Profinite categories and semidirect products*

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Abstract. After developing a theory of implicit operations and proving an analogue of Reiterman's theorem for categories, this paper addresses two complementary questions for semidirect products and two-sided semidirect products of pseudovarieties of semigroups: to determine when a pseudoidentity is valid in it, and to find a basis of pseudoidentities. The first question involves looking into the structure of relatively free profinite objects whereas, for the second question, a general approach is presented which is sufficiently powerful to allow the calculation of many semidirect products. A systematic translation of bases of pseudoidentities of pseudovarieties of categories into characterizations of semidirect products of pseudovarieties of semigroups is given. The latter characterizations, of a syntactic nature, are not effective but may in many cases be reduced to effective characterizations. Several known results are derived as examples—including a syntactic proof of a generalization of the Delay Theorem—and further new applications are obtained using these techniques.

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1. Introduction

The language of graphs has been used in the theory of finite semigroups since its early days. It was introduced with significant applications by Brzozowski and Simon [14] in connection with the study of the lowest levels of a concatenation hierarchy of star-free rational languages. Right from the start, the use of graphs appeared to be intimately connected with the calculation of semidirect products of pseudovarieties of semigroups. In the early 1980's, with the progress achieved through Knast's work [19, 20] (again related with hierarchies of star-free languages), several authors took a new look at the use of graphs in finite semigroup theory. Straubing [29] and Thérien and Weiss [30] started using systematically congruences on graphs to compute semidirect products but it was only Tilson [31] who provided a convenient theoretical framework for their approach.

Tilson introduced the language of categories into the theory of finite semigroups and developed a theory of varieties and pseudovarieties of categories in the tradition of classical Universal Algebra. While Tilson's language of categories allowed a considerable step forward in the theory by simplifying many statements, it somewhat lacked an ingredient to fully realize its power. For the description of pseudovarieties of categories, Tilson cannot use in general free objects (for they fail to be finite) and so, instead of equational descriptions, he needs to use "ultimate equational" descriptions, a tool that appears to be rather impractical.

Inspired by a successful approach to a similar problem for pseudovarieties of semigroups (cf. Reiterman [25] and Banaschewski [12], for the beginnings of the theory, and Almeida [4] for applications), we complement Tilson's work by constructing profinite free categories and thus obtain "pseudoequational" descriptions of pseudovarieties of categories. This seems to be the missing ingredient in Tilson's theory.[1]

Our aim in this paper is to use these profinite constructions to advance in the study of the semidirect product of pseudovarieties of semigroups. The main—and complementary—questions which we address in relation to such a semidirect product $\mathbf{V} * \mathbf{W}$ are the description of the pseudoidentities which are valid in $\mathbf{V} * \mathbf{W}$, and the computation of a basis of pseudoidentities for $\mathbf{V} * \mathbf{W}$.

As a first step in our approach to the description by pseudoidentities of the semidirect product $\mathbf{V} * \mathbf{W}$ of two pseudovarieties of semigroups, we were led to an extension of the semidirect product to an operation $\mathbf{U} \circledast \mathbf{W}$ where $\mathbf{U}$ is a pseudovariety of semigroupoids. This turns out to be a rather natural operation which enjoys some nice properties and which extends several operations that have been of interest in finite semigroup theory. Further extensions can be found in Jones and Pustejovsky [18]. In particular, we describe the free profinite object on $\mathbf{U} \circledast \mathbf{W}$, and we study the properties of the mapping $\mathbf{U} \mapsto$

[1] P. R. Jones [17] has independently arrived at similar results.

$\mathbf{U} \circledast \mathbf{W}$.

We then return to the study of semidirect products $\mathbf{V} * \mathbf{W}$ of pseudovarieties of semigroups. By means of a general compactness theorem, we are able to use Tilson's Derived Category Theorem to give a basis of pseudoidentities for $\mathbf{V} * \mathbf{W}$. While such a basis is not in general effective, it can be often used to obtain a decision criterion for membership in $\mathbf{V} * \mathbf{W}$. Several new examples of application of this approach are included. Many others can be dealt with similarly and we expect to present some of them in forthcoming papers. The reader may also wish to feel the potential of our techniques by trying out her/his own examples. The case of the bilateral semidirect product is also considered, leading to similar results and applications. Also similar results, but with easier proofs, hold for (Birkhoff) varieties.

We also include a new simple proof of a generalization of Tilson's Delay Theorem for pseudovarieties of the form $\mathbf{V} \circledast \mathbf{D}$. It portrays quite transparently the two main ingredients in our approach: the syntactical manipulation of the general basis for $\mathbf{V} \circledast \mathbf{D}$, complemented with a combinatorial (Ramsey type) property of finite semigroups. For the sake of completeness, we also discuss some important applications of the Delay Theorem and some related results concerning the operation $\mathbf{V} \mapsto \mathbf{V} \circledast \mathbf{D}$. For instance, we show that the lattice of pseudovarieties of categories is naturally embedded in the lattice of pseudovarieties of semigroups.

2. Implicit operations and pseudoidentities for categories

By a (directed) *graph* we mean a (nonempty) partial algebra G with a partition $G = V(G) \mathbin{\mathring{\cup}} E(G)$ into two sorts of elements, respectively *vertices* and *edges*, and two unary operations $\alpha, \omega : E(G) \to V(G)$. For $c, d \in V(G)$, denote by $G(c, d)$ the set of all edges $s \in G$ such that $\alpha s = c$ and $\omega s = d$. A *graph morphism* is just a function $\varphi : G \to H$ between two graphs G and H respecting sorts and operations, i.e., such that $\varphi V(G) \subseteq V(H)$, $\varphi E(G) \subseteq E(H)$, and, for any $s \in E(G)$, $\varphi(\alpha s) = \alpha(\varphi s)$ and $\varphi(\omega s) = \omega(\varphi s)$. A *subgraph* G of a graph H is a graph G contained in H such that the inclusion $G \hookrightarrow H$ is a graph morphism. The *product* $\prod_{i \in I} G_i$ of a family $(G_i)_{i \in I}$ of graphs is the graph $G = \prod_{i \in I} V(G_i) \mathbin{\mathring{\cup}} \prod_{i \in I} E(G_i)$, where the latter two $\prod_{i \in I}$ denote cartesian products of sets, with $\alpha(s_i)_{i \in I} = (\alpha s_i)_{i \in I}$ and $\omega(s_i)_{i \in I} = (\omega s_i)_{i \in I}$ for $s_i \in E(G_i)$ $(i \in I)$.

A (small) *category* is a graph C with a unary operation $1\_ : V(C) \to E(C)$ (chosing an *identity* for each vertex) and an associative partial binary operation on $E(C)$ (usually called *multiplication* or *composition*) such that:

i) $\alpha 1_c = \omega 1_c = c$ for every $c \in V(C)$;

ii) for all $s, t \in E(C)$, st is defined if and only if $\omega s = \alpha t$ and, in this case, $\alpha(st) = \alpha s$ and $\omega(st) = \omega t$;

iii) for every $s \in E(C)$, $1_{\alpha s} s = s = s 1_{\omega s}$.

We view a monoid M as a category $\{v\} \overset{\circ}{\cup} M$ by adjoining a vertex v and taking $\alpha m = \omega m = v$ and the multiplication of edges as in M. On the other hand, for a category C and any $c \in V(C)$, $C(c, c)$ is a monoid under multiplication of edges; this is called a *local submonoid* of the category C.

A *morphism of categories* (or *functor*) is a function $\varphi : C \to D$ respecting sorts and operations in the same sense as in the graph case. We say that a morphism $\varphi : C \to D$ is:

– a *quotient morphism* if φ is surjective and the restriction $\varphi|_{V(C)}$ is injective;

– *faithful* if, for each $c, d \in V(C)$, $\varphi|_{C(c,d)}$ is injective;

– an *embedding* if φ is injective;

– an *isomorphism* if φ is bijective.

The notions of *subcategory* and *product* are analogous to their graph counterparts. If $\varphi : C \to D$ is a surjective morphism of categories and $\psi : A \to D$ is a graph morphism, then there may be no graph morphism $\psi' : A \to C$ such that $\varphi \circ \psi' = \psi$. However, if $\varphi : C \to D$ is a morphism of categories with the above factorization property, then there is a subcategory C' of C such that $\varphi|_{C'} : C' \to D$ is a quotient morphism (see the construction in the next paragraph). We say that a category C *divides* a category D and we write $C \prec D$ if there are a category E, a quotient morphism $E \to C$ and a faithful morphism $E \to D$.

$$\begin{tikzcd} E \arrow[r, "\text{faithful}"] \arrow[d, "\text{quotient}"] & D \\ C & \end{tikzcd}$$

Note that, if $C_i \prec D_i$ $(i \in I)$, then $\prod_{i \in I} C_i \prec \prod_{i \in I} D_i$.

Given an arbitrary morphism $\varphi : C \to D$ of categories, we define a category C/φ by taking $V(C/\varphi) = V(C)$, $E(C/\varphi) = E(C)/\!\sim$, where $\sim$ is the equivalence relation given by

$$s \sim t \quad \text{if} \quad \alpha s = \alpha t,\ \omega s = \omega t \text{ and } \varphi s = \varphi t,$$

$\alpha(s/\sim) = \alpha s$, $\omega(s/\sim) = \omega s$, 1_c in C/φ is $1_c/\sim$, and $s/\sim \cdot t/\sim = st/\sim$. Then the mappings

$$\begin{array}{rcl} \varphi_1 : C & \longrightarrow & C/\varphi \\ c \in V(C) & \longmapsto & c \\ s \in E(C) & \longmapsto & s/\sim \end{array} \quad \text{and} \quad \begin{array}{rcl} \varphi_2 : C/\varphi & \longrightarrow & D \\ c \in V(C) & \longmapsto & \varphi c \\ s/\sim \in E(C/\varphi) & \longmapsto & \varphi s \end{array}$$

are category morphisms, respectively a quotient morphism and a faithful morphism, and $\varphi = \varphi_2 \circ \varphi_1$. In particular, $C/\varphi \prec C$ and $C/\varphi \prec D$.

A *pseudovariety of categories* is a class $\mathbf{V}$ of finite categories which is closed under taking divisors and finitary products. In particular, the class $\mathbf{C}$ of all finite categories is a pseudovariety. For a pseudovariety $\mathbf{V}$ of monoids, define $g\mathbf{V}$ to be the pseudovariety of categories generated by the members of $\mathbf{V}$ (viewed as categories). It is an easy exercise to show that $g\mathbf{V}$ consists of all finite categories which divide some element of $\mathbf{V}$. Consider also the class $\ell\mathbf{V}$ of all finite categories C all of whose local submonoids $C(c,c)$ $(c \in V(C))$ lie in $\mathbf{V}$. The class $\ell\mathbf{V}$ is also a pseudovariety of categories. We say that $\mathbf{V}$ is *local* if $g\mathbf{V} = \ell\mathbf{V}$.

By a *profinite graph* (respectively *profinite category*) we mean a projective limit of finite graphs (resp. categories), finite structures being regarded as discrete topological spaces. More generally, for a class $\mathcal{C}$ of finite graphs (or categories, semigroups, monoids, ...), a topological graph (resp. category, semigroup, monoid, ...) is said to be *pro-$\mathcal{C}$* if it is a projective limit of members of $\mathcal{C}$.

In particular, a profinite graph (resp. profinite category) is a *topological graph* (resp. *topological category*), i.e., a graph (resp. category) endowed with a topology such that all operations are continuous. Let A be a profinite graph. We say that a profinite category C is *A-generated via* $\varphi_C : A \to C$ if φ_C is a continuous morphism of graphs such that the closed subcategory of C generated by $\varphi_C A$ is C itself.

The *fiber product* $C \times_A D$ of two A-generated profinite categories is the closed subcategory of the product $C \times D$ generated by ψA where $\psi a = (\varphi_C a, \varphi_D a)$ for any $a \in A$, i.e., $V(C \times_A D) = \{\psi a : a \in V(A)\}$ and, for any $a, a' \in V(A)$, $C \times_A D(\psi a, \psi a') = C(\varphi_C a, \varphi_C a') \times D(\varphi_D a, \varphi_D a')$. Then $C \times_A D$ is also an A-generated profinite category via the mapping $\varphi_{C \times_A D} = \psi$ and the component projections $p_1 : C \times_A D \to C$ and $p_2 : C \times_A D \to D$ are category morphisms such that $p_1 \circ \varphi_{C \times_A D} = \varphi_C$ and $p_2 \circ \varphi_{C \times_A D} = \varphi_D$.

Fix a pseudovariety $\mathbf{V}$ of categories. In view of the fiber product construction, for each profinite graph A, the A-generated members of $\mathbf{V}$ form a directed system and thus give rise to a projective limit which we denote by $\overline{\Omega}_A\mathbf{V}$ (more precisely, the projective limit is given by a continuous morphism $\iota : A \to \overline{\Omega}_A\mathbf{V}$). The category $\overline{\Omega}_A\mathbf{V}$ is easily seen to be a compact zero-dimensional category. Just as in the case of the construction of projective limits of finite algebras (see, e.g., [8]), one can also prove the following three propositions.

Proposition 2.1 (First universal property of projective limits). *For every pro-$\mathbf{V}$ category C and every continuous graph morphism $\varphi : A \to C$, there is a unique continuous morphism of categories $\overline{\varphi} : \overline{\Omega}_A\mathbf{V} \to C$ such that $\overline{\varphi} \circ \iota = \varphi$.*

$$\begin{tikzcd}
A \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & \overline{\Omega}_A\mathbf{V} \arrow[d, dashed, "\overline{\varphi}"] \\
 & C
\end{tikzcd}$$

Proposition 2.2 (Second universal property of projective limits). *Let P be a topological category and suppose $\{\psi_C\}$ is a nonempty family of continuous category morphisms $\psi_C : P \to C$, one per each continuous graph morphism $\varphi_C : A \to C \in \mathbf{V}$, such that, if $h : C \to C'$ is a category morphism such that $h \circ \varphi_C = \varphi_{C'}$, then $h \circ \psi_C = \psi_{C'}$.*

$$\begin{tikzcd}
A \arrow[r, "\iota"] \arrow[dr, "\varphi_C"'] & \overline{\Omega}_A\mathbf{V} \arrow[d, "\overline{\varphi_C}"'] \arrow[dr, "\overline{\varphi_{C'}}"'] & P \arrow[l, dashed, "\theta"] \arrow[dl, "\psi_C"] \arrow[d, "\psi_{C'}"] \\
 & C \arrow[r, "h"'] & C'
\end{tikzcd}$$

Then there is a unique continuous category morphism $\theta : P \to \overline{\Omega}_A\mathbf{V}$ such that $\overline{\varphi_C} \circ \theta = \psi_C$ for all φ_C.

Proposition 2.3. *All finite categories which are continuous quotient morphic images of $\overline{\Omega}_A\mathbf{V}$ are A-generated members of $\mathbf{V}$.*

While the canonical graph morphism $\iota : A \to \overline{\Omega}_A\mathbf{V}$ is not necessarily injective on $E(A)$, it is always injective on $V(A)$ as is shown below. In case A is finite, we can say a little more.

Proposition 2.4. *Let A be a finite graph. Then $\overline{\Omega}_A\mathbf{V}$ is the projective limit of all A-generated members C of $\mathbf{V}$ under graph morphisms $A \to C$ which are injective on $V(A)$. In particular, the canonical mapping $\iota : A \to \overline{\Omega}_A\mathbf{V}$ is injective on $V(A)$.*

Proof. Let P denote the projective limit indicated in the statement, with associated graph morphism $\kappa : A \to P$. We show that $P \simeq \overline{\Omega}_A\mathbf{V}$ by proving that the family of graph morphisms $A \to C \in \mathbf{V}$ which are injective on $V(A)$ is cofinal in the family of all those graph morphisms without that extra restriction. For this purpose, in view of Proposition 2.2, it suffices to show that $\kappa : A \to P$ has the extension property of Proposition 2.1, i.e., given any graph morphism $\varphi : A \to C$ with $C \in \mathbf{V}$, there is a unique continuous morphism of categories $\overline{\varphi} : P \to C$ such that $\overline{\varphi} \circ \kappa = \varphi$.

Consider the category A^* with $V(A^*) = V(A)$ and whose edges are the *paths* (i.e., finite, possibly empty, sequences of consecutive edges) in A under (consecutive) path concatenation. This is the *free category on* A in the sense that, for any graph morphism $\psi : A \to D$ into a category D, there is a unique extension $\hat{\psi}$ to a category morphism $A^* \to D$.

In particular, there is a unique extension $\hat{\varphi} : A^* \to C$ of the given graph morphism $\varphi : A \to C$, which we may factorize as $\hat{\varphi} = \hat{\varphi}_2 \circ \hat{\varphi}_1$ where $\hat{\varphi}_1 : A^* \to A^*/\hat{\varphi}$ is a quotient morphism and $\hat{\varphi}_2 : A^*/\hat{\varphi} \to C$ is a faithful morphism. Let $C' = A^*/\hat{\varphi}$. Then, since A is a finite graph and, therefore, the category C' is finite, and $C' \prec C$, we have $C' \in \mathbf{V}$. The mapping $\varphi' = \hat{\varphi}_1|_A : A \to C'$ is a graph morphism which is injective on $V(A)$. Whence, by definition of P, there is a continuous morphism $\overline{\varphi'} : P \to C'$ such that $\overline{\varphi'} \circ \kappa = \varphi'$. Finally, take $\overline{\varphi} = \hat{\varphi}_2 \circ \overline{\varphi'}$ to obtain the desired continuous morphism. This argument is illustrated in the following commutative diagram

$$\begin{tikzcd}
A \arrow[rr, "\kappa"] \arrow[d, hook] \arrow[dr, dashed, "\varphi'"'] \arrow[drr, "\varphi"] & & P \arrow[dl, dashed, "\overline{\varphi'}"] \arrow[d, dashed, "\overline{\varphi}"] \\
A^* \arrow[r, dashed, "\hat{\varphi}_1"'] \arrow[rr, bend right, "\hat{\varphi}"'] & C' \arrow[r, dashed, "\hat{\varphi}_2"'] & C
\end{tikzcd}$$

The uniqueness of the morphism $\overline{\varphi}$ follows from the fact that the category P is A-generated via κ, which in turn follows from its definition as a profinite limit of A-generated categories.□

Corollary 2.5. *Let A be a profinite graph. Then the canonical mapping $\iota : A \to \overline{\Omega}_A\mathbf{V}$ is injective on $V(A)$.*

Proof. Let c and d be two distinct vertices of A. Then there exists a continuous morphism $\varphi : A \to B$ into a finite graph B such that $\varphi c \neq \varphi d$. By Proposition 2.1, φ induces a continuous category morphism $\varphi' : \overline{\Omega}_A\mathbf{V} \to \overline{\Omega}_B\mathbf{V}$ such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "\iota_A"] \arrow[d, "\varphi"] & \overline{\Omega}_A\mathbf{V} \arrow[d, "\varphi'"] \\
B \arrow[r, "\iota_B"] & \overline{\Omega}_B\mathbf{V}
\end{tikzcd}$$

Since ι_B is injective by Proposition 2.4 and $\varphi c \neq \varphi d$, we deduce that $\iota_A c \neq \iota_A d$.□

Using Proposition 2.1, we show how to view the elements of $\overline{\Omega}_A\mathbf{V}$ as certain operations on the members of $\mathbf{V}$. For a profinite graph A and a topological category C, denote by

$\mathcal{G}(A,C)$ the set of all continuous graph morphisms $A \to C$. By an *A-ary implicit operation on* $\mathbf{V}$ we mean a family $(\pi_C)_{C\in\mathbf{V}}$ such that:

i) each $\pi_C : \mathcal{G}(A,C) \to C$ is a function;

ii) for any morphism of categories $h : C \to C'$ with $C, C' \in \mathbf{V}$, the following diagram commutes

$$\begin{tikzcd} \mathcal{G}(A,C) \arrow[r, "\pi_C"] \arrow[d, "h\circ\_"] & C \arrow[d, "h"] \\ \mathcal{G}(A,C') \arrow[r, "\pi_{C'}"] & C' \end{tikzcd}$$

Notice that condition (ii) implies that the π_C ($C \in \mathbf{V}$) cannot assume both vertex and edge values, since $\mathbf{V}$ is closed under finitary product. Thus, we could already distinguish *vertex* and *edge implicit operations*. This distinction will come for free in a somewhat more elegant way from Proposition 2.6 below.

Now, for each $\pi \in \overline{\Omega}_A\mathbf{V}$ and each $C \in \mathbf{V}$, we define a function $\pi_C : \mathcal{G}(A,C) \to C$ by $\pi_C(\varphi) = \overline{\varphi}(\pi)$ where $\overline{\varphi}$ is given by Proposition 2.1. To show that $(\pi_C)_{C\in\mathbf{V}}$ is an A-ary implicit operation on $\mathbf{V}$, we need to show that, if $h : C \to C'$ is a morphism of categories with $C, C' \in \mathbf{V}$, then, for any $\varphi \in \mathcal{G}(A,C)$,

$$\pi_{C'}(h\circ\varphi) = h(\pi_C(\varphi)), \text{ i.e., } \overline{h\circ\varphi}(\pi) = (h\circ\overline{\varphi})(\pi),$$

and this follows from the uniqueness statement in Proposition 2.1 since $h\circ\overline{\varphi} : \overline{\Omega}_A\mathbf{V} \to C'$ is a continuous morphism of categories such that $h\circ\overline{\varphi}\circ\iota = h\circ\varphi$.

$$\begin{tikzcd} A \arrow[r, "\iota"] \arrow[dr, "\varphi"'] & \overline{\Omega}_A\mathbf{V} \arrow[d, "\overline{\varphi}"] \arrow[dr, "h\circ\overline{\varphi}=\overline{h\circ\varphi}"] & \\ & C \arrow[r, "h"] & C' \end{tikzcd}$$

Proposition 2.6. *The mapping $\pi \mapsto (\pi_C)_{C\in\mathbf{V}}$ is a bijection between $\overline{\Omega}_A\mathbf{V}$ and the set of all A-ary implicit operations on $\mathbf{V}$.*

Proof. By definition of the projective limit, the continuous morphisms $\psi : \overline{\Omega}_A\mathbf{V} \to C \in \mathbf{V}$ are sufficient to separate the points of $\overline{\Omega}_A\mathbf{V}$. Now, for $\pi \in \overline{\Omega}_A\mathbf{V}$, $\pi_C(\psi\circ\iota) = \psi(\pi)$, so that, for $\pi, \rho \in \overline{\Omega}_A\mathbf{V}$, $(\pi_C)_{C\in\mathbf{V}} = (\rho_C)_{C\in\mathbf{V}}$ implies $\pi = \rho$, showing that the given mapping is injective.

Now, let $(\pi_C)_{C\in\mathbf{V}}$ be an A-ary implicit operation on $\mathbf{V}$. For each $C \in \mathbf{V}$ and each $\varphi \in \mathcal{G}(A,C)$, consider the subset $\overline{\varphi}^{-1}(\pi_C(\varphi))$ of $\overline{\Omega}_A\mathbf{V}$, where $\overline{\varphi}$ is given by Proposition

2.1. Since $\overline{\varphi}$ is continuous, $\overline{\varphi}^{-1}(\pi_C(\varphi))$ is a nonempty clopen subset of $\overline{\Omega}_A\mathbf{V}$: indeed, since $(\pi_C)_{C\in\mathbf{V}}$ is an implicit operation, $\pi_C(\varphi) = \pi_D(\varphi)$ where D is the subcategory of C generated by φA, and so $\pi_C(\varphi)$ belongs to the image of $\overline{\varphi}$, which is D. Since $\overline{\Omega}_A\mathbf{V}$ is compact, to show that there is $\rho \in \overline{\Omega}_A\mathbf{V}$ such that $\rho \in \overline{\varphi}^{-1}(\pi_C(\varphi))$, i.e., $\rho_C(\varphi) = \overline{\varphi}(\rho) = \pi_C(\varphi)$, for all $C \in \mathbf{V}$ and $\varphi \in \mathcal{G}(A, C)$, it suffices to show that every finite collection $\overline{\varphi_1}^{-1}(\pi_{C_1}(\varphi_1)), \ldots, \overline{\varphi_n}^{-1}(\pi_{C_n}(\varphi_n))$ has nonempty intersection. To prove this fact, consider the continuous morphism $\varphi : A \to C_1 \times \cdots \times C_n$ into the product category which is given by $\varphi a = (\varphi_1 a, \ldots, \varphi_n a)$. Then, since $(\pi_C)_{C\in\mathbf{V}}$ is an implicit operation, $\pi_C(\varphi) = \big(\pi_{C_1}(\varphi_1), \ldots, \pi_{C_n}(\varphi_n)\big)$ and so $\overline{\varphi}^{-1}(\pi_C(\varphi)) = \bigcap_{i=1}^n \overline{\varphi_i}^{-1}(\pi_{C_i}(\varphi_i))$, establishing the claim and, therefore, the surjectiveness of the given mapping.□

In view of Proposition 2.6, we identify the elements of $\overline{\Omega}_A\mathbf{V}$ with the associated A-ary implicit operations on $\mathbf{V}$. An *A-pseudoidentity for* $\mathbf{V}$ is a formal equality $\pi = \rho$ between two edges of $\overline{\Omega}_A\mathbf{V}$ with $\alpha\pi = \alpha\rho$ and $\omega\pi = \omega\rho$. We say that a category C in $\mathbf{V}$ *satisfies* the pseudoidentity $\pi = \rho$ and we write $C \models \pi = \rho$ if $\pi_C = \rho_C$. For a set Σ of pseudoidentities for $\mathbf{V}$, we denote by $[\![\Sigma]\!]_{\mathbf{V}}$ the class of all C in $\mathbf{V}$ which satisfy all the pseudoidentities in Σ; the set Σ is then said to be a *basis* (of $\mathbf{V}$-*pseudoidentities*) for $[\![\Sigma]\!]_{\mathbf{V}}$. In case $\mathbf{V}$ is the pseudovariety $\mathbf{C}$ of all finite categories, we sometimes write simply $[\![\Sigma]\!]$ instead of $[\![\Sigma]\!]_{\mathbf{C}}$.

Let $\mathbf{W}$ be a subpseudovariety of $\mathbf{V}$. Then, by Proposition 2.2, for each profinite graph A, there is a unique continuous morphism $\theta : \overline{\Omega}_A\mathbf{V} \to \overline{\Omega}_A\mathbf{W}$ such that $\theta \circ \iota_{\mathbf{V}} = \iota_{\mathbf{W}}$ where $\iota_{\mathbf{V}} : A \to \overline{\Omega}_A\mathbf{V}$ and $\iota_{\mathbf{W}} : A \to \overline{\Omega}_A\mathbf{W}$ are the morphisms defining the projective limits; it is not hard to show that θ is in fact a quotient morphism. We call θ the *canonical projection* $\overline{\Omega}_A\mathbf{V} \to \overline{\Omega}_A\mathbf{W}$; note that, in terms of implicit operations, it is given by $(\pi_C)_{C\in\mathbf{V}} \mapsto (\pi_C)_{C\in\mathbf{W}}$. In view of Proposition 2.1, the A-pseudoidentity $\pi = \rho$ is satisfied by $\mathbf{W}$ if and only if $\theta(\pi) = \theta(\rho)$.

The following is an extension to categories of Reiterman's theorem on the definability of pseudovarieties of finite algebras by pseudoidentities [25]. It might be of interest to extend it further to a more general setting of partial algebraic systems.

Theorem 2.7. *A subclass $\mathbf{W}$ of $\mathbf{V}$ is a pseudovariety of categories if and only if there is a set Σ of pseudoidentities over finite graphs for $\mathbf{V}$ such that $\mathbf{W} = [\![\Sigma]\!]_{\mathbf{V}}$.*

Proof. Using the fact that implicit operations commute with category morphisms, it is easy to check that $[\![\Sigma]\!]_{\mathbf{V}}$ is closed under taking divisors (where the restriction that the two sides of a pseudoidentity are coterminal edges intervenes) and finitary products and, therefore, it is a pseudovariety.

For the converse, consider a subpseudovariety $\mathbf{W}$ of $\mathbf{V}$ and let Σ be the set of all pseudoidentities for $\mathbf{V}$ over finite graphs A (taken from a set containing one graph per isomorphism class of finite graphs) which are satisfied by $\mathbf{W}$. Let $\mathbf{W}' = [\![\Sigma]\!]_{\mathbf{V}}$. Then clearly $\mathbf{W}$ is contained in $\mathbf{W}'$. On the other hand, if $C \in \mathbf{W}'$, let A be a subgraph of C generating C (say $A = C$). Then, by definition of projective limit, there is a continuous quotient morphism $\overline{\Omega}_A\mathbf{W}' \to C$. Now, consider the canonical projections

$$\overline{\Omega}_A\mathbf{V} \xrightarrow{\theta_1} \overline{\Omega}_A\mathbf{W}' \xrightarrow{\theta_2} \overline{\Omega}_A\mathbf{W}.$$

If $\pi, \rho \in \overline{\Omega}_A\mathbf{V}$ and $\theta_2(\theta_1(\pi)) = \theta_2(\theta_1(\rho))$, then the pseudoidentity $\pi = \rho$ is valid in $\mathbf{W}$ and so it belongs to Σ; whence $\pi = \rho$ is valid in $\mathbf{W}'$ so that $\theta_1(\pi) = \theta_1(\rho)$. Hence the mapping θ_2 is injective and, therefore, it is an isomorphism of profinite categories. Finally, by Proposition 2.3, since C is a continuous quotient morphic image of $\overline{\Omega}_A\mathbf{W}$, we conclude that $C \in \mathbf{W}$. This proves that $\mathbf{W} = [\![\Sigma]\!]_{\mathbf{V}}$.□

Note that, if we also allow pseudoidentities involving vertices, then these do not necessarily define pseudovarieties of categories since they are not preserved in general by division via faithful morphisms, unless they are *trivial* (i.e., both sides are the same implicit operation). However, they are preserved by embeddings, quotients and products and we might look for an analogue of Theorem 2.7 with a different definition of pseudovariety using vertex pseudoidentities. Now, a category satisfies a nontrivial "vertex pseudoidentity" if and only if it has only one vertex, i.e., it is a monoid. So all we would be adding by allowing such "vertex pseudoidentities" would be the pseudovarieties of monoids, and these are better treated as pseudovarieties of (total) algebras.

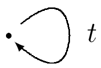
Let us consider some natural examples of implicit operations on $\mathbf{V}$. For a profinite graph A and a fixed edge $s \in E(A)$, let $s_C : \mathcal{G}(A, C) \to C$ be defined by $s_C(\varphi) = \varphi(s)$ for each $C \in \mathbf{V}$ and $\varphi \in \mathcal{G}(A, C)$. Then $(s_C)_{\mathbf{V}}$ is an A-ary implicit operation on $\mathbf{V}$ which we call an *(edge-) projection*. The elements of the subcategory $\Omega_A\mathbf{V}$ of $\overline{\Omega}_A\mathbf{V}$ generated by $\iota(A)$ are called the *A-ary explicit operations on* $\mathbf{V}$. By standard techniques of Universal Algebra, it may be shown that, for a finite graph A, $\Omega_A\mathbf{V}$ is the *free category over A relative to* $\mathbf{V}$ in the sense that, for any graph morphism $\varphi : A \to C$, there is a unique category morphism $\hat{\varphi} : \Omega_A\mathbf{V} \to C$ such that $\hat{\varphi} \circ \iota = \varphi$. Note that the projection in $C \in \mathbf{V}$ of $\Omega_A\mathbf{V}$ under any onto morphism $\overline{\Omega}_A\mathbf{V} \to C$ is the whole C. Since $\overline{\Omega}_A\mathbf{V}$ has the initial topology for all such morphisms, we conclude that $\Omega_A\mathbf{V}$ is a dense subcategory of $\overline{\Omega}_A\mathbf{V}$ and so $\overline{\Omega}_A\mathbf{V}$ is precisely the *pro-$\mathbf{V}$ completion* of $\Omega_A\mathbf{V}$, i.e., the completion with respect to the uniform structure determined by the onto morphisms $\Omega_A\mathbf{V} \to C \in \mathbf{V}$ (which may be taken to be quotient morphisms in case A is finite).

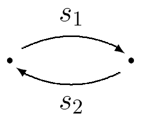
Given $C \in \mathbf{V}$ and a *loop* $s \in C$ (i.e., an edge such that $\alpha s = \omega s$), we denote by s^ω the unique power of s of positive exponent which is an idempotent. We thus define an element a^ω of $\overline{\Omega}_A\mathbf{V}$ where A is the graph with only one vertex v and one loop a: for $C \in \mathbf{V}$, an element φ of $\mathcal{G}(A, C)$ is given by the choice of a loop s of C; take $(a^\omega)_C(\varphi) = s^\omega$.

Another way of producing implicit operations is to compose given implicit operations. To be precise, let A and B be finite graphs, let $\pi \in \overline{\Omega}_A\mathbf{V}$ and let $\delta : A \to \overline{\Omega}_B\mathbf{V}$ be a graph morphism. Then, for each $C \in \mathbf{V}$ which is B-generated via $\varphi : B \to C$, we have a graph morphism $\overline{\varphi} \circ \delta : A \to C$ and so a continuous category morphism $\overline{\overline{\varphi} \circ \delta} : \overline{\Omega}_A\mathbf{V} \to C$. For another $\varphi' : B \to C'$ and a category morphism $h : C \to C'$ such that $\varphi' = h \circ \varphi$, clearly $\overline{\overline{\varphi'} \circ \delta} = h \circ \overline{\overline{\varphi} \circ \delta}$ since $\varphi' \circ \delta = h \circ \varphi \circ \delta$. Hence, by the universal property of $\overline{\Omega}_B\mathbf{V}$, we have a continuous category morphism $\varepsilon : \overline{\Omega}_A\mathbf{V} \to \overline{\Omega}_B\mathbf{V}$ such that $\varepsilon \circ \iota = \delta$. The following commutative diagram illustrates this construction.

$$\begin{tikzcd}
A \arrow[r, "\iota"] \arrow[d, "\delta"] & \overline{\Omega}_A\mathbf{V} \arrow[dl, dashed, "\varepsilon"'] \arrow[dd, dashed, "\overline{\overline{\varphi}\circ\delta}"] \arrow[ddr, dashed, "\overline{\overline{\varphi'}\circ\delta}"] & \\
\overline{\Omega}_B\mathbf{V} \arrow[dr, dashed, "\overline{\varphi}"] \arrow[drr, dashed, "\overline{\varphi'}"] & & \\
B \arrow[u] \arrow[r, "\varphi"] \arrow[rr, bend right, "\varphi'"'] & C \arrow[r, "h"] & C'
\end{tikzcd}$$

We may view $\varepsilon(\pi)$ as the B-ary implicit operation on $\mathbf{V}$ obtained by substituting each $a \in A$ in π by the corresponding B-ary implicit operation $\delta(a)$. As an example, let A and B be the graphs

$A :$ 

$B :$ 

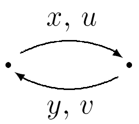
let $\pi = t^\omega$ and let $\delta : A \to \overline{\Omega}_B\mathbf{V}$ be given by $\delta(t) = s_1s_2$. Then, with the above notation, $\varepsilon(\pi) = (s_1s_2)^\omega$ is the B-ary implicit operation such that, for $C \in \mathbf{V}$ and $\varphi \in \mathcal{G}(B, C)$, $(\varepsilon(\pi))_C(\varphi) = \big(\varphi(s_1)\varphi(s_2)\big)^\omega$.

For example, we may rephrase Knast's Theorem on graphs [20] in the language of pseudoidentities. Recall that $\mathbf{J}$ denotes the pseudovariety of all finite $\mathcal{J}$-trivial semigroups. Since $\mathbf{J}$ is a monoidal pseudovariety, in the sense that $S \in \mathbf{J}$ implies $S^1 \in \mathbf{J}$, there is no harm in also denoting by $\mathbf{J}$ the pseudovariety of all finite $\mathcal{J}$-trivial monoids.

Theorem 2.8 (Knast). *The pseudovariety $g\mathbf{J}$ is defined by the pseudoidentity*

$$(xy)^\omega xv(uv)^\omega = (xy)^\omega(uv)^\omega$$

over the following graph:



Call a graph A *strongly connected* if, for all vertices c and d, there is a directed path from c to d. The maximal strongly connected subgraphs of a graph are called its *bonded components*.

Let $\mathbf{Sl}$ denote the pseudovariety of all finite semilattices, which is generated by the 2-element multiplicative monoid $\{1,0\}$. It is easy to see that $\overline{\Omega}_A g\mathbf{Sl}$ is a finite category. In fact, more generally, if C is a finite category and $\mathbf{V}(C)$ is the pseudovariety it generates, then $\overline{\Omega}_A \mathbf{V}(C)$ is finite for every finite graph A. In particular, since $\Omega_A g\mathbf{Sl}$ is dense in $\overline{\Omega}_A g\mathbf{Sl}$, $\Omega_A g\mathbf{Sl}$ coincides with $\overline{\Omega}_A g\mathbf{Sl}$.

An explicit construction of $\Omega_A g\mathbf{Sl}$ is borrowed from [5] and is obtained as follows. Define A' to be the category whose vertices are those of A and whose edges from $p \in V(A)$ to $q \in V(A)$ are the triples $(p, c(w), q)$ with $w \in E(A^*)$ a path from p to q, and where $c(w)$ denotes the subgraph of A consisting of all edges and vertices used by the path w. The composite of two edges $(p, c(u), q)$ and $(q, c(v), r)$ is the edge $(p, c(uv), r) = (p, c(u) \cup c(v), q)$. Then the function $(p, c(w), q) \mapsto E(c(w))$ defines a faithful morphism $\varphi : A' \to \mathcal{P}(E(A))$ where $\mathcal{P}(E(A))$ is a semilattice under union. Hence $A' \in g\mathbf{Sl}$ and so there is a quotient morphism $\psi : \Omega_A g\mathbf{Sl} \to A'$. But the composite $\varphi \circ \psi : \Omega_A g\mathbf{Sl} \to \mathcal{P}(E(A))$ is faithful by [5, Prop. 2.3]. Hence ψ is an isomorphism.

Let now A be a finite graph and $\mathbf{V}$ a pseudovariety of categories containing $g\mathbf{Sl}$. Then there is a unique continuous quotient mophism $\theta : \overline{\Omega}_A \mathbf{V} \to A'$ such that $\theta(a) = (\alpha a, c(a), \omega a)$ for each $a \in E(A)$. For a subset U of $\overline{\Omega}_A \mathbf{V}$, we define its *content* to be the graph $c(U) = F \cup \alpha F \cup \omega F$ where $F = \varphi(\theta(U))$. It follows that $c(U)$ is the smallest subgraph B of A such that $U \subseteq \overline{\Omega}_B \mathbf{V}$. We say that a pseudoidentity $\pi = \rho$ for $\mathbf{V}$ is *bonded* if the graph $c(\{\pi, \rho\})$ (also called the *content* of $\pi = \rho$) is strongly connected. For example, the pseudoidentity in Knast's Theorem is bonded.

Theorem 2.7 may be improved as follows. The pseudovariety consisting of all finite *trivial categories* (i.e., with no more than one edge between any two vertices or, equivalently, a divisor of a trivial monoid) is called the *trivial pseudovariety*; it is defined by the pseudoidentity $u = v$ over the graph $\cdot \xrightarrow{u,v} \cdot$, which is not bonded. This is the only exceptional case in the following sense.

Theorem 2.9. *If* $\mathbf{W}$ *is a nontrivial subpseudovariety of a pseudovariety* $\mathbf{V}$ *of categories containing* $g\mathbf{Sl}$*, then* $\mathbf{W} = [\![\Sigma]\!]_{\mathbf{V}}$ *for some set* Σ *of bonded pseudoidentities for* $\mathbf{V}$*.*

Proof. Let Σ be the set of all bonded pseudoidentities for $\mathbf{V}$ over finite graphs (from a suitably chosen set) which are valid in $\mathbf{W}$, and consider the pseudovariety given by $\mathbf{W}' = [\![\Sigma]\!]_{\mathbf{V}}$. Then $\mathbf{W}'$ is a pseudovariety containing $\mathbf{W}$. To prove the equality of these two pseudovarieties, it suffices to show that, if $C \in \mathbf{W}'$, then $C \in \mathbf{W}$, i.e., C satisfies all pseudoidentities over finite graphs which are valid in $\mathbf{W}$ (cf. Theorem 2.7).

So, let $\pi = \rho$ be such a pseudoidentity. Then, since $c(\{\pi, \rho\})$ is a finite graph, by arguments of Tilson [31, section 10] which are easily adapted and will not be reproduced here, we may factorize π and ρ in the form $\pi = \pi_0 t_1 \pi_1 \dots t_n \pi_n$ and $\rho = \rho_0 t_1 \rho_1 \dots t_n \rho_n$, as products of implicit operations, so that each pseudoidentity $\pi_i = \rho_i$ is bonded, its content is a bonded component of the content of $\pi = \rho$, and each t_i is an edge from a vertex of $c(\{\pi_{i-1}, \rho_{i-1}\})$ to a vertex of $c(\{\pi_i, \rho_i\})$ $(i = 1, \dots, n)$. By identifying all vertices from $c(\{\pi_0 t_1 \dots \pi_{i-1} t_i, \rho_0 t_1 \dots \rho_{i-1} t_i\})$ to ωt_i and all edges to $1_{\omega t_i}$ and proceeding similarly with the graph $c(\{t_{i+1}\pi_{i+1} \dots t_n \pi_n, t_{i+1}\rho_{i+1} \dots t_n \rho_n\})$ (identification which amounts to composition of implicit operations as described earlier), we conclude that the bonded pseudoidentity $\pi_i = \rho_i$ is valid in $\mathbf{W}$. Hence it is also valid in C. Since this is true for $i = 0, \dots, n$, it follows that $\pi = \rho$ is valid in C, as desired.□

The restriction that $g\mathbf{Sl} \subseteq \mathbf{V}$ causes no problems in the applications of Theorem 2.9 since usually $\mathbf{V}$ is taken to be the pseudovariety of all finite categories. As a corollary, we obtain Tilson's Bonded Component Theorem [31, Theor. 13.6].

Corollary 2.10 (Tilson). *A finite category belongs to a nontrivial pseudovariety* $\mathbf{V}$ *if and only if all of its bonded components belong to* $\mathbf{V}$.

Proof. Since $\mathbf{V}$ is nontrivial, by Theorem 2.9, to check whether a finite category C belongs to $\mathbf{V}$, it suffices to check whether it satisfies certain bonded pseudoidentities. Since the image under a graph morphism of a strongly connected subgraph is also strongly connected, the verification of all such pseudoidentities takes place within the bonded components of C and the result follows.□

By a *semigroupoid* we mean a graph S endowed with an associative partial operation on $E(S)$ satisfying condition (ii) of the definition of category at the beginning of this section. The notions of *morphism, quotient morphism, faithful morphism, division* and *product* for semigroupoids are identical to their category counterparts. Another relevant operation is the *coproduct* which, for two semigroupoids S and T is given by their disjoint union, for vertices, for edges, and for the operations.

A *pseudovariety of semigroupoids* is a class of finite semigroupoids which is closed under taking divisors, finitary products and finitary coproducts, and contains the semigroupoid with one vertex and one edge. The last requirement in the definition of pseudovariety only excludes some rather particular cases and is asserted as a necessary and sufficient condition to have morphisms from any graph into members of the class. This condition is not given explicitly in [31] where empty products are also considered. The class of all finite semigroupoids will be denoted by $\mathbf{Sd}$.

Local identities do not intervene in the results of this section, up to and excluding Theorem 2.9. Therefore these results carry over unchanged to the realm of semigroupoids. In particular, every pseudovariety of semigroupoids is defined by some set of semigroupoid pseudoidentities.

Any category pseudoidentity $u = v$ may be viewed as a semigroupoid pseudoidentity: without change if neither side is a local identity; if $v = 1_q$ is an identity at the vertex q, then $u = v$ is interpreted for semigroupoids as meaning the two pseudoidentities $ux = x$ and $yu = y$ where x and y are new edges added to the graph $c(u)$ satisfying $\alpha x = \omega y = q$. Of course, this interpretation may change bonded pseudoidentities into non-bonded pseudoidentities. However, if this happens, then every local semigroup $S(q, q)$ of a semigroupoid satisfying such pseudoidentities is a group and this is a rather special situation which does not cause difficulties.

Given a semigroupoid S, the set of loops $S(c, c)$ may be empty; but, otherwise, it is a semigroup under edge composition which is called a *local subsemigroup* of S. Every semigroup S may be viewed as a one-vertex semigroupoid such that S arises from it in this way.

For a pseudovariety $\mathbf{V}$ of semigroups (respectively semigroupoids), we denote by $\mathbf{V_M}$ (resp. $\mathbf{V_C}$) the pseudovariety of all monoids (resp. categories) which, viewed as semigroups (resp. semigroupoids), lie in $\mathbf{V}$. For a pseudovariety $\mathbf{V}$ of monoids (resp. categories), we denote by $\mathbf{V_S}$ (resp. $\mathbf{V_{Sd}}$) the pseudovariety generated by the elements of $\mathbf{V}$ viewed as semigroups (resp. semigroupoids). Then a pseudovariety $\mathbf{V}$ of semigroups is monoidal if and only if it is of the form $\mathbf{V_S}$ for some pseudovariety $\mathbf{V}$ of monoids, in which case $(\mathbf{V_M})_\mathbf{S} = \mathbf{V}$. Similarly, we say that a pseudovariety of semigroupoids $\mathbf{V}$ is *categorical* if it is of the form $\mathbf{V_{Sd}}$ for some pseudovariety $\mathbf{V}$ of categories; equivalently, $\mathbf{V}$ is categorical if and only if $(\mathbf{V_C})_\mathbf{Sd} = \mathbf{V}$. Moreover, the operators $\__\mathbf{S}$ and $\__\mathbf{Sd}$ behave well with respect to lattice operations and semidirect product (cf. [4] for the semigroup case; in the semigroupoid case, the semidirect product is to be replaced by the operation defined in the next section). We will therefore drop the subscripts $\mathbf{S}$ and $\mathbf{Sd}$ without harm and we will often deal indifferently with pseudovarieties of monoids (resp. categories) or monoidal pseudovarieties of semigroups (resp. categorical pseudovarieties of semigroupoids).

For a semigroupoid S, define a category S^c which is obtained from S by adding a local identity 1_v at each vertex v which does not already have one. One can easily check that

$$\text{if a semigroupoid } S \text{ divides a semigroupoid } T, \text{ then } S^c \prec T^c \text{ as categories.} \tag{1}$$

In particular, the division relation as categories or as semigroupoids is the same for categories. Moreover, for a pseudovariety $\mathbf{V}$ of categories and a semigroupoid S,

$$S \in \mathbf{V_{Sd}} \quad \text{if and only if} \quad S^c \in \mathbf{V}. \tag{2}$$

At the pseudovariety level, we may define an associated operation mapping each pseudovariety $\mathbf{V}$ of semigroupoids to the pseudovariety $\mathbf{CV}$ of categories generated by all S^c

with $S \in \mathbf{V}$. It is easy to verify that

$$\text{for every pseudovariety } \mathbf{V} \text{ of categories, } \mathbf{C}(\mathbf{V_{Sd}}) = \mathbf{V}. \tag{3}$$

For a pseudovariety $\mathbf{V}$ of semigroups, we also denote by $g\mathbf{V}$ the pseudovariety of semigroupoids generated by the elements of $\mathbf{V}$, viewed as semigroupoids. It consists of all finite semigroupoids which divide some element of $\mathbf{V}$.

We also define $\ell\mathbf{V}$ to be the class of all finite semigroupoids all of whose local subsemigroups lie in $\mathbf{V}$. Clearly if Σ is a basis of pseudoidentities for $\mathbf{V}$, then these same pseudoidentities, viewed as semigroupoid pseudoidentities over one-vertex graphs define $\ell\mathbf{V}$. We say that a pseudovariety $\mathbf{V}$ of semigroups is *local* if $\ell\mathbf{V} = g\mathbf{V}$. Equivalently, $\mathbf{V}$ is local if $g\mathbf{V}$ is defined by pseudoidentities over one-vertex graphs.

As has been observed by Karl Auinger [10], locality of a pseudovariety $\mathbf{V}$ of monoids and locality of $\mathbf{V_S}$ are not equivalent notions. It is well-known that every non-trivial pseudovariety of groups is local as a pseudovariety of monoids ([29, Theor. 10.2] and [31, Theor. 17.3]). However, the semigroupoid

$$\begin{tikzcd} \bullet \arrow[r, bend left, "s"] \arrow[r, bend right, "se"'] & \bullet \arrow[loop right, "e=e^2"] \end{tikzcd} \tag{4}$$

is in $\ell\mathbf{I}$ for the trivial pseudovariety $\mathbf{I}$ of semigroups but cannot belong to $g\mathbf{G}$ for the pseudovariety $\mathbf{G}$ of all finite groups since it fails the pseudoidentity $xy^\omega = x$ over the graph

$$\begin{tikzcd} \bullet \arrow[r, "x"] & \bullet \arrow[loop right, "y"] \end{tikzcd}$$

which is valid in $\mathbf{G}$. Hence no pseudovariety of groups is local as a pseudovariety of semigroups.

The situation just portrayed for pseudovarieties of groups is rather unique. To analyze it in more detail, note that, for a pseudovariety $\mathbf{V}$ of monoids, in view of (1) and (2), $g(\mathbf{V_S}) = (g\mathbf{V})_{\mathbf{Sd}}$ whereas it may happen that $\ell(\mathbf{V_S}) \supsetneqq (\ell\mathbf{V})_{\mathbf{Sd}}$. The possibility of inequality stems from the fact that, if $S \in (\ell\mathbf{V})_{\mathbf{Sd}}$, i.e., $S^c \in \ell\mathbf{V}$, then certainly for each local subsemigroup T of S, T^1 is a subsemigroup of the local submonoid of S^c at the same vertex, but does not necessarily coincide with it (cf. the example given by (4)) for T may have a neutral element which is not a local identity in the sense of condition (i) of the definition of category. However, if $\mathbf{V} \not\subseteq \mathbf{G}$, then $\mathbf{V}$ contains $\mathbf{Sl}$ and so, if $T = T^1 \in \mathbf{V}$, then adding a new neutral element to T yields still a monoid in $\mathbf{V}$. Hence, if $\mathbf{V} \not\subseteq \mathbf{G}$, then $\ell(\mathbf{V_S}) = (\ell\mathbf{V})_{\mathbf{Sd}}$. Taking also into account (3), it follows that, if $\mathbf{V}$ is a pseudovariety of monoids not contained in $\mathbf{G}$, then $\mathbf{V}$ is local if and only if $\mathbf{V_S}$ is local as a pseudovariety of semigroups.

It may also be observed that $\ell(\mathbf{V_M}) = (\ell\mathbf{V})_\mathbf{C}$. for every pseudovariety $\mathbf{V}$ of monoids, while using (1) we obtain $g(\mathbf{V_M}) = (g\mathbf{V})_\mathbf{C}$ in case $\mathbf{V}$ is monoidal. The latter equation fails for instance for the non-monoidal pseudovariety $\mathbf{N}$ of all finite nilpotent semigroups. (See also the end of section 7).

3. An extension of the semidirect product of pseudovarieties

Given two semigroups S and T, a *relational morphism* $R : S \to T$ is any subsemigroup R of the product $S \times T$ such that the projection $R \to S$ is onto. We often think of a relational morphism as a relation which, in particular, allows us to consider composites. The composite

$$R' \circ R = \{(s,u) : \text{ there exists } t \text{ such that } (s,t) \in R \text{ and } (t,u) \in R'\}$$

of two relational morphisms $R : S \to T$ and $R' : T \to U$ is a relational morphism $S \to U$. A relational morphism $R : S \to T$ is said to be a *division* if the projection $R \to T$ is injective. Notice that S divides T if and only if there is a division $S \to T$.

The *derived semigroupoid* D_R of a relational morphism $R : S \to T$ is defined by:

- $V(D_R) = T^1$;
- the edges of D_R are equivalence classes $[t, s_1, t_1]$ of triples in $T^1 \times R$ under the equivalence relation given by

$$\begin{aligned}(t, s_1, t_1) \sim_R (t', s_2, t_2) \quad & \text{if } t = t',\ tt_1 = tt_2, \text{ and,} \\ & \text{for every } s \text{ such that } (s,t) \in R \cup \{(1,1)\},\ ss_1 = ss_2; \end{aligned} \tag{5}$$

- $\alpha[t, s_1, t_1] = t$, $\omega[t, s_1, t_1] = tt_1$;
- $[t, s_1, t_1] \cdot [tt_1, s_2, t_2] = [t, s_1s_2, t_1t_2]$.

The following proposition gathers a few useful properties of derived semigroupoids.

Proposition 3.1. *a) Let $R : S \to T$ and $R' : S \to T'$ be two relational morphisms and suppose that $\psi : T \to T'$ is a homomorphism such that $\psi \circ R \subseteq R'$. Then $D_R \prec D_{R'}$.*

b) Let $R : S \to T$ and $R' : S' \to T$ be two relational morphisms and suppose that $R'' : S \to S'$ is a division such that $R' \circ R'' = R$. Then $D_R \prec D_{R'}$.

c) Let $R_1 : S_1 \to T_1$ and $R_2 : S_2 \to T_2$ be relational morphisms and let $R : S_1 \times S_2 \to T_1 \times T_2$ be the associated relational morphism (via the natural bijection $(S_1 \times T_1) \times (S_2 \times T_2) \to (S_1 \times S_2) \times (T_1 \times T_2)$). Then D_R is a quotient of $D_{R_1} \times D_{R_2}$.

Proof. Except for the missing assumption $\psi \circ R \supseteq R'$, (a) is the semigroupoid counterpart of Proposition 4.4 in [31]; the reader may easily verify that the extra assumption is not used in Tilson's proof. Part (b) is the semigroupoid version of Proposition 4.5 in [31]. For part (c), the correspondence

$$\Big([t_1, r_1], [t_2, r_2]\Big) \longmapsto \Big[(t_1, t_2), (r_1, r_2)\Big]$$

is easily checked to define a quotient morphism $D_{R_1} \times D_{R_2} \to D_R$.□

Let $\mathbf{V}$ be a class of finite semigroupoids and let $\mathbf{W}$ be a class of finite semigroups. We denote by $\mathbf{V} \circledast \mathbf{W}$ the class of all finite semigroups S such that there is a relational morphism $R : S \to T$ with $T \in \mathbf{W}$ and $D_R \in \mathbf{V}$.

Proposition 3.2. *Let $\mathbf{V}$ be a pseudovariety of semigroupoids and let $\mathbf{W}$ be a pseudovariety of semigroups. Then $\mathbf{V} \circledast \mathbf{W}$ is a pseudovariety of semigroups.*

Proof. By Proposition 3.1(b), (b) again, and (c), respectively, $\mathbf{V} \circledast \mathbf{W}$ is closed under taking homomorphic images, subsemigroups, and finitary direct products.□

Recall that, for two pseudovarieties of semigroups, $\mathbf{V}$ and $\mathbf{W}$, $\mathbf{V} * \mathbf{W}$ denotes the pseudovariety of semigroups generated by all semidirect products of the form $S * T$ with $S \in \mathbf{V}$ and $T \in \mathbf{W}$. Tilson's Derived Category Theorem may now be used to show that the operation $\circledast$ is, in a sense, an extension of the semidirect product on pseudovarieties of semigroups. The following result is just a reformulation of Tilson's Theorem B.2 [31]. A proof of it appears in the next section.

Theorem 3.3. *Let $\mathbf{V}$ and $\mathbf{W}$ be pseudovarieties of semigroups. Then*

$$\mathbf{V} * \mathbf{W} = (g\mathbf{V}) \circledast \mathbf{W}.$$

We proceed with some useful properties of the operation $\circledast$, some of which will be applied in the next section. The first one is just an observation which follows immediately from the definition of $\circledast$.

Proposition 3.4. *Let $\mathbf{V}$ be any class of finite semigroupoids and let $(\mathbf{W}_i)_{i\in I}$ be any family of classes of finite semigroups. Then $\mathbf{V} \circledast (\bigcup_{i\in I} \mathbf{W}_i) = \bigcup_{i\in I} (\mathbf{V} \circledast \mathbf{W}_i)$.*□

We recall that, for a pseudovariety $\mathbf{W}$ of semigroups and a profinite set A, the projective limit $\overline{\Omega}_A \mathbf{W}$ of all "A-generated" members of $\mathbf{W}$ is profinitely freely generated by A relatively to $\mathbf{W}$. The elements of $\overline{\Omega}_A \mathbf{W}$ are naturally identified with "A-ary implicit operations" on $\mathbf{W}$ (cf. [4, 8]). We denote by $\mathbf{S}$ the pseudovariety of all finite semigroups.

Proposition 3.5. *Let* $\mathbf{V}$ *be a pseudovariety of semigroupoids, let* $\mathbf{W}$ *be a finitely generated pseudovariety of semigroups, let* S *be a finite semigroup, let* A *be a set such that* S *is* A*-generated, and let* $\gamma : \overline{\Omega}_A\mathbf{S} \to \overline{\Omega}_A\mathbf{W}$ *be the canonical projection. Then the following are equivalent:*

i) $S \in \mathbf{V} \circledast \mathbf{W}$*;*

ii) for every onto homomorphism $\varphi : \overline{\Omega}_A\mathbf{S} \to S$*,* $D_{\gamma\circ\varphi^{-1}} \in \mathbf{V}$*;*

iii) there exists an onto homomorphism $\varphi : \overline{\Omega}_A\mathbf{S} \to S$ *such that* $D_{\gamma\circ\varphi^{-1}} \in \mathbf{V}$.

Proof. Since there are onto homomorphisms $\overline{\Omega}_A\mathbf{S} \to S$, the only nontrivial part of the proof is (i)⇒(ii). So, assume that $S \in \mathbf{V} \circledast \mathbf{W}$ and let $\varphi : \overline{\Omega}_A\mathbf{S} \to S$ be an onto homomorphism. Let $R : S \to T$ be a relational morphism such that $T \in \mathbf{W}$ and $D_R \in \mathbf{V}$ and consider the following diagram, where p_1 and p_2 denote the component projections:

$$\begin{tikzcd}
\overline{\Omega}_A\mathbf{S} \arrow[rr, "\gamma"] \arrow[d, "\varphi"'] \arrow[dr, dashed, "\delta"] & & \overline{\Omega}_A\mathbf{W} \arrow[d, dashed, "\psi"] \\
S & R \arrow[l, "p_1"] \arrow[r, "p_2"'] & T
\end{tikzcd}$$

The existence of a homomorphism δ such that $p_1 \circ \delta = \varphi$ follows from the facts that p_1 is onto, R is finite and $\overline{\Omega}_A\mathbf{S}$ is free profinite. Similarly, the existence of the homomorphism ψ such that $\psi \circ \gamma = p_2 \circ \delta$ is obtained by noting in addition that $T \in \mathbf{W}$. Now, consider the relational morphism $R' = \gamma \circ \varphi^{-1}$. From the commutativity of the diagram, it follows that $\psi \circ R' \subseteq R$. Hence, by Proposition 3.1(a), $D_{R'} \prec D_R$ and so $D_{R'} \in \mathbf{V}$.□

Proposition 3.6. *Let* $\mathbf{W}$ *be a pseudovariety of semigroups and let* $(\mathbf{V}_i)_{i\in I}$ *be a family of pseudovarieties of semigroupoids. Then* $(\bigcap_{i\in I} \mathbf{V}_i) \circledast \mathbf{W} = \bigcap_{i\in I}(\mathbf{V}_i \circledast \mathbf{W})$.

Proof. In view of Proposition 3.4, it suffices to consider the case when $\mathbf{W}$ is finitely generated. Moreover, the inclusion from left to right is trivial in any case.

Conversely, let $S \in \bigcap_{i\in I}(\mathbf{V}_i \circledast \mathbf{W})$. Let A be a set such that S is A-generated, $\varphi : \overline{\Omega}_A\mathbf{S} \to S$ be an onto homomorphism and let $\gamma : \overline{\Omega}_A\mathbf{S} \to \overline{\Omega}_A\mathbf{W}$ be the canonical homomorphism. Then, by Proposition 3.5, $D_{\gamma\circ\varphi^{-1}} \in \mathbf{V}_i$ for all $i \in I$, so that $D_{\gamma\circ\varphi^{-1}} \in \bigcap_{i\in I} \mathbf{V}_i$ and, therefore, again by the same proposition, $S \in (\bigcap_{i\in I} \mathbf{V}_i) \circledast \mathbf{W}$.□

Lemma 3.7. *Let* $\mathbf{W}$ *be a finitely generated pseudovariety of semigroups,* S *a finite semigroup,* A *a set such that* S *is* A*-generated, and let* $u = v$ *be a semigroupoid pseudoidentity over a finite graph* G*. Let* $\varphi : \overline{\Omega}_A\mathbf{S} \to S$ *be an onto homomorphism and let* $\gamma : \overline{\Omega}_A\mathbf{S} \to \overline{\Omega}_A\mathbf{W}$ *be the canonical homomorphism. Then* $D_{\gamma\circ\varphi^{-1}} \models u = v$ *if and only* *if, for any $\pi_q \in (\overline{\Omega}_A\mathbf{S})^1$ $(q \in V(G))$ and $\rho_s \in \overline{\Omega}_A\mathbf{S}$ $(s \in E(G))$, the following condition holds:*

$$\mathbf{W} \models \pi_{\alpha s}\rho_s = \pi_{\omega s} \text{ for all } s \in E(G) \quad \Longrightarrow \quad \varphi(\pi_{\alpha u}\varepsilon(u)) = \varphi(\pi_{\alpha u}\varepsilon(v)) \tag{6}$$

where $\varepsilon : \overline{\Omega}_G\mathbf{Sd} \to \overline{\Omega}_A\mathbf{S}$ is the continuous morphism of semigroupoids such that $\varepsilon(s) = \rho_s$ for every $s \in E(G)$.

Proof. We start by rewriting (6) as

$$\gamma(\pi_{\alpha s}\rho_s) = \gamma\pi_{\omega s} \text{ for all } s \in E(G) \quad \Longrightarrow \quad \varphi(\pi_{\alpha u}\varepsilon(u)) = \varphi(\pi_{\alpha u}\varepsilon(v)). \tag{7}$$

The left side of (7) in turn means that

$$[\gamma\pi_{\alpha s}, \varphi\rho_s, \gamma\rho_s] \in D_{\gamma\circ\varphi^{-1}}(\gamma\pi_{\alpha s}, \gamma\pi_{\omega s}) \quad \text{for all } s \in E(G),$$

i.e., it says that the mapping

$$\begin{array}{rcl} \delta : G & \longrightarrow & D_{\gamma\circ\varphi^{-1}} \\ q \in V(G) & \longmapsto & \gamma\pi_q \\ s \in E(G) & \longmapsto & [\gamma\pi_{\alpha s}, \varphi\rho_s, \gamma\rho_s] \end{array} \tag{8}$$

is a graph morphism. By Proposition 2.1, there is a unique continuous morphism of semigroupoids $\overline{\delta} : \overline{\Omega}_G\mathbf{Sd} \to D_{\gamma\circ\varphi^{-1}}$ such that $\overline{\delta} \circ \iota = \delta$. Whence $D_{\gamma\circ\varphi^{-1}} \models u = v$ if and only if, for all choices of $\pi_q \in (\overline{\Omega}_A\mathbf{S})^1$ and $\rho_s \in \overline{\Omega}_A\mathbf{S}$, $\overline{\delta}u = \overline{\delta}v$ for the associated continuous morphism $\overline{\delta}$.

To conclude the proof, we just need to relate the action of $\overline{\delta}$ on edges with the mapping ε. We claim that, for any $\mu \in E(\overline{\Omega}_G\mathbf{Sd})$,

$$\overline{\delta}\mu = [\gamma\pi_{\alpha\mu}, \varphi\varepsilon(\mu), \gamma\varepsilon(\mu)]. \tag{9}$$

To prove (9), consider the mapping $\delta' : \overline{\Omega}_G\mathbf{Sd} \to D_{\gamma\circ\varphi^{-1}}$ which coincides with $\overline{\delta}$ on vertices and such that, for $\mu \in E(\overline{\Omega}_G\mathbf{Sd})$, $\delta'\mu$ is defined to be the right side of (9). By the definition of $\overline{\delta}$, it suffices to show that δ' is a continuous morphism of semigroupoids such that $\delta' \circ \iota = \delta$. Since φ and γ are continuous homomorphisms of semigroups and ε is a continuous morphism of semigroupoids, δ' is clearly a continuous morphism of semigroupoids. For $s \in E(G)$, we have

$$\delta' \circ \iota(s) = [\gamma\pi_{\alpha s}, \varphi\varepsilon(s), \gamma\varepsilon(s)] = [\gamma\pi_{\alpha s}, \varphi\rho_s, \gamma\rho_s] = \delta s,$$

which proves the claim.

By (9), we conclude that

$$\overline{\delta}u = \overline{\delta}v \quad \Longleftrightarrow \quad [\gamma\pi_{\alpha u}, \varphi\varepsilon(u), \gamma\varepsilon(u)] = [\gamma\pi_{\alpha v}, \varphi\varepsilon(v), \gamma\varepsilon(v)],$$

which in turn means that, for any $\nu \in (\overline{\Omega}_A\mathbf{S})^1$,

$$\gamma\nu = \gamma\pi_{\alpha u} \quad \Longrightarrow \quad \varphi(\nu\varepsilon(u)) = \varphi(\nu\varepsilon(v)).$$

Summarizing, we have shown that (8) defines a graph morphism such that $\overline{\delta}u = \overline{\delta}v$ if and only if

$$\left.\begin{array}{l}\gamma(\pi_{\alpha s}\rho_s) = \gamma\pi_{\omega s} \text{ for all } s \in E(G)\\ \gamma\pi_{\alpha u} = \gamma\nu\end{array}\right\} \quad \Longrightarrow \quad \varphi(\nu\varepsilon(u)) = \varphi(\nu\varepsilon(v)).$$

In view of the universal quantifiers applying to both conditions, this condition is clearly equivalent to (7), which finishes the proof.□

Proposition 3.8. *Let* $\mathbf{W}$ *be a finitely generated pseudovariety of semigroups,* S *a finite semigroup,* A *a set such that* S *is* A*-generated,* $\mathbf{U}$ *the pseudovariety of semigroups generated by* S*, and let* $u = v$ *be a semigroupoid pseudoidentity over a finite graph* G*. Let* $\beta : \overline{\Omega}_A\mathbf{S} \to \overline{\Omega}_A\mathbf{U}$ *and* $\gamma : \overline{\Omega}_A\mathbf{S} \to \overline{\Omega}_A\mathbf{W}$ *be the canonical homomorphisms. Then the following are equivalent:*

i) $D_{\gamma\circ\varphi^{-1}} \models u = v$ *for every onto homomorphism* $\varphi : \overline{\Omega}_A\mathbf{S} \to S$*;*

ii) $D_{\gamma\circ\varphi^{-1}} \models u = v$ *for some onto homomorphism* $\varphi : \overline{\Omega}_A\mathbf{S} \to S$*;*

iii) $D_{\gamma\circ\beta^{-1}} \models u = v$*;*

iv) for any $\pi_q \in (\overline{\Omega}_A\mathbf{S})^1$ *(*$q \in V(G)$*) and* $\rho_s \in \overline{\Omega}_A\mathbf{S}$ *(*$s \in E(G)$*), the following condition holds:*

$$\mathbf{W} \models \pi_{\alpha s}\rho_s = \pi_{\omega s} \text{ for all } s \in E(G) \quad \Longrightarrow \quad S \models \pi_{\alpha u}\varepsilon(u) = \pi_{\alpha u}\varepsilon(v)$$

where $\varepsilon : \overline{\Omega}_G\mathbf{Sd} \to \overline{\Omega}_A\mathbf{S}$ *is the continuous morphism of semigroupoids such that* $\varepsilon(s) = \rho_s$ *for every* $s \in E(G)$*.*

Proof. The equivalence between (i) and (ii) is the particular case of the equivalence between conditions (ii) and (iii) of Proposition 3.5 corresponding to the pseudovariety of semigroupoids defined by the pseudoidentity $u = v$.

The equivalence between the conditions (i), (iii), and (iv) follows from Lemma 3.7 applied to the homomorphisms φ and β since S and $\overline{\Omega}_A\mathbf{U}$ satisfy the same pseudoidentities.□

4. An extension of the semidirect product of semigroups

We now introduce an extension of the semidirect product of semigroups which will be used to give a natural set of generators for a pseudovariety of the form $\mathbf{V} \circledast \mathbf{W}$ and to describe the relatively free profinite semigroup $\overline{\Omega}_A(\mathbf{V} \circledast \mathbf{W})$.

Let C be a topological semigroupoid and S a topological semigroup. For convenience of notation, edge composition in C will be denoted additively. Suppose $\varphi : S^1 \times C \to C$ is a continuous mapping such that, denoting $\varphi(s, c)$ by sc, the following conditions hold:

i) $S^1 V(C) \subseteq V(C)$;

ii) $s\, C(c, d) \subseteq C(sc, sd)$ for any $s \in S^1$ and $c, d \in V(C)$;

iii) $(s_1 s_2)c = s_1(s_2 c)$ for any $s_1, s_2 \in S^1$ and $c \in C$;

iv) $s(t_1 + t_2) = (st_1) + (st_2)$ for any $s \in S^1$ and $t_1, t_2 \in E(C)$ with $\omega t_1 = \alpha t_2$;

v) $1c = c$ for any $c \in C$.

We then say that φ describes a continuous *(monoidal) action* of S on C. In case C is just a topological graph, we also say that φ is a continuous *(monoidal) action* if conditions (i), (ii), (iii) and (v) hold.

Given C, S and φ as above and a fixed vertex $v_0 \in V(C)$, we denote by $C \circledast_{v_0,\varphi} S$ the set of all pairs of the form (e, s) with $s \in S$ and $e \in C(v_0, sv_0)$. We endow $C \circledast_{v_0,\varphi} S$ with the following binary operation:

$$(e_1, s_1)\,(e_2, s_2) = (e_1 + s_1 e_2, s_1 s_2).$$

It is easy to verify that this operation defines a structure of topological semigroup on $C \circledast_{v_0,\varphi} S$. Such a semigroup is called a *semidirect product* of C with S. Whenever it is clear from the context which vertex v_0 and action φ are being considered, we will write $C \circledast S$ instead of $C \circledast_{v_0,\varphi} S$. (This construction has already been considered by Margolis and Pin [22].) Observe that if C is a semigroupoid with a single vertex, then $C \circledast S$ is a semidirect product of semigroups in the usual sense.

For most of this paper, we will concentrate on the discrete case. We start by studying the derived semigroupoid D_p of the natural projection

$$\begin{aligned} p : C \circledast_{v_0,\varphi} S &\longrightarrow S \\ (e, s) &\longmapsto s \end{aligned}$$

Proposition 4.1. *The derived semigroupoid D_p divides C.*

Proof. The derived semigroupoid D_p has set of vertices S^1 and its edges (from s to ss_1) may be viewed as equivalence classes of triples (s, e_1, s_1) with $s \in S^1$ and $(e_1, s_1) \in C \circledast_{v_0,\varphi} S$ under the relation $\sim$ defined by

$$(s, e_1, s_1) \sim (s', e_2, s_2) \qquad \begin{array}{l}\text{if } s = s',\ ss_1 = ss_2 \text{ and, for every edge } e : v_0 \to sv_0 \\ \text{(or } e = 1_{v_0}, \text{ in case } sv_0 = v_0),\ e + se_1 = e + se_2.\end{array} \tag{10}$$

Composition of edges is given by

$$[s, e_1, s_1]\,[ss_1, e_2, s_2] = [s, e_1 + s_1e_2, s_1s_2].$$

Consider another category defined in a slightly modified way. The category Z has set of vertices S^1 and the edges (from s to ss_1) are the equivalence classes of triples (s, e_1, s_1) with $s \in S^1$ and $(e_1, s_1) \in C \circledast_{v_0,\varphi} S$ under the relation $\approx$ defined by

$$(s, e_1, s_1) \approx (s', e_2, s_2) \quad \text{if } s = s',\ ss_1 = ss_2 \text{ and } se_1 = se_2.$$

Denoting the $\approx$-class of an edge (s, e_1, s_1) by $[[s, e_1, s_1]]$, the composition given by

$$[[s, e_1, s_1]]\,[[ss_1, e_2, s_2]] = [[s, e_1 + s_1e_2, s_1s_2]]$$

is easily checked to be well-defined.

Since $\approx \subseteq \sim$, the edge correspondence $[[s, e_1, s_1]] \mapsto [s, e_1, s_1]$ yields a well-defined quotient morphism $Z \to D_p$.

Consider next the mapping $\psi : Z \to C$ which sends each vertex s to the vertex sv_0 and each edge $[[s, e_1, s_1]]$ to the edge se_1. Then, for two edges $[[s, e_1, s_1]]$ and $[[ss_1, e_2, s_2]]$,

$$\begin{aligned}\psi\Big([[s, e_1, s_1]]\,[[ss_1, e_2, s_2]]\Big) &= \psi[[s, e_1 + s_1e_2, s_1s_2]] = s(e_1 + s_1e_2) \\ &= se_1 + (ss_1)e_2 = \psi[[s, e_1, s_1]] + \psi[[ss_1, e_2, s_2]],\end{aligned}$$

which shows that ψ is a semigroupoid morphism. To show that ψ is faithful, suppose that the coterminal edges $[[s, e_1, s_1]]$ and $[[s, e_2, s_2]]$ of Z have the same image under ψ. Then $se_1 = se_2$ and so $[[s, e_1, s_1]] = [[s, e_2, s_2]]$.□

Proposition 4.2. *Let $R : S \to T$ be a relational morphism of semigroups and let $D_R \prec C$ where C is a semigroupoid. Then $S \prec C^{T^1} \circledast_{w_0,\varphi} T$ for some $w_0 \in V(C^{T^1})$ and some monoidal action of T on C^{T^1}.*

Proof. Let Z be a semigroupoid, $\psi : Z \to C$ a faithful morphism, and $\theta : Z \to D_R$ a quotient morphism. Consider the action

$$\begin{aligned}\varphi : T^1 \times C^{T^1} &\longrightarrow C^{T^1} \\ (t, f) &\longmapsto tf\end{aligned}$$

where $(u)(tf) = (ut)f$ for $u \in T$, viewing the elements of the semigroupoid power C^{T^1} as functions acting on the right (which assume either only vertex or only edge values). Let w_0 be the vertex of C^{T^1} defined by $(t)w_0 = \psi(\theta^{-1}(t))$ for each $t \in T^1$ and consider the associated semidirect product $C^{T^1} \circledast_{w_0,\varphi} T$. We show that it is divided by S by constructing a specific division $U : S \to C^{T^1} \circledast_{w_0,\varphi} T$, namely the relation

$$U = \Big\{(s,f,t) \in S \times C^{T^1} \times T : (\forall t' \in T^1)\,(\exists e \in E(Z)) \left\{ \begin{array}{rcl} \theta e &=& [t',s,t] \\ \psi e &=& (t')f \end{array} \right. \Big\}.$$

To conclude the proof it thus suffices to show that U is a division $S \to C^{T^1} \circledast T$.

We first verify that U is a subsemigroup of $S \times C^{T^1} \circledast T$. Given $(s,f,t) \in U$ and $t' \in T^1$, let $e \in E(Z)$ be such that $\theta e = [t',s,t]$ and $\psi e = (t')f$. Then $\theta(\alpha e) = \alpha(\theta e) = t'$ and so $\alpha e = \theta^{-1}(t')$ and $\alpha((t')f) = \alpha(\psi e) = \psi(\alpha e) = \psi(\theta^{-1}(t'))$. On the other hand, $\theta(\omega e) = \omega(\theta e) = t't$ yields $\omega e = \theta^{-1}(t't)$ and so $\omega((t')f) = \omega(\psi e) = \psi(\omega e) = \psi(\theta^{-1}(t't)) = (t't)w_0 = (t')(tw_0)$. Since $t' \in T^1$ is arbitrary, this shows that $\alpha(f) = w_0$ and $\omega(f) = tw_0$, that is $U \subseteq S \times C^{T^1} \circledast_{w_0,\varphi} T$. Next, let (s_1,f_1,t_1) and (s_2,f_2,t_2) be two elements of U and let $t' \in T^1$. By definition of U, there exist $e_1, e_2 \in E(Z)$ such that

$$\left\{ \begin{array}{rcl} \theta e_1 &=& [t',s_1,t_1] \\ \psi e_1 &=& (t')f_1 \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{rcl} \theta e_2 &=& [t't_1,s_2,t_2] \\ \psi e_2 &=& (t't_1)f_2 \end{array} \right.$$

Since θ is a quotient morphism, $\omega e_1 = \alpha e_2$ and so e_1e_2 is an edge of Z such that

$$\left\{ \begin{array}{rcl} \theta(e_1e_2) &=& [t',s_1,t_1]\,[t't_1,s_2,t_2] = [t',s_1s_2,t_1t_2] \\ \psi(e_1e_2) &=& (t')f_1 + (t't_1)f_2 = (t')(f_1 + t_1f_2). \end{array} \right.$$

Hence $(s_1s_2, f_1 + t_1f_2, t_1t_2) \in U$.

Next, we show that the projection $U \to S$ is onto. Let $s \in S$. Then there is some $t \in T$ such that $(s,t) \in R$. For each $t' \in T^1$, since θ is onto, there is some $e_{t'} \in E(Z)$ such that $\theta e_{t'} = [t',s,t]$. Let $f \in C^{T^1}$ be defined by $(t')f = \psi e_{t'}$. Then, by construction, $(s,f,t) \in U$.

Finally, we check that the projection $U \to C^{T^1} \circledast T$ is injective. Indeed, for any $(s,f,t) \in U$, there is some $e \in E(Z)$ such that $\theta e = [1,s,t]$ and $\psi e = (1)f$. Hence, if $(s,f,t), (s',f,t) \in U$, then $[1,s,t] = [1,s',t]$ since ψ is faithful. But, by the definition of the derived category, taking the pair $(1,1)$ in condition (5), it follows that $1s = 1s'$, which concludes the proof.□

Propositions 4.1 and 4.2 extend Tilson's Derived Category Theorem. It is however only fair to note that the proofs of the propositions are inspired in Tilson's proofs. At the pseudovariety level, we obtain the following result which contains Theorem 3.3.

Theorem 4.3. *Let* $\mathbf{V}$ *be a pseudovariety of semigroupoids,* $\mathbf{V}_0$ *a subclass of* $\mathbf{V}$ *such that every member of* $\mathbf{V}$ *divides some member of* $\mathbf{V}_0$*, and* $\mathbf{W}$ *a pseudovariety of semigroups. Then* $\mathbf{V} \circledast \mathbf{W}$ *is the pseudovariety of semigroups generated by the semidirect products of the form* $C \circledast S$ *with* $C \in \mathbf{V}_0$ *and* $S \in \mathbf{W}$*.*

Proof. Suppose $S \in \mathbf{V} \circledast \mathbf{W}$. Then there is a relational morphism $R : S \to T$ such that $T \in \mathbf{W}$ and $D_R \prec C$ for some $C \in \mathbf{V}_0$. By Proposition 4.2, S divides a semidirect product of the form $C^{T^1} \circledast T$. So, it suffices to check that the semidirect products of the form $C \circledast T$ with $C \in \mathbf{V}$ and $T \in \mathbf{W}$ belong to $\mathbf{V} \circledast \mathbf{W}$. This follows from Proposition 4.1 since the derived semigroupoid D_p of the projection $p : C \circledast T \to T$ divides C and, therefore, also belongs to $\mathbf{V}$.□

Aiming at characterizing the pseudoidentities valid in $\mathbf{V} \circledast \mathbf{W}$, we proceed by giving a representation of the relatively free profinite semigroups $\overline{\Omega}_A(\mathbf{V} \circledast \mathbf{W})$ which extends results of [7].

The *Cayley graph* of $\overline{\Omega}_A\mathbf{W}$ is the graph with set of vertices $(\overline{\Omega}_A\mathbf{W})^1$, set of edges $(\overline{\Omega}_A\mathbf{W})^1 \times A$, $\alpha(w, a) = w$ and $\omega(w, a) = wa$; we denote it by $\Gamma_A\mathbf{W}$ and we observe that it is a profinite graph. We have an action of $\overline{\Omega}_A\mathbf{W}$ on $\Gamma_A\mathbf{W}$ given by $w \cdot w' = ww'$ and $w\,(w', a) = (ww', a)$.

In the following, $\mathbf{V}$ denotes a pseudovariety of semigroupoids, $\mathbf{W}$ a pseudovariety of semigroups, and A a profinite set.

Lemma 4.4. *For any profinite set* A*, the natural action*

$$\varphi : (\overline{\Omega}_A\mathbf{W})^1 \times \overline{\Omega}_{\Gamma_A\mathbf{W}}\mathbf{V} \longrightarrow \overline{\Omega}_{\Gamma_A\mathbf{W}}\mathbf{V}$$

induced by the action of $\overline{\Omega}_A\mathbf{W}$ *on* $\Gamma_A\mathbf{W}$ *yields a semidirect product* $\overline{\Omega}_{\Gamma_A\mathbf{W}}\mathbf{V} \circledast_{1,\varphi} \overline{\Omega}_A\mathbf{W}$ *which is a pro-*$\mathbf{V} \circledast \mathbf{W}$ *semigroup.*

Proof. By the universal property of the projective limit defining $\overline{\Omega}_{\Gamma_A\mathbf{W}}\mathbf{V}$, for each $w \in (\overline{\Omega}_A\mathbf{W})^1$, there is a unique continuous morphism $\varphi_w : \overline{\Omega}_{\Gamma_A\mathbf{W}}\mathbf{V} \to \overline{\Omega}_{\Gamma_A\mathbf{W}}\mathbf{V}$ such that $\varphi_w(w') = ww'$ and $\varphi_w(w', a) = (ww', a)$ for $w, w' \in (\overline{\Omega}_A\mathbf{W})^1$ and $a \in A$. The action φ is defined by $\varphi(w, \pi) = \varphi_w(\pi)$.

Let $\mathbf{W}_1 \subseteq \mathbf{W}_2 \subseteq \cdots$ and $\mathbf{V}_1 \subseteq \mathbf{V}_2 \subseteq \cdots$ be chains of finitely generated pseudovarieties whose unions are, respectively, $\mathbf{W}$ and $\mathbf{V}$. Let $A_1 \leftarrow A_2 \leftarrow \cdots$ be a projective sequence of mappings between finite sets yielding A as a projective limit. Then, it is easy to verify that

- $\overline{\Omega}_A\mathbf{W} = \varprojlim \overline{\Omega}_{A_i}\mathbf{W}_i$;
- $\Gamma_A\mathbf{W} = \varprojlim \Gamma_{A_i}\mathbf{W}_i$;

– $\overline{\Omega}_{\Gamma_A \mathbf{W}}\mathbf{V} = \varprojlim \overline{\Omega}_{\Gamma_{A_i}\mathbf{W}_i}\mathbf{V}_i$.

Moreover, the natural projections $\overline{\Omega}_A\mathbf{W} \to \overline{\Omega}_{A_i}\mathbf{W}_i$ and $\overline{\Omega}_{\Gamma_A \mathbf{W}}\mathbf{V} \to \overline{\Omega}_{\Gamma_{A_i}\mathbf{W}_i}\mathbf{V}_i$, give rise to commutative diagrams of actions and projections

$$\begin{tikzcd}
(\overline{\Omega}_A\mathbf{W})^1 \times \overline{\Omega}_{\Gamma_A \mathbf{W}}\mathbf{V} \arrow[r, "\varphi"] \arrow[d] & \overline{\Omega}_{\Gamma_A \mathbf{W}}\mathbf{V} \arrow[d] \\
(\overline{\Omega}_{A_i}\mathbf{W}_i)^1 \times \overline{\Omega}_{\Gamma_{A_i}\mathbf{W}_i}\mathbf{V}_i \arrow[r, "\varphi_i"] & \overline{\Omega}_{\Gamma_{A_0}\mathbf{W}_i}\mathbf{V}_i
\end{tikzcd}$$

Since each φ_i, as a mapping between finite (discrete) sets, is certainly continuous, it follows that φ is continuous. Hence $\overline{\Omega}_{\Gamma_A \mathbf{W}}\mathbf{V} \circledast_{1,\varphi} \overline{\Omega}_A\mathbf{W}$ is a topological semigroup. But, the commutativity of the above diagram also implies that the same projections give rise to continuous homomorphisms

$$\overline{\Omega}_{\Gamma_A \mathbf{W}}\mathbf{V} \circledast_{1,\varphi} \overline{\Omega}_A\mathbf{W} \longrightarrow \overline{\Omega}_{\Gamma_{A_i}\mathbf{W}_i}\mathbf{V}_i \circledast_{1,\varphi_i} \overline{\Omega}_{A_i}\mathbf{W}_i.$$

Since these homomorphisms clearly suffice to separate points and $\overline{\Omega}_{\Gamma_A \mathbf{W}}\mathbf{V} \circledast_{1,\varphi} \overline{\Omega}_A\mathbf{W}$ is a compact semigroup, we deduce that this semigroup is in fact the projective limit of the semigroups $\overline{\Omega}_{\Gamma_{A_i}\mathbf{W}_i}\mathbf{V}_i \circledast_{1,\varphi_i} \overline{\Omega}_{A_i}\mathbf{W}_i$ (cf. [7, Lemma 1.2]).□

This leads to the following representation theorem.

Theorem 4.5. *For any profinite set A, the mapping $a \in A \mapsto \big((1,a),a\big)$ induces an embedding of topological semigroups*

$$\overline{\Omega}_A(\mathbf{V} \circledast \mathbf{W}) \hookrightarrow \overline{\Omega}_{\Gamma_A \mathbf{W}}\mathbf{V} \circledast_{1,\varphi} \overline{\Omega}_A\mathbf{W}$$

into the semidirect product of Lemma 4.4.

Proof. By Lemma 4.4, $\overline{\Omega}_{\Gamma_A \mathbf{W}}\mathbf{V} \circledast_{1,\varphi} \overline{\Omega}_A\mathbf{W}$, and therefore also any closed subsemigroup, is a pro-$\mathbf{V} \circledast \mathbf{W}$ semigroup. Hence, it suffices to show that any continuous function $\psi_0 : A \to S$ with $S \in \mathbf{V} \circledast \mathbf{W}$ extends to a continuous homomorphism $\overline{\psi}_0 : \overline{\Omega}_{\Gamma_A \mathbf{W}}\mathbf{V} \circledast_{1,\varphi} \overline{\Omega}_A\mathbf{W} \to S$ in the sense that $\overline{\psi}_0\big((1,a),a\big) = \psi_0(a)$ for any $a \in A$.

So, consider a continuous function $\psi_0 : A \to S$ with $S \in \mathbf{V} \circledast \mathbf{W}$. By Theorem 4.3, there exist $C \in \mathbf{V}$, $T \in \mathbf{W}$, a semidirect product $C \circledast T$ with distinguished vertex v_0, a subsemigroup S' of $C \circledast T$ and an onto homomorphism $S' \to S$. The function ψ_0 lifts to a continuous function $\psi_0' : A \to S'$ (since A is a profinite set and, in case A is finite, there is clearly such a lifting). We obtain an induced continuous mapping $\psi : A \to C \circledast T$, say given by $\psi a = (\psi_1 a, \psi_2 a)$ for each $a \in A$.

By the universal properties of $\overline{\Omega}_A\mathbf{W}$ and $\overline{\Omega}_{\Gamma_A\mathbf{W}}\mathbf{V}$, there are continuous morphisms, respectively of semigroups and semigroupoids,

$$\begin{array}{lll}\overline{\psi}_2 : \overline{\Omega}_A\mathbf{W} \to T & \text{such that} & \overline{\psi}_2 a = \psi_2 a \text{ for all } a \in A \\ \overline{\psi}_1 : \overline{\Omega}_{\Gamma_A\mathbf{W}}\mathbf{V} \to C & \text{such that} & \left\{\begin{array}{rcll} \overline{\psi}_1 w &=& (\overline{\psi}_2 w)v_0 & \text{for all } w \in (\overline{\Omega}_A\mathbf{W})^1 \\ \overline{\psi}_1(w,a) &=& (\overline{\psi}_2 w)(\psi_1 a) & \text{and } a \in A \end{array}\right.\end{array}$$

Note that, for every edge e of the semigroupoid $\overline{\Omega}_{\Gamma_A\mathbf{W}}\mathbf{V}$ and every $w \in (\overline{\Omega}_A\mathbf{W})^1$,

$$\overline{\psi}_1(we) = (\overline{\psi}_2 w)(\overline{\psi}_1 e)$$

since, in case e is actually an edge of $\Gamma_A\mathbf{W}$, say $e : w' \xrightarrow{(w',a)} w'a$,

$$\begin{aligned}\overline{\psi}_1(we) &= \overline{\psi}_1(ww',a) = \overline{\psi}_2(ww')(\psi_1 a) = (\overline{\psi}_2 w)\Big((\overline{\psi}_2 w')(\psi_1 a)\Big) \\ &= (\overline{\psi}_2 w)\left(\overline{\psi}_1(w',a)\right) = (\overline{\psi}_2 w)(\overline{\psi}_1 e).\end{aligned}$$

Consider the mapping

$$\begin{aligned}\overline{\psi} : \overline{\Omega}_{\Gamma_A\mathbf{W}}\mathbf{V} \circledast_{1,\varphi} \overline{\Omega}_A\mathbf{W} &\longrightarrow C \circledast T \\ (e,w) &\longmapsto (\overline{\psi}_1 e, \overline{\psi}_2 w)\end{aligned}$$

Then $\overline{\psi}$ is clearly continuous and

$$\begin{aligned}\overline{\psi}(e,w)\cdot\overline{\psi}(e',w') &= (\overline{\psi}_1 e,\overline{\psi}_2 w)(\overline{\psi}_1 e',\overline{\psi}_2 w') \\ &= \Big(\overline{\psi}_1 e + (\overline{\psi}_2 w)(\overline{\psi}_1 e'), \overline{\psi}_2(ww')\Big) = \Big(\overline{\psi}_1 e + \overline{\psi}_1(we'), \overline{\psi}_2(ww')\Big) \\ &= \Big(\overline{\psi}_1(e+we'), \overline{\psi}_2(ww')\Big) = \overline{\psi}(e+we', ww') = \overline{\psi}\Big((e,w)\cdot(e',w')\Big),\end{aligned}$$

so that $\overline{\psi}$ is a continuous homomorphism such that $\overline{\psi}\Big((1,a),a\Big) = \psi a$ for all $a \in A$. To complete the proof of the theorem, take $\overline{\psi}_0$ to be $\overline{\psi}$ followed by the mapping $S' \to S$, which is everywhere defined since the image of $\overline{\psi}$ is contained in S'.□

As announced earlier, we now explore the above representation theorem to characterize the pseudoidentities of $\mathbf{V}\circledast\mathbf{W}$. Composing the natural mapping $\overline{\Omega}_A\mathbf{S} \to \overline{\Omega}_A(\mathbf{Sd}\circledast\mathbf{W})$ (which is actually the identity) with the representation of Theorem 4.5, we obtain a continuous homomorphism

$$\eta : \overline{\Omega}_A\mathbf{S} \longrightarrow \overline{\Omega}_{\Gamma_A\mathbf{W}}\mathbf{Sd} \circledast \overline{\Omega}_A\mathbf{W}.$$

We denote the first component of the image of $u \in \overline{\Omega}_A\mathbf{S}$ by $\Phi^{\mathbf{W}}(u)$ and the second component, obtained by restriction of u to $\mathbf{W}$, by $u_{\mathbf{W}}$. As a consequence of the representation theorem, we obtain the following result.

Theorem 4.6. *A pseudoidentity $u = v$ is valid in $\mathbf{V} \circledast \mathbf{W}$ if and only if $\mathbf{W} \models u = v$ and $\mathbf{V} \models \Phi^{\mathbf{W}}(u) = \Phi^{\mathbf{W}}(v)$.*

Proof. Consider the projection $\psi : \overline{\Omega}_A\mathbf{S} \to \overline{\Omega}_A(\mathbf{V} \circledast \mathbf{W})$, where A is a finite set such that $u, v \in \overline{\Omega}_A\mathbf{S}$ and the following commutative diagram of continuous homomorphisms

$$\begin{tikzcd}
\overline{\Omega}_A\mathbf{S} \arrow[r, "\eta"] \arrow[d, "\psi"] & \overline{\Omega}_{\Gamma_A\mathbf{W}}\mathbf{Sd} \circledast \overline{\Omega}_A\mathbf{W} \arrow[d, "\theta"] \\
\overline{\Omega}_A(\mathbf{V} \circledast \mathbf{W}) \arrow[r, hook, "\mu"] & \overline{\Omega}_{\Gamma_A\mathbf{W}}\mathbf{V} \circledast \overline{\Omega}_A\mathbf{W}
\end{tikzcd}$$

where θ is induced by the quotient morphism $\theta_1 : \overline{\Omega}_{\Gamma_A\mathbf{W}}\mathbf{Sd} \to \overline{\Omega}_{\Gamma_A\mathbf{W}}\mathbf{V}$ by means of the formula $\theta(e, w) = (\theta_1 e, w)$. Then $\mathbf{V} \circledast \mathbf{W} \models u = v$ if and only if $\psi u = \psi v$. Since μ is an embedding, $\psi u = \psi v$ if and only if $\theta\eta u = \theta\eta v$ which in turn is equivalent to $\mathbf{W} \models u = v$ and $\theta_1\Phi^{\mathbf{W}}(u) = \theta_1\Phi^{\mathbf{W}}(v)$, i.e., $\mathbf{V} \models \Phi^{\mathbf{W}}(u) = \Phi^{\mathbf{W}}(v)$.□

As a particular instance of Theorem 4.6, consider the case $\mathbf{W} = \mathbf{D}_n$ where $\mathbf{D}_n$ is the pseudovariety of semigroups defined by the identity $xy_1 \ldots y_n = y_1 \ldots y_n$. For $u \in \Omega_A\mathbf{S}$, $\Phi^{\mathbf{D}_n}(u)$ is the edge of $\overline{\Omega}_{\Gamma_A\mathbf{D}_n}\mathbf{Sd}$ from 1 to $u_{\mathbf{D}_n}$ obtained by successively reading the prefixes of u of length $\leq n$ and then the successive factors of u of length n. This allows us to recognize Theorem 4.6 as an extension of a theorem of Almeida and Azevedo [6] characterizing the pseudoidentities valid in semidirect products of the form $\mathbf{V} * \mathbf{D}_n$.

For a pseudovariety $\mathbf{V}$ of semigroups, of monoids, of semigroupoids or of categories, we denote by $\mathcal{P}s\,\mathbf{V}$ the lattice of all pseudovarieties of the same kind of structures contained in $\mathbf{V}$.

The next result is a further application of Theorem 4.6, of which a particular case plays a special role in section 7.

Theorem 4.7. *Let $\mathbf{W}$ be a pseudovariety of semigroups. Then $\mathbf{V} \mapsto \mathbf{V} \circledast \mathbf{W}$ defines a complete lattice homomorphism $\circledast\mathbf{W} : \mathcal{P}s\,\mathbf{Sd} \to \mathcal{P}s\,\mathbf{S}$.*

Proof. By Proposition 3.6, the mapping $\circledast\mathbf{W}$ preserves arbitrary meets. So, it remains to show that it also preserves arbitrary joins. Let $(\mathbf{V}_i)_{i\in I}$ be a family of pseudovarieties of semigroupoids. Since $\circledast\mathbf{W}$ clearly preserves order, certainly $\bigvee_{i\in I}(\mathbf{V}_i \circledast \mathbf{W}) \subseteq (\bigvee_{i\in I}\mathbf{V}_i) \circledast \mathbf{W}$. To prove the reverse inclusion, we use Reiterman's Theorem: it suffices to show that, if a pseudoidentity $u = v$ is valid in all $\mathbf{V}_i \circledast \mathbf{W}$ $(i \in I)$, then $(\bigvee_{i\in I}\mathbf{V}_i) \circledast \mathbf{W}$ also satisfies $u = v$. By Theorem 4.6, the hypothesis in the preceding condition means that $\mathbf{W} \models u = v$ and $\mathbf{V}_i \models \Phi^{\mathbf{W}}(u) = \Phi^{\mathbf{W}}(v)$ for all $i \in I$. Hence $\bigvee_{i\in I}\mathbf{V}_i \models \Phi^{\mathbf{W}}(u) = \Phi^{\mathbf{W}}(v)$ and so $(\bigvee_{i\in I}\mathbf{V}_i) \circledast \mathbf{W} \models u = v$ again by Theorem 4.6.□

To conclude this section, we observe that, using Theorem 4.5, one may show that

$$\mathbf{V} \circledast (\mathbf{W}_1 * \mathbf{W}_2) = (\mathbf{V} \circledast \mathbf{W}_1) * \mathbf{W}_2 \tag{11}$$

for all pseudovarieties $\mathbf{V}$ of semigroupoids and $\mathbf{W}_1$ and $\mathbf{W}_2$ of semigroups. The details are too technical to justify including here a proof of this marginal result which extends the associative property of the semidirect product of semigroups.

5. A compactness theorem and a general application

Let $\varepsilon : u = v$ be a pseudoidentity for semigroups, with $u, v \in \overline{\Omega}_X\mathbf{S}$. Given $\vec{\pi} = (\pi_x)_{x\in X} \in (\overline{\Omega}_A\mathbf{S})^X$, let $\varepsilon(\vec{\pi}) : u(\vec{\pi}) = v(\vec{\pi})$ where $u(\vec{\pi})$ and $v(\vec{\pi})$ are the composite implicit operations in $\overline{\Omega}_A\mathbf{S}$ which, informally, are obtained by substituting, respectively, in u and v each variable $x \in X$ by the corresponding component π_x of $\vec{\pi}$. For a set Σ of pseudoidentities over the set of variables X, let $\Sigma(\vec{\pi}) = \{\varepsilon(\vec{\pi}) : \varepsilon \in \Sigma\}$. The following compactness result is crucial in the applications given in the sequel.

Theorem 5.1. *Let Σ and Γ be two sets of pseudoidentities of semigroups over the countable set of variables $\{x_0, x_1, \ldots\}$ where Γ involves only a finite number of those variables. Let $\mathbf{W}$ be an arbitrary pseudovariety of semigroups and let S be a finite semigroup. Then there exists a finitely generated subpseudovariety $\mathbf{W}_0$ of $\mathbf{W}$ such that the following two conditions are equivalent:*

i) for every finite set A and every $\vec{\pi} \in (\overline{\Omega}_A\mathbf{S})^{\mathbb{N}}$,

$$\mathbf{W} \models \Sigma(\vec{\pi}) \quad\Longrightarrow\quad S \models \Gamma(\vec{\pi});$$

ii) for every finite set A and every $\vec{\pi} \in (\overline{\Omega}_A\mathbf{S})^{\mathbb{N}}$,

$$\mathbf{W}_0 \models \Sigma(\vec{\pi}) \quad\Longrightarrow\quad S \models \Gamma(\vec{\pi}).$$

Proof. For every subpseudovariety $\mathbf{W}_0$ of $\mathbf{W}$, clearly $\mathbf{W} \models \Sigma(\vec{\pi})$ implies $\mathbf{W}_0 \models \Sigma(\vec{\pi})$. Hence (ii) implies (i) for all $\mathbf{W}_0 \subseteq \mathbf{W}$. So, we only need to "construct" $\mathbf{W}_0$ such that (i) reduces to (ii).

Let n be the number of elements of a generating set of S. We claim that condition (i) (and so also (ii)) is equivalent to the same condition with a fixed set A_0 with n elements. Indeed, for an arbitrary A, if $\vec{\pi} \in (\overline{\Omega}_A\mathbf{S})^{\mathbb{N}}$ and ε is pseudoidentity over $\{x_0, x_1, \ldots\}$, then $S \models \varepsilon(\vec{\pi})$ if and only if S satisfies all pseudoidentities which are obtained from $\varepsilon(\vec{\pi})$ by replacing each variable by one of a finite number of words over A_0, namely words giving expressions for all the elements of S in terms of the chosen n generators. On the other hand, if δ is a pseudoidentity over $\{x_0, x_1, \ldots\}$ and $\mathbf{W}$ satisfies $\delta(\vec{\pi})$, then $\mathbf{W}$ also satisfies every pseudoidentity which is obtained from $\delta(\vec{\pi})$ by substituting each variable by a word over A_0. The claim follows from these observations.

So, let us fix a set A with n elements. Let $\mathbf{U}$ be the pseudovariety generated by S and consider the two canonical projections $\beta : \overline{\Omega}_A\mathbf{S} \to \overline{\Omega}_A\mathbf{U}$ and $\gamma : \overline{\Omega}_A\mathbf{S} \to \overline{\Omega}_A\mathbf{W}$. Assume that (i) is valid.

We next observe that the set

$$C = \{\vec{\pi} \in (\overline{\Omega}_A\mathbf{S})^{\mathbb{N}} : S \not\models \Gamma(\vec{\pi})\}$$

of counter-examples for $S \models \Gamma(\vec{\pi})$ is a compact subset of $(\overline{\Omega}_A\mathbf{S})^{\mathbb{N}}$, where this space is viewed as a power space of the compact space $\overline{\Omega}_A\mathbf{S}$. For this purpose, let m be such that Γ only involves variables from the set $\{x_0, x_1, \ldots, x_m\}$ and consider the projection $p_m : (\overline{\Omega}_A\mathbf{S})^{\mathbb{N}} \to (\overline{\Omega}_A\mathbf{S})^m$ on the first m components and the mapping

$$\begin{aligned} \varphi_\Gamma : (\overline{\Omega}_A\mathbf{U})^m &\longrightarrow \mathcal{P}(\overline{\Omega}_A\mathbf{U} \times \overline{\Omega}_A\mathbf{U}) \\ \vec{\pi} &\longmapsto \{(u(\vec{\pi}), v(\vec{\pi})) : (u = v) \in \Gamma\} \end{aligned}$$

between finite sets. Then

$$C = (\varphi_\Gamma \circ \beta^m \circ p_m)^{-1}\Big\{P \in \mathcal{P}(\overline{\Omega}_A\mathbf{U} \times \overline{\Omega}_A\mathbf{U}) : P \text{ is not contained in the diagonal}\Big\}$$

and so C is compact.

Now consider the mapping

$$\begin{aligned} \gamma_\Sigma : (\overline{\Omega}_A\mathbf{S})^{\mathbb{N}} &\longrightarrow (\overline{\Omega}_A\mathbf{W} \times \overline{\Omega}_A\mathbf{W})^\Sigma \\ \vec{\pi} &\longmapsto \Big(\big(\gamma(u(\vec{\pi})), \gamma(v(\vec{\pi}))\big)\Big)_{(u=v)\in\Sigma} \end{aligned}$$

which is continuous since composition with a given implicit operation and γ are continuous mappings. Since C is compact, so is $\gamma_\Sigma(C)$. By (i), $\gamma_\Sigma(C)$ has empty intersection with the (also compact) "diagonal"

$$\Delta = \{(w, w) : w \in \overline{\Omega}_A\mathbf{W}\}^\Sigma.$$

But $(\overline{\Omega}_A\mathbf{W} \times \overline{\Omega}_A\mathbf{W})^\Sigma$ is a compact zero-dimensional space and so there is some clopen neighbourhood K of Δ such that $K \cap \gamma_\Sigma(C) = \emptyset$. By definition of the product topology, we may take K of the form $K = K_1 \cup \cdots \cup K_p$ where each K_i is of the form $\prod_{\varepsilon\in\Sigma} K_{\varepsilon i}$ with each $K_{\varepsilon i}$ a clopen subset of $\overline{\Omega}_A\mathbf{W} \times \overline{\Omega}_A\mathbf{W}$ and $K_{\varepsilon i} = \overline{\Omega}_A\mathbf{W} \times \overline{\Omega}_A\mathbf{W}$ for all $\varepsilon \neq \varepsilon_1, \ldots, \varepsilon_r$. Hence there is some clopen partition Q of $\overline{\Omega}_A\mathbf{W}$ such that, for every class V of Q, $V \times V$ is contained in some $K_{\varepsilon_i j}$. By Hunter's Lemma [16, Lemma 4], each class V of Q is saturated by some clopen congruence of $\overline{\Omega}_A\mathbf{W}$ (say its syntactical congruence) and so there is a clopen congruence θ of $\overline{\Omega}_A\mathbf{W}$ which refines the partition Q. In particular, $\theta^\Sigma \subseteq K$.

Let $\mathbf{W}_0$ be the subpseudovariety of $\mathbf{W}$ generated by the finite semigroup $\overline{\Omega}_A\mathbf{W}/\theta$. Then, by construction, for every $\vec{\pi} \in C$, $\gamma_\Sigma(\vec{\pi}) \notin \theta^\Sigma$, i.e., there is some $\varepsilon \in \Sigma$ such that $\gamma\varepsilon(\vec{\pi}) \notin \theta$ which in turn implies that $\overline{\Omega}_A\mathbf{W} \not\models \varepsilon(\vec{\pi})$. Hence $\mathbf{W}_0 \models \Sigma(\vec{\pi})$ implies $\vec{\pi} \notin C$ which shows that $\mathbf{W}_0$ satisfies (ii).□

A simple extension of Theorem 5.1 is obtained by observing that for some of the variables we may admit substituting them by 1. To establish it, one may easily adapt the above proof by taking $(\overline{\Omega}_A\mathbf{S})^1$ instead of $\overline{\Omega}_A\mathbf{S}$ in the corresponding components of the product space $(\overline{\Omega}_A\mathbf{S})^{\mathbb{N}}$. It is in this form that the result will be used below.

The general application of Theorem 5.1 we propose herein is an automatic translation of definitions of $g\mathbf{V}$ in terms of pseudoidentities of semigroupoids as a membership criterion for the semidirect product $\mathbf{V} * \mathbf{W}$ which is phrased in terms of pseudoidentities of semigroups. Besides Theorem 5.1, our translation depends on Tilson's Derived Category Theorem and the results of section 3. It is worth observing that a similar result and applications hold for *varieties* rather than pseudovarieties, with a proof along the same lines except for the fact that, since varieties have free objects within themselves, there is no need or place for a compactness argument.

Theorem 5.2. *Let $\mathbf{V}$ be a pseudovariety of semigroupoids and let $\mathbf{W}$ be a pseudovariety of semigroups. If $\mathbf{V} = [\![\Sigma]\!]$ where Σ is a set of pseudoidentities of semigroupoids over finite graphs, then a finite semigroup S lies in $\mathbf{V} \circledast \mathbf{W}$ if and only if, for every $(u = v) \in \Sigma$, say with $u, v \in E(\overline{\Omega}_A\mathbf{Sd})$, and for all $\pi_q \in (\overline{\Omega}_B\mathbf{S})^1$ $(q \in V(A))$ and $\rho_s \in \overline{\Omega}_B\mathbf{S}$ $(s \in E(A))$, the following condition holds:*

$$\mathbf{W} \models \pi_{\alpha s}\rho_s = \pi_{\omega s} \text{ for all } s \in E(A) \quad \Longrightarrow \quad S \models \pi_{\alpha u}\varepsilon(u) = \pi_{\alpha u}\varepsilon(v) \tag{12}$$

where $\varepsilon : \overline{\Omega}_A\mathbf{Sd} \to \overline{\Omega}_B\mathbf{S}$ is the continuous morphism of semigroupoids such that $\varepsilon(s) = \rho_s$ for all $s \in E(A)$.

Proof. Fix a finite semigroup S. Let B be a set such that S is B-generated.

We start by proving the theorem in case $\Sigma \subseteq \overline{\Omega}_G\mathbf{Sd} \times \overline{\Omega}_G\mathbf{Sd}$ for some finite graph G. By Theorem 5.1, since the left sides of the implications (12) may all be viewed as substitutions of pseudoidentities of the form $x_{\alpha s}y_s = x_{\omega s}$ associated with the edges of the graph G, there is a finitely generated subpseudovariety $\mathbf{W}_0$ of $\mathbf{W}$ such that the validity of condition (12) for $\mathbf{W}$ and S under all the indicated quantifiers is equivalent to the validity of the following condition with similar quantifiers, except that B is fixed as above:

$$\mathbf{W}_0 \models \pi_{\alpha s}\rho_s = \pi_{\omega s} \text{ for all } s \in E(A) \quad \Longrightarrow \quad S \models \pi_{\alpha u}\varepsilon(u) = \pi_{\alpha u}\varepsilon(v) \tag{13}$$

In view of the equality $\mathbf{V} \circledast \mathbf{W} = \bigcup_i \mathbf{V} \circledast \mathbf{W}_i$ where the union runs over all finitely generated subpseudovarieties of $\mathbf{W}$, we may assume that $\mathbf{W}_0$ also satisfies: $S \in \mathbf{V} \circledast \mathbf{W}$ if and only if $S \in \mathbf{V} \circledast \mathbf{W}_0$.[2]

Let $\mathbf{U}$ be the pseudovariety of semigroups generated by S. Consider the canonical projections $\beta : \overline{\Omega}_B\mathbf{S} \to \overline{\Omega}_B\mathbf{U}$ and $\gamma : \overline{\Omega}_B\mathbf{S} \to \overline{\Omega}_B\mathbf{W}_0$ and consider the derived category $D_{\gamma\circ\beta^{-1}}$. Then, by Propositions 3.5 and 3.8, $S \in \mathbf{V} \circledast \mathbf{W}_0$ if and only if $D_{\gamma\circ\beta^{-1}} \in \mathbf{V}$. Hence it suffices to show that S satisfies (13) for all choices of π_q ($q \in V(A)$) and $\rho_s \in \overline{\Omega}_B\mathbf{S}$ ($s \in E(A)$) if and only if $D_{\gamma\circ\beta^{-1}} \models u = v$ and this is given by Proposition 3.8. The proof of the case when the pseudoidentities of Σ can be written over a fixed finite graph is thus complete. Also note that, for the "only if" part of the statement, in the above argument we do not need the extra assumption on Σ (since the implication (ii)$\Rightarrow$(i) in Theorem 5.1 is verified in any case).

Let us finally consider the general case. If $S \in \mathbf{V} \circledast \mathbf{W}$, then S satisfies the condition of the theorem by the remark at the end of the preceding paragraph. Conversely, suppose S satisfies condition (12) with the indicated quantifiers. Then, for every finite subset Γ of Σ, S satisfies (12) where the pseudoidentity $u = v$ runs over all elements of Γ. By the already proved case, we deduce that $S \in [\![\Gamma]\!] \circledast \mathbf{W}$ for every finite subset Γ of Σ. Since, by Proposition 3.6, $\mathbf{V}$ is the intersection of the pseudovarieties of semigroupoids $[\![\Gamma]\!]$ for all such Γ, it follows that $S \in \mathbf{V} \circledast \mathbf{W}$, as desired.□

Combining with Theorem 3.3, we obtain pseudoidentity bases for semidirect products of pseudovarieties.

Theorem 5.3. *Let* $\mathbf{V}$ *and* $\mathbf{W}$ *be pseudovarieties of semigroups. If* Σ *is a basis of pseudoidentities of semigroupoids for* $g\mathbf{V}$*, then* $\mathbf{V} * \mathbf{W}$ *is defined by the pseudoidentities of the form*

$$\pi_{\alpha u}\varepsilon(u) = \pi_{\alpha u}\varepsilon(v)$$

where $(u = v) \in \Sigma$*, with say* $u, v \in E(\overline{\Omega}_A\mathbf{Sd})$*,* $\pi_q \in (\overline{\Omega}_B\mathbf{S})^1$ *(*$q \in V(A)$*) and* $\rho_s \in \overline{\Omega}_B\mathbf{S}$ *(*$s \in E(A)$*),* $\mathbf{W} \models \pi_{\alpha s}\rho_s = \pi_{\omega s}$ *for all* $s \in E(A)$ *and* $\varepsilon : \overline{\Omega}_A\mathbf{Sd} \to \overline{\Omega}_B\mathbf{S}$ *is the continuous morphism of semigroupoids such that* $\varepsilon(s) = \rho_s$ *for all* $s \in E(A)$*.*□

In case $\mathbf{V}$ is local, the formulation of the preceding result is simpler.

Corollary 5.4. *Let* $\mathbf{V}$ *and* $\mathbf{W}$ *be pseudovarieties of semigroups with* $\mathbf{V}$ *local and let* Σ *be a basis of pseudoidentities for* $\mathbf{V}$*. Then a finite semigroup* S *lies in* $\mathbf{V} * \mathbf{W}$ *if and only if, for every* $(u = v) \in \Sigma$ *with* $u, v \in \overline{\Omega}_r\mathbf{S}$ *and all* $\vec{\rho} \in (\overline{\Omega}_A\mathbf{S})^r$*,* $\pi \in (\overline{\Omega}_A\mathbf{S})^1$*,*

$$\mathbf{W} \models \pi\rho_1 = \cdots = \pi\rho_r = \pi \quad\Longrightarrow\quad S \models \pi u(\vec{\rho}) = \pi v(\vec{\rho}). \tag{14}$$

[2] We emphasize that nothing is being claimed here about the possibility of effectively constructing $\mathbf{W}_0$.

Proof. This is just a reformulation of Theorem 5.2 for the case of a local pseudovariety $\mathbf{V}$.□

Recall that $\mathbf{A} = [\![x^{\omega+1} = x^\omega]\!]$, $\mathbf{Com} = [\![xy = yx]\!]$, $\mathbf{G} = [\![x^\omega = 1]\!]$ and $\mathbf{Sl} = [\![x^2 = x,\, xy = yx]\!]$ denote the pseudovarieties of all finite semigroups which are, respectively, aperiodic, commutative, groups, and semilattices.

Particular instances of the general translation scheme of Theorem 5.2 and Corollary 5.4 are formulated in the following corollary which uses the bases of pseudoidentities for $g\mathbf{Sl}$, $g\mathbf{J}$ and $g\mathbf{Com}$ obtained, respectively, by Brzozowski and Simon [14], Knast [20] (cf. Theorem 2.8) and Thérien and Weiss [30].

In the case of $\mathbf{G}$, which is not local as a pseudovariety of semigroups, the natural basis of pseudoidentities, $x^\omega = 1$, translates for semigroups as $x^\omega y = yx^\omega = y$. We then use the remark that, for pseudovarieties $\mathbf{V}$ of monoids and $\mathbf{W}$ of semigroups, one may define a semidirect product $\mathbf{V} * \mathbf{W}$ as in section 3[3], yielding, for $\mathbf{V}$ and $\mathbf{W}$ pseudovarieties of semigroups, with $\mathbf{V}$ monoidal, $\mathbf{V} * \mathbf{W} = \mathbf{V_M} * \mathbf{W}$. In fact, this is the kind of semidirect product considered by Tilson [31]. Then one easily verifies that all of the above carries over without difficulties to this mixed kind of semidirect product and, therefore, to compute a semidirect product $\mathbf{V} * \mathbf{W}$ with $\mathbf{V}$ monoidal, we may as well work with a basis of pseudoidentities for $g(\mathbf{V_M})$ and, in particular, we may treat nontrivial pseudovarieties of groups as though they were local, as pseudovarieties of semigroups, provided we take bases for pseudoidentities taking into account that variables may assume the value 1. Otherwise, as has been observed by Karl Auinger [10], one may obtain incorrect results. In view of the discussion at the end of section 2, we remind the reader that this word of warning is only necessary when a pseudovariety of groups is involved as the first factor in a semidirect product.

Corollary 5.5. *Let* $\mathbf{V}$ *be an arbitrary pseudovariety of semigroups and let* S *be a finite semigroup.*

a) S *lies in* $\mathbf{Sl} * \mathbf{V}$ *if and only if, for all implicit operations* x, y *and* z *on* $\mathbf{S}$ *(where* x *and* y *may be* 1*), the following condition holds:*

$$\mathbf{V} \models xy = xz = x \quad \Longrightarrow \quad S \models xy^2 = xy,\ xyz = xzy. \tag{15}$$

b) S *lies in* $\mathbf{G} * \mathbf{V}$ *if and only if, for all implicit operations* x *and* y *on* $\mathbf{S}$ *(where* x *may be* 1*),*

$$\mathbf{V} \models xy = x \quad \Longrightarrow \quad S \models xy^\omega = x.$$

c) S *lies in* $\mathbf{A} * \mathbf{V}$ *if and only if, for all implicit operations* x *and* y *on* $\mathbf{S}$ *(where* x *may be* 1*),*

$$\mathbf{V} \models xy = x \quad \Longrightarrow \quad S \models xy^{\omega+1} = xy^\omega.$$

[3] which amounts to defining $\mathbf{V} * \mathbf{W}$ as $\mathbf{V_S} * \mathbf{W}$.

d) S lies in $\mathbf{J} * \mathbf{V}$ if and only if, for all implicit operations a, b, x, y, u and v on $\mathbf{S}$ (where a and b may be 1),

$$\mathbf{V} \models ax = au = b,\ by = bv = a \quad \Longrightarrow \quad S \models a(xy)^\omega xv(uv)^\omega = a(xy)^\omega (uv)^\omega.$$

e) S lies in $\mathbf{Com} * \mathbf{V}$ if and only if, for all implicit operations a, b, x, y and z on $\mathbf{S}$ (where a and b may be 1),

$$\mathbf{V} \models ax = az = b,\ by = a \quad \Longrightarrow \quad S \models axyz = azyx.$$

6. The case of the two-sided semidirect product

An analogue of Tilson's derived category for the study of two-sided semidirect products $\mathbf{V} ** \mathbf{W}$ of pseudovarieties is provided by the kernel category. Both the two-sided semidirect product and the kernel category were introduced by Rhodes and Tilson [26] (see also [27]).

Theorem 5.1 above can be applied in this context as well, to give a systematic translation of a basis of pseudoidentities for the pseudovariety $g\mathbf{V}$ into a basis of pseudoidentities for $\mathbf{V} ** \mathbf{W}$. The translation process closely parallels that described in sections 3 and 5, and we only indicate its main steps.

Let $R : S \to T$ be a relational morphism of semigroups. The *kernel semigroupoid* K_R is constructed as follows:

- $V(K_R) = T^1 \times T^1$;
- for (t, t') and (u, u') in $T^1 \times T^1$, the edges $(t, t') \to (u, u')$ are the equivalence classes $[t, s_1, t_1, u']$ of the quadruples (t, s_1, t_1, u') in $T^1 \times R \times T^1$ such that $tt_1 = u$ and $t' = t_1 u'$ under the equivalence relation defined by

$$(t, s_1, t_1, u') \sim (t, s_2, t_2, u') \qquad \text{if } tt_1 = tt_2,\ t_1u' = t_2u', \text{ and,}$$
$$\text{for all } (\bar{s}, t), (\bar{s}', u') \in R \cup \{(1,1)\},\ \bar{s}s_1\bar{s}' = \bar{s}s_2\bar{s}'.$$

- Composition of edges $[t, s_1, t_1, u']$ and $[u, s_2, t_2, v']$ with $tt_1 = u$ and $u' = t_2v'$ is defined by

$$[t, s_1, t_1, u']\,[u, s_2, t_2, v'] = [t, s_1s_2, t_1t_2, v'].$$

The kernel semigroupoid construction enjoys properties analogous to those of the derived semigroupoid indicated in section 3. See [26] for further details. We will only state here the analogue of Proposition 3.8 for kernels.

Proposition 6.1. *Let* $\mathbf{W}$ *be a finitely generated pseudovariety of semigroups,* S *a finite semigroup,* A *a generating set for* S*,* $\mathbf{U}$ *the pseudovariety of semigroups generated by* S*, and let* $u = v$ *be a semigroupoid pseudoidentity over a finite graph* G*. Let* $\beta : \overline{\Omega}_A\mathbf{S} \to \overline{\Omega}_A\mathbf{U}$ *and* $\gamma : \overline{\Omega}_A\mathbf{S} \to \overline{\Omega}_A\mathbf{W}$ *be the canonical homomorphisms. Then the following are equivalent:*

i) $K_{\gamma\circ\varphi^{-1}} \models u = v$ *for any onto homomorphism* $\varphi : \overline{\Omega}_A\mathbf{S} \to S$*;*

ii) $K_{\gamma\circ\varphi^{-1}} \models u = v$ *for some onto homomorphism* $\varphi : \overline{\Omega}_A\mathbf{S} \to S$*;*

iii) $K_{\gamma\circ\beta^{-1}} \models u = v$*;*

iv) for all $\pi_q, \pi'_q \in (\overline{\Omega}_A\mathbf{S})^1$ *(*$q \in V(G)$*) and* $\rho_s \in \overline{\Omega}_A\mathbf{S}$ *(*$s \in E(G)$*), the following condition holds:*

$$\mathbf{W} \models \begin{cases} \pi_{\alpha s}\rho_s &= \pi_{\omega s} \\ \pi'_{\alpha s} &= \rho_s\pi'_{\omega s} \end{cases} \text{for all } s \in E(G) \quad \Longrightarrow \quad \mathbf{U} \models \pi_{\alpha u}\varepsilon(u)\pi'_{\omega u} = \pi_{\alpha u}\varepsilon(v)\pi'_{\omega u} \tag{16}$$

where $\varepsilon : \overline{\Omega}_G\mathbf{Sd} \to \overline{\Omega}_A\mathbf{S}$ *is the continuous morphism of semigroupoids such that* $\varepsilon(s) = \rho_s$ *for each* $s \in E(G)$*.*

We may also define, for a pseudovariety $\mathbf{V}$ of semigroupoids and a pseudovariety $\mathbf{W}$ of semigroups, the class $\mathbf{V} \circledast\!\!\circledast \mathbf{W}$ to consist of all finite semigroups S for which there is a relational morphism $R : S \to T$ with $T \in \mathbf{W}$ and $K_R \in \mathbf{V}$. Just as in section 3, we conclude that $\mathbf{V} \circledast\!\!\circledast \mathbf{W}$ is a pseudovariety of semigroups. The Kernel Theorem [26] yields $\mathbf{V} ** \mathbf{W} = (g\mathbf{V}) \circledast\!\!\circledast \mathbf{W}$ for pseudovarieties $\mathbf{V}$ and $\mathbf{W}$ of semigroups.

Finally, adapting the arguments of section 5, via Theorem 5.1, Proposition 6.1 and the Kernel Theorem, we obtain the following result.

Theorem 6.2. *Let* $\mathbf{V}$ *be a pseudovariety of semigroupoids and let* $\mathbf{W}$ *be a pseudovariety of semigroups. If* $\mathbf{V} = [\![\Sigma]\!]$ *where* Σ *is a set of pseudoidentities of semigroupoids over finite graphs, then a finite semigroup* S *lies in* $\mathbf{V} \circledast\!\!\circledast \mathbf{W}$ *if and only if, for every* $(u = v) \in \Sigma$*, say with* $u, v \in E(\overline{\Omega}_A\mathbf{Sd})$*, and for all* $\pi_q, \pi'_q \in (\overline{\Omega}_B\mathbf{S})^1$ *(*$q \in V(A)$*) and* $\rho_s \in \overline{\Omega}_B\mathbf{S}$ *(*$s \in E(A)$*), the following condition holds:*

$$\mathbf{W} \models \begin{cases} \pi_{\alpha s}\rho_s &= \pi_{\omega s} \\ \pi'_{\alpha s} &= \rho_s\pi'_{\omega s} \end{cases} \text{for all } s \in E(A) \quad \Longrightarrow \quad S \models \pi_{\alpha u}\varepsilon(u)\pi'_{\omega u} = \pi_{\alpha u}\varepsilon(v)\pi'_{\omega u},$$

where $\varepsilon : \overline{\Omega}_A\mathbf{Sd} \to (\overline{\Omega}_B\mathbf{S})^1$ *is the continuous morphism of categories such that* $\varepsilon(s) = \rho_s$ *for all* $s \in E(A)$*.*

Theorem 6.3. *Let* $\mathbf{V}$ *and* $\mathbf{W}$ *be arbitrary pseudovarieties of semigroups. If* $g\mathbf{V} = [\![\Sigma]\!]$ *where* Σ *is a set of pseudoidentities of semigroupoids over finite graphs, then* $\mathbf{V} ** \mathbf{W}$ *is defined by the pseudoidentities of the form*

$$\pi_{\alpha u}\varepsilon(u)\pi'_{\omega u} = \pi_{\alpha u}\varepsilon(v)\pi'_{\omega u}$$

where $(u = v) \in \Sigma$, *with say* $u, v \in E(\overline{\Omega}_A\mathbf{Sd})$, $\pi_q, \pi'_q \in (\overline{\Omega}_B\mathbf{S})^1$ $(q \in V(A))$ *and* $\rho_s \in \overline{\Omega}_B\mathbf{S}$ $(s \in E(A))$, $\mathbf{W} \models \pi_{\alpha s}\rho_s = \pi_{\omega s}, \pi'_{\alpha s} = \rho_s\pi'_{\omega s}$ *for all* $s \in E(A)$, *and* $\varepsilon\colon \overline{\Omega}_A\mathbf{Sd} \to \overline{\Omega}_B\mathbf{S}$ *is the continuous morphism of semigroupoids such that* $\varepsilon(s) = \rho_s$ *for every* $s \in E(A)$.

Corollary 6.4. *Let* $\mathbf{V}$ *and* $\mathbf{W}$ *be pseudovarieties of semigroups with* $\mathbf{V}$ *local, and let* Σ *be a basis of pseudoidentities for* $\mathbf{V}$. *Then a finite semigroup* S *lies in* $\mathbf{V}**\mathbf{W}$ *if and only if, for every* $(u = v) \in \Sigma$, *say with* $u, v \in \overline{\Omega}_r\mathbf{S}$, *and for all* $\vec{\rho} \in (\overline{\Omega}_A\mathbf{S})^r$, $\pi, \pi' \in (\overline{\Omega}_A\mathbf{S})^1$,

$$\mathbf{W} \models \begin{cases} \pi &= \pi\rho_1 = \ldots = \pi\rho_r \\ \pi' &= \rho_1\pi' = \ldots = \rho_r\pi' \end{cases} \quad\Longrightarrow\quad S \models \pi u(\vec{\rho})\pi' = \pi v(\vec{\rho})\pi'.$$

Corollary 6.5. *Let* $\mathbf{V}$ *be a pseudovariety of semigroups and let* S *be a finite semigroup.*

a) S *lies in* $\mathbf{Sl}**\mathbf{V}$ *if and only if, for all implicit operations* x, x', y, z *on* $\mathbf{S}$ *(where* x *and* x' *may be* 1*), the following condition holds:*

$$\mathbf{V} \models xy = xz = x,\ yx' = zx' = x' \quad\Longrightarrow\quad S \models xy^2x' = xyx',\ xyzx' = xzyx'.$$

b) S *lies in* $\mathbf{G} ** \mathbf{V}$ *if and only if, for all implicit operations* x, x', y *on* $\mathbf{S}$ *(where* x *and* x' *may be* 1*), the following condition holds:*

$$\mathbf{V} \models xy = x,\ yx' = x' \quad\Longrightarrow\quad S \models xy^\omega x' = xx'.$$

c) S *lies in* $\mathbf{A} ** \mathbf{V}$ *if and only if, for all implicit operations* x, x', y *on* $\mathbf{S}$ *(where* x *and* x' *may be* 1*), the following condition holds:*

$$\mathbf{V} \models xy = x,\ yx' = x' \quad\Longrightarrow\quad S \models xy^{\omega+1}x' = xy^\omega x'.$$

d) S *lies in* $\mathbf{J} ** \mathbf{V}$ *if and only if, for all implicit operations* a, a', b, b', x, y, u, v *on* $\mathbf{S}$ *(where* a, a', b *and* b' *may be* 1*), the following condition holds:*

$$\mathbf{V} \models \begin{cases} ax = au = b, & xb' = ub' = a' \\ by = bv = a, & ya' = va' = b' \end{cases} \quad\Longrightarrow\quad S \models a(xy)^\omega xv(uv)^\omega b' = a(xy)^\omega(uv)^\omega b'.$$

e) S *lies in* $\mathbf{Com} ** \mathbf{V}$ *if and only if, for all implicit operations* a, a', b, b', x, y, u, v *on* $\mathbf{S}$ *(where* a, a', b *and* b' *may be* 1*), the following condition holds:*

$$\mathbf{V} \models \begin{cases} ax = az = b, & xb' = zb' = a' \\ by = a, & ya' = b' \end{cases} \quad\Longrightarrow\quad S \models axyzb' = azyxb'.$$

The results of section 4 can also be extended to two-sided semidirect products. The statements and proofs of the extensions are left out since they are easily adapted from the one-sided case.

7. The Delay Theorem and related results

Denote by $\mathbf{D}$ the pseudovariety defined by the pseudoidentity $xy^\omega = y^\omega$. For an implicit operation π, denote by $t_n(\pi)$ the longest word w of length at most n which is a suffix of π, i.e., such that $\pi = \pi' w$ for some implicit operation π. It is well-known that the relation $t_n(\pi) = t_n(\rho)$ characterizes those pseudoidentities $\pi = \rho$ which are valid in the pseudovariety $\mathbf{D}_n = [\![xy_1 \dots y_n = y_1 \dots y_n]\!]$ and that $\mathbf{D} = \bigcup_{n \geq 1} \mathbf{D}_n$. Hence $\mathbf{D} \models \pi = \rho$ if and only if, for every $n \geq 1$, $t_n(\pi) = t_n(\rho)$.

We start by stating and proving a combinatorial property of finite semigroups that will be of use in the sequel.

Lemma 7.1. *Let B be a finite set, let $\pi \in \overline{\Omega}_B\mathbf{S} \setminus \Omega_B\mathbf{S}$ and let S be a finite semigroup. Then there is a factorization $\pi = \pi'\rho w v$ where w and v are words, ρ is not explicit, and $S \models \rho = w = w^2$.*

Proof. For each $n \geq 1$, let $\pi = \pi_n u_n$ where $u_n = t_n(\pi)$ and, for $m > n$, write $u_m = w_{m,n} u_n$. Consider the canonical projection $\varphi : \overline{\Omega}_B\mathbf{S} \to \overline{\Omega}_B\mathbf{U}$ where $\mathbf{U}$ is the pseudovariety generated by S. Let $\mathcal{P}_2(X)$ denote the set of all two-element subsets of a set X and let $\mathbb{Z}_+$ denote the set of all positive integers.

Since the set $\overline{\Omega}_B\mathbf{U}$ is finite, the mapping

$$\begin{aligned} \mathcal{P}_2(\mathbb{Z}_+) &\longrightarrow \overline{\Omega}_B\mathbf{U} \\ \{n, m\} &\longmapsto \varphi w_{m,n} \quad (n < m) \end{aligned}$$

gives a finite colouring of $\mathcal{P}_2(\mathbb{Z}_+)$. By Ramsey's Theorem, there is an infinite subset X of $\mathbb{Z}_+$ such that all elements of $\mathcal{P}_2(X)$ get the same colour.

Let $X = \{n_1, n_2, \dots\}$ with $n_1 < n_2 < \cdots$. Take $v = u_{n_1}$, $w = w_{n_2,n_1}$ and, for each $i > 2$, let $\pi_{n_2} = \pi'_i \rho_i$ where $\rho_i = w_{n_i,n_2}$. By compactness of $\overline{\Omega}_B\mathbf{S}$, there is in $\overline{\Omega}_B\mathbf{S} \times \overline{\Omega}_B\mathbf{S}$ some accumulation point (π', ρ) of the sequence $\big((\pi'_i, \rho_i)\big)_{i>2}$. By continuity of multiplication in $\overline{\Omega}_B\mathbf{S}$, we have $\pi = \pi'\rho w v$. Since

$$\varphi w = \varphi w_{n_2,n_1} = \varphi w_{n_3,n_1} = \varphi w_{n_3,n_2} \cdot \varphi w_{n_2,n_1} = \varphi w \cdot \varphi w$$

and $\varphi \rho_i = \varphi w_{n_i,n_2} = \varphi w$ for all $i > 2$, we conclude that $S \models \rho = w = w^2$.□

For a semigroup S, let S_E be the category whose vertices are the idempotents of S and whose edges $e \to f$ are the triples of the form (e, esf, f) with $s \in S$; the composite of two edges (e, s, f) and (f, t, g) is the edge (e, st, g).

The following two results extend a theorem due to Tilson [31, Theor. 17.1] which is known as the *Delay Theorem* for reasons that are made clear in Tilson's work. Tilson's theorem in turn is a reformulation of a result obtained by Straubing [29] stating that, for a monoidal pseudovariety $\mathbf{V}$ and a semigroup S whose longest strict $\leq_{\mathcal{R}}$-chain has n elements, $S \in \mathbf{V} * \mathbf{D}$ if and only if $S \in \mathbf{V} * \mathbf{D}_n$. Formulated in Tilson's terminology, Straubing also added the equivalent condition $S_E \in g\mathbf{V}$. Tilson's proof of the hard part of the Delay Theorem consists in showing that, fixing a finite alphabet B and an onto homomorphism $\varphi : B^+ \to S$ and letting $\gamma : B^+ \to \Omega_B \mathbf{D}_n$ be the natural homomorphism, then the only nontrivial bonded component of the derived semigroupoid $D_{\gamma\circ\varphi^{-1}}$ (whose vertices are the idempotents of $\Omega_B \mathbf{D}_n$) divides S_E. The result then follows by applying Propositions 4.2 and 7.2. We prove here only the equivalence giving the characterization of membership of S in $\mathbf{V} * \mathbf{D}$ in terms of the condition $S_E \in g\mathbf{V}$, extended to the case when instead of $g\mathbf{V}$, an arbitrary nontrivial pseudovariety of categories is considered.

Proposition 7.2. *Let* $\mathbf{V}$ *be a pseudovariety of semigroupoids and let* S *be an element of* $\mathbf{V} \circledast \mathbf{D}$. *Then* $S_E \in \mathbf{V}_{\mathbf{C}}$.

Proof. Let Σ be a basis of semigroupoid pseudoidentities for $\mathbf{V}$. Then, by Theorem 5.2, $\mathbf{V} \circledast \mathbf{D}$ is defined by the pseudoidentities of the form

$$\pi_{\alpha u}\varepsilon(u) = \pi_{\alpha u}\varepsilon(v) \qquad \begin{array}{l}\text{with } (u = v) \in \Sigma, \text{ say written over a graph } A,\\ \text{and } \mathbf{D} \models \pi_{\alpha s}\rho_s = \pi_{\omega s} \text{ for any } s \in E(A)\end{array}$$

where the π_q are semigroup implicit operations or 1, the ρ_s are semigroup implicit operations, and $\varepsilon : \overline{\Omega}_A\mathbf{Sd} \to \overline{\Omega}_B\mathbf{S}$ is the continuous semigroupoid morphism such that $\varepsilon(s) = \rho_s$ for every $s \in E(A)$. In particular, if we choose each π_q to be an idempotent e_q and take $\rho_s = e_{\alpha s}x_s e_{\omega s}$ where the x_s ($s \in E(A)$) are variables, then we conclude that a necessary condition for S to belong to $\mathbf{V} \circledast \mathbf{D}$ is that $S_E \in \mathbf{V}$. Since S_E is a category, it follows that $S_E \in \mathbf{V}_{\mathbf{C}}$.□

Theorem 7.3. *Let* $\mathbf{V}$ *be a nontrivial pseudovariety of categories. Then a finite semigroup* S *lies in* $\mathbf{V} \circledast \mathbf{D}$ *if and only if* $S_E \in \mathbf{V}$.

Proof. If S belongs to $\mathbf{V} \circledast \mathbf{D}$, then $S_E \in \mathbf{V}$ by Proposition 7.2. Conversely, suppose that $S_E \in \mathbf{V}$. By Theorem 2.9, $\mathbf{V}$ admits a basis Σ of bonded pseudoidentities. Let $(u = v) \in \Sigma$ and let $A = c(\{u, v\})$ which, by the choice of Σ, is a strongly connected finite graph. Take any $\pi_q \in (\overline{\Omega}_B\mathbf{S})^1$ ($q \in V(A)$) and $\rho_s \in \overline{\Omega}_B\mathbf{S}$ ($s \in E(A)$) such that $\mathbf{D} \models \pi_{\alpha s}\rho_s = \pi_{\omega s}$ for every $s \in E(A)$. We must show that $S \models \pi_{\alpha u}\varepsilon(u) = \pi_{\alpha u}\varepsilon(v)$ where $\varepsilon : \overline{\Omega}_A\mathbf{C} \to (\overline{\Omega}_B\mathbf{S})^1$ is the continuous category morphism such that $\varepsilon(s) = \rho_s$ for every $s \in E(A)$.

If $\mathbf{D} \models \pi\rho = \pi'$ and π' is a word, then so are π and ρ and the words $\pi\rho$ and π' are equal. Since A is strongly connected and the ρ_s are not 1, it follows that no π_q with $q \in V(A)$

is explicit. By Lemma 7.1, for each $q \in V(A)$, there is a factorization $\pi_q = \pi_{q,1}e_q\pi_{q,2}$ where $\pi_{q,2}$ is a word, e_q is a non-explicit implicit operation (whence idempotent on $\mathbf{D}$), and there is a word w_q which is a suffix of e_q and such that $S \models e_q = w_q = w_q^2$. Hence

$$\mathbf{D} \models e_{\alpha s}\pi_{\alpha s,2}\rho_s = \pi_{\alpha s,1}e_{\alpha s} \cdot e_{\alpha s}\pi_{\alpha s,2}\rho_s = \pi_{\alpha s}\rho_s = \pi_{\omega s} = \pi_{\omega s,1}e_{\omega s}\pi_{\omega s,2} = e_{\omega s}\pi_{\omega s,2}$$

and so $t_n(e_{\alpha s}\pi_{\alpha s,2}\rho_s) = t_n(e_{\omega s}\pi_{\omega s,2})$ for every n. In particular, $e_{\alpha s}\pi_{\alpha s,2}\rho_s = \mu_s w_{\omega s}\pi_{\omega s,2}$ for some implicit operation μ_s since the word $w_{\omega s}\pi_{\omega s,2}$ is a suffix of $e_{\omega s}\pi_{\omega s,2}$. Let $\varepsilon' : \overline{\Omega}_A\mathbf{C} \to (\overline{\Omega}_B\mathbf{S})^1$ be the continuous category morphism defined by $\varepsilon'(s) = \mu_s w_{\omega s}$ for each $s \in E(A)$.

We claim that, for every $\xi \in E(\overline{\Omega}_A\mathbf{C})$,

$$S \models \varepsilon'(\xi)\pi_{\omega\xi,2} = e_{\alpha\xi}\pi_{\alpha\xi,2}\,\varepsilon(\xi). \tag{17}$$

The validity of (17) for $\xi \in E(A)$ follows directly from the definition of $\varepsilon'(\xi)$. To prove (17) in general, we observe that, by continuity of ε, ε', α and ω, it suffices to consider $\xi \in E(A^*)$; for such ξ, we proceed by induction on the length of the path ξ in the graph A. Suppose then that ξ satisfies (17) and consider $s \in E(A)$ such that $\omega s = \alpha\xi$. Then

$$\begin{aligned} S \models \varepsilon'(s\xi)\pi_{\omega(s\xi),2} &= \varepsilon'(s)\varepsilon'(\xi)\pi_{\omega\xi,2} = \varepsilon'(s)e_{\alpha\xi}\pi_{\alpha\xi,2}\,\varepsilon(\xi) \quad \text{by (17)} \\ &= \varepsilon'(s)\pi_{\omega s,2}\,\varepsilon(\xi) \quad \text{since } S \models e_{\alpha\xi} = e_{\omega s} = w_{\omega s} = w_{\omega s}^2 \\ &= e_{\alpha s}\pi_{\alpha s,2}\,\varepsilon(s)\varepsilon(\xi) = e_{\alpha s}\pi_{\alpha s,2}\,\varepsilon(s\xi), \end{aligned}$$

so that $s\xi$ also satisfies (17). This proves the validity of (17) for all $\xi \in E(\overline{\Omega}_A\mathbf{C})$.

Hence, for the given $u, v \in E(\overline{\Omega}_A\mathbf{C})$, since $S_E \models u = v$, we have $S \models \varepsilon'(u) = \varepsilon'(v)$ (as $S \models \varepsilon'(s) = e_{\alpha s}\varepsilon'(s)e_{\omega s}$) and so

$$\begin{aligned} S \models \pi_{\alpha u}\varepsilon(u) &= \pi_{\alpha u,1}e_{\alpha u}\pi_{\alpha u,2}\varepsilon(u) = \pi_{\alpha u,1}\varepsilon'(u)\pi_{\omega u,2} \\ &= \pi_{\alpha u,1}\varepsilon'(v)\pi_{\omega u,2} = \pi_{\alpha u,1}e_{\alpha u}\pi_{\alpha u,2}\varepsilon(v) = \pi_{\alpha u}\varepsilon(v). \end{aligned}$$

This shows that $S \in \mathbf{V} \circledast \mathbf{D}$ and concludes the proof of the theorem.□

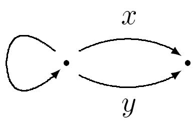
The above proof actually holds for an arbitrary pseudovariety $\mathbf{V}$ of semigroupoids admitting a basis of bonded ($\mathbf{Sd}$-)pseudoidentities. However, it is easy to verify that this condition implies that $\mathbf{V_C}$ is nontrivial and so the extended version of Theorem 7.3 thus obtained follows from the above version together with the following corollary.

Corollary 7.4. *Let $\mathbf{V}$ be a pseudovariety of semigroupoids such that $\mathbf{V_C}$ is nontrivial. Then $\mathbf{V} \circledast \mathbf{D} = \mathbf{V_C} \circledast \mathbf{D}$.*

Proof. The inclusion $\mathbf{V_C} \circledast \mathbf{D} \subseteq \mathbf{V} \circledast \mathbf{D}$ is clear in any case. Since $\mathbf{V_C}$ is nontrivial the converse follows immediately from Proposition 7.2 and Theorem 7.3.□

Recall that, for a pseudovariety $\mathbf{V}$, $\mathbf{LV}$ denotes the pseudovariety of all finite semigroups S whose subsemigroups of the form eSe with $e^2 = e \in S$ belong to $\mathbf{V}$.

The equality $\mathbf{V} \circledast \mathbf{D} = \mathbf{V_C} \circledast \mathbf{D}$ may fail in case $\mathbf{V_C}$ is trivial. For example, consider the pseudovariety $\mathbf{V}$ defined by the pseudoidentity $x = y$ over the following graph:



Then $\mathbf{V_C} \circledast \mathbf{D} = \mathbf{D}$ whereas, using Theorem 5.2, it is not hard to show that $\mathbf{V} \circledast \mathbf{D} = \mathbf{LI}$ where $\mathbf{I} = [\![x = y]\!]$.

As two extreme particular cases of Theorem 7.3, applying Theorem 3.3, we obtain the following result, whose part (a) is the Delay Theorem while part (b) is an immediate consequence of the definitions and Theorem 7.3.

Corollary 7.5. *Let* $\mathbf{V}$ *be a monoidal pseudovariety of semigroups and let* S *be a finite semigroup.*

a) If $\mathbf{V}$ *is nontrivial, then, for every finite semigroup* S*,* $S \in \mathbf{V} * \mathbf{D}$ *if and only if* $S_E \in g\mathbf{V}$*.*

b) $S \in \mathbf{LV}$ *if and only if* $S_E \in \ell\mathbf{V}$*. Hence* $(\ell\mathbf{V}) \circledast \mathbf{D} = \mathbf{LV}$.□

We next recall Tilson's *consolidation operation.* For a category (or semigroupoid) C, denote by C_{Cd} the semigroup which is obtained from the set $E(C)$ of edges of C by adding a zero, if necessary, and setting equal to zero all composites of edges which are undefined in C. The reader may find in [31, Prop. 16.1] an elementary proof of the observation that the categories C and $(C_{\mathrm{Cd}})_E$ divide each other.

As a sort of inverse of the consolidation operation at the pseudovariety level, we introduce the following operation: for a pseudovariety $\mathbf{V}$ of semigroups, $c\mathbf{V}$ denotes the class of all finite categories C such that $C_{\mathrm{Cd}} \in \mathbf{V}$. Let B_Q denote the consolidation of the category C with set of vertices Q and exactly one edge (q_1, q_2) in each $C(q_1, q_2)$. Then B_Q is a Brandt aperiodic semigroup with multiplication of nonzero elements given by

$$(q_1, q_2)\,(q_3, q_4) = \begin{cases} (q_1, q_4) & \text{if } q_2 = q_3 \\ 0 & \text{otherwise} \end{cases}$$

It is well-known that all B_Q with $2 \leq |Q| < \infty$ generate the same pseudovariety of semigroups.

Proposition 7.6. *Let* $\mathbf{V}$ *be a pseudovariety of semigroups such that* $B_2 \in \mathbf{V}$*. Then* $c\mathbf{V}$ *is a pseudovariety of categories.*

Proof. Suppose that $D \in c\mathbf{V}$ and let $\varphi : C \to D$ be a faithful morphism of categories. Then the following is a semigroup embedding:

$$\begin{array}{rcl} C_{\mathrm{Cd}} & \longrightarrow & D_{\mathrm{Cd}} \times B_{V(C)} \\ s & \longmapsto & \begin{cases} (\varphi s, \alpha s, \omega s) & \text{if } s \in E(C) \\ (0,0) & \text{otherwise} \end{cases} \end{array}$$

Hence, if C is finite, since $B_2 \in \mathbf{V}$, we conclude that $C_{\mathrm{Cd}} \in c\mathbf{V}$. It is also a routine matter to show that every quotient morphism of categories $C \to D$ induces an onto homomorphism of semigroups $C_{\mathrm{Cd}} \to D_{\mathrm{Cd}}$ and that $(C \times D)_{\mathrm{Cd}} \prec C_{\mathrm{Cd}} \times D_{\mathrm{Cd}}$.□

With a similar argument, we obtain the following result which reformulates [30, Theor. 1.5].

Corollary 7.7. *Let* $\mathbf{V}$ *be a pseudovariety of semigroups containing* B_2*. Then, for a finite semigroupoid* S, $S \in g\mathbf{V}$ *if and only if* $S_{\mathrm{Cd}} \in \mathbf{V}$*.*

Proof. Suppose $S \in g\mathbf{V}$, say $S \prec T$ with $T \in \mathbf{V}$. Then, by the argument in the proof of Proposition 7.6, $S_{\mathrm{Cd}} \prec T_{\mathrm{Cd}} \times B_{V(S)} = T \times B_{V(S)} \in \mathbf{V}$, which shows that $S_{\mathrm{Cd}} \in \mathbf{V}$. For the converse, just note that, by definition of S_{Cd}, the natural semigroupoid morphism $S \to S_{\mathrm{Cd}}$ is faithful.□

The following result is an immediate corollary of Theorem 7.3.

Corollary 7.8. *Let* $\mathbf{V}$ *be a nontrivial pseudovariety of categories. Then* $c(\mathbf{V} \circledast \mathbf{D}) = \mathbf{V}$.□

If $\mathbf{V}$ is trivial, then $\mathbf{V} \circledast \mathbf{D} = \mathbf{D}$ and $C_{Cd} \in \mathbf{D}$ implies that C is a trivial category with just one vertex. Hence the conclusion of Corollary 7.8 would remain true in all cases if we would replace $c(\mathbf{V} \circledast \mathbf{D})$ by the pseudovariety of categories it generates.

Combining Corollary 7.8 with Theorem 4.7, we deduce that the theory of pseudovarieties of categories is in a sense, part of the theory of pseudovarieties of semigroups.

Corollary 7.9. *The mapping* $\circledast\mathbf{D} : \mathcal{P}s\,\mathbf{C} \to \mathcal{P}s\,\mathbf{S}$ *defined by* $\mathbf{V} \mapsto \mathbf{V} \circledast \mathbf{D}$ *is a lattice embedding.*□

It would be of interest to determine the image of $\circledast\mathbf{D}$. A somewhat related question concerns the study of the operator c. In particular, we do not know whether c respects joins and whether $(c\mathbf{V}) \circledast \mathbf{D} = \mathbf{V}$, say for pseudovarieties of semigroups $\mathbf{V} \supseteq \mathbf{LSl}$. The following are but a few elementary observations in this direction.

Lemma 7.10. *Let S be a finite semigroup. Then $(S_E)_{Cd} \prec S \times B_I$ where I denotes the set of idempotents of S.*

Proof. Consider the following mapping of $(S_E)_{Cd}$ into a Rees quotient of $S \times B_I$:

$$\begin{aligned} (S_E)_{Cd} &\longrightarrow S \times B_I \,/\, S \times \{0\} \\ (e,s,f) &\longmapsto (s,e,f) \\ 0 &\longmapsto 0 \end{aligned}$$

Then it is immediate to check that this is an embedding.□

Corollary 7.11. *Let $\mathbf{V}$ be a pseudovariety of semigroups containing B_2. Then $\mathbf{V} \subseteq (c\mathbf{V}) \circledast \mathbf{D}$.*

Proof. Let S be an element of $\mathbf{V}$. By Lemma 7.10, $(S_E)_{Cd} \in \mathbf{V}$. Hence $S_E \in c\mathbf{V}$ and so $S \in (c\mathbf{V}) \circledast \mathbf{D}$ by Theorem 7.3.□

More precisely, we have the following result.

Proposition 7.12. *Let $\mathbf{V}$ be a pseudovariety of semigroups and $\mathbf{W}$ a categorical pseudovariety of semigroupoids.*
a) If $\mathbf{V} \subseteq \mathbf{W} \circledast \mathbf{D}$ then $c\mathbf{V} \subseteq \mathbf{W}$.
b) If $\mathbf{W} \circledast \mathbf{D} \subseteq \mathbf{V}$ then $\mathbf{W} \subseteq c\mathbf{V}$.
c) If $B_2 \in \mathbf{V}$ and $c\mathbf{V} \subseteq \mathbf{W}$, then $\mathbf{V} \subseteq \mathbf{W} \circledast \mathbf{D}$.

Proof. (a) and (b) follow from Corollary 7.8 and (c) from Corollary 7.11.□

Again, as particular cases of Corollary 7.8 worth mentioning we have the following.

Corollary 7.13. *Let $\mathbf{V}$ be a monoidal pseudovariety of semigroups and let C be a finite category.*
a) If $\mathbf{V}$ is nontrivial, then $C \in g\mathbf{V}$ if and only if $C_{\mathrm{Cd}} \in \mathbf{V} * \mathbf{D}$.
b) $C \in \ell\mathbf{V}$ if and only if $C_{\mathrm{Cd}} \in \mathbf{LV}$.□

In particular, we obtain the following result due to Tilson [31, Theor. 17.3].

Corollary 7.14. *Let $\mathbf{V}$ be a monoidal pseudovariety of semigroups. Then $\mathbf{V}$ is local if and only if $\mathbf{V} * \mathbf{D} = \mathbf{LV}$.*

Proof. In view of Corollary 7.13, it suffices to consider the case where $\mathbf{V}$ is the trivial pseudovariety $\mathbf{I} = [\![x = y]\!]$. Since $\mathbf{I} * \mathbf{D} = \mathbf{D} \neq \mathbf{LI}$, we observe that $\mathbf{I}$ is not local, i.e., that there is a nontrivial finite category C all of whose submonoids are trivial; the following picture describes such a category:

$$\begin{tikzcd} u \arrow[loop left, "1_u"] \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & v \arrow[loop right, "1_v"] \end{tikzcd} \qquad \square$$

The first author has already pointed out in [3] that, while Theorem 7.5 has been traditionally used as a means of calculating the semidirect product $\mathbf{V}*\mathbf{D}$ (for a monoidal pseudovariety $\mathbf{V}$) via the calculation of $g\mathbf{V}$, one may turn it around, as in Corollary 7.13, to compute $g\mathbf{V}$ if an independent calculation of $\mathbf{V}*\mathbf{D}$ can be found, which has been often the case [1, 3]. So, to apply Theorem 5.2 to compute $\mathbf{V}*\mathbf{W}$ in case $g\mathbf{V}$ is not already known, it may be useful to start off by calculating $\mathbf{V}*\mathbf{D}$.

In view of Corollary 7.4, one may ask whether also $\mathbf{V}*\mathbf{D} = \mathbf{V_M}*\mathbf{D}$ for every pseudovariety $\mathbf{V}$ of semigroups. In fact, this equality has been observed by the first author [4, Thm. 10.7.1] for pseudovarieties of commutative semigroups. However, it is not true in general. For instance, if $\mathbf{V}$ is a monoidal pseudovariety which is not local (such as $\mathbf{J}$ and $\mathbf{Com}$), then

$$\mathbf{LV}*\mathbf{D} = \mathbf{LV} \neq \mathbf{V}*\mathbf{D} = (\mathbf{LV})_\mathbf{M}*\mathbf{D}$$

where the first equality follows from the relation (11) at the end of section 4; alternatively, one may use Theorem 5.2 to prove it or even verify it more or less directly as in [4, Prop. 10.6.13]. Equivalently,

$$(g\mathbf{LV})_\mathbf{C} = \ell\mathbf{V} \neq g\mathbf{V} = g((\mathbf{LV})_\mathbf{M}).$$

Therefore, for non-monoidal $\mathbf{V}$, some other type of approach may be needed to compute $\mathbf{V}*\mathbf{D}$. A simplified case occurs when $\mathbf{V_M}$ is local, for then

$$\mathbf{L}(\mathbf{V_M}) = \mathbf{V_M}*\mathbf{D} \subseteq \mathbf{V}*\mathbf{D} \subseteq \mathbf{L}(\mathbf{V_M})*\mathbf{D} = \mathbf{L}(\mathbf{V_M})$$

and so $\mathbf{V}*\mathbf{D} = \mathbf{V_M}*\mathbf{D} = \mathbf{L}(\mathbf{V_M}) = \mathbf{LV}$.

8. Examples of calculations of semidirect products

We now show with some concrete examples how to use the theorems of sections 5 and 6 to compute semidirect products of pseudovarieties. Some of the examples herein were previously known, although previous proofs did not have such a definitely syntactical flavour.

For a pseudovariety $\mathbf{V}$ of semigroups, denote by $\mathbf{DV}$ the pseudovariety of all finite semigroups all of whose regular $\mathcal{D}$-classes are subsemigroups which lie in $\mathbf{V}$. The left-right dual of $\mathbf{D}_n$ will be denoted by $\mathbf{K}_n$; thus, for instance, $\mathbf{K}_1 = [\![xy = x]\!]$. The pseudovariety $\mathbf{DK}_1$ consisting of all $\mathcal{R}$-trivial semigroups will be denoted by $\mathbf{R}$. By $\mathbf{DO}$

we mean the pseudovariety of all finite semigroups whose regular $\mathcal{D}$-classes are orthodox subsemigroups; this pseudovariety has been extensively studied by Azevedo [11] (see also [6, 4]).

The application of Theorems 5.2 and 6.3 to the calculation of $\mathbf{V} * \mathbf{W}$ and $\mathbf{V} ** \mathbf{W}$ presupposes some information on the solutions of equations of the form $\pi_v \rho_s = \pi_{v'}$ and $\pi'_v = \rho_s \pi'_{v'}$ in $\overline{\Omega}_B \mathbf{V}$. Semidirect products of the form $\mathbf{V} * \mathbf{J}$ and $\mathbf{V} ** \mathbf{J}$ are easier to compute in general, since the structure of the $\overline{\Omega}_B \mathbf{J}$ is very well known [2]. In Theorems 8.4 and 8.5 below, we show that for such products, all pseudovarieties $\mathbf{V}$ behave as though they were local.

We start with some technical lemmas.

Lemma 8.1. *Let $\mathbf{W}$ be a pseudovariety of semigroups containing $\mathbf{J}$ and let $\pi, \rho \in \overline{\Omega}_B \mathbf{S}$ be such that $\mathbf{W} \models \pi\rho = \pi$. Then there is a factorization $\pi = \pi' e$ such that e is idempotent on $\mathbf{J}$ and $c(\rho) \subseteq c(e)$.*

Proof. By a result due to Azevedo [11], every implicit operation on $\mathbf{S}$ has a factorization into a finite number of factors, each of which is either regular when restricted to $\mathbf{DS}$ or explicit; such a factorization may be simplified by noting that regular factors (on $\mathbf{DS}$) absorb adjacent factors which involve only variables that appear in the absorbing regular factor. Moreover, after this simplification, the factorization is unique up to equality over $\mathbf{J}$ of regular factors in corresponding positions (cf. Almeida [2]).

Since $\mathbf{J} \subseteq \mathbf{W}$ and $\mathbf{W} \models \pi\rho = \pi$, by uniqueness of the factorization over $\mathbf{J}$, the last factor in the factorization of π must be an idempotent e over $\mathbf{J}$ whose content contains that of ρ, which establishes the lemma.□

Lemma 8.2. *Let A be a strongly connected graph, let $\mathbf{W}$ be a pseudovariety containing $\mathbf{J}$, and let $\pi_q, \rho_s \in \overline{\Omega}_B \mathbf{S}$ ($q \in V(A)$, $s \in E(A)$) be such that $\mathbf{W} \models \pi_{\alpha s} \rho_s = \pi_{\omega s}$ for every $s \in E(A)$. Then for a fixed $v \in V(A)$, there is a factorization $\pi_v = \pi'_v e$ such that e is idempotent on $\mathbf{J}$ and $c(\rho_s) \subseteq c(e)$ for every $s \in E(A)$.*

Proof. Since A is strongly connected, from $\mathbf{W} \models \pi_{\alpha s} \rho_s = \pi_{\omega s}$ for every $s \in E(A)$, we deduce that all the implicit operations π_v with $v \in V(A)$ are $\mathcal{R}$-equivalent over $\mathbf{W}$ and, therefore, equal over $\mathbf{J}$. By Lemma 8.1, taking $\pi = \pi_v$ and $\rho = \rho_{s_1} \dots \rho_{s_r}$ where $s_1 \dots s_r$ is a path in A which starts in v and goes at least once through each edge, we obtain the desired factorization.□

The following combinatorial-compactness argument extends an idea which was already used by the first author in [3, Lemma 5.3].

Lemma 8.3. *Let $\pi \in \overline{\Omega}_B \mathbf{S}$ be such that $\mathbf{J} \models \pi^2 = \pi$. Then there is a factorization $\pi = \lambda \mu^\omega \nu$ such that $c(\mu) = c(\pi)$.*

Proof. Let $(w_n)_{n\geq 1}$ be a sequence of words converging to π in $\overline{\Omega}_B\mathbf{S}$. By [2, Lemma 3.1 and Theor. 3.1], we may assume that each w_n has a factorization $w_n = w_{n1}\dots w_{nn}$ such that $c(w_{ni}) = c(\pi)$.

Given a finite semigroup S, then, by a standard combinatorial argument [15, Prop. III.9.2], provided n is sufficiently large, there are indices $i, j \in \{1,\dots,n\}$ such that $i < j$ and

$$S \models w_n = w_{n1}\dots w_{n,i-1}(w_{ni}\dots w_{n,j-1})^\omega w_{nj}\dots w_{nn}.$$

By compactness of $\overline{\Omega}_B\mathbf{S}$, it follows that there are implicit operations λ_S, μ_S and ν_S such that $S \models \pi = \lambda_S\mu_S^\omega\nu_S$ and $c(\mu_S) = c(\pi)$.

Consider the net

$$\{(\lambda_S,\mu_S,\nu_S) : S \in \mathbf{S} \text{ with } S\ B\text{-generated}\} \tag{18}$$

where the indexing set is ordered by $S \leq T$ if there is a homomorphism $T \to S$ which respects the choice of generators. Since $\overline{\Omega}_B\mathbf{S}$ is compact, there are implicit operations λ, μ and ν such that some subnet of the net (18) converges to (λ,μ,ν) in the topological product $(\overline{\Omega}_B\mathbf{S})^3$. Then, by continuity of the content function, $c(\mu) = c(\pi)$. We claim that $\pi = \lambda\mu^\omega\nu$. To prove the claim, it suffices to show that, for every B-generated $S \in \mathbf{S}$, $S \models \pi = \lambda\mu^\omega\nu$. Indeed, since some subnet of (18) converges to (λ,μ,ν) and the multiplication and the operation $x \mapsto x^\omega$ are continuous, there is some B-generated finite semigroup T such $S \leq T$ and $S \models \lambda_T\mu_T^\omega\nu_T = \lambda\mu^\omega\nu$. Since $T \models \lambda_T\mu_T^\omega\nu_T = \pi$ and S is a homomorphic image of T, it follows that $S \models \lambda\mu^\omega\nu = \pi$ which proves the claim.□

For $\vec{\pi} = (\pi_1,\dots,\pi_r) \in (\overline{\Omega}_B\mathbf{S})^r$, write $c(\vec{\pi})$ for $c(\pi_1)\cup\dots\cup c(\pi_r)$.

Theorem 8.4. *Let $\mathbf{V}$ be a nontrivial monoidal pseudovariety of semigroups. If Σ is a basis of pseudoidentities for $\mathbf{V}$, then $\mathbf{V} * \mathbf{J}$ is defined by the pseudoidentities of the form*

$$e\pi u(\vec{\rho}) = e\pi v(\vec{\rho}) \text{ with } e \text{ idempotent, } c(\pi)\cup c(\vec{\rho}) \subseteq c(e) \text{ and } (u = v) \in \Sigma. \tag{19}$$

Proof. Suppose first that $\mathbf{V}$ does not consist entirely of groups. Then $\mathbf{V}$ satisfies no nontrivial identities of the form $u = 1$ and so, in view of Theorems 2.9 and 5.2, $\mathbf{V} * \mathbf{J}$ is defined by the pseudoidentities of the form

$$\pi_{\alpha u}\varepsilon(u) = \pi_{\alpha u}\varepsilon(v) \tag{20}$$

where $u = v$ is a bonded category pseudoidentity of $g\mathbf{V}$, $A = c(\{u,v\})$, $\pi_q, \rho_s \in \overline{\Omega}_B\mathbf{S}$ ($q \in V(A)$, $s \in E(A)$) are such that $\mathbf{J} \models \pi_{\alpha s}\rho_s = \pi_{\omega s}$ for all $s \in E(A)$, and $\varepsilon\colon \overline{\Omega}_A\mathbf{C} \to (\overline{\Omega}_B\mathbf{S})^1$ is the continuous category morphism defined by $\varepsilon(s) = \rho_s$ for all $s \in E(A)$.

Fix a nontrivial bonded category pseudoidentity $u_0 = v_0$ valid in $g\mathbf{V}$, let $A = c(\{u_0,v_0\})$ and let $q_0 = \alpha u_0$. Let A' be the graph obtained from A by identifying all the vertices, let $\varphi: A \to A'$ be the associated graph morphism, and let $\hat{\varphi}: \overline{\Omega}_A\mathbf{C} \to \overline{\Omega}_{A'}\mathbf{C}$ be its continuous morphic extension.

Let $\pi_q, \rho_s \in \overline{\Omega}_B\mathbf{S}$ ($q \in V(A)$, $s \in E(A)$) be such that $\mathbf{J} \models \pi_{\alpha s}\rho_s = \pi_{\omega s}$ for all $s \in E(A)$. Since A is strongly connected, we have, for each $q \in V(A)$, $\pi_q \,\mathcal{R}\, \pi_{q_0}$ in $\overline{\Omega}_B\mathbf{J}$, and hence $\mathbf{J} \models \pi_q = \pi_{q_0}$ for all $q \in V(A)$. In particular,

$$\mathbf{J} \models \pi_{q_0} = \pi_{q_0}\rho_s \quad \text{for all } s \in E(A). \tag{21}$$

Let $\varepsilon: \overline{\Omega}_A\mathbf{C} \to (\overline{\Omega}_B\mathbf{S})^1$ and $\varepsilon': \overline{\Omega}_{A'}\mathbf{C} \to (\overline{\Omega}_B\mathbf{S})^1$ be the continuous category morphisms defined by $\varepsilon(s) = \varepsilon'(s) = \rho_s$ for all $s \in E(A)$. Then $\varepsilon = \varepsilon' \circ \hat{\varphi}$. If we assume that S satisfies all pseudoidentities of the form (20) where, in addition, $u = v$ is a one-vertex category pseudoidentity satisfied by $g\mathbf{V}$, then $S \models \pi_{q_0}\varepsilon'(\hat{\varphi}u) = \pi_{q_0}\varepsilon'(\hat{\varphi}v)$ because of (21). So $S \models \pi_{q_0}\varepsilon(u) = \pi_{q_0}\varepsilon(v)$.

Thus, in case $\mathbf{V} \not\subseteq \mathbf{G}$, for the definition of $\mathbf{V} * \mathbf{J}$ by pseudoidentities of the form (20), we need only consider pseudoidentities $u = v$ over one-vertex graphs which hold in $g\mathbf{V}$. On the other hand, since every nontrivial pseudovariety of groups is local, the same restriction is clearly valid for $\mathbf{V} \subseteq \mathbf{G}$ in view of the remarks preceding Corollary 5.5. But, as was already observed in section 5 (following Theorem 5.3), those one-vertex pseudoidentities define the pseudovariety of categories $\ell\mathbf{V}$. So, if S is a finite semigroup, then $S \in \mathbf{V} * \mathbf{J}$ if and only if $D_{\gamma\circ\beta^{-1}} \in \ell\mathbf{V}$. Here γ and β are as in the proof of Theorem 5.2. Returning to the proof of Theorem 5.2, we conclude that $\mathbf{V} * \mathbf{J}$ is defined by the pseudoidentities of the form (20) induced by pseudoidentities in Σ, viewed as category pseudoidentities.

Finally, we show how to reduce a pseudoidentity of the form (20) with $(u = v) \in \Sigma$ to one of the form (19). By Lemma 8.2, $\pi_{\alpha u} = \pi'_{\alpha u}f$ for some $\pi'_{\alpha u}, f \in \overline{\Omega}_B\mathbf{S}$ with f idempotent on $\mathbf{J}$ and $c(\rho_s) \subseteq c(f)$ for all $s \in E(A)$ where $A = c(\{u, v\})$. By Lemma 8.3, there is a factorization $f = \lambda e \pi$ where e is idempotent (on all of $\mathbf{S}$) and $c(f) = c(e)$ (so that, in particular, $c(\pi) \subseteq c(e)$). Let $\vec{\rho} \in (\overline{\Omega}_B\mathbf{S})^{|E(A)|}$ have components the ρ_s ($s \in E(A)$). Then $\varepsilon(u) = u(\vec{\rho})$ and $\varepsilon(v) = v(\vec{\rho})$, so that (20) becomes

$$\pi'_{\alpha u}\lambda \cdot e\pi u(\vec{\rho}) = \pi'_{\alpha u}\lambda \cdot e\pi v(\vec{\rho}) \tag{22}$$

with e idempotent and $c(\pi) \cup c(\vec{\rho}) \subseteq c(e)$. To finish up the proof, just observe that (19) is a particular case of (22) (with, say $\pi'_{\alpha u} = \lambda = e$) which implies (22).$\square$

The analogous statement regarding the two-sided semidirect product is as follows.

Theorem 8.5. *Let* $\mathbf{V}$ *be a nontrivial monoidal pseudovariety of semigroups and let* Σ *be a basis of pseudoidentities for* $\mathbf{V}$. *Then* $\mathbf{V} ** \mathbf{J}$ *is defined by the pseudoidentities of the form*

$$e\pi u(\vec{\rho})\pi' e' = e\pi v(\vec{\rho})\pi' e' \tag{23}$$

with e *and* e' *idempotents,* $c(\pi) \cup c(\vec{\rho}) \subseteq c(e)$, $c(\pi') \cup c(\vec{\rho}) \subseteq c(e')$, *and* $(u = v) \in \Sigma$.

Proof. Consider first the case where $\mathbf{V} \not\subseteq \mathbf{G}$. By Theorems 2.9 and 6.3, we find that $\mathbf{V} ** \mathbf{J}$ is defined by the pseudoidentities of the form

$$\pi_{\alpha u}\varepsilon(u)\pi'_{\omega u} = \pi_{\alpha u}\varepsilon(v)\pi'_{\omega u} \tag{24}$$

where $u = v$ is a bonded category pseudoidentity of $g\mathbf{V}$, $A = c(\{u, v\})$, $\pi_q, \pi'_q, \rho_s \in \overline{\Omega}_B\mathbf{S}$ ($q \in V(A)$, $s \in E(A)$) are such that $\mathbf{J} \models \pi_{\alpha s}\rho_s = \pi_{\omega s}, \pi'_{\alpha s} = \rho_s\pi'_{\omega s}$ for all $s \in E(A)$, and $\varepsilon\colon \overline{\Omega}_A\mathbf{C} \to (\overline{\Omega}_B\mathbf{S})^1$ is the continuous category morphism defined by $\varepsilon(s) = \rho_s$ for all $s \in E(A)$.

Let $u_0 = v_0$ be a nontrivial bonded category pseudoidentity valid in $g\mathbf{V}$, let $A = c(\{u_0, v_0\})$ and let $q_0 = \alpha u_0$. Let also $\pi_q, \pi'_q, \rho_s \in \overline{\Omega}_B\mathbf{S}$ ($q \in V(A)$, $s \in E(A)$) be such that $\mathbf{J} \models \pi_{\alpha s}\rho_s = \pi_{\omega s}, \pi'_{\alpha s} = \rho_s\pi'_{\omega s}$ for all $s \in E(A)$. As in the proof of Theorem 8.4, it follows that $\mathbf{J} \models \pi_{q_0}\rho_s = \pi_{q_0}, \pi'_{q_0} = \rho_s\pi'_{q_0}$. In particular, for the definition of $\mathbf{V} ** \mathbf{J}$ by pseudoidentities of the form (24), we need only consider the pseudoidentities $u = v$ over one-vertex graphs which hold in $g\mathbf{V}$, and more precisely, it suffices to consider the pseudoidentities $(u = v) \in \Sigma$, viewed as one-vertex category pseudoidentities. This conclusion also holds if $\mathbf{V}$ is a nontrivial pseudovariety of groups just as in the one-sided case.

The translation of the pseudoidentities of the form (24) for such $(u = v)$ into pseudoidentities of the form (23) is effected as in Theorem 8.4, using right-left duals of Lemmas 8.1 and 8.2.□

Corollary 8.6. *For any nontrivial monoidal pseudovariety* $\mathbf{V}$ *of semigroups, we have* $\mathbf{V} * \mathbf{J} = \mathbf{V} * \mathbf{R}$.

Proof. The inclusion $\mathbf{V} * \mathbf{J} \subseteq \mathbf{V} * \mathbf{R}$ is obvious. For the reverse inclusion, just note that all pseudoidentities of the form (19) are satisfied by $\mathbf{V} * \mathbf{R}$, say by Theorem 5.2.□

Here are a few simple applications of Theorem 8.4. They are both fairly elementary and do not require the full power of Theorem 8.4.

Corollary 8.7 [13]. $\mathbf{Sl} * \mathbf{J} = \mathbf{R}$.

Proof. It is well-known, and amounts to an easy calculation, that $\mathbf{R} * \mathbf{R} = \mathbf{R}$ (this may also be obtained using Theorem 8.6). Hence $\mathbf{R} \subseteq \mathbf{Sl} * \mathbf{R} = \mathbf{Sl} * \mathbf{J} \subseteq \mathbf{R} * \mathbf{R} = \mathbf{R}$ and the result follows.□

Theorem 8.8. *Let* $\mathbf{V}$ *be a nontrivial monoidal pseudovariety of semigroups contained in* $\mathbf{DS}$*, and let* Σ *be a basis of pseudoidentities for* $\mathbf{V}$*, containing the pseudoidentity* $(xy)^\omega = ((xy)^\omega(yx)^\omega(xy)^\omega)^\omega$ *(which defines* $\mathbf{DS}$*). Then* *a)* $\mathbf{V} * \mathbf{J}$ *is defined by the pseudoidentities of the form* $w^\omega u(\vec{y}) = w^\omega v(\vec{y})$, *where* $w \in \overline{\Omega}_A\mathbf{S}$ $(u = v) \in \Sigma \cap (\overline{\Omega}_r\mathbf{S} \times \overline{\Omega}_r\mathbf{S})$, $\vec{y} \in (\overline{\Omega}_A\mathbf{S})^r$ *and* $c(\vec{y}) \subseteq c(w)$. *Equivalently,*

$$\begin{aligned}\mathbf{V} * \mathbf{J} \;=\; [\![&(y_1 \cdots y_r x)^\omega u(y_1, \ldots, y_r) = (y_1 \cdots y_r x)^\omega v(y_1, \ldots, y_r) : \\ &(u = v) \in \Sigma \cap (\overline{\Omega}_r\mathbf{S} \times \overline{\Omega}_r\mathbf{S})]\!].\end{aligned}$$

b) $\mathbf{V} ** \mathbf{J}$ *is defined by the pseudoidentities of the form* $w_1^\omega v w_2^\omega = w_1^\omega v w_2^\omega$, *where* $w_1, w_2 \in \overline{\Omega}_A\mathbf{S}$, $(u = v) \in \Sigma \cap (\overline{\Omega}_r\mathbf{S} \times \overline{\Omega}_r\mathbf{S})$, $\vec{y} \in (\overline{\Omega}_A\mathbf{S})^r$ *and* $c(\vec{y}) \subseteq c(w_1) \cap c(w_2)$. *Equivalently,*

$$\begin{aligned}\mathbf{V} ** \mathbf{J} \;=\; [\![&(y_1 \cdots y_r x)^\omega u(y_1, \ldots, y_r)(z y_1 \cdots y_r)^\omega = \\ &(y_1 \cdots y_r x)^\omega v(y_1, \ldots, y_r)(z y_1 \cdots y_r)^\omega : \\ &(u = v) \in \Sigma \cap (\overline{\Omega}_r\mathbf{S} \times \overline{\Omega}_r\mathbf{S})]\!].\end{aligned}$$

Proof. We give only the proof of statement (a). The proof of statement (b) is similar.

By Theorem 8.4, $\mathbf{V} * \mathbf{J}$ is defined by the pseudoidentities of the form

$$a^\omega b u(\vec{y}) = a^\omega b v(\vec{y}), \tag{25}$$

where $a, b \in \overline{\Omega}_A\mathbf{S}$, $(u = v) \in \Sigma \cap (\overline{\Omega}_r\mathbf{S} \times \overline{\Omega}_r\mathbf{S})$, $\vec{y} \in (\overline{\Omega}_A\mathbf{S})^r$ and $c(b) \cup c(\vec{y}) \subseteq c(a)$.

Let $\mathbf{W}$ be the pseudovariety defined by the pseudoidentities of the form $w^\omega u(\vec{y}) = w^\omega v(\vec{y})$, where $w \in \overline{\Omega}_A\mathbf{S}$, $(u = v) \in \Sigma \cap (\overline{\Omega}_r\mathbf{S} \times \overline{\Omega}_r\mathbf{S})$, $\vec{y} \in (\overline{\Omega}_A\mathbf{S})^r$ and $c(\vec{y}) \subseteq c(w)$. Let also

$$\begin{aligned}\mathbf{W}' \;=\; [\![&(y_1 \cdots y_r x)^\omega u(y_1, \ldots, y_r) = (y_1 \cdots y_r x)^\omega v(y_1, \ldots, y_r) : \\ &(u = v) \in \Sigma \cap (\overline{\Omega}_r\mathbf{S} \times \overline{\Omega}_r\mathbf{S}).]\!]\end{aligned}$$

The pseudoidentities defining $\mathbf{W}'$ are special cases of those defining $\mathbf{W}$, themselves special cases of pseudoidentities of the form (25). So $\mathbf{V} * \mathbf{J} \subseteq \mathbf{W} \subseteq \mathbf{W}'$.

By hypothesis, the pseudoidentity $(st)^\omega = ((st)^\omega (ts)^\omega (st)^\omega)^\omega$ is in Σ. So, letting $x = (st)^\omega$, $y_1 = s$ and $y_2 = t$, we find that $\mathbf{W}'$ satisfies $(st)^\omega = ((st)^\omega (ts)^\omega (st)^\omega)^\omega$, and hence $\mathbf{W}' \subseteq \mathbf{DS}$.

We must now verify that the pseudoidentities of the form (25) are valid in $\mathbf{W}'$. Let $a, b \in \overline{\Omega}_A\mathbf{S}$, $(u = v) \in \Sigma \cap (\overline{\Omega}_r\mathbf{S} \times \overline{\Omega}_r\mathbf{S})$, $\vec{y} \in (\overline{\Omega}_A\mathbf{S})^r$ and $c(b) \cup c(\vec{y}) \subseteq c(a)$. In $\mathbf{DS}$, we have (cf. [4, section 8.1])

$$\begin{aligned}a^\omega b = a^\omega b (a^\omega b)^\omega \;&=\; a^\omega b ((a^\omega b)^\omega y_1 \cdots y_r (a^\omega b)^\omega)^\omega \\ &=\; \Big(a^\omega b ((a^\omega b)^\omega y_1 \cdots y_r (a^\omega b)^\omega)^\omega\Big)\Big(y_1 \cdots y_r (a^\omega b)^\omega\Big)^\omega.\end{aligned}$$

Therefore, in $\mathbf{W}'$,

$$\begin{aligned} a^\omega bu(\vec{y}) &= \Big(a^\omega b((a^\omega b)^\omega y_1 \cdots y_r (a^\omega b)^\omega)^\omega\Big)\Big(y_1 \cdots y_r (a^\omega b)^\omega\Big)^\omega u(\vec{y}) \\ &= \Big(a^\omega b((a^\omega b)^\omega y_1 \cdots y_r (a^\omega b)^\omega)^\omega\Big)\Big(y_1 \cdots y_r (a^\omega b)^\omega\Big)^\omega v(\vec{y}) \\ &= a^\omega bv(\vec{y}). \end{aligned}$$

This concludes the proof.□

Easy applications of this result are presented in the next statement.

Corollary 8.9. *The following formulæ hold.*

$$\begin{aligned} \mathbf{DS} * \mathbf{J} &= \mathbf{DS} ** \mathbf{J} = \mathbf{DS} \\ \mathbf{DA} * \mathbf{J} &= \mathbf{DA} ** \mathbf{J} = \mathbf{DA} \\ \mathbf{DO} * \mathbf{J} &= \mathbf{DO} ** \mathbf{J} = \mathbf{DO} \\ \mathbf{DG} ** \mathbf{J} &= [\![(xyz)^\omega (xy)^\omega (txy)^\omega = (xyz)^\omega (yx)^\omega (txy)^\omega]\!] \\ &= [\![u^\omega v^\omega w^\omega = u^\omega v'^\omega w^\omega : c(v) = c(v') \subseteq c(u) \cap c(w)]\!] \\ \mathbf{J} ** \mathbf{J} &= (\mathbf{DG} ** \mathbf{J}) \cap \mathbf{A} \\ \mathbf{Sl} ** \mathbf{J} &= [\![(xyz)^\omega xy(txy)^\omega = (xyz)^\omega yx(txy)^\omega,\ (xz)^\omega x^2 (tx)^\omega = (xz)^\omega x(tx)^\omega]\!] \\ &= [\![u^\omega vw^\omega = u^\omega v'w^\omega : c(v) = c(v') \subseteq c(u) \cap c(w)]\!] \\ \mathbf{G} ** \mathbf{J} &= [\![(xz)^\omega x^\omega (tx)^\omega = (xz)^\omega (tx)^\omega]\!] \\ \mathbf{G} * \mathbf{J} &= [\![(xz)^\omega x^\omega = (xz)^\omega]\!].\square \end{aligned}$$

Note that $\mathbf{Sl}**\mathbf{J}$ is equal to $\Diamond_2\mathbf{J}$, which is the first level of an important subhierarchy of the dot-depth hierarchy (cf. [32]). For the semidirect product $\mathbf{G} * \mathbf{J}$, we can be a bit more precise.

Corollary 8.10. $\mathbf{G} * \mathbf{J} = [\![(xy)^\omega x^\omega = (xy)^\omega]\!] = [\![(xy)^\omega (yx)^\omega = (xy)^\omega]\!] = \mathbf{D}(\mathbf{G} \vee \mathbf{K}_1) = \mathbf{DG} * \mathbf{J}$.

Proof. The first equality is part of Corollary 8.9. If a and b are inverses of each other in a semigroup S which lies in the third pseudovariety, then $(ab)^\omega (ba)^\omega = (ab)^\omega$, which implies that a and b are $\mathcal{L}$-equivalent and so regular $\mathcal{D}$-classes of S are left groups; whence $S \in \mathbf{D}(\mathbf{G} \vee \mathbf{K}_1)$. Next, if $S \in \mathbf{D}(\mathbf{G} \vee \mathbf{K}_1)$, then, for all $a, b \in S$, $(ab)^\omega a^\omega$ and $(ab)^\omega$ are two $\mathcal{R}$-equivalent idempotents and, therefore they are equal. To conclude the proof, just observe that, by Theorem 8.8(a), $\mathbf{DG} * \mathbf{J}$ satisfies the pseudoidentity $(xyz)^\omega (yx)^\omega = (xyz)^\omega (xy)^\omega$ and take $z = (xy)^\omega$ to deduce the pseudoidentity $(xy)^\omega (yx)^\omega = (xy)^\omega$.□

For a pseudovariety $\mathbf{H}$ of groups, let $\overline{\mathbf{H}}$ be the pseudovariety of all finite semigroups all of whose subgroups lie in $\mathbf{H}$.

Theorem 8.11. *For every nontrivial pseudovariety* $\mathbf{H}$ *of groups,* $\mathbf{H}*\mathbf{J} = \mathbf{D}(\mathbf{H}\vee\mathbf{K}_1) = \mathbf{D}(\mathbf{G}\vee\mathbf{K}_1)\cap\overline{\mathbf{H}}$.

Proof. Since all subgroups are situated in regular $\mathcal{D}$-classes, the second equality is clear. Now, by Theorem 8.4 and the usual arguments when $\mathbf{J}$ is involved, $\mathbf{H}*\mathbf{J}$ is defined by the pseudoidentities of the form

$$e\pi = e\rho \quad \text{where } e \text{ is idempotent over } \mathbf{J},\ c(\pi)\cup c(\rho)\subseteq c(e) \text{ and } \mathbf{H}\models \pi=\rho. \tag{26}$$

In particular, all groups in $\mathbf{H}*\mathbf{J}$ lie in $\mathbf{H}$. Since clearly $\mathbf{H}*\mathbf{J}\subseteq\mathbf{G}*\mathbf{J}$, it remains to be shown that $\mathbf{D}(\mathbf{H}\vee\mathbf{K}_1)$ satisfies every pseudoidentity in (26).

Take any pseudoidentity in (26), say with $\pi=\pi(x_1,\ldots,x_n)$ and $\rho=\rho(x_1,\ldots,x_n)$. Then, since each $\mathcal{R}$-class in a regular $\mathcal{D}$-class of a semigroup $S\in\mathbf{D}(\mathbf{G}\vee\mathbf{K}_1)$ is a group, in such a semigroup we have $eu = eue$ whenever u is a word such that $c(u)\subseteq c(e)$. It follows that, in S,

$$e\pi(x_1,\ldots,x_n) = \pi(ex_1,\ldots,ex_n) = \rho(ex_1,\ldots,ex_n) = e\rho(x_1,\ldots,x_n),$$

where the second equality is valid since, in S, $ex_1,\ldots,ex_n$ always assume values in the same group which, by hypothesis, lies in $\mathbf{H}$.□

Corollary 8.12. *Let* $\mathbf{H}$ *be a nontrivial pseudovariety of groups. Then, for a pseudovariety* $\mathbf{V}$ *of semigroups,* $\mathbf{V}*\mathbf{J} = \mathbf{D}(\mathbf{H}\vee\mathbf{K}_1)$ *if and only if* $\mathbf{H}\subseteq\mathbf{V}$ *and* $\mathbf{V}\subseteq\mathbf{D}(\mathbf{H}\vee\mathbf{K}_1)$.

Proof. By Corollary 8.6, Theorem 8.11 and the associativity of the semidirect product of pseudovarieties, we have

$$\mathbf{D}(\mathbf{H}\vee\mathbf{K}_1)*\mathbf{J} = (\mathbf{H}*\mathbf{R})*\mathbf{R} = \mathbf{H}*(\mathbf{R}*\mathbf{R}) = \mathbf{H}*\mathbf{R} = \mathbf{D}(\mathbf{H}\vee\mathbf{K}_1).$$

In view of Theorem 8.11, this shows that $\mathbf{V}*\mathbf{J} = \mathbf{D}(\mathbf{H}\vee\mathbf{K}_1)$ if $\mathbf{H}\subseteq\mathbf{V}$ and $\mathbf{V}\subseteq\mathbf{D}(\mathbf{H}\vee\mathbf{K}_1)$. Conversely, if $\mathbf{V}*\mathbf{J}=\mathbf{D}(\mathbf{H}\vee\mathbf{K}_1)$, then clearly $\mathbf{V}\subseteq\mathbf{V}*\mathbf{J}\subseteq\mathbf{D}(\mathbf{H}\vee\mathbf{K}_1)\subseteq\overline{\mathbf{H}}$. If $\mathbf{H}'=\mathbf{V}\cap\mathbf{G}$, then $\mathbf{V}\subseteq\mathbf{D}(\mathbf{H}\vee\mathbf{K}_1)\cap\overline{\mathbf{H}'} = \mathbf{D}(\mathbf{H}'\vee\mathbf{K}_1)$ and so $\mathbf{V}*\mathbf{J}\subseteq\mathbf{D}(\mathbf{H}'\vee\mathbf{K}_1)$ by the already proved part of the statement. Consequently, $\mathbf{H}'=\mathbf{H}$ and $\mathbf{H}\subseteq\mathbf{V}$.□

A somewhat simpler situation arises when the second factor in the semidirect product is the pseudovariety $\mathbf{Sl}$.

Theorem 8.13. *a)* $\mathbf{G}*\mathbf{Sl} = [\![xyx^\omega = xy,\ x^{\omega+1}=x]\!]$.

b) $\mathbf{G}**\mathbf{Sl} = [\![xyx^\omega zx = xyzx,\ xyx^{\omega+1} = x^{\omega+1}yx = xyx,\ x^{\omega+2}=x^2]\!]$.

Proof. We prove only (b), the case of (a) being similar.

By Corollary 6.4, a finite semigroup S lies in $\mathbf{G} ** \mathbf{Sl}$ if and only if

$$\mathbf{Sl} \models xy = x, x' = yx' \quad \Longrightarrow \quad S \models xy^\omega x' = xx',$$

that is,

$$\mathbf{G} ** \mathbf{Sl} = [\![uw^\omega v = uv : c(w) \subseteq c(u) \cap c(v)]\!].$$

In particular, $\mathbf{G} ** \mathbf{Sl}$ satisfies $xyx^\omega zx = xyzx$, $xyx^{\omega+1} = x^{\omega+1}yx = xyx$, and $x^{\omega+2} = x^2$. Let now S be a finite semigroup satisfying these pseudoidentities. We want to show that $S \in \mathbf{G} ** \mathbf{Sl}$.

Let $u, v \in \overline{\Omega}_A\mathbf{S}$ and let $a \in c(u) \cap c(v)$. Then $u = u_1au_2$, $v = v_1av_2$ and

$$S \models uv = u_1au_2v_1av_2 = u_1au_2a^\omega v_1av_2 = ua^\omega v.$$

Suppose now that for all $u, v \in \overline{\Omega}_A\mathbf{S}$, and for all words w of length less than k,

$$c(w) \subseteq c(u) \cap c(v) \quad \Longrightarrow \quad S \models uv = uw^\omega v. \tag{27}$$

Let $a \in c(u) \cap c(v)$. Then

$$\begin{aligned}
S \models uv &= uw^\omega w^\omega v \qquad \text{by (27)}\\
&= uw^\omega a^\omega w^\omega v \qquad \text{by (27)}\\
&= uw^{\omega-1}(wa)a^{\omega-1}w^{\omega-1}(wa)a^{\omega-1}v\\
&= uw^{\omega-1}(wa)a^{\omega-1}(wa)^\omega w^{\omega-1}(wa)a^{\omega-1}v \qquad \text{by hypothesis on } S\\
&= uw^\omega a^\omega (wa)^\omega w^\omega a^\omega v\\
&= u(wa)^\omega v \qquad \text{by (27)}.
\end{aligned}$$

Thus, by induction, $S \models uw^\omega v = uv$ for all words w such that $c(w) \subseteq c(u) \cap c(v)$. By continuity, this implies that $S \models uw^\omega v = uv$ for all implicit operations w such that $c(w) \subseteq c(u) \cap c(v)$.□

The following corollary may be proved in just the same way as Theorem 8.11.

Corollary 8.14. *For every nontrivial pseudovariety* $\mathbf{H}$ *of groups,*

a) $\mathbf{H} * \mathbf{Sl} = (\mathbf{G} * \mathbf{Sl}) \cap \overline{\mathbf{H}}$.

b) $\mathbf{H} ** \mathbf{Sl} = (\mathbf{G} ** \mathbf{Sl}) \cap \overline{\mathbf{H}}$.□

For the particular case where the second factor in the semidirect product is a pseudovariety of groups, Corollary 5.4 has a rather simple formulation. Recall that, in this case the one-sided and two-sided semidirect products coincide (cf. [26]).

Theorem 8.15. *Let* $\mathbf{V}$ *be a local monoidal pseudovariety of semigroups with a given basis* Σ *of pseudoidentities and let* $\mathbf{H}$ *be a pseudovariety of groups. Then a finite semigroup* S *lies in* $\mathbf{V} * \mathbf{H}$ *if and only if, for every* $(u = v) \in \Sigma$ *with* $u, v \in \overline{\Omega}_r\mathbf{S}$ *and all* $\vec{\pi} \in (\overline{\Omega}_A\mathbf{S})^r$,

$$\mathbf{H} \models \pi_1 = \cdots = \pi_r = 1 \quad \Longrightarrow \quad S \models u(\vec{\pi}) = v(\vec{\pi}). \tag{28}$$

Proof. Note that $\mathbf{V} * \mathbf{H}$ is a monoidal pseudovariety. Hence $S \in \mathbf{V} * \mathbf{H}$ if and only if $S^1 \in \mathbf{V} * \mathbf{H}$. Now, in the left side of (14) for $\mathbf{W} = \mathbf{H}$, we have a condition equivalent to the left side of (28) which, therefore, imposes no conditions on ρ. In particular, we may take $\rho = 1$ since the preceding observation reduces the problem to the monoid case.□

Let $\mathbf{V} \textcircled{m} \mathbf{H}$ denote the pseudovariety of semigroups generated by all monoids M such that there is a homomorphism $\varphi : M \to G \in \mathbf{H}$ satisfying $\varphi^{-1}(1) \in \mathbf{V}$. Equivalently, $S \in \mathbf{V} \textcircled{m} \mathbf{H}$ if and only if there is a relational morphism $\psi : S^1 \to G \in \mathbf{H}$ such that $\psi^{-1}(1) \in \mathbf{V}$. For a finite semigroup S, let $K_{\mathbf{H}}(S)$ be the subset of S^1 obtained by taking the intersection of all submonoids of the form $\psi^{-1}(1)$ where $\psi : S^1 \to G \in \mathbf{H}$ is a relational morphism. Then yet another definition of $\mathbf{V} \textcircled{m} \mathbf{H}$ is obtained by stating that $S \in \mathbf{V} \textcircled{m} \mathbf{H}$ if and only if $K_{\mathbf{H}}(S) \in \mathbf{V}$. It is also important to note that, for a finite semigroup S, there is always some $G \in \mathbf{H}$ and some relational morphism $\psi : S' \to G$ such that $\psi^{-1}(1) = K_{\mathbf{H}}(S)$ [24].

While, for every pseudovariety $\mathbf{V}$ of semigroups, it is easy to show that $\mathbf{V} * \mathbf{H} \subseteq \mathbf{V} \textcircled{m} \mathbf{H}$, in the local monoidal case we have equality. This is stated in the following result which follows from the elementary observation that, for a monoidal pseudovariety $\mathbf{V}$, a finite semigroup S lies in $\mathbf{V} \textcircled{m} \mathbf{H}$ if and only if there is a relational morphism $R : S^1 \to G \in \mathbf{H}$ such that $D_R \in \ell\mathbf{V}$ (cf. [21]). A more syntactical argument is sketched below.

Proposition 8.16. *Let* $\mathbf{V}$ *be a local monoidal pseudovariety of semigroups. Then* $\mathbf{V} * \mathbf{H} = \mathbf{V} \textcircled{m} \mathbf{H}$ *for every pseudovariety* $\mathbf{H}$ *of groups.*□

Another proof of Proposition 8.16 is obtained by observing that the validity of (28) for all $\vec{\pi} \in (\overline{\Omega}_A\mathbf{S})^r$ amounts to saying that $K_{\mathbf{H}}(S) \models u = v$. For this purpose, just note that every relational morphism $S^1 \to G \in \mathbf{H}$ is of the form $\gamma \circ \beta^{-1}$ for some continuous homomorphisms $\beta : (\overline{\Omega}_A\mathbf{S})^1 \to S^1$ and $\gamma : (\overline{\Omega}_A\mathbf{S})^1 \to G$. Indeed this follows easily from the universal property of $(\overline{\Omega}_A\mathbf{S})^1$.

Taking into account a recent result of Ribes and Zalesskiĭ [28], we obtain, for example, the following decidability result.

Corollary 8.17. *Let* $\mathbf{V}$ *be a local monoidal pseudovariety of semigroups which has a decidable membership problem and let* $\mathbf{G}_p$ *be the pseudovariety of all finite* p*-groups. Then* $\mathbf{V} * \mathbf{G}_p$ *also has a decidable membership problem.*

Proof. In [28], Ribes and Zalesskiĭ exhibit, for such pseudovarieties $\mathbf{H}$ of groups, an algorithm to compute $K_{\mathbf{H}}(S)$ for a given finite semigroup S. Thus, to decide whether $S \in \mathbf{V} * \mathbf{H}$, compute the monoid $K_{\mathbf{H}}(S)$ and check whether it belongs to $\mathbf{V}$.□

It would be useful to translate the Ribes and Zalesskiĭ algorithms into specific bases of pseudoidentities for various semidirect products of the form $\mathbf{V} * \mathbf{H}$. For example, under the conditions of Corollary 8.17, do we have

$$\mathbf{Sl} * \mathbf{H} = \mathbf{ECom} \cap \overline{\mathbf{H}} \tag{29}$$

where $\mathbf{EV}$ denotes the pseudovariety consisting of all finite semigroups whose idempotents generate subsemigroups in $\mathbf{V}$? This relation is known to hold when $\mathbf{H} = \mathbf{G}$ [9] and to fail when $\mathbf{H}$ is the pseudovariety of all finite abelian groups [23]. Using inverse semigroups, Stuart Margolis (private communication) has further shown that there are other counter-examples to (29), besides the obvious case when $\mathbf{H}$ is the trivial pseudovariety $\mathbf{I} = [\![x = 1]\!]$. While $\mathbf{R} * \mathbf{G} = \mathbf{ER}$ [15], Margolis's examples show that $\mathbf{ER} \cap \overline{\mathbf{I}} \not\subseteq \mathbf{R} * \mathbf{H}$ unless $\mathbf{H} = \mathbf{G}$.

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