathbf{Z} \cap \mathbf{M} \rangle = \mathbf{Z}$. Therefore $\langle \mathbf{X} \rangle \subseteq \mathbf{Z}^\dagger$, and hence $\mathbf{X} \subseteq \mathbf{Z}^\dagger \cap \mathbf{M}$. Thus $(\mathbf{Z} \cap \mathbf{M})^\dagger$ exists, and it is equal to $\mathbf{Z}^\dagger \cap \mathbf{M}$. $\square$

We also observe the following rules for the computation of the operation $\mathbf{Z} \mapsto \mathbf{Z}^\dagger$.

Proposition 4.3 *Let $\mathbf{Z}$ be a pseudovariety of band semigroups. If $(\mathbf{Z}^{\kappa_\ell})^\dagger$ and $(\mathbf{Z}^{\kappa_r})^\dagger$ exist, then $\mathbf{Z}^\dagger$ exists, and $\mathbf{Z}^\dagger = (\mathbf{Z}^{\kappa_\ell})^\dagger \cap (\mathbf{Z}^{\kappa_r})^\dagger$.*

Proof. It is clear that $((\mathbf{Z}^{\kappa_\ell})^\dagger \cap (\mathbf{Z}^{\kappa_r})^\dagger) \cap \mathbf{B} = \mathbf{Z}^{\kappa_\ell} \cap \mathbf{Z}^{\kappa_r} = \mathbf{Z}$. If $\mathbf{X} \in Ps(\mathbf{DA})$ and $\mathbf{X} \cap \mathbf{B} = \mathbf{Z}$, then $\mathbf{X}^{\mathbf{K}} \cap \mathbf{B} = \mathbf{Z}^{\kappa_\ell}$ and $\mathbf{X}^{\mathbf{D}} \cap \mathbf{B} = \mathbf{Z}^{\kappa_r}$ by Proposition 2.4, so $\mathbf{X}^{\mathbf{K}} \subseteq (\mathbf{Z}^{\kappa_\ell})^\dagger$ and $\mathbf{X}^{\mathbf{D}} \subseteq (\mathbf{Z}^{\kappa_r})^\dagger$, thus concluding the proof. $\square$

Proposition 4.4 *Let* $\mathbf{Z}$ *be a pseudovariety of band monoids. If* $\mathbf{Z}^{\uparrow}$ *exists (in* $Ps_{\mathbf{M}}(\mathbf{DA})$*), then* $(L\mathbf{Z} \cap \mathbf{B})^{\uparrow}$ *exists, and* $(L\mathbf{Z} \cap \mathbf{B})^{\uparrow} = L(\mathbf{Z}^{\uparrow}) \cap \mathbf{DA}$.

Proof. It follows from Proposition 2.3 that $L(\mathbf{Z}^{\uparrow}) \cap \mathbf{B} = L(\mathbf{Z}^{\uparrow} \cap \mathbf{B}) \cap \mathbf{B} = L\mathbf{Z} \cap \mathbf{B}$. Let now $\mathbf{X} \in Ps(\mathbf{DA})$ such that $\mathbf{X} \cap \mathbf{B} = L\mathbf{Z} \cap \mathbf{B}$. Then $\mathbf{X} \cap \mathbf{M} \cap \mathbf{B} = L\mathbf{Z} \cap \mathbf{B} \cap \mathbf{M} = \mathbf{Z} \cap \mathbf{B} = \mathbf{Z}$. By hypothesis, it follows that the pseudovariety of monoids $\mathbf{X} \cap \mathbf{M}$ is contained in $\mathbf{Z}^{\uparrow}$, and hence $\mathbf{X} \subseteq L(\mathbf{Z}^{\uparrow})$. □

Let $L_{\ell}\mathbf{J}$ be the class of all semigroups S such that $eS \in \mathbf{J}$ for each idempotent of S. It is easily verified that $L_{\ell}\mathbf{J} = L\mathbf{J} \cap \mathbf{R}$. The definition and properties of $L_r\mathbf{J}$ are the left-right dual of those of $L_{\ell}\mathbf{J}$. We have the following formulæ.

Corollary 4.5 *We have*

$$\mathbf{LNB}^{\kappa_r \uparrow} = L\mathbf{L} \cap \mathbf{DA}, \qquad \mathbf{RNB}^{\kappa_\ell \uparrow} = L\mathbf{R} \cap \mathbf{DA},$$
$$\mathbf{B}_2'^{\uparrow} = L\mathbf{J} \cap \mathbf{DA},$$
$$\mathbf{LNB}^{\uparrow} = L_{\ell}\mathbf{J}, \qquad \textit{and} \quad \mathbf{RNB}^{\uparrow} = L_r\mathbf{J}.$$

4.3 Computing $\mathbf{Z}^{\uparrow}$ when $\mathbf{Z} = \mathbf{Z}^{\kappa_\ell}$ or $\mathbf{Z} = \mathbf{Z}^{\kappa_r}$

In view of the structure of the lattices $Ps(\mathbf{B})$ and $Ps_{\mathbf{M}}(\mathbf{B})$, it follows from the previous propositions that, in order to compute $\mathbf{Z}^{\uparrow}$ for each pseudovariety $\mathbf{Z}$ in either of these lattices, it suffices to compute $\mathbf{Z}^{\uparrow}$ for each monoidal pseudovariety $\mathbf{Z}$ of the form $\mathbf{Z}^{\kappa_\ell}$ or $\mathbf{Z}^{\kappa_r}$, that is for $\mathbf{Z} = [\![G_n = I_n]\!]_{\mathbf{B}}$ and $\mathbf{Z} = [\![\overline{G_n} = \overline{I_n}]\!]_{\mathbf{B}}$ for each $n \geq 2$.

Let us set the following notation. We let $\beta \colon \hat{F}_A(\mathbf{DA}) \to \hat{F}_A(\mathbf{B}) = F_A(\mathbf{B})$ be the natural projection (for any alphabet A). If $A = \{x_1, \ldots, x_n\}$ is a n-letter alphabet with $n \geq 2$, we let $\varphi \colon \hat{F}_A(\mathbf{DA}) \to \hat{F}_A(\mathbf{DA})$ be the continuous endomorphism given by

$$\begin{aligned}
\varphi(x_1) &= (x_1^{\omega} x_2^{\omega} x_1^{\omega})^{\omega} \\
\varphi(x_2) &= x_2^{\omega} \\
\varphi(x_k) &= (x_k^{\omega}\, (\varphi(\overline{G_{k-1}} G_{k-1}))^{\omega}\, x_k^{\omega})^{\omega} \quad \text{for } k \geq 3.
\end{aligned}$$

Whenever needed, we extend φ and β to morphisms defined on $\hat{F}_A(\mathbf{DA})^1$ by letting $\varphi(1) = 1$ and $\beta(1) = 1$.

Proposition 4.6 *Let* $n \geq 2$. *If* $\mathbf{Z} = [\![G_n = I_n]\!]_{\mathbf{B}}$ *(resp.* $\mathbf{Z} = [\![\overline{G_n} = \overline{I_n}]\!]_{\mathbf{B}}$*), then we have* $\mathbf{Z}^{\uparrow} = [\![\varphi(G_n) = \varphi(I_n)]\!]_{\mathbf{DA}}$ *(resp.* $\mathbf{Z}^{\uparrow} = [\![\varphi(\overline{G_n}) = \varphi(\overline{I_n})]\!]_{\mathbf{DA}}$*).*

The proof of Proposition 4.6 will be obtained after a series of lemmas.

Lemma 4.7 *For each* $n \geq 2$, $\beta(\varphi(G_n)) = \beta(G_n)$ *and* $\beta(\varphi(\overline{G_n})) = \beta(\overline{G_n})$.

Proof. By definition of $\mathbf{B}$, we have $\beta(x^{\omega}) = \beta(x^2) = \beta(x)$ for each $x \in \hat{F}_A(\mathbf{DA})$. The proof of the lemma is by induction. For $n = 2$, we have

$$\beta(\varphi(G_2)) = x_2^{\omega}(x_1^{\omega} x_2^{\omega} x_1^{\omega})^{\omega} = \beta(x_2 x_1) = \beta(G_2).$$

Similarly, $\beta(\varphi(\overline{G_2})) = \beta(\overline{G_2})$. Let now $n > 2$. Then

$$\begin{aligned}
\beta(\varphi(G_n)) = \beta(\varphi(x_n \overline{G_{n-1}})) &= \beta((x_n^{\omega}\, (\varphi(\overline{G_{n-1}} G_{n-1}))^{\omega}\, x_n^{\omega})^{\omega} \varphi(\overline{G_{n-1}})) \\
&= \beta(x_n\, \overline{G_{n-1}}\, G_{n-1}\, x_n\, \overline{G_{n-1}})
\end{aligned}$$

by the induction hypothesis. It follows from Proposition 2.1 that

$$\beta(\varphi(G_n)) \;=\; \beta(x_n\,\overline{G_{n-1}}) \;=\; \beta(G_n).$$

The equality $\beta(\varphi(\overline{G_n})) = \beta(\overline{G_n})$ is established in the same fashion. $\square$

Lemma 4.8 *Let $x \in \hat{F}_A(\mathbf{DA})$ and let k be the maximal index such that $x_k \in c(x)$. If $k \geq 3$, then $\varphi(x)$ is an idempotent of content $\{x_1, \ldots, x_k\}$. If $c(x) = \{x_1, x_2\}$ or $c(x) = \{x_1\}$, then $\varphi(x)$ is an idempotent of content $\{x_1, x_2\}$, and if $c(x) = \{x_2\}$, then $\varphi(x)$ is an idempotent of content $\{x_2\}$.*

Proof. In view of the definition of φ and Proposition 1.8, the result is clear if x is a word. In general, x is the limit of a sequence of words of equal content, so $\varphi(x)$ has the required properties by continuity. $\square$

Lemma 4.9 *Let $x \in \hat{F}_A(\mathbf{DA})$. If $x = z_1 a r_1$ is the left basic factorization of x and $x = r_2 b z_2$ is the right basic factorization of x, then $\varphi(x) = \varphi(z_1 a b z_2)$.*

Proof. By definition of the basic factorizations of x, the contents of x, $z_1 a$ and $b z_2$ are equal. Therefore $\varphi(x)$, $\varphi(z_1 a)$ and $\varphi(b z_2)$ are idempotent and $\mathcal{D}$-equivalent by Lemma 4.8 and Proposition 1.8. Since in addition $\varphi(x) \leq_{\mathcal{R}} \varphi(z_1 a)$ and $\varphi(x) \leq_{\mathcal{L}} \varphi(b z_2)$, the result follows by definition of $\mathbf{DA}$. $\square$

Lemma 4.10 *Let $x, y \in \hat{F}_A(\mathbf{DA})$. If $\beta(x) = \beta(y)$, then $\varphi(x) = \varphi(y)$.*

Proof. Let us first observe that $\beta(x) = \beta(y)$ implies $c(x) = c(y)$. The proof is now by induction on the common cardinality of $c(x)$ and $c(y)$. If $c(x)$ is empty, then by convention $\varphi(x) = 1$.

Let now x have non empty content, let $x = z_1 a r_1$ be the left basic factorization of x and let $x = r_2 b z_2$ be the right basic factorization of x. Since $\beta(x) = \beta(z_1) a \beta(r_1)$ and β preserves contents, we have $0(\beta(x)) = \beta(z_1)$ and $\bar{0}(\beta(x)) = a$ (using the notation of Section 2.2). Similarly, we have $1(\beta(x)) = \beta(z_2)$ and $\bar{1}(\beta(x)) = b$.

Let $y = y_1 a' r_1'$ and $y = r_2' b' y_2$ be respectively the left and right basic factorizations of y. By Proposition 2.1, we have $0(\beta(x)) = 0(\beta(y))$, $\bar{0}(\beta(x)) = \bar{0}(\beta(y))$, $1(\beta(x)) = 1(\beta(y))$ and $\bar{1}(\beta(x)) = \bar{1}(\beta(y))$ since $\beta(x) = \beta(y)$. Therefore we have $\beta(z_1) = \beta(y_1)$, $a = a'$, $b = b'$ and $\beta(z_2) = \beta(y_2)$. By the induction hypothesis, this implies in particular $\varphi(z_1) = \varphi(y_1)$ and $\varphi(z_2) = \varphi(y_2)$. It now suffices to apply Lemma 4.9 to conclude the proof. $\square$

Corollary 4.11 *There exists a continuous morphism $\gamma\colon F_A(\mathbf{B}) \to \hat{F}_A(\mathbf{DA})$ such that $\gamma \circ \beta = \varphi$.*

Lemma 4.12 *Let $u, v \in \hat{F}_A(\mathbf{DA})$ and $r, s \in F_A(\mathbf{B})$. If $[\![\beta(u) = \beta(v)]\!]_{\mathbf{B}} \subseteq [\![r = s]\!]_{\mathbf{B}}$, then $[\![u = v]\!]_{\mathbf{DA}} \subseteq [\![\gamma(r) = \gamma(s)]\!]_{\mathbf{DA}}$.*

Proof. The proof relies on the finiteness of $F_A(\mathbf{B})$. Since $[\![\beta(u) = \beta(v)]\!]_{\mathbf{B}} \subseteq [\![r = s]\!]_{\mathbf{B}}$, there exists a finite sequence of elements of $F_A(\mathbf{B})$

$$b_0 = r,\ b_1,\ \ldots,\ b_{k+1} = s$$

such that, for all $0 \leq i \leq k$,

$$\{b_i,\ b_{i+1}\} = \{\ell_i\ \zeta_i(\beta(u))\ r_i,\ \ell_i\ \zeta_i(\beta(v))\ r_i\},$$

where $\ell_i, r_i \in F_A(\mathbf{B}) \cup \{1\}$ and ζ_i is an endomorphism of $F_A(\mathbf{B})$.

Then, for each $0 \leq i \leq k$,

$$\{\gamma(b_i),\ \gamma(b_{i+1})\} = \{\gamma(\ell_i)\ \gamma(\zeta_i(\beta(u)))\ \gamma(r_i),\ \gamma(\ell_i)\ \gamma(\zeta_i(\beta(v)))\ \gamma(r_i)\}.$$

Now each $\gamma \circ \zeta_i \circ \beta$ is a continuous endomorphism of $\hat{F}_A(\mathbf{DA})$, Therefore, in $[\![u = v]\!]_{\mathbf{DA}}$, we have $\gamma(r) = \gamma(s)$, that is, $[\![u = v]\!]_{\mathbf{DA}} \subseteq [\![\gamma(r) = \gamma(s)]\!]_{\mathbf{DA}}$. □

Proof of Proposition 4.6. Let $n \geq 2$ and let $\mathbf{Z} = [\![G_n = I_n]\!]_{\mathbf{B}}$. We want to show that $\mathbf{Z}^\uparrow = [\![\varphi(G_n) = \varphi(I_n)]\!]_{\mathbf{B}}$. Let $\mathbf{X}$ be a subpseudovariety of $\mathbf{DA}$ such that $\mathbf{X} \cap \mathbf{B} = \mathbf{Z}$. Then we have

$$\begin{aligned} \mathbf{Z} &= [\![\{\beta(u) = \beta(v) \mid \mathbf{X} \text{ satisfies } u = v\}]\!]_{\mathbf{B}} \\ &= \bigcap_{\mathbf{X} \text{ satisfies } u = v} [\![\beta(u) = \beta(v)]\!]_{\mathbf{B}} \end{aligned}$$

But $\mathbf{Z} = [\![G_n = I_n]\!]_{\mathbf{B}}$ is meet-irreducible in $Ps(\mathbf{B})$ (see Figure 1), so there exist u, v in $\hat{F}_A(\mathbf{DA})$ (for some alphabet A) such that $\mathbf{X}$ satisfies the pro-identity $u = v$ and such that $\mathbf{Z} = [\![\beta(u) = \beta(v)]\!]_{\mathbf{B}}$. Now we have

$$\mathbf{Z} = [\![\beta(u) = \beta(v)]\!]_{\mathbf{B}} = [\![G_n = I_n]\!]_{\mathbf{B}}$$

so by Lemma 4.12 we have

$$\mathbf{X} \subseteq [\![u = v]\!]_{\mathbf{DA}} \subseteq [\![\gamma(G_n) = \gamma(I_n)]\!]_{\mathbf{DA}}.$$

Since G_n and I_n are words, we have $G_n = \beta(G_n)$ in $F_A(\mathbf{B})$, and hence $\gamma(G_n) = \gamma(\beta(G_n)) = \varphi(G_n)$. Similarly, $\gamma(I_n) = \varphi(I_n)$. Therefore $\mathbf{X} \subseteq [\![\varphi(G_n) = \varphi(I_n)]\!]_{\mathbf{DA}}$.

Of course, by Lemma 4.7, $[\![\varphi(G_n) = \varphi(I_n)]\!]_{\mathbf{DA}} \cap \mathbf{B} = \mathbf{Z}$, so the Proposition is proved. □

Remark. With the same proof as Proposition 4.6, we can also prove the equalities for $n \geq 3$

$$\begin{aligned} [\![G_n = H_n]\!]^\uparrow_{\mathbf{B}} &= [\![\varphi(G_n) = \varphi(H_n)]\!]_{\mathbf{DA}} \\ [\![\overline{G_n} = \overline{H_n}]\!]^\uparrow_{\mathbf{B}} &= [\![\varphi(\overline{G_n}) = \varphi(\overline{H_n})]\!]_{\mathbf{DA}} \end{aligned}$$

4.4 The general case

Propositions 4.1, 4.2, 4.3, 4.4 and 4.6 prove the following theorem.

Theorem 4.13 *Let $\mathbf{Z}$ be a pseudovariety of band semigroups (resp. monoids). Then the class of elements $\mathbf{X}$ of $Ps(\mathbf{DA})$ (resp. $Ps_{\mathbf{M}}(\mathbf{DA})$) such that $\mathbf{X} \cap \mathbf{B} = \mathbf{Z}$, is an interval of the form $[\mathbf{Z}, \mathbf{Z}^\uparrow]$, and the pseudovariety $\mathbf{Z}^\uparrow$ is effectively computable.*

It is interesting to note the following properties of the operation $\mathbf{Z} \mapsto \mathbf{Z}^\uparrow$.

Proposition 4.14 *Let $\mathbf{V}$ and $\mathbf{W}$ be subpseudovarieties of $\mathbf{B}$.*

(1) $(\mathbf{V}^{\kappa_\ell})^\uparrow = ((\mathbf{V}^{\kappa_\ell})^\uparrow)^{\mathbf{K}}$ *and* $(\mathbf{V}^{\kappa_r})^\uparrow = ((\mathbf{V}^{\kappa_r})^\uparrow)^{\mathbf{D}}$.

(2) *If* $\mathbf{V} \subseteq \mathbf{W}$, *then* $\mathbf{V}^{\uparrow} \subseteq \mathbf{W}^{\uparrow}$.

(3) *If* $\mathbf{SL} \subseteq \mathbf{V}_i$ *for all* $i \in I$, *then* $(\bigcap_{i\in I} \mathbf{V}_i)^{\uparrow} = \bigcap_{i\in I}(\mathbf{V}_i^{\uparrow})$.

Proof. (1) follows from Proposition 2.4 since we have

$$((\mathbf{V}^{\kappa_\ell})^{\uparrow})^{\mathbf{K}} \cap \mathbf{B} \;=\; ((\mathbf{V}^{\kappa_\ell})^{\uparrow} \cap \mathbf{B})^{\kappa_\ell} \;=\; (\mathbf{V}^{\kappa_\ell})^{\kappa_\ell} \;=\; \mathbf{V}^{\kappa_\ell}.$$

(2) From $\mathbf{V} \subseteq \mathbf{W}$, it follows that $\mathbf{V}^{\kappa_\ell} \subseteq \mathbf{W}^{\kappa_\ell}$. If $\mathbf{SL} \subseteq \mathbf{V}$, then one of the following holds:

- $\mathbf{V}^{\kappa_\ell} = [\![G_n = I_n]\!]_{\mathbf{B}}$ and $\mathbf{W}^{\kappa_\ell} = [\![G_m = H_m]\!]_{\mathbf{B}}$ for some $m > n \geq 2$;
- $\mathbf{V}^{\kappa_\ell} = [\![G_n = I_n]\!]_{\mathbf{B}}$ and $\mathbf{W}^{\kappa_\ell} = [\![G_m = I_m]\!]_{\mathbf{B}}$ for some $m \geq n \geq 2$;
- $\mathbf{V}^{\kappa_\ell} = [\![G_n = H_n]\!]_{\mathbf{B}}$ and $\mathbf{W}^{\kappa_\ell} = [\![G_m = I_m]\!]_{\mathbf{B}}$ for some $m \geq n \geq 3$;
- $\mathbf{V}^{\kappa_\ell} = [\![G_n = H_n]\!]_{\mathbf{B}}$ and $\mathbf{W}^{\kappa_\ell} = [\![G_m = H_m]\!]_{\mathbf{B}}$ for some $m \geq n \geq 3$.

In all cases, Proposition 4.6 and Lemma 4.12 show that $(\mathbf{V}^{\kappa_\ell})^{\uparrow}$ is contained in $(\mathbf{W}^{\kappa_\ell})^{\uparrow}$. Similarly, $(\mathbf{V}^{\kappa_r})^{\uparrow} \subseteq (\mathbf{W}^{\kappa_r})^{\uparrow}$. But Proposition 4.3 gives

$$\mathbf{V}^{\uparrow} = (\mathbf{V}^{\kappa_\ell})^{\uparrow} \cap (\mathbf{V}^{\kappa_r})^{\uparrow} \qquad \text{and} \qquad \mathbf{W}^{\uparrow} = (\mathbf{W}^{\kappa_\ell})^{\uparrow} \cap (\mathbf{W}^{\kappa_r})^{\uparrow}$$

so $\mathbf{V}^{\uparrow} \subseteq \mathbf{W}^{\uparrow}$.

If $\mathbf{V}$ or $\mathbf{W}$ does not contain $\mathbf{SL}$, the result is easily verified by Propositions 4.1 and 4.6.

(3) We have $(\bigcap_i \mathbf{V}_i)^{\uparrow} \subseteq \bigcap_i(\mathbf{V}_i^{\uparrow})$ by (2). For the converse containment, it suffices to observe that $\bigcap_i(\mathbf{V}_i^{\uparrow}) \cap \mathbf{B} = \bigcap_i(\mathbf{V}_i^{\uparrow} \cap \mathbf{B}) = \bigcap_i \mathbf{V}_i$. □

Let us observe that we have now obtained a new proof of Reilly's and Zhang's result, Theorem 2.2 above. We have indeed already verified that the classes of the partition induced by the mapping $\mathbf{V} \mapsto \mathbf{V} \cap \mathbf{B}$ are intervals, and that $\mathbf{V} \subseteq \mathbf{W}$ in $Ps(\mathbf{DA})$ implies $\mathbf{V} \cap \mathbf{B} \subseteq \mathbf{W} \cap \mathbf{B}$ and $(\mathbf{V} \cap \mathbf{B})^{\uparrow} \subseteq (\mathbf{W} \cap \mathbf{B})^{\uparrow}$ (Proposition 4.14 above). Now a general lattice-theoretic result of Pastijn and Petrich [16, Lemma 4.13] shows immediately that the map $\mathbf{V} \mapsto \mathbf{V} \cap \mathbf{B}$ is a complete lattice morphism from $Ps(\mathbf{DA})$ onto $Ps(\mathbf{B})$.

Finally, our results give a simple decision procedure for membership in $\mathbf{Z}^{\uparrow}$, for each pseudovariety of bands $\mathbf{Z}$. If $\mathbf{Z}$ is $\mathbf{B}$ or $\mathbf{SL}$, if $\mathbf{Z}$ is contained in $\mathbf{ReB}$ or if $\mathbf{Z}$ is of the form $[\![G_n = I_n]\!]_{\mathbf{B}}$, $[\![G_n = H_n]\!]_{\mathbf{B}}$, $[\![\overline{G_n} = \overline{I_n}]\!]_{\mathbf{B}}$ or $[\![\overline{G_n} = \overline{H_n}]\!]_{\mathbf{B}}$, then we computed explicitly $\mathbf{Z}^{\uparrow}$, and we gave a finite set of defining pro-identities for it, which can be effectively checked. If $\mathbf{Z}$ is not of one of the above forms, then $\mathbf{Z}^{\kappa_\ell}$ and $\mathbf{Z}^{\kappa_r}$ are of these forms, and can be effectively computed. But $\mathbf{Z}^{\uparrow} = (\mathbf{Z}^{\kappa_\ell})^{\uparrow} \cap (\mathbf{Z}^{\kappa_r})^{\uparrow}$, so we also have a finite defining set of computable pro-identities for $\mathbf{Z}^{\uparrow}$.

These remarks are summarized in the following statement.

Theorem 4.15 *The mapping* $\mathbf{V} \mapsto \mathbf{V} \cap \mathbf{B}$ *is a complete lattice morphism, from* $Ps(\mathbf{DA})$ *onto* $Ps(\mathbf{B})$. *Each class of the partition it induces on* $Ps(\mathbf{DA})$ *is an interval.*

Let $\mathbf{Z}$ *be a pseudovariety of bands. Then* $\mathbf{Z}$ *is the least element of its class, and the greatest element of the class of* $\mathbf{Z}$ *is effectively computable and has decidable membership problem.*

5 The k-generated semigroups of DA

For each $k \geq 1$, let $\mathbf{V}_k$ be the pseudovariety generated by the k-generated elements of $\mathbf{DA}$. The aim of this section is to study the $\mathbf{V}_k$, and in particular to determine their position on the "chart" of the lattice of subpseudovarieties of $\mathbf{DA}$, that is, to determine $\mathbf{V}_k \cap \mathbf{B}$.

Observe that a subpseudovariety $\mathbf{V}$ of $\mathbf{DA}$ contains $\mathbf{V}_k$ if and only if it satisfies no non-trivial pro-identity $x = y$ where $x, y \in \hat{F}_k(\mathbf{DA})$, that is, no non-trivial k-variable pro-$\mathbf{DA}$-identity. In particular, $\mathbf{V}_k \subseteq \mathbf{V}$ if and only if the canonical projection from $\hat{F}_k(\mathbf{DA})$ onto $\hat{F}_k(\mathbf{V})$ is a homeomorphism.

The *dual* of a semigroup S is the semigroup S^δ given by the product $s \cdot t = ts$ for all $s, t \in S$. The dual of a pseudovariety $\mathbf{V}$ is the pseudovariety $\mathbf{V}^\delta$ consisting of all the duals of the elements of S. Finally, we say that $\mathbf{V}$ is *self-dual* if $\mathbf{V} = \mathbf{V}^\delta$. It is clear that a pseudovariety which is generated by a class of semigroups closed under taking duals, is self-dual. In particular, the $\mathbf{V}_k$ (≥ 1) are self-dual.

The results of Section 2.2 immediately give us a lower bound for the $\mathbf{V}_k$, since the free band over k generators lies in $\mathbf{V}_k$.

Proposition 5.1 *For each $k \geq 2$, $\mathbf{B}'_k \subseteq \mathbf{V}_k$.*

We also easily derive the following:

Proposition 5.2 $\mathbf{V}_1 \subseteq \mathbf{N} \cap \mathbf{Com}$ *and* $\mathbf{V}_1 \cap \mathbf{B} = \mathbf{I}$.

Proof. It is immediate that each finite cyclic aperiodic semigroup is nilpotent and commutative, so $\mathbf{V}_1 \subseteq \mathbf{N} \cap \mathbf{Com}$. But $\mathbf{N} \cap \mathbf{B} = \mathbf{I}$, so $\mathbf{V}_1 \cap \mathbf{B}$ is trivial. $\square$

It would be natural to expect that, for each k, $\mathbf{V}_k \cap \mathbf{B}$ is the pseudovariety of bands generated by the k-generated free band, namely $\mathbf{B}'_k$. In fact, we will see that this is true for $k \geq 3$, but that $\mathbf{V}_2 \cap \mathbf{B}$ is, surprisingly, larger than $\mathbf{B}'_2$.

5.1 An upper bound for $\mathbf{V}_k$

In order to determine the position of $\mathbf{V}_k$ ($k \geq 2$), we now give an upper bound result.

Theorem 5.3 *Let $k \geq 1$. If $\mathbf{V}$ is a subpseudovariety of $\mathbf{DA}$ containing $\mathbf{SL}$, and if $\mathbf{V}_k \subseteq \mathbf{V}$, then $\mathbf{V}_{k+1} \subseteq \mathbf{V}^{\mathbf{K}} \vee \mathbf{V}^{\mathbf{D}}$.*

Proof. Let S be a $(k+1)$-generated element of $\mathbf{DA}$, say with generators $x_1, \ldots, x_{k+1}$. Let α_L (resp. α_R) be the relation on S defined by the Green relation $\mathcal{L}$ (resp. $\mathcal{R}$) on the minimal ideal of S, and by the identity elsewhere. By elementary facts of semigroup theory (Green's lemma), α_L and α_R are congruences. Let $S_\ell = S/\alpha_L$ and $S_r = S/\alpha_R$. Then the minimal ideals of these semigroups are respectively $\mathcal{L}$-trivial and $\mathcal{R}$-trivial. Moreover, two elements of S which are α_L- and α_R-equivalent must be equal (since S is aperiodic, so its minimal ideal is a rectangular band). Therefore S is a subdirect product of S_ℓ and S_r.

So it suffices to show that $S_\ell \in \mathbf{V}^{\mathbf{D}}$ and $\mathbf{S}_r \in \mathbf{V}^{\mathbf{K}}$. We show here that $S_\ell \in \mathbf{V}^{\mathbf{D}}$. The proof that $\mathbf{S}_r \in \mathbf{V}^{\mathbf{K}}$ is similar. Let I_j be the ideal of S_ℓ generated by the generator x_j, and let $I = \bigcap_{j=1}^{k+1} I_j$. In $\hat{F}_{k+1}(\mathbf{DA})$, there is exactly one regular $\mathcal{D}$-class of elements having $k+1$ letters in their content. So I contains exactly one regular $\mathcal{D}$-class K, which is the minimal ideal of S_ℓ. But K is $\mathcal{L}$-trivial, so $I \in \mathbf{D}$, and it suffices to show that $S_\ell/I \in \mathbf{V}$.

Since I is the intersection of the I_j, the quotient S_ℓ/I is a subdirect product of the S_ℓ/I_j ($1 \leq j \leq k+1$), and it suffices to show that each S_ℓ/I_j lies in $\mathbf{V}$. But S_ℓ/I_j is either a k-generated semigroup (if some element of $\langle x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{k+1}\rangle$ falls in I_j), or it is a k-generated semigroup with a zero adjoined.

In the first case, $S_\ell/I_j \in \mathbf{V}_k$, so $S_\ell/I_j \in \mathbf{V}$ and we are done. In the second case, we have $S_\ell/I_j = T \cup \{0\}$, where $T \in \mathbf{V}_k \subseteq \mathbf{V}$. Then S_ℓ/I_j is the quotient of $T \times \{0,1\}$ by the ideal $T \times \{0\}$. Since $\{0,1\} \in \mathbf{SL}$, it follows that $S_\ell/I_j \in \mathbf{V}$, as required. This concludes the proof. $\square$

Corollary 5.4 $\mathbf{V}_2 \subseteq \mathbf{R} \vee \mathbf{L}$*, and for* $k \geq 2$*,* $\mathbf{V}_{k+1} \subseteq \mathbf{V}_k^{\mathbf{K}} \vee \mathbf{V}_k^{\mathbf{D}}$*.*

5.2 Determining the position of $\mathbf{V}_2$

The main result of this section is the following.

Theorem 5.5 $\mathbf{V}_2 \cap \mathbf{B} = \mathbf{B}_3$*.*

The proof requires the following technical lemma.

Lemma 5.6 $\mathbf{R} = \mathbf{J}^{\mathbf{K}}$ *does not satisfy* $(a^\omega b a^\omega b^2)^\omega = (a^\omega b^2 a^\omega b)^\omega$*.*

Proof. The iterated left basic factorizations of $(a^\omega b a^\omega b^2)^\omega$ and $(a^\omega b^2 a^\omega b)^\omega$ are respectively

$$\begin{aligned}(a^\omega b a^\omega b^2)^\omega &= a^\omega b\ a^\omega b\ ba\ a^\omega b\ a^\omega b\ ba\ \cdots \\ (a^\omega b^2 a^\omega b)^\omega &= a^\omega b\ ba\ a^\omega b\ a^\omega b\ ba\ a^\omega b\ \cdots\end{aligned}$$

The lemma follows, since these factorizations differ in $\mathbf{J}$ (at their second factor). $\square$

Corollary 5.7 *There exist 2-generated elements of* $\mathbf{R}$ *which do not lie in* $L_\ell\mathbf{J}$*.*

Proof. This is immediate, once we notice that $L_\ell\mathbf{J}$ is defined by the pro-identities $x^\omega x = x^\omega$ and $(x^\omega y x^\omega z)^\omega = (x^\omega z x^\omega y)^\omega$. $\square$

Proof of Theorem 5.5. By Proposition 5.1, we have $\mathbf{B}_2' \subsetneq \mathbf{V}_2$. Moreover, by Corollary 5.4, $\mathbf{V}_2 \subseteq \mathbf{R} \vee \mathbf{L}$. But $\mathbf{R} \vee \mathbf{L} = \mathbf{J}^{\mathbf{K}} \vee \mathbf{J}^{\mathbf{D}} \subseteq \mathbf{J}^{\mathbf{KD}} \cap \mathbf{J}^{\mathbf{DK}}$, and by Proposition 2.4 $(\mathbf{J}^{\mathbf{KD}} \cap \mathbf{J}^{\mathbf{DK}}) \cap \mathbf{B} = \mathbf{SL}^{\kappa_\ell\kappa_r} \cap \mathbf{SL}^{\kappa_r\kappa_\ell} = \mathbf{B}_3$, so $\mathbf{V}_2 \cap \mathbf{B} \subseteq \mathbf{B}_3$. Since, in addition, $\mathbf{V}_2$ is self-dual, it follows from the structure of the lattice of band pseudovarieties that $\mathbf{V}_2 \cap \mathbf{B}$ is equal to either $\mathbf{B}_2'$ or $\mathbf{B}_3$.

If $\mathbf{V}_2 \cap \mathbf{B} = \mathbf{B}_2'$, then $(\mathbf{V}_2 \cap \mathbf{R}) \cap \mathbf{B} = \mathbf{B}_2' \cap \mathbf{SL}^{\kappa_\ell} = \mathbf{LNB}$, which implies, by Corollary 4.5 that $\mathbf{V}_2 \cap \mathbf{R} \subseteq L_\ell\mathbf{J}$. But this contradicts Corollary 5.7, so $\mathbf{V}_2 \cap \mathbf{B} \neq \mathbf{B}_2'$, and hence, $\mathbf{V}_2 \cap \mathbf{B} = \mathbf{B}_3$. $\square$

5.3 Position of the $\mathbf{V}_k$ for $k \geq 3$

By Proposition 5.1 and Corollary 5.4, we find that $\mathbf{B}_3' \subseteq \mathbf{V}_3$ and $\mathbf{V}_3 \subseteq \mathbf{V}_2^{\mathbf{K}} \vee \mathbf{V}_2^{\mathbf{D}}$. But $\mathbf{V}_2^{\mathbf{K}} \vee \mathbf{V}_2^{\mathbf{D}} \subseteq \mathbf{V}_2^{\mathbf{KD}} \cap \mathbf{V}_2^{\mathbf{DK}}$, and by Theorem 5.5

$$(\mathbf{V}_2^{\mathbf{KD}} \cap \mathbf{V}_2^{\mathbf{DK}}) \cap \mathbf{B} = (\mathbf{V}_2 \cap \mathbf{B})^{\kappa_\ell\kappa_r} \cap (\mathbf{V}_2 \cap \mathbf{B})^{\kappa_r\kappa_\ell} = \mathbf{B}_3^{\kappa_\ell\kappa_r} \cap \mathbf{B}_3^{\kappa_r\kappa_\ell} = \mathbf{B}_4.$$

Since $\mathbf{V}_3$ is self-dual, it follows that $\mathbf{V}_3$ is equal to one of $\mathbf{B}_3'$ and $\mathbf{B}_4$. In order to get the exact position of $\mathbf{V}_3$, we need the following lemma.

Proposition 5.8 $\mathbf{V}_3 \subseteq L(\mathbf{J}^{\mathbf{DK}})$*.*

Proof. Since $\mathbf{V}_3$ is generated by the finite continuous homomorphic images of $\hat{F}_3(\mathbf{DA})$, it suffices to verify that $\hat{F}_3(\mathbf{DA})$ satisfies a defining set of pro-identities of $L(\mathbf{J}^{\mathbf{DK}}) \cap \mathbf{DA}$. By Propositions 1.5 and 1.6, we need to verify that $\varphi_z(x^\omega y) = \varphi_z(x^\omega)$ for all $x, y, z \in \hat{F}_3(\mathbf{DA})$ such that $\mathbf{J}^{\mathbf{D}}$ satisfies $x^\omega = y$. Notice that this last condition implies that $c(x) = c(y)$, y is regular, and $\mathbf{J}^{\mathbf{DK}}$ satisfies $x^\omega y = x^\omega$. Let $A = \{a, b, c\}$ be the alphabet of generators of $\hat{F}_3(\mathbf{DA})$.

Observe that $\varphi_z(x^\omega y) = \varphi_z(x^\omega)\varphi_z(y)$ and $\varphi_z(x^\omega) = \varphi_z(x)^\omega$ are both regular, have the same content, and have the same (infinite) iterated left basic factorization. So we need to show that they also have the same iterated right basic factorization, since the iterated central basic factorization of regular elements of $\hat{F}_A(\mathbf{DA})$ is entirely determined by their iterated right and left basic factorizations.

Let us first assume that $|c(x)| \leq 2$. It follows from Corollary 5.4 that $\mathbf{V}_2$ is contained in $\mathbf{J}^{\mathbf{DK}}$, so we have $x^\omega y = x^\omega$ in $\hat{F}_3(\mathbf{DA})$. The equation $\varphi_z(x^\omega y) = \varphi_z(x^\omega)$ follows trivially.

Next, assume that $|c(z)| = 3$. Then $\varphi_z(a) = z^\omega a z^\omega = z^\omega$, and also $\varphi_z(b) = \varphi_z(c) = z^\omega$. So φ_z is constant on $\hat{F}_3(\mathbf{DA})$, and the expected equality again holds trivially.

If $|c(z)| = 2$, say $c(z) = \{b, c\}$, and if u is a regular element of $\hat{F}_3(\mathbf{DA})$ with content A, then the n-th iteration of the right basic factorization of $\varphi_z(u)$ is of the form $\varphi_z(u) = w_n(az^\omega)^n$ for each n. Let indeed $u = u_1 a v_1$ be the factorization of u showing the longest suffix v_1 of u with content in $\{b, c\}$. Since u is regular, u_1 is regular as well and $c(u_1) = c(u) = A$, so we can iterate this factorization arbitrarily many times: for each $n \geq 1$,

$$u = u_n a v_n \cdots a v_1, \quad \text{with } v_i \in \hat{F}_{\{b,c\}}(\mathbf{DA}) \cup \{1\} \text{ for each } i.$$

Then we have

$$\varphi_z(u) = \varphi_z(u_n)\varphi_z(a)\varphi_z(v_n)\cdots\varphi_z(a)\varphi_z(v_1) = \varphi_z(u_n)a\varphi_z(v_n)\cdots a\varphi_z(v_1)$$

since $\varphi_z(w) = z^\omega \varphi_z(w) z^\omega$ for all $w \in \hat{F}_3(\mathbf{DA})$. In addition, we have $\varphi_z(v_i) = z^\omega$ for each i, since $\varphi_z(b) = \varphi_z(c) = z^\omega$. Thus the n-th iteration of the right basic factorization of $\varphi_z(u)$ is $\varphi_z(u) = \varphi_z(u_n)\ (az^\omega)^n$ as announced. In particular, if $c(x) = c(y) = A$, then $\varphi_z(x^\omega y)$ and $\varphi_z(x^\omega)$ have identical iterated right basic factorizations, and hence $\varphi_z(x^\omega y) = \varphi_z(x^\omega)$.

The last case is that where $c(x) = A$ and $|c(z)| = 1$, say $c(z) = \{c\}$. Of course, $\mathbf{DA}$ and $\mathbf{J}^{\mathbf{KD}}$ are supermonoidal, so $\hat{F}_3(\mathbf{DA} \cap \mathbf{M}) = \hat{F}_3(\mathbf{DA}) \cup \{1\}$ by Proposition 1.4. Let ρ be the continuous endomorphism of $\hat{F}_3(\mathbf{DA} \cap \mathbf{M})$ defined by $\rho(a) = a$, $\rho(b) = b$ and $\rho(c) = 1$. Note that $z^\omega = c^\omega$ and that $\varphi_z(d) = \varphi_z(\rho(d))$ for each letter $d \in \{a, b, c\}$. Therefore $\varphi_z \circ \rho = \varphi_z$. Since $\mathbf{J}^{\mathbf{KD}}$ satisfies $x^\omega y = x^\omega$, $\mathbf{J}^{\mathbf{KD}}$ also satisfies $\rho(x)^\omega \rho(y) = \rho(x)^\omega$. But $|c(\rho(x))| = |c(\rho(y))| = 2$ so by the case first dealt with, $\varphi_z(\rho(x^\omega y)) = \varphi_z(\rho(x^\omega))$, that is, $\varphi_z(x^\omega y) = \varphi_z(x^\omega)$. □

Corollary 5.9 $\mathbf{V}_3 \cap \mathbf{B} = \mathbf{B}'_3$.

Proof. By the above proposition,

$$\mathbf{V}_3 \cap \mathbf{B} \subseteq L(\mathbf{J}^{\mathbf{DK}}) \cap \mathbf{B} = L(\mathbf{J}^{\mathbf{DK}} \cap \mathbf{B}) = L(\mathbf{SL}^{\kappa_r \kappa_\ell}).$$

By left-right symmetry, it follows that

$$\mathbf{V}_3 \cap \mathbf{B} \subseteq L(\mathbf{SL}^{\kappa_r \kappa_\ell}) \cap L(\mathbf{SL}^{\kappa_\ell \kappa_r}) = L(\mathbf{SL}^{\kappa_r \kappa_\ell} \cap \mathbf{SL}^{\kappa_\ell \kappa_r}) = \mathbf{B}'_3.$$

But we already know that $\mathbf{B}'_3 \subseteq \mathbf{V}_3$, so $\mathbf{V}_3 \cap \mathbf{B} = \mathbf{B}'_3$. □

We can now prove the final theorem of this section.

Theorem 5.10 $\mathbf{V}_k \cap \mathbf{B} = \mathbf{B}'_k$ *for each* $k \geq 3$.

Proof. We just verified the result for $k = 3$. By Proposition 5.1, $\mathbf{B}'_k$ is contained in $\mathbf{V}_k \cap \mathbf{B}$ for each k. Now Corollary 5.4 shows that $\mathbf{V}_{k+1} \subseteq \mathbf{V}_k^{\mathbf{K}} \vee \mathbf{V}_k^{\mathbf{D}} \subseteq \mathbf{V}_k^{\mathbf{KD}} \cap \mathbf{V}_k^{\mathbf{DK}}$. So, proceeding by induction on $k \geq 3$, we find that

$$\begin{aligned}
\mathbf{V}_{k+1} \cap \mathbf{B} \ \subseteq \ \mathbf{V}_k^{\mathbf{KD}} \cap \mathbf{V}_k^{\mathbf{DK}} \cap \mathbf{B} &= (\mathbf{V}_k \cap \mathbf{B})^{\kappa_\ell \kappa_r} \cap (\mathbf{V}_k \cap \mathbf{B})^{\kappa_r \kappa_\ell} \\
&= (\mathbf{B}'_k)^{\kappa_\ell \kappa_r} \cap (\mathbf{B}'_k)^{\kappa_r \kappa_\ell} \\
&= \mathbf{B}'_{k+1}.
\end{aligned}$$

$\square$

Corollary 5.11 *There exists a band, which does not divide a direct product of* 2*-generated bands, yet divides a direct product of* 2*-generated semigroups in* $\mathbf{DA}$*. In contrast, if* $k \geq 3$*, any band which divides a direct product of* k*-generated semigroups in* $\mathbf{DA}$*, also divides a direct product of* k*-generated bands.*

5.4 A counter-example to the formula $\mathbf{V_K} \cap \mathbf{B} = (\mathbf{V} \cap \mathbf{B})_{\kappa_\ell}$

In Proposition 2.4, we proved the following formulæ:

$$\mathbf{V}^{\mathbf{K}} \cap \mathbf{B} = (\mathbf{V} \cap \mathbf{B})^{\kappa_\ell} \quad \text{and} \quad \mathbf{V}^{\mathbf{D}} \cap \mathbf{B} = (\mathbf{V} \cap \mathbf{B})^{\kappa_r}.$$

We can now present a counterexample to the analogous formulæ concerning $\mathbf{V_K}$ and $\mathbf{V_D}$.

Proposition 5.12 *Let* $\mathbf{V}$ *be a pseudovariety in the interval* $[\mathbf{SL}, \mathbf{DA}]$ *and let* $k \geq 1$*. If* $\mathbf{V}_{k+1} \subseteq \mathbf{V}^{\mathbf{D}}$ *(resp.* $\mathbf{V}_{k+1} \subseteq \mathbf{V}^{\mathbf{K}}$*), then* $\mathbf{V}_k \subseteq \mathbf{V}$*.*

Proof. Suppose that $\mathbf{V}_{k+1} \subseteq \mathbf{V}^{\mathbf{D}}$ and $\mathbf{V}_k$ is not contained in $\mathbf{V}$. Then $\mathbf{V}$ satisfies a non-trivial pro-identity $u = v$ with $u, v \in \hat{F}_k(\mathbf{DA})$. Since $\mathbf{SL} \subseteq \mathbf{V}$, we have $c(u) = c(v)$. Let now $a \notin c(u)$ and let $x = ua, y = va \in \hat{F}_{k+1}(\mathbf{DA})$. Then $\mathbf{V}$ satisfies $x^\omega = y^\omega$, and hence $\mathbf{V}^{\mathbf{D}}$ satisfies $x^\omega y^\omega = y^\omega$ by Prop 1.6. But the left basic factorizations of $x^\omega y^\omega$ and y^ω, namely

$$\begin{aligned}
x^\omega y^\omega &= u\, a\, (x^\omega y^\omega) \\
y^\omega &= v\, a\, y^\omega
\end{aligned}$$

differ by definition, so $x^\omega y^\omega$ and y^ω are distinct elements of $\hat{F}_{k+1}(\mathbf{DA})$. Thus $\mathbf{V}_{k+1} \not\subseteq \mathbf{V}^{\mathbf{D}}$, a contradiction. $\square$

Corollary 5.13 *For each* $k \geq 1$*,* $\mathbf{V}_k \subseteq (\mathbf{V}_{k+1})_{\mathbf{D}} \cap (\mathbf{V}_{k+1})_{\mathbf{K}}$.

Corollary 5.14 $(\mathbf{V}_3)_{\mathbf{K}} \cap \mathbf{B} \neq (\mathbf{V}_3 \cap \mathbf{B})_{\kappa_\ell}$ *and* $(\mathbf{V}_3)_{\mathbf{D}} \cap \mathbf{B} \neq (\mathbf{V}_3 \cap \mathbf{B})_{\kappa_r}$.

Proof. By Corollary 5.9, we have $(\mathbf{V}_3 \cap \mathbf{B})_{\kappa_\ell} = (\mathbf{B}'_3)_{\kappa_\ell}$. The structure of the lattice of band varieties (Fig. 1) shows that $(\mathbf{B}'_3)_{\kappa_\ell} = (\mathbf{B}'_2)^{\kappa_r}$, so $(\mathbf{V}_3 \cap \mathbf{B})_{\kappa_\ell} = (\mathbf{B}'_2)^{\kappa_r}$. Now, by Theorem 5.5 and Corollary 5.13, $\mathbf{B}_3 = \mathbf{V}_2 \cap \mathbf{B} \subseteq (\mathbf{V}_3)_{\mathbf{K}} \cap \mathbf{B}$. But $\mathbf{B}_3$ is not contained in $(\mathbf{B}'_2)^{\kappa_r}$ (see Fig. 1). So $(\mathbf{V}_3)_{\mathbf{K}} \cap \mathbf{B} \neq (\mathbf{V}_3 \cap \mathbf{B})_{\kappa_\ell}$. The other statement is proved by duality. $\square$

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