Semigroup Forum
Springer-Verlag New York Inc. Version of October 2, 1997

On radical congruence systems[1]

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Communicated by J.-E. Pin

Résumé

Nous caractérisons les systèmes radicaux de congruences sur les semigroupes finis, qui sont en même temps des systèmes de plus grandes congruences sur une pseudovariété fixée.

Abstract

We characterize all the radical congruence systems for finite semigroups which are systems of greatest congruences over some pseudovariety.

A *congruence system* is a system of congruences $\rho = (\rho(S))_S$ defined on all finite semigroups S. *Radical congruence systems* for finite semigroups were introduced by Auinger, Hall, Reilly and Zhang [3], following work by Petrich and Reilly [9] and Reilly and Zhang [13]. They are the congruence systems which satisfy the following four properties (in this paper all semigroups are assumed to be finite, unless otherwise indicated):

C1 For each S, $\rho(S/\rho(S))$ is the trivial congruence on $S/\rho(S)$.
C2 For all semigroups S and T, $\rho(S)\times\rho(T)$ is contained in $\rho(S\times T)$.
C3 If S is a subsemigroup of T, then the restriction of $\rho(T)$ to S, $\rho(T)_{|S}$, is contained in $\rho(S)$.
C4 If $\varphi: S\to T$ is an onto morphism, then $s\,\rho(S)\,s'$ implies $\varphi(s)\,\rho(T)\,\varphi(s')$.

In [3], it was shown that the systems of greatest congruences over the following pseudovarieties form radical congruence systems: $\mathbf{N}$, $\mathbf{D}$, $\mathbf{K}$, $\mathbf{LI}$, $\mathbf{IE}$, $\mathbf{D}\vee\mathbf{G}$, $\mathbf{K}\vee\mathbf{G}$ and $\mathbf{LG}$, respectively the pseudovarieties of nilpotent semigroups, definite semigroups, reverse definite semigroups, locally trivial semigroups, unipotent semigroups, the joins of $\mathbf{D}$ and $\mathbf{K}$ with the pseudovariety $\mathbf{G}$ of groups, and the pseudovariety of semigroups that are locally groups. These results have important consequences on the decidability of the pseudovarieties of the form $\mathbf{V} \textcircled{m} \mathbf{W}$, and on the solution sets of equations of the form $\mathbf{V} \textcircled{m} \mathbf{X} = \mathbf{W}$ ($\mathbf{W}$ fixed) when $\mathbf{V} \in \{\mathbf{N},\mathbf{K},\mathbf{D},\mathbf{LI},\mathbf{IE},\mathbf{K}\vee\mathbf{G},\mathbf{D}\vee\mathbf{G},\mathbf{LG}\}$ ([3], see also Section 2.2 below). For these values of $\mathbf{V}$, the operations $\mathbf{X}\mapsto\mathbf{V} \textcircled{m} \mathbf{X}$ on the lattice of pseudovarieties play a particularly important role, as can be seen in [3, 8, 9, 17].

In this paper, we characterize all those radical congruence systems which are, like the above examples, systems of greatest congruences over some pseudovariety. All semigroups considered in this paper are finite.

[1] The second author gratefully acknowledges support from Australian Research Council Grant A69231516, and from the Projet de Recherche Coordonnée "Mathématique et Informatique".

1. Preliminaries

Let us first review some elementary definitions and results on pseudovarieties and congruences.

1.1 Pseudovarieties and Mal'cev products

A pseudovariety of semigroups is a class of finite semigroups which is closed under taking subsemigroups, homomorphic images and finite direct products. The following classes, each a pseudovariety, will play an important role in this paper (for the description by pseudoidentities, see for instance [10, 1, 2]):

- $\mathbf{S} = [\![x = x]\!]$, the class of all finite semigroups;
- $\mathbf{I} = [\![x = y]\!]$, the trivial pseudovariety, consisting only of the 1-element semigroups;
- $\mathbf{G} = [\![x^\omega y = yx^\omega = y]\!]$, the pseudovariety of all finite groups;
- For each pseudovariety $\mathbf{H}$ contained in $\mathbf{G}$ (we say that $\mathbf{H}$ is a *pseudovariety of groups*), $\overline{\mathbf{H}}$ is the class of semigroups all of whose subgroups lie in $\mathbf{H}$;
- $\mathbf{SL} = [\![x^2 = x,\ xy = yx]\!]$, the pseudovariety of idempotent and commutative semigroups, that is, semilattices;
- $\mathbf{CS} = [\![(xyx)^\omega = x^\omega,\ x^\omega x = x]\!]$, the pseudovariety of completely simple semigroups;
- $\mathbf{Null} = [\![xy = zt]\!]$, the pseudovariety of null semigroups;
- $\mathbf{N} = [\![x^\omega y = yx^\omega = x^\omega]\!]$, the pseudovariety of nilpotent semigroups;
- $\mathbf{D} = [\![yx^\omega = x^\omega]\!]$, the pseudovariety of definite semigroups;
- $\mathbf{K} = [\![x^\omega y = x^\omega]\!]$, the pseudovariety of reverse definite semigroups;
- $\mathbf{LI} = [\![x^\omega yx^\omega = x^\omega]\!]$, the pseudovariety of locally trivial semigroups;
- $\mathbf{IE} = [\![x^\omega = y^\omega]\!]$, the pseudovariety of unipotent semigroups;
- $\mathbf{D} \vee \mathbf{G} = [\![y^\omega x^\omega = x^\omega]\!]$;
- $\mathbf{K} \vee \mathbf{G} = [\![x^\omega y^\omega = x^\omega]\!]$ (see Section 3.1);
- $\mathbf{LG} = [\![(x^\omega yx^\omega)^\omega = x^\omega]\!]$, the pseudovariety of semigroups that are locally groups. If $\mathbf{H}$ is a pseudovariety of groups, then $\mathbf{LH} = \mathbf{LG} \cap \overline{\mathbf{H}}$.

If $\mathbf{V}$ and $\mathbf{W}$ are pseudovarieties of semigroups, we denote by $\mathbf{V} ⓜ \mathbf{W}$ the Mal'cev product of $\mathbf{V}$ and $\mathbf{W}$, that is, the pseudovariety generated by the semigroups S such that there exists a morphism $\beta\colon S \to W$ with $W \in \mathbf{W}$ and $\beta^{-1}(e) \in \mathbf{V}$ for each idempotent e of W. It is easy to verify [18] that a semigroup $T \in \mathbf{V} ⓜ \mathbf{W}$ if and only if there exist morphisms $T \overset{\alpha}{\leftarrow} S \overset{\beta}{\rightarrow} W$ such that α is onto, $W \in \mathbf{W}$, and $\beta^{-1}(e) \in \mathbf{V}$ for each idempotent e of W.

Lemma 1.1. *Let $\mathbf{V}$ be a pseudovariety.*
(1) *A semigroup S lies in $\mathbf{V} ⓜ \mathbf{N}$ if and only if the ideal generated by the regular elements of S lies in $\mathbf{V}$.*
(2) *If $\mathbf{V} \subseteq \mathbf{LG}$, then $\mathbf{V} \subseteq (\mathbf{V} \cap \mathbf{CS}) ⓜ \mathbf{N}$.*

Proof. For Statement (1), see [12]. For Statement (2), it suffices to observe that $\mathbf{LG}$ is the class of semigroups in which the regular elements form the minimum ideal, and hence form a completely simple subsemigroup. The result then follows from (1). ∎

1.2 Congruences

The following lemma, which can be found in [4], will be useful in the sequel.

Lemma 1.2. *Let S be a semigroup and let $(\rho_i)_{i \in I}$ be a family of congruences on S. Then the least equivalence relation on S containing the ρ_i is a congruence.* ∎

This lemma will be used in the following well known form:

Corollary 1.3. *Let S be a semigroup and let $\mathcal{P}$ be a partition of S. Then there exists a greatest congruence on S which is contained in $\mathcal{P}$.* ∎

2. Radical and weakly radical congruence systems

The definition of a radical congruence system was given in the introduction. In view of the special role played by Condition C1, we introduce the notion of *weakly radical congruence systems*: these are the congruence systems which satisfy Conditions C2, C3 and C4.

From Condition C4, we obtain $\rho(S \times T) \subseteq \rho(S) \times \rho(T)$, by considering the projections of $S \times T$ onto S and T. Thus, when in conjunction with C4, Condition C2 can be replaced by
C′2 For all semigroups S and T, $\rho(S) \times \rho(T) = \rho(S \times T)$.

Examples 2.1. The universal (respectively trivial) congruence system, where $\rho(S)$ is the universal (respectively trivial) congruence on S for each semigroup S, is a radical congruence system. Other examples and counter-examples are given further in the paper.

2.1 Universal class of a weakly radical congruence system

Let ρ be a congruence system. A semigroup S is *ρ-universal* if $\rho(S)$ is the universal congruence on S. As in [3], the class of ρ-universal semigroups is called the *universal class* of ρ, and is denoted $\mathbf{V}(\rho)$.

Lemma 2.2. *Let ρ be a weakly radical congruence system.*
(1) *A semigroup S is ρ-universal if and only if there exists a semigroup T and an idempotent e of T such that S is isomorphic to the $\rho(T)$-class of e.*
(2) *$\mathbf{V}(\rho)$ is a pseudovariety.*

Proof. (1) If S is ρ-universal, then S is the $\rho(S)$-class of any one of its idempotents. Conversely, suppose that there exists a semigroup T and an idempotent e of T such that S is isomorphic to the $\rho(T)$-class C_e of e. Then the restriction of $\rho(T)$ to C_e is universal, so by Condition C3, C_e is ρ-universal. It now follows from C4 that S itself is ρ-universal.

(2) This follows immediately from the definition of weakly radical congruence systems: If S and T are ρ-universal, then $\rho(S \times T) = \rho(S) \times \rho(T)$ is the universal congruence on $S \times T$. Similarly, if S is a subsemigroup of T and if T is ρ-universal, then $\rho(S)$, which contains the restriction of $\rho(T)$ to S, is the universal congruence. Finally, if $\varphi\colon S \to T$ is an onto morphism and if S is ρ-universal, then $\rho(T)$, which contains the φ-image of $\rho(S)$, is universal as well. ∎

We say that a congruence σ on a semigroup S is *over* a pseudovariety $\mathbf{V}$ if each idempotent class of σ (a subsemigroup of S) is in $\mathbf{V}$. A congruence system ρ is *over* $\mathbf{V}$ if each congruence $\rho(S)$ is over $\mathbf{V}$. In particular, if ρ is a weakly radical congruence system, then it is a system of congruences over its universal class $\mathbf{V}(\rho)$.

We give some restriction on the possible values of $\mathbf{V}(\rho)$.

Proposition 2.3. *If ρ is a weakly radical congruence system and if* $\mathbf{SL} \subseteq \mathbf{V}(\rho)$*, then ρ is the universal congruence system.*

Proof. Let S be a semigroup and let $T = \{0, 1\}$. Then T is a semilattice, so T is ρ-universal. It follows that $\rho(S \times T) = \rho(S) \times \rho(T)$ is such that $(s,t)\,\rho(S \times T)\,(s',t')$ if and only if $s\,\rho(S)\,s'$. Let U be the quotient of $S \times T$ by the ideal $S \times \{0\}$, that is, $U = (S \times \{1\}) \cup \{0\}$, and let φ be the natural morphism from $S \times T$ onto U. Then, for each $s \in S$, we have $\varphi(s,1)\,\rho(U)\,\varphi(s,0)$, i.e. $(s,1)\,\rho(U)\,0$. Thus U is ρ-universal. But S is isomorphic to a subsemigroup of U, so S is ρ-universal as well: we have proved that every finite semigroup is ρ-universal, and hence ρ is the universal congruence system. ∎

Corollary 2.4. *If ρ is a weakly radical congruence system which is not the universal congruence system, then*
(1) $\mathbf{V}(\rho) \subseteq \mathbf{LG}$*;*
(2) $\mathbf{V}(\rho) \cap \mathbf{CS} \subseteq \mathbf{V}(\rho) \subseteq (\mathbf{V}(\rho) \cap \mathbf{CS}) \circledm \mathbf{N}$.

Proof. (1) $\mathbf{LG}$ is the largest pseudovariety which does not contain all semilattices.

(2) This follows from (1) and Lemma 1.1. ∎

Proposition 2.5. *Let ρ be a weakly radical congruence system with* $\mathbf{V}(\rho) \neq \mathbf{I}$*. Then*
(1) $\mathbf{Null} \subseteq \mathbf{V}(\rho)$*;*
(2) *$\rho(S)$ is non-trivial for any non-trivial nilpotent semigroup S.*

Proof. (1) Recall that $\mathbf{Null}$ is generated by the semigroup $N = \{a, 0\}$, where $a^2 = 0$. Let S be a non-trivial member of $\mathbf{V}(\rho)$. Then, for all $s, t \in S$, we have $(s,a)\,\rho(S \times N)\,(t,a)$. For each $s \in S$, let U_s be the quotient of $S \times N$ by the ideal $(S \times N) \setminus \{(s,a)\}$, that is, $U_s = \{(s,a), 0\}$, and let φ_s be the natural morphism from $S \times N$ onto U_s. Let $t \in S$ be different from s. Then, $(s,a) = \varphi_s(s,a)\ \rho(U_s)\ \varphi_s(t,a) = 0$, so U_s is ρ-universal. But U_s is isomorphic to N, so $N \in \mathbf{V}(\rho)$, and hence $\mathbf{Null} \subseteq \mathbf{V}(\rho)$.

(2) Let S be a non-trivial nilpotent semigroup and let $b \in S$ be such that $b \neq 0$ and $bS = Sb = \{0\}$ (say b lies in some 0-minimal ideal of S). Within $S \times N$, we consider the subsemigroup $U = (S \times \{0\}) \cup \{(0,a)\}$. Finally, we consider the morphism φ defined on U by

$$\begin{array}{rcl} \varphi : U & \longrightarrow & S \\ (s,0) & \longmapsto & s \\ (0,a) & \longmapsto & b \end{array}$$

Then $(0,a)\,\rho(S \times N)\,(0,0)$ by (1), so $(0,a)\,\rho(U)\,(0,0)$. Now this implies that $\varphi(0,a)\,\rho(S)\,\varphi(0,0)$; that is, $b\,\rho(S)\,0$. ∎

Corollary 2.6. *If ρ is a radical congruence system and* $\mathbf{V}(\rho) \neq \mathbf{I}$*, then* $\mathbf{N} \subseteq \mathbf{V}(\rho)$.

Proof. Since ρ is a radical congruence system, ρ satisfies Condition C1, and for each nilpotent semigroup S, the congruence $\rho(S/\rho(S))$ is trivial. So $S/\rho(S)$ is trivial, by Proposition 2.5. ∎

Example 2.7. Hall and Zhang exhibited an example of a radical congruence system whose universal class is $\mathbf{K}$, but which is not the system of greatest congruences over $\mathbf{K}$ [5]. For each finite semigroup S, the relation $\rho(S)$ is defined on S by

$$a\,\rho(S)\,b \quad \text{if and only if} \quad exa = exb \text{ for all } e \in E(S),\ x \in S.$$

It can be verified that $\rho = (\rho(S))_S$ is a radical congruence system and that $\mathbf{V}(\rho) = \mathbf{K}$. To verify that ρ is not the system of greatest congruences over $\mathbf{K}$, it suffices to consider the monoid $S = \{1, a, b\}$ given by $aS = a$ and $bS = b$: then $\rho(S)$ is the trivial congruence, whereas the partition $\{\{1\}, \{a, b\}\}$ corresponds to a congruence on S over $\mathbf{K}$.

2.2 Systems of greatest congruences over a pseudovariety

We first observe that systems of greatest congruences over a pseudovariety always satisfy some of the conditions defining weakly radical congruence systems.

Proposition 2.8. *Let $\mathbf{V}$ be a pseudovariety such that every semigroup S admits a greatest congruence over $\mathbf{V}$, and let ρ be the resulting congruence system. Then ρ satisfies Conditions C2 and C3.*

Moreover, ρ satisfies Condition C1 if and only if $\mathbf{V} \circledm \mathbf{V} = \mathbf{V}$.

Proof. Let S be a subsemigroup of a semigroup T. Then the restriction of $\rho(T)$ to S is trivially a congruence over $\mathbf{V}$, and hence it is contained in the greatest congruence on S over $\mathbf{V}$, namely $\rho(S)$. Thus ρ satisfies Condition C3.

Let now S and T be semigroups. Then $\rho(S) \times \rho(T)$ is a congruence on $S \times T$ which is over $\mathbf{V}$, so it is contained in $\rho(S \times T)$. Thus ρ satisfies Condition C2.

Let us first assume that ρ satisfies Condition C1. If $S \in \mathbf{V} \circledm \mathbf{V}$, then there exist morphisms $S \xleftarrow{\alpha} T \xrightarrow{\beta} V$ such that α is onto, $V \in \mathbf{V}$, and $\beta^{-1}(e) \in \mathbf{V}$ for each idempotent e of V. Then the congruence induced by β on T is over $\mathbf{V}$, and hence is contained in $\rho(T)$. This implies that $T/\rho(T)$ is a quotient of $\beta(T) \subseteq V$, and hence $T/\rho(T) \in \mathbf{V}$. Therefore $\rho(T/\rho(T))$ is simultaneously trivial and universal, that is, $T/\rho(T)$ is trivial. Thus T is ρ-universal, i.e. $T \in \mathbf{V}$. Finally, this implies $S = \alpha(T) \in \mathbf{V}$.

Conversely, let us assume that $\mathbf{V} = \mathbf{V} \circledm \mathbf{V}$. Let β and γ be the natural morphisms $\beta\colon S \to S/\rho(S) = T$ and $\gamma\colon T \to T/\rho(T)$. Then for each idempotent e of $T/\rho(T)$, the inverse image $\beta^{-1}(\gamma^{-1}(e))$ lies in $\mathbf{V} \circledm \mathbf{V}$, since $\rho(T)$ is over $\mathbf{V}$ and since the restriction of $\rho(S)$ to $\beta^{-1}(\gamma^{-1}(e))$ is over $\mathbf{V}$ as well. It follows that the congruence on S induced by the morphism $\gamma \circ \beta$ is over $\mathbf{V} \circledm \mathbf{V} = \mathbf{V}$, and hence is contained in $\rho(S)$. Thus γ is an isomorphism, that is, $\rho(T) = \rho(S/\rho(S))$ is trivial, which completes the proof. ∎

Note however that not all systems of greatest congruences over a pseudovariety are weakly radical congruence systems. We give two examples.

Examples 2.9. (1) From Corollary 1.3 it is easily seen that each semigroup admits a greatest congruence over right zero semigroups. This congruence system is not a weakly radical congruence system, from Proposition 2.5 (1).

(2) Every semigroup also admits a greatest congruence over $\mathbf{I}$. This system of congruences, say ρ, is not a weakly radical congruence system either. Let $S = \{a, b, 0\}$ be the 3-element null semigroup. Then $a\,\rho(S)\,b$. Let T be the quotient of S by the ideal $\{b, 0\}$ and let φ be the natural morphism of S onto T. If ρ satisfied Condition C4, then $T = \{a, 0\}$ would satisfy $a = \varphi(a)\,\rho(T)\,\varphi(b) = 0$, a contradiction.

Corollary 2.10. *Let $\mathbf{V}$ be a pseudovariety such that every semigroup S admits a greatest congruence over $\mathbf{V}$, and let ρ be the resulting congruence system. If ρ is a radical congruence system, then $\mathbf{V} = \mathbf{S}$ or $\mathbf{V} = (\mathbf{V} \cap \mathbf{CS})$ ⓜ $\mathbf{N}$.*

Proof. By the previous example, we have $\mathbf{V} \neq \mathbf{I}$, and then by Corollary 2.6, $\mathbf{N} \subseteq \mathbf{V}$. Let us assume that $\mathbf{V} \neq \mathbf{S}$. Then, from Corollary 2.4 (2), we have $\mathbf{V} \subseteq (\mathbf{V} \cap \mathbf{CS})$ ⓜ $\mathbf{N}$. But $\mathbf{V}$ ⓜ $\mathbf{V} = \mathbf{V}$ (by Proposition 2.8), so $\mathbf{V} = (\mathbf{V} \cap \mathbf{CS})$ ⓜ $\mathbf{N}$ as required. ∎

If $\mathbf{V}$ is a pseudovariety such that each semigroup S admits a greatest congruence over $\mathbf{V}$, and if this system of congruences is a weakly radical congruence system, then Mal'cev products with $\mathbf{V}$ are more easily computed. The results below slightly extend those of [3].

Lemma 2.11. *Let $\mathbf{V}$ be a pseudovariety such that each semigroup S admits a greatest congruence over $\mathbf{V}$, and let ρ be the resulting congruence system. If ρ is a weakly radical congruence system, then $S \in \mathbf{V}$ ⓜ $\mathbf{W}$ if and only if $S/\rho(S) \in \mathbf{W}$, for any semigroup S and any pseudovariety $\mathbf{W}$.*

Proof. Since $\rho(S)$ is a congruence over $\mathbf{V}$, $S/\rho(S) \in \mathbf{W}$ implies $S \in \mathbf{V}$ ⓜ $\mathbf{W}$. Conversely, let us assume that $S \in \mathbf{V}$ ⓜ $\mathbf{W}$. Then there exist morphisms $S \stackrel{\alpha}{\leftarrow} T \stackrel{\beta}{\rightarrow} W$ such that α is onto, $W \in \mathbf{W}$, and $\beta^{-1}(e) \in \mathbf{V}$ for each idempotent e of W. Since the congruence induced by β on T is over $\mathbf{V}$, it is contained in $\rho(T)$. This implies that $T/\rho(T)$ is a quotient of W, and hence $T/\rho(T) \in \mathbf{W}$. Now Condition C4 implies that $S/\rho(S)$ is a quotient of $T/\rho(T)$, so $S/\rho(S) \in \mathbf{W}$. ∎

Corollary 2.12. *Let $\mathbf{V}$ be a pseudovariety such that each semigroup S admits a greatest congruence over $\mathbf{V}$, and let ρ be the resulting congruence system. If ρ is a weakly radical congruence system, then*
(1) for any pseudovariety $\mathbf{W}$, a semigroup S lies in $\mathbf{V}$ ⓜ $\mathbf{W}$ if and only if there exists an onto morphism $\beta: S \rightarrow W$ with $W \in \mathbf{W}$ and $\beta^{-1}(e) \in \mathbf{V}$ for all idempotents $e \in \mathbf{W}$;
(2) if $\mathbf{V}$ and $\mathbf{W}$ are decidable pseudovarieties, then $\mathbf{V}$ ⓜ $\mathbf{W}$ is decidable.

Proof. Statement (1) is an immediate consequence of Lemma 2.11. This implies the following: in order to check membership of S in $\mathbf{V}$ ⓜ $\mathbf{W}$, it suffices to check, for each quotient of S that lies in $\mathbf{W}$, whether the idempotent classes lie in $\mathbf{V}$. But a finite semigroup has finitely many congruences, and for each of them, the idempotent classes are effectively computable. Thus membership in $\mathbf{V}$ ⓜ $\mathbf{W}$ is decidable. ∎

Following again the notations of [3] for radical congruence systems, we define the *radical class* of a weakly radical congruence system ρ, denoted $RC(\rho)$, to be the class of all semigroups of the form $S/\rho(S)$.

For each class $\mathbf{C}$ of semigroups, $\mathrm{psv}(\mathbf{C})$ denotes the pseudovariety generated by $\mathbf{C}$.

Proposition 2.13. *Let* $\mathbf{V}$ *be a pseudovariety such that each finite semigroup* S *admits a greatest congruence over* $\mathbf{V}$*, and let us assume that the resulting congruence system* ρ *is a weakly radical congruence system. Let* $\mathbf{W}$ *and* $\mathbf{W}'$ *be pseudovarieties.*
(1) *The following conditions are equivalent.*

(a) $\mathbf{V} \; ⓜ \; \mathbf{W} = \mathbf{V} \; ⓜ \; \mathbf{W}'$

(b) $\mathbf{W} \cap RC(\rho) = \mathbf{W}' \cap RC(\rho)$

(c) $\mathrm{psv}(\mathbf{W} \cap RC(\rho)) = \mathrm{psv}(\mathbf{W}' \cap RC(\rho))$.

(2) *If* ρ *is a radical congruence system, then the set of solutions of the equation* $\mathbf{V} \; ⓜ \; \mathbf{X} = \mathbf{W}$ *is either empty, or is the interval* $[\mathrm{psv}(\mathbf{W} \cap RC(\rho)), \mathbf{W}]$.

Proof. (1) By Lemma 2.11, $S \in \mathbf{V} \; ⓜ \; \mathbf{W}$ if and only if $S/\rho(S) \in \mathbf{W} \cap RC(\rho)$. So (a) implies (b). Trivially, (b) implies (c). Suppose (c). Then $\mathrm{psv}(\mathbf{W} \cap RC(\rho))$ is contained in $\mathbf{W} \cap \mathbf{W}'$. Therefore, if $S \in \mathbf{V} \; ⓜ \; \mathbf{W}$, then $S/\rho(S) \in \mathbf{W} \cap RC(\rho) \subseteq \mathbf{W}'$, and hence $S \in \mathbf{V} \; ⓜ \; \mathbf{W}'$. Thus (c) implies (a).

(2) We refer the reader to [3]. ■

2.3 Some important radical congruence systems

In [3], it is proved that each finite semigroup admits a greatest congruence over $\mathbf{N}$, $\mathbf{K}$, $\mathbf{D}$, $\mathbf{LI}$, $\mathbf{IE}$, $\mathbf{K} \vee \mathbf{G}$, $\mathbf{D} \vee \mathbf{G}$ and $\mathbf{LG}$, and that the resulting congruence systems are radical congruence systems. It may be useful to give "closed" formulæ for these congruences.

The existence of these greatest congruences, the formulas giving them, and the fact that they satisfy Condition C4, can also be found in [6] as well as in [3, Lemma 5.1], in a different terminology.

Proposition 2.14. *The greatest congruences over* $\mathbf{N}$, $\mathbf{K}$, $\mathbf{D}$, $\mathbf{LI}$, $\mathbf{IE}$, $\mathbf{K} \vee \mathbf{G}$, $\mathbf{D} \vee \mathbf{G}$ *and* $\mathbf{LG}$ *on a semigroup* S *are given by the following formulæ:*

$$s \sim_N t \iff \begin{cases} \text{for each regular } \mathcal{D}\text{-class } D \text{ and for each idempotent} \\ e \text{ of } D\text{, then } es, et \notin D \text{ or } es = et\text{, and } se, te \notin D \\ \text{or } se = te\text{;} \end{cases}$$

$$s \sim_K t \iff \begin{cases} \text{for each regular } \mathcal{D}\text{-class } D \text{ and for each idempotent} \\ e \text{ of } D\text{, then } es, et \notin D \text{ or } es = et\text{;} \end{cases}$$

$$s \sim_D t \iff \begin{cases} \text{for each regular } \mathcal{D}\text{-class } D \text{ and for each idempotent} \\ e \text{ of } D\text{, then } se, te \notin D \text{ or } se = te\text{;} \end{cases}$$

$$s \sim_{LI} t \iff \begin{cases} \text{for each regular } \mathcal{D}\text{-class } D \text{ and for all idempotents} \\ e, f \text{ of } D\text{, then } esf, etf \notin D \text{ or } esf = etf\text{;} \end{cases}$$

$$s \sim_{IE} t \iff \begin{cases} \text{for each regular } \mathcal{D}\text{-class } D \text{ and for each idempotent} \\ e \text{ of } D\text{, then } es, et \notin D \text{ or } es \mathrel{\mathcal{H}} et\text{, and } se, te \notin D \\ \text{or } se \mathrel{\mathcal{H}} te\text{;} \end{cases}$$

$$s \sim_{KG} t \iff \begin{cases} \textit{for each regular } \mathcal{D}\textit{-class } D \textit{ and for each idempotent} \\ e \textit{ of } D, \textit{ then } es, et \notin D \textit{ or } es\ \mathcal{H}\ et; \end{cases}$$
$$s \sim_{DG} t \iff \begin{cases} \textit{for each regular } \mathcal{D}\textit{-class } D \textit{ and for each idempotent} \\ e \textit{ of } D, \textit{ then } se, te \notin D \textit{ or } se\ \mathcal{H}\ te; \end{cases}$$
$$s \sim_{LG} t \iff \begin{cases} \textit{for each regular } \mathcal{D}\textit{-class } D \textit{ and for all idempotents} \\ e, f \textit{ of } D, \textit{ then } esf, etf \notin D \textit{ or } esf\ \mathcal{H}\ etf. \end{cases}$$

Proof. We give a full proof of the result only for $\sim_{KG}$. The other relations are dealt with similarly.

Let us first verify that $\sim_{KG}$ is a congruence. Let $s, t \in S$ with $s \sim_{KG} t$ and let $u, v \in S \cup \{1\}$: we want to verify that $usv \sim_{KG} utv$. Let D be a regular $\mathcal{D}$-class and let $e \in E(S) \cap D$. If $eu \notin D$, then $eusv, eutv \notin D$. If $eu \in D$, there exist $u' \in S$, $e' \in E(S)$ such that $e'\ \mathcal{L}\ eu$ and $eu = u'e'$. If $e's \notin D$, then we know that $e't \notin D$, so $eusv = u'e'sv$ and $eutv = u'e'tv$ are not in D. If $e's \in D$, then $e's\ \mathcal{H}\ e't$. As long as we remain within D, right and left inner translations preserve $\mathcal{H}$-classes by Green's Lemma, so either $eusv, eutv \notin D$, or $eusv\ \mathcal{H}\ eutv$.

We now verify that $\sim_{KG}$ is over $\mathbf{K} \vee \mathbf{G}$: let $e \in E(S)$ and let $s \in S$ with $e \sim_{KG} s$. Let D be the $\mathcal{D}$-class of e. Since $e = ee \in D$, we have $es\ \mathcal{H}\ e$. This expresses exactly that the $\sim_{KG}$-class of e is in $\mathbf{K} \vee \mathbf{G}$.

Finally, let $\equiv$ be a congruence on S over $\mathbf{K} \vee \mathbf{G}$ and let $s \equiv t$. Let D be a regular $\mathcal{D}$-class of S and let e be an idempotent in D. Let us assume that $es \in D$. Then there exists $u\ \mathcal{L}\ e$ such that $e = esu$ (u is an inverse of es). In particular, $us = ues$ is idempotent. We have $e = esu \equiv etu$. Since $\equiv$ is over $\mathbf{K} \vee \mathbf{G}$, it follows that $e\ \mathcal{L}\ etu$, and hence $e\ \mathcal{H}\ etu\ \mathcal{R}\ et$. Thus $e\ \mathcal{H}\ (etu)^3 = et(uet)^2u$ and so $(uet)^2 \in D$ and $uet \in D$: hence uet lies in a group $\mathcal{H}$-class of D. Let h be the idempotent power of uet: then $h\ \mathcal{H}\ uet\ \mathcal{L}\ et$ and $ues \equiv uet \equiv h$. But $\equiv$-equivalent idempotents are $\mathcal{L}$-related, so $ues\ \mathcal{H}\ h$. In turn, this implies that $es\ \mathcal{H}\ et$. By symmetry, $et \in D$ implies $et\ \mathcal{H}\ es$ and so it follows that $s \equiv t$ implies $s \sim_{KG} t$. ■

3. Some new systems of greatest congruences

We construct systems of greatest congruences over certain pseudovarieties of $\mathbf{LG}$ which extend the above examples. In Section 4, we will prove that they are weakly radical congruence systems, and give a simple necessary and sufficient condition that they be radical congruence systems.

3.1 Some elementary computations

The following technical lemma will be useful in the sequel. Some of the results therein belong to "folklore" (see for instance [11]). Almost all can also be found in [14]. We include the proof anyway for the sake of completeness, and also to justify Corollary 3.2 below. Recall that $\mathbf{LG}$ is exactly the class of all finite semigroups in which the regular elements all lie in the minimum ideal.

Lemma 3.1. (1) $\mathbf{N} ⓜ \mathbf{G} = \mathbf{G} ⓜ \mathbf{N} = \mathbf{N} \vee \mathbf{G} = \mathbf{IE}$.
(2) $\mathbf{K} ⓜ \mathbf{G} = \mathbf{G} ⓜ \mathbf{K} = \mathbf{K} \vee \mathbf{G} = [\![x^\omega y^\omega = x^\omega]\!]$.
(3) $\mathbf{D} ⓜ \mathbf{G} = \mathbf{G} ⓜ \mathbf{D} = \mathbf{D} \vee \mathbf{G} = [\![y^\omega x^\omega = x^\omega]\!]$.
(4) $\mathbf{LI} ⓜ \mathbf{G} = \mathbf{LI} \vee \mathbf{G} = [\![x^\omega y^\omega x^\omega = x^\omega]\!]$ *is strictly contained in* $\mathbf{G} ⓜ \mathbf{LI} = \mathbf{LG}$.

Proof. (1) $\mathbf{G} ⓜ \mathbf{N}$ is generated by semigroups in which the regular elements form the minimum ideal, and that minimum ideal is a group: such semigroups have only one idempotent, and hence lie in **IE**. On the other hand, $\mathbf{N} ⓜ \mathbf{G}$ is generated by semigroups S admitting a congruence ρ such that $S/\rho \in \mathbf{G}$ and the (only) idempotent class of ρ is in $\mathbf{N}$. Such semigroups also have only one idempotent, and hence lie in **IE**. So we have the inclusions $\mathbf{N} \vee \mathbf{G} \subseteq (\mathbf{N} ⓜ \mathbf{G}) \cup (\mathbf{G} ⓜ \mathbf{N}) \subseteq \mathbf{IE}$. Let now $S \in \mathbf{IE}$, let e be the idempotent of S and let I be the minimum ideal of S, that is, the $\mathcal{H}$-class of e. Let $\varphi: S \to I$ be given by $\varphi(s) = es$, and let $\psi: S \to S/I$ be the canonical projection. Since es is $\mathcal{H}$-equivalent to e, we have $e(st) = (es)t = (ese)t$ for each $t \in S$, so φ is a morphism. Finally, let us assume that $\varphi(s) = \varphi(t)$ and $\psi(s) = \psi(t)$ for some $s, t \in S$. If $s \notin I$, then $\psi(s) = \psi(t)$ implies that $t \notin I$ and $s = t$. If $s \in I$, then $\psi(s) = 0$, so $t \in I$. But I is a group with unit e, so $s = es = \varphi(s) = \varphi(t) = et = t$. Thus, in all cases, we have $s = t$. Therefore, S embeds in the direct product $I \times S/I$, and hence $S \in \mathbf{G} \vee \mathbf{N}$, which proves (1).

(2) $\mathbf{K} ⓜ \mathbf{G}$ is generated by semigroups S on which there exists a congruence ρ such that $S/\rho \in \mathbf{G}$ and the (unique) idempotent class of ρ is in $\mathbf{K}$. Since the idempotents of S all lie in that idempotent class, they are all $\mathcal{L}$-equivalent, that is, S satisfies $x^\omega y^\omega = x^\omega$. On the other hand, $\mathbf{G} ⓜ \mathbf{K}$ is generated by semigroups S on which there exists a congruence ρ such that $S/\rho \in \mathbf{K}$ and the idempotent classes of ρ are groups (so $e\rho \subseteq H_e$ for each idempotent $e \in S$). Since $\mathbf{K}$ satisfies $x^\omega y = x^\omega$, we find that in S, $x^\omega y^\omega \, \rho \, x^\omega$, whence $x^\omega y^\omega \; \mathcal{H} \; x^\omega$. Thus the idempotents of S are all $\mathcal{L}$-equivalent, and hence S satisfies $x^\omega y^\omega = x^\omega$. So we have the inclusions $\mathbf{K} \vee \mathbf{G} \subseteq (\mathbf{K} ⓜ \mathbf{G}) \cup (\mathbf{G} ⓜ \mathbf{K}) \subseteq [\![x^\omega y^\omega = x^\omega]\!]$. Let now $S \in [\![x^\omega y^\omega = x^\omega]\!]$, let I be the minimum ideal of S, let e be an idempotent of S (so $e \in I$), let H_e be the $\mathcal{H}$-class of e, and let $\varphi: S \to H_e$ be given by $\varphi(s) = es$. Since $es \in I$, we have $es = es(es)^\omega$. In addition, $e = e^\omega (es)^\omega = (es)^\omega$, so $ese = es$. Therefore φ is a morphism. Let $\sim$ be the congruence on S defined by $s \sim t$ if and only if $s = t$ or $s, t \in I$ and $s \; \mathcal{H} \; t$. Then $S/\!\sim \in \mathbf{K}$. We let $\psi: S \to S/\!\sim$ be the natural morphism. We now verify that $\varphi(s) = \varphi(t)$ and $\psi(s) = \psi(t)$ implies $s = t$. Indeed, if $s \notin I$, then $\psi(s) = \psi(t)$ implies $t \notin I$ and $s = t$. If $s \in I$, then $\psi(s) = \psi(t)$ implies that $t \in I$ and $t \; \mathcal{H} \; s$. Let f be the idempotent of the common $\mathcal{H}$-class of s and t: then $\varphi(s) = \varphi(t)$ implies $s = fs = (fe)s = f(es) = f(et) = ft = t$. Thus, S imbeds in $H_e \times S/\!\sim$, and hence $S \in \mathbf{G} \vee \mathbf{K}$. This completes the proof of Statement (2).

(3) Statement (3) is the left-right dual of Statement (2).

(4) $\mathbf{LI} ⓜ \mathbf{G}$ is generated by semigroups S on which there exists a congruence ρ such that $S/\rho \in \mathbf{G}$ and the (unique) idempotent class of ρ is in $\mathbf{LI}$. Since the idempotents of S all lie in that idempotent class, they form a rectangular band, that is, S satisfies $x^\omega y^\omega x^\omega = x^\omega$. Conversely, let us assume that S satisfies $x^\omega y^\omega x^\omega = x^\omega$, let I be the minimum ideal of S, let e be an idempotent of S (so $e \in I$), let H_e be the $\mathcal{H}$-class of e, and let $\varphi: S \to H_e$ be given by $\varphi(s) = ese$. Since $es, se \in I$, we have $es = es(es)^\omega$ and $se = (se)^\omega se$. So $este = es(es)^\omega (te)^\omega te = es(es)^\omega e(te)^\omega te = esete$ for all $s, t \in S$. Therefore φ is a morphism. Let $\sim$ be the congruence on S defined by $s \sim t$ if and only if $s = t$ or $s, t \in I$ and $s \; \mathcal{H} \; t$. Then $S/\!\sim \in \mathbf{LI}$. We let $\psi: S \to S/\!\sim$ be the projection morphism. Again, we verify that $\varphi(s) = \varphi(t)$ and $\psi(s) = \psi(t)$ implies $s = t$. Indeed, if $s \notin I$, then $\psi(s) = \psi(t)$implies $t \notin I$ and $s = t$.

If $s \in I$, then $\psi(s) = \psi(t)$ implies that $t \in I$ and $t\ \mathcal{H}\ s$. Let f be the idempotent of the common $\mathcal{H}$-class of s and t: then $\varphi(s) = \varphi(t)$ implies $s = fsf = (fef)s(fef) = f(ese)f = f(ete)f = (fef)t(fef) = ftf = t$. Thus, S imbeds in $H_e \times S/\!\sim$, and hence $S \in \mathbf{G} \vee \mathbf{LI}$. Since clearly $\mathbf{LI} \vee \mathbf{G} \subseteq \mathbf{LI} \textcircled{m} \mathbf{G}$, we have obtained the two first equalities.

$\mathbf{G} \textcircled{m} \mathbf{LI}$ is generated by semigroups S on which there exists a congruence ρ such that $S/\rho \in \mathbf{LI}$ and each idempotent class of ρ is a group. Thus S/ρ satisfies $x^\omega y x^\omega = x^\omega$ and so S satisfies $(x^\omega y x^\omega)^\omega = x^\omega$, that is, $S \in \mathbf{LG}$. Conversely, if $S \in \mathbf{LG}$ and I is its minimum ideal, let $\sim$ be the congruence on S defined by $s \sim t$ if $s = t$ or $s, t \in I$ and $s\ \mathcal{H}\ t$. Then $S/\!\sim \in \mathbf{LI}$ and each idempotent class of $\sim$ is an $\mathcal{H}$-class in I, that is, a group. So $\mathbf{G} \textcircled{m} \mathbf{LI} = \mathbf{LG}$.

In order to verify the strict containment of $\mathbf{LI} \vee \mathbf{G}$ in $\mathbf{LG}$, it suffices to consider the case of a non-orthodox completely simple semigroup. ■

With the same proofs, one also obtains the following.

Corollary 3.2. *Let $\mathbf{H}$ be a non-trivial pseudovariety of groups.*
(1) $\mathbf{N} \textcircled{m} \mathbf{H} = \mathbf{H} \textcircled{m} \mathbf{N} = \mathbf{N} \vee \mathbf{H} = \mathbf{IE} \cap \overline{\mathbf{H}}$.
(2) $\mathbf{K} \textcircled{m} \mathbf{H} = \mathbf{H} \textcircled{m} \mathbf{K} = \mathbf{K} \vee \mathbf{H} = [\![x^\omega y^\omega = x^\omega]\!] \cap \overline{\mathbf{H}}$.
(3) $\mathbf{D} \textcircled{m} \mathbf{H} = \mathbf{H} \textcircled{m} \mathbf{D} = \mathbf{D} \vee \mathbf{H} = [\![y^\omega x^\omega = x^\omega]\!] \cap \overline{\mathbf{H}}$.
(4) $\mathbf{LI} \textcircled{m} \mathbf{H} = \mathbf{LI} \vee \mathbf{H} = [\![x^\omega y^\omega x^\omega = x^\omega]\!] \cap \overline{\mathbf{H}}$ *is strictly contained in* $\mathbf{H} \textcircled{m} \mathbf{LI} = \mathbf{LH}$.
■

3.2 Existence of certain systems of greatest congruences

In a group G, a congruence is entirely determined by its unique idempotent class, which is a normal subgroup of G. Let $\mathbf{H}$ be a pseudovariety of groups: a finite group admits a greatest congruence over $\mathbf{H}$ if and only if it admits a greatest normal subgroup in $\mathbf{H}$.

Proposition 3.3. *Let $\mathbf{H}$ be a pseudovariety of groups such that $\mathbf{H} = \mathbf{H} \textcircled{m} \mathbf{H}$. Then every group admits a greatest congruence over $\mathbf{H}$.*

Proof. We want to prove that each group G has a greatest normal subgroup $G_\mathbf{H}$ in $\mathbf{H}$. Observe that if $G \in \mathbf{H}$, then $G_\mathbf{H} = G$, and if G has no non-trivial normal subgroups in $\mathbf{H}$, then $G_\mathbf{H} = 1$. In particular, if G is simple, then $G_\mathbf{H}$ exists. We proceed by induction on the order of G. If $|G| \leq 3$, then G is simple, so $G_\mathbf{H}$ exists. Let us now assume that $|G| > 3$. If G has no non-trivial normal subgroup in $\mathbf{H}$, then $G_\mathbf{H}$ exists, and is trivial. Otherwise, let N be a non-trivial normal subgroup of G in $\mathbf{H}$. Let $\varphi\colon G \to G/N$ be the natural morphism. Since $|G/N| < |G|$, $K = (G/N)_\mathbf{H}$ exists: we let $H = \varphi^{-1}(K)$. Then H is normal in G. Since $\mathbf{H} = \mathbf{H} \textcircled{m} \mathbf{H}$, $H/N = K \in \mathbf{H}$ and $N \in \mathbf{H}$, we have $H \in \mathbf{H}$. Finally, if H' is a normal subgroup of G in $\mathbf{H}$, then $\varphi(H')$ is a normal subgroup of G/N in $\mathbf{H}$, and hence $\varphi(H') \subseteq K$. Thus $H' \subseteq H$, and hence H is the greatest normal subgroup of G in $\mathbf{H}$. ■

Examples 3.4. Not every group has a greatest normal abelian subgroup: if $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ is the group of quaternions, all proper subgroups are abelian and normal, but there is no greatest abelian subgroup.

The pseudovarieties $\mathbf{G}_p$ (where p is prime and $\mathbf{G}_p$ consists of all p-groups), $\mathbf{G}_\Pi$ (where Π is a set of prime numbers and $\mathbf{G}_\Pi$ is the pseudovariety of Π-groups), and the pseudovariety of all solvable groups, are closed under extension, that is, under Mal'cev product with themselves. If p is a fixed prime, the intersection of the p-Sylows of a group G constitutes its greatest normal subgroup in $\mathbf{G}_p$.

On the other hand, the pseudovariety $\mathbf{G}_n$ of all nilpotent groups is not closed under extension. Yet each finite group has a greatest normal nilpotent subgroup, called the Fitting subgroup [16].

Theorem 3.5. *If $\mathbf{H}$ is a pseudovariety of groups such that each group admits a greatest congruence over $\mathbf{H}$, then each semigroup admits a greatest congruence over $\mathbf{N} \vee \mathbf{H}$, $\mathbf{K} \vee \mathbf{H}$, $\mathbf{D} \vee \mathbf{H}$ and $\mathbf{LH}$.*

Before we prove Theorem 3.5, let us recall a few facts about completely simple semigroups (see [7]). Every completely simple semigroup S admits a normalized Rees matrix representation $S = \mathcal{M}(G, I, \Lambda, P)$, where $G \in \mathbf{G}$, I and Λ are finite sets and P is a $(\Lambda \times I)$-matrix with entries in G, and with a full row and a full column of 1. The group G is called the *structure* group of S. The product in S is given by

$$(i, g, \lambda)(j, h, \mu) = (i, gp_{\lambda,j}h, \mu).$$

Moreover, the congruences on S are characterized by the following result [7].

Let N be a normal subgroup of G and let $\sim_I$ and $\sim_\Lambda$ be equivalence relations on I and Λ. We say that the triple $(N, \sim_I, \sim_\Lambda)$ is *admissible* if

$$\begin{aligned} i \sim_I j &\implies \forall \lambda \in \Lambda,\ p_{\lambda,i} = p_{\lambda,j} \pmod N \\ \lambda \sim_\Lambda \mu &\implies \forall i \in I,\ p_{\lambda,i} = p_{\mu,i} \pmod N \end{aligned}$$

Lemma 3.6. *Let S be a completely simple semigroup and let $\mathcal{M}(G, I, \Lambda, P)$ be a normalized Rees matrix representation of S.*
(1) *For each admissible triple $(N, \sim_I, \sim_\Lambda)$, the relation σ on S given by*

$$(i, g, \lambda)\, \sigma\, (j, h, \mu) \iff g = h \pmod N,\ i \sim_I j \text{ and } \lambda \sim_\Lambda \mu$$

is a congruence on S.
(2) *Every congruence on S is obtained in this fashion.*
As a result, there is a greatest congruence on S for which the first component of the associated triple is a fixed normal subgroup N: it is the congruence for which

$$\begin{aligned} i \sim_I j &\iff \forall \lambda \in \Lambda,\ p_{\lambda,i} = p_{\lambda,j} \pmod N \\ \lambda \sim_\Lambda \mu &\iff \forall i \in I,\ p_{\lambda,i} = p_{\mu,i} \pmod N \end{aligned}$$

■

This lemma is used to prove Theorem 3.5 in the case of a completely simple semigroup.

Lemma 3.7. *If $\mathbf{H}$ is a pseudovariety of groups such that each group admits a greatest congruence over $\mathbf{H}$, then each completely simple semigroup admits a greatest congruence over $\mathbf{N} \vee \mathbf{H}$, $\mathbf{K} \vee \mathbf{H}$, $\mathbf{D} \vee \mathbf{H}$ and $\mathbf{LH}$.*

Proof. Let $S = \mathcal{M}(G, I, \Lambda, P)$ be a completely simple semigroup given by a normalized Rees matrix representation. Let H be the greatest normal subgroup of G in $\mathbf{H}$ and let $\sim_I$ and $\sim_\Lambda$ be the equivalence relations on I and Λ given by

$$\begin{aligned} i \sim_I j &\iff \forall \lambda \in \Lambda,\ p_{\lambda,i} = p_{\lambda,j} \pmod{H} \\ \lambda \sim_\Lambda \mu &\iff \forall i \in I,\ p_{\lambda,i} = p_{\mu,i} \pmod{H} \end{aligned}$$

Let σ be the congruence on S corresponding to the triple $(H, \sim_I, \sim_\Lambda)$. Then σ is trivially a congruence over $\mathbf{LH}$.

Let τ be a congruence on S over $\mathbf{LH}$, and let $(N, \equiv_I, \equiv_\Lambda)$ be the associated admissible triple. Then every idempotent class of τ is a completely simple semigroup with structure group N, so $N \in \mathbf{H}$, and hence $N \subseteq H$. Therefore

$$\begin{aligned} (i,g,\lambda)\,\tau\,(j,h,\mu) &\implies \begin{cases} g = h \pmod{N};\ \text{for all } \lambda' \in \Lambda,\ i' \in I,\ p_{\lambda,i'} = \\ p_{\mu,i'} \pmod{N} \text{ and } p_{\lambda',i} = p_{\lambda',j} \pmod{N} \end{cases} \\ &\implies \begin{cases} g = h \pmod{H};\ \text{for all } \lambda' \in \Lambda,\ i' \in I,\ p_{\lambda,i'} = \\ p_{\mu,i'} \pmod{H} \text{ and } p_{\lambda',i} = p_{\lambda',j} \pmod{H} \end{cases} \\ &\implies (i,g,\lambda)\,\sigma\,(j,h,\mu) \end{aligned}$$

Thus σ is the greatest congruence on S over $\mathbf{LH}$.

The construction of the greatest congruences on S over $\mathbf{K} \vee \mathbf{H}$, (respectively $\mathbf{D} \vee \mathbf{H}$, $\mathbf{N} \vee \mathbf{H}$) is similar: it suffices to modify the above construction by choosing $\sim_I$ (respectively $\sim_\Lambda$, $\sim_I$ and $\sim_\Lambda$) to be trivial. ∎

A simple examination of these constructions reveals the following.

Corollary 3.8. *Let S be a completely simple semigroup. The join of the greatest congruences on S over $\mathbf{K} \vee \mathbf{H}$ and $\mathbf{D} \vee \mathbf{H}$ is the greatest congruence over $\mathbf{LH}$. Their intersection is the greatest congruence over $\mathbf{N} \vee \mathbf{H}$.* ∎

We can now prove Theorem 3.5.

Proof of Theorem 3.5. Let S be a finite semigroup and let ν be the greatest congruence on S over $\mathbf{LG}$. For each idempotent ν-class N, $N \in \mathbf{LG}$, so all the regular elements of N are contained in its minimum ideal, say D. Moreover $D \in \mathbf{CS}$: we let $\sigma(D)$ be the greatest congruence on D over $\mathbf{LH}$ (respectively $\mathbf{K} \vee \mathbf{H}$, $\mathbf{D} \vee \mathbf{H}$, $\mathbf{N} \vee \mathbf{H}$). For each idempotent $\sigma(D)$-class Q, we let $P_{N,Q}$ be the set

$$P_{N,Q} = \{s \in N \mid ss^\omega \in Q\}.$$

Then the $P_{N,Q}$ are pairwise equal or disjoint, and we let $\mathcal{P}$ be the partition of S whose classes are the $P_{N,Q}$ and $S \setminus \bigcup_{N,Q} P_{N,Q}$. Finally, we let ρ be the greatest congruence on S contained in $\mathcal{P}$: such a congruence exists by Corollary 1.3.

Observe that each idempotent of S lies in some $P_{N,Q}$, so its ρ-class C is contained in $P_{N,Q}$. But $P_{N,Q} \subseteq N$, so $C \in \mathbf{LG}$. Moreover the minimum ideal of C is contained in the set of regular elements of N, namely D, so it is contained in Q. Therefore the minimum ideal of C lies in $\mathbf{LH}$ (respectively $\mathbf{K} \vee \mathbf{H}$, $\mathbf{D} \vee \mathbf{H}$, $\mathbf{N} \vee \mathbf{H}$), and hence so does C. Thus, ρ is a congruence over $\mathbf{LH}$ (respectively $\mathbf{K} \vee \mathbf{H}$, $\mathbf{D} \vee \mathbf{H}$, $\mathbf{N} \vee \mathbf{H}$).

Let now τ be a congruence on S over $\mathbf{LH}$ (respectively $\mathbf{K} \vee \mathbf{H}$, $\mathbf{D} \vee \mathbf{H}$, $\mathbf{N} \vee \mathbf{H}$). In order to verify that τ is contained in ρ, it suffices to verify that τ is contained in the partition $\mathcal{P}$. Let $s,t \in S$ such that $s \,\tau\, t$. Since τ is over **LG**, we have $s \,\nu\, t$. If the ν-class of s and t is not idempotent, then $s,t \in S \setminus \bigcup_{N,Q} P_{N,Q}$. So now we assume that the ν-class of s and t, say N', is idempotent. Let D' be the minimum ideal of N'. The restriction of τ to D' is over **LH** (respectively $\mathbf{K} \vee \mathbf{H}$, $\mathbf{D} \vee \mathbf{H}$, $\mathbf{N} \vee \mathbf{H}$), so it is contained in $\sigma(D')$. But $s \,\tau\, t$ implies $ss^\omega \,\tau\, tt^\omega$, and $ss^\omega, tt^\omega \in D'$, so we have $ss^\omega \,\sigma(D')\, tt^\omega$. If the $\sigma(D')$-class of ss^ω, tt^ω is not idempotent, then $s,t \in S \setminus \bigcup_{N,Q} P_{N,Q}$. Otherwise, this $\sigma(D')$-class, say Q', is idempotent, and $s,t \in P_{N',Q'}$. Thus τ is contained in $\mathcal{P}$, and hence in ρ. Thus ρ is the greatest congruence on S over **LH** (respectively $\mathbf{K} \vee \mathbf{H}$, $\mathbf{D} \vee \mathbf{H}$, $\mathbf{N} \vee \mathbf{H}$). ■

4. Greatest congruences which form weakly radical congruence systems

4.1 The greatest congruences of Section 3.2 form weakly radical congruence systems

Theorem 4.1. *If* **H** *is a pseudovariety of groups such that each group admits a greatest congruence over* **H**, *then the systems of greatest congruences over* $\mathbf{N} \vee \mathbf{H}$, $\mathbf{K} \vee \mathbf{H}$, $\mathbf{D} \vee \mathbf{H}$ *and* **LH** *are weakly radical congruence systems.*

Proof. Let $\rho = (\rho(S))_S$ be the system of greatest congruences over **LH** (respectively $\mathbf{K} \vee \mathbf{H}$, $\mathbf{D} \vee \mathbf{H}$, $\mathbf{N} \vee \mathbf{H}$). By Proposition 2.8, it suffices to verify that ρ satisfies Condition C4, that is, to verify that if $\varphi\colon S \to T$ is an onto morphism, then $x \,\rho(S)\, y$ implies $\varphi(x) \,\rho(T)\, \varphi(y)$ for all $x,y \in S$.

Let us first assume that S is a group. Then T is a group as well. Let H (respectively K) be the greatest normal subgroup of S (respectively T) in **H**. Since φ is onto, $\varphi(H)$ is normal in T, and $\varphi(H) \in \mathbf{H}$, so $\varphi(H) \subseteq K$. Now $\rho(S)$ is a congruence on a group, so it is in fact a congruence over $\mathbf{LH} \cap \mathbf{G}$ (respectively $(\mathbf{K} \vee \mathbf{H}) \cap \mathbf{G}$, $(\mathbf{D} \vee \mathbf{H}) \cap \mathbf{G}$, $(\mathbf{N} \vee \mathbf{H}) \cap \mathbf{G}$), that is, a congruence over **H**. Thus, $\rho(S)$ is the greatest congruence on S over **H**, and is given by

$$x \,\rho(S)\, y \quad \Longleftrightarrow \quad xy^{-1} \in H.$$

A similar formula characterizes $\rho(T)$, and we have

$$\begin{aligned} x \,\rho(S)\, y &\Longrightarrow xy^{-1} \in H \\ &\Longrightarrow \varphi(x)\varphi(y)^{-1} \in \varphi(H) \\ &\Longrightarrow \varphi(x)\varphi(y)^{-1} \in K \\ &\Longrightarrow \varphi(x) \,\rho(T)\, \varphi(y). \end{aligned}$$

Let us now assume that $S = \mathcal{M}(G, I, \Lambda, P)$ is a completely simple semigroup, given by a normalized Rees matrix representation. Then T also is in **CS**. Let $(H, \sim_I, \sim_\Lambda)$ be the admissible triple associated with $\rho(S)$, and let $(N, \equiv_I, \equiv_\Lambda)$ be the admissible triple associated with the congruence induced by φ on S. Then

$$\begin{aligned} i \equiv_I j &\Longrightarrow \forall \lambda \in \Lambda,\ p_{\lambda,i} = p_{\lambda,j} \pmod N \\ \lambda \equiv_\Lambda \mu &\Longrightarrow \forall i \in I,\ p_{\lambda,i} = p_{\mu,i} \pmod N). \end{aligned}$$

Moreover, $\mathcal{M}(G/N, I/\equiv_I, \Lambda/\equiv_\Lambda, Q)$, with $q_{[\lambda],[i]} = p_{\lambda,i}N$, is a normalized Rees matrix representation of T. In this representation, φ is given by $\varphi(i,g,\lambda) = ([i], gN, [\lambda])$. Let H' be the greatest normal subgroup of G/N in $\mathbf{H}$ and let $\psi\colon G \to G/N$ be the natural morphism. Then the previous case shows that, for all $g, g' \in G$, $g = g' \pmod H$ implies $gN = g'N \pmod{H'}$. So

$$\begin{aligned}
(i,g,\lambda)\,\rho(S)\,(j,h,\mu) &\implies \begin{cases} g = h \pmod H; \text{ for all } \lambda' \in \Lambda,\ i' \in \\ I,\ p_{\lambda,i'} = p_{\mu,i'} \pmod H \text{ and } p_{\lambda',i} = \\ p_{\lambda',j} \pmod H \end{cases}\\
&\implies \begin{cases} gN = hN \pmod{H'}; \text{ for all } \lambda' \in \Lambda, \\ i' \in I,\ p_{\lambda,i'}N = p_{\mu,i'}N \pmod{H'} \text{ and} \\ p_{\lambda',i}N = p_{\lambda',j}N \pmod{H'} \end{cases}\\
&\implies \varphi(i,g,\lambda)\,\rho(T)\,\varphi(j,h,\mu).
\end{aligned}$$

Finally, we consider the general case. In order to show that $x\,\rho(S)\,y$ implies $\varphi(x)\,\rho(T)\,\varphi(y)$, it suffices to verify that $x\,\rho(S)\,y$ implies that $\varphi(x)$ and $\varphi(y)$ lie in the same class of the partition $\mathcal{P}$ used to define $\rho(T)$ (see the proof of Theorem 3.5). Indeed, if the relation $\{(\varphi(x),\varphi(y)) \mid x\,\rho(S)\,y\}$ is contained in $\mathcal{P}$, then so is its transitive closure. But it is not difficult to verify that this transitive closure is in fact a congruence on T, so it is contained in the greatest congruence contained in $\mathcal{P}$, namely $\rho(T)$.

Let $\nu(S)$ and $\nu(T)$ be the greatest congruences on S and T respectively over **LG**. Recall that the classes of $\mathcal{P}$ are

- the $P_{N,Q}$, where N is an idempotent $\nu(T)$-class of T and Q is an idempotent $\rho(D)$-class of the minimum ideal D of N (which is in **CS**), and
- $T \setminus \bigcup_{N,Q} P_{N,Q}$.

Let $x, y \in S$ be such that $x\,\rho(S)\,y$. Then $x\,\nu(S)\,y$, and hence $\varphi(x)\,\nu(T)\,\varphi(y)$, since ν is a radical congruence system (see Section 2.3). If the $\nu(T)$-class of $\varphi(s)$ and $\varphi(t)$ is not idempotent, then $\varphi(s), \varphi(t) \in T \setminus \bigcup_{N,Q} P_{N,Q}$. So we now assume that the $\nu(T)$-class of $\varphi(s)$ and $\varphi(t)$, say N', is idempotent, and let D' be its minimum ideal. Then $\varphi(x)\varphi(x)^\omega, \varphi(y)\varphi(y)^\omega \in D'$. Let C be the minimum ideal of the subsemigroup $\varphi^{-1}(D')$ of S. Then $C \in \mathbf{CS}$ and $\varphi(C) = D'$ (see for instance [15]). Moreover, for each idempotent e of C, we have

$$\begin{array}{cccl}
ex^\omega & \rho(S) & ey^\omega & \\
ex^\omega & \rho(C) & ey^\omega & \text{since } \rho \text{ satisfies C2} \\
\varphi(ex^\omega) & \rho(D') & \varphi(ey^\omega) & \text{from the previous case.} \\
\varphi(x^\omega e) & \rho(D') & \varphi(y^\omega e) & \text{similarly.}
\end{array}$$

Choosing e such that $\varphi(e) = \varphi(x^\omega)$, it follows that

$$\begin{array}{rl}
 & \varphi(x^\omega)\ \rho(D')\ \varphi(x^\omega y^\omega)\ \rho(D')\ \varphi(y^\omega x^\omega), \\
\text{so} & \varphi(x^\omega)\ \rho(D')\ \varphi(y^\omega x^\omega)\varphi(x^\omega y^\omega) = \varphi(y^\omega x^\omega x^\omega y^\omega) \\
\text{whence} & \varphi(x^\omega)\ \rho(D')\ \varphi((y^\omega x^\omega x^\omega y^\omega)^\omega) = \varphi(y^\omega).
\end{array}$$

In the same fashion, we show that $\varphi(exx^\omega e)\,\rho(D')\,\varphi(eyy^\omega e)$ for each idempotent e of C. Choosing again e such that $\varphi(e) = \varphi(x^\omega)$, we get

$$\varphi(xx^\omega)\,\rho(D')\,\varphi(x^\omega yy^\omega x^\omega)\,\rho(D')\,\varphi(y^\omega yy^\omega y^\omega) = \varphi(yy^\omega).$$

Thus, if the $\rho(D')$-class of $\varphi(xx^\omega)$ and $\varphi(yy^\omega)$, say Q', is not idempotent, then $\varphi(x), \varphi(y) \in T \setminus \bigcup_{N,Q} P_{N,Q}$. If, on the other hand, Q' is an idempotent class, then $\varphi(x), \varphi(y) \in P_{N',Q'}$. This completes the proof. ∎

Corollary 4.2. *If $\mathbf{H}$ is a pseudovariety of groups such that each group admits a greatest congruence over $\mathbf{H}$, then the systems of greatest congruences over $\mathbf{N} \vee \mathbf{H}$, $\mathbf{K} \vee \mathbf{H}$, $\mathbf{D} \vee \mathbf{H}$ and $\mathbf{LH}$ are radical congruence systems if and only if $\mathbf{H} \mathbin{\textcircled{m}} \mathbf{H} = \mathbf{H}$.*

Proof. By Proposition 2.8 and Theorem 4.1, the system of greatest congruences over $\mathbf{N} \vee \mathbf{H}$ (respectively $\mathbf{K} \vee \mathbf{H}$, $\mathbf{D} \vee \mathbf{H}$, $\mathbf{LH}$) is a radical congruence system if and only if $\mathbf{N} \vee \mathbf{H}$ (respectively $\mathbf{K} \vee \mathbf{H}$, $\mathbf{D} \vee \mathbf{H}$, $\mathbf{LH}$) is closed under Mal'cev product with itself. It is easily verified that this is the case if and only if $\mathbf{H} \mathbin{\textcircled{m}} \mathbf{H} = \mathbf{H}$. ∎

We also obtain the following corollary to Corollary 2.12 and Theorem 4.1.

Corollary 4.3. *Let $\mathbf{W}$ be a decidable pseudovariety, and let $\mathbf{H}$ be a decidable pseudovariety of groups such that each group admits a greatest congruence over $\mathbf{H}$ (e.g. $\mathbf{H}$ is decidable and $\mathbf{H} \mathbin{\textcircled{m}} \mathbf{H} = \mathbf{H}$). Then the pseudovarieties $(\mathbf{N} \vee \mathbf{H}) \mathbin{\textcircled{m}} \mathbf{W}$, $(\mathbf{K} \vee \mathbf{H}) \mathbin{\textcircled{m}} \mathbf{W}$, $(\mathbf{D} \vee \mathbf{H}) \mathbin{\textcircled{m}} \mathbf{W}$ and $\mathbf{LH} \mathbin{\textcircled{m}} \mathbf{W}$ are decidable for each prime p.* ∎

4.2 All systems of greatest congruences over a pseudovariety which are radical congruence systems

We first determine all possible values (modulo group theory) of $\mathbf{V} \cap \mathbf{CS}$ when $\mathbf{V}$ is such that every semigroup admits a greatest congruence over $\mathbf{V}$.

Lemma 4.4. *Let $\mathbf{V}$ be a pseudovariety such that each completely simple semigroup admits a greatest congruence over $\mathbf{V}$, and let $\mathbf{H} = \mathbf{V} \cap \mathbf{G}$. Then every group admits a greatest congruence over $\mathbf{H}$ and $\mathbf{V} \cap \mathbf{CS}$ is equal to one of $\mathbf{H} = (\mathbf{N} \vee \mathbf{H}) \cap \mathbf{CS}$, $(\mathbf{K} \vee \mathbf{H}) \cap \mathbf{CS}$, $(\mathbf{D} \vee \mathbf{H}) \cap \mathbf{CS}$ or $(\mathbf{LH}) \cap \mathbf{CS}$.*

Proof. Let $\mathbf{LZ} = [\![xy = x]\!]$ and $\mathbf{RZ} = [\![yx = x]\!]$. Four cases may arise:

- If $\mathbf{LZ}, \mathbf{RZ} \not\subseteq \mathbf{V}$, then $\mathbf{V} \cap \mathbf{CS} \subseteq \mathbf{G}$, and hence $\mathbf{V} \cap \mathbf{CS} = \mathbf{H}$.
- If $\mathbf{LZ} \subseteq \mathbf{V}$ and $\mathbf{RZ} \not\subseteq \mathbf{V}$, then $\mathbf{V} \cap \mathbf{CS} \subseteq \mathbf{LZ} \vee \mathbf{H}$, and hence $\mathbf{V} \cap \mathbf{CS} = \mathbf{LZ} \vee \mathbf{H} = (\mathbf{K} \vee \mathbf{H}) \cap \mathbf{CS}$.
- If $\mathbf{RZ} \subseteq \mathbf{V}$ and $\mathbf{LZ} \not\subseteq \mathbf{V}$, then dually $\mathbf{V} \cap \mathbf{CS} = (\mathbf{D} \vee \mathbf{H}) \cap \mathbf{CS}$.
- If $\mathbf{LZ} \vee \mathbf{RZ} \subseteq \mathbf{V}$, then $\mathbf{V} \cap \mathbf{CS}$ contains $(\mathbf{D} \vee \mathbf{H}) \cap \mathbf{CS}$ and $(\mathbf{K} \vee \mathbf{H}) \cap \mathbf{CS}$.

So, for each completely simple semigroup S, the greatest congruence on S over $\mathbf{V}$ contains the greatest congruences on S over $\mathbf{K} \vee \mathbf{H}$ and $\mathbf{D} \vee \mathbf{H}$. By Corollary 3.8, the join of these two congruences is the greatest congruence on S over $\mathbf{LH}$. So $\mathbf{LH} \cap \mathbf{CS} \subseteq \mathbf{V} \cap \mathbf{CS}$. But $\mathbf{V} \cap \mathbf{CS}$ is trivially contained in $\mathbf{CS} \cap \overline{\mathbf{H}} = \mathbf{LH} \cap \mathbf{CS}$, so $\mathbf{V} \cap \mathbf{CS} = \mathbf{LH} \cap \mathbf{CS}$. ∎

Theorem 4.5. *The only pseudovarieties $\mathbf{V}$ such that each semigroup admits a greatest congruence over $\mathbf{V}$ and such that the resulting congruence system is a radical congruence system are $\mathbf{S}$ and the pseudovarieties of the form $\mathbf{N} \vee \mathbf{H}$, $\mathbf{K} \vee \mathbf{H}$, $\mathbf{D} \vee \mathbf{H}$ and $\mathbf{LH}$ where $\mathbf{H}$ is a pseudovariety of groups such that each group admits a greatest normal subgroup in $\mathbf{H}$ and such that $\mathbf{H} \mathbin{\textcircled{m}} \mathbf{H} = \mathbf{H}$.*

Proof. If $\mathbf{V} \neq \mathbf{S}$, then by Corollary 2.10, we have $\mathbf{V} = (\mathbf{V} \cap \mathbf{CS}) \, ⓜ \, \mathbf{N}$. Now Lemma 4.4 shows that $\mathbf{V} \cap \mathbf{CS}$ is one of $(\mathbf{N} \vee \mathbf{H}) \cap \mathbf{CS}$, $(\mathbf{K} \vee \mathbf{H}) \cap \mathbf{CS}$, $(\mathbf{D} \vee \mathbf{H}) \cap \mathbf{CS}$ or $\mathbf{LH} \cap \mathbf{CS}$, with $\mathbf{H} = \mathbf{V} \cap \mathbf{G}$ and each group admits a greatest normal subgroup in $\mathbf{H}$. It is easily verified (say with the help of Lemma 1.1) that $((\mathbf{N}\vee\mathbf{H})\cap\mathbf{CS}) \, ⓜ \, \mathbf{N} = \mathbf{N}\vee\mathbf{H}$, $((\mathbf{K}\vee\mathbf{H})\cap\mathbf{CS}) \, ⓜ \, \mathbf{N} = \mathbf{K}\vee\mathbf{H}$, $((\mathbf{D}\vee\mathbf{H})\cap\mathbf{CS}) \, ⓜ \, \mathbf{N} = \mathbf{D} \vee \mathbf{H}$ and $(\mathbf{LH} \cap \mathbf{CS}) \, ⓜ \, \mathbf{N} = \mathbf{LH}$. Moreover, by Proposition 2.8, $\mathbf{V} \, ⓜ \, \mathbf{V} = \mathbf{V}$, so $\mathbf{H} \, ⓜ \, \mathbf{H} = \mathbf{H}$. This concludes the proof of the theorem, in view of Corollary 4.2. ∎

5. Open problems

We have, in a way, completed the work of [3], by determining all the pseudovarieties $\mathbf{V}$ such that each semigroup admits a greatest congruence over $\mathbf{V}$ and such that these congruences form a radical congruence system. There remain however several open questions regarding radical congruence systems.

For instance, we do not know any pseudovariety $\mathbf{V}$ such that every semigroup admits a greatest congruence over $\mathbf{V}$ and such that the resulting system of congruences is a weakly radical congruence system, other than $\mathbf{S}$ and those listed in Theorem 4.1. Would it be possible to list them all? We know that $\mathbf{V} \cap \mathbf{CS} \subset \mathbf{V} \subseteq (\mathbf{V} \cap \mathbf{CS}) \, ⓜ \, \mathbf{N}$, and that if $\mathbf{H} = \mathbf{G} \cap \mathbf{V}$, then $\mathbf{V} \cap \mathbf{CS} \in \{\mathbf{H}, \mathbf{LZ}\vee\mathbf{H}, \mathbf{RZ}\vee\mathbf{H}, \mathbf{CS}\cap\overline{\mathbf{H}}\}$. The least pseudovariety satisfying these Conditions is $\mathbf{Null}\vee\mathbf{H}$, but it is easy to show that not every semigroup admits a greatest congruence over $\mathbf{Null}\vee\mathbf{H}$. Let indeed $S = \{a, b, ab, 0\}$ be given by $a^2 = b^2 = ba = 0$. Then the ideals $\{a, ab, 0\}$ and $\{b, ab, 0\}$ are in $\mathbf{Null}$, so the greatest congruence on S over $\mathbf{Null} \vee \mathbf{H}$, if it exists, is universal. But $S \notin \mathbf{Null} \vee \mathbf{H}$, a contradiction. Is it possible to show that, under these hypotheses, $\mathbf{N} \subseteq \mathbf{V}$?

We also know very little about the classification of the (weakly) radical congruence systems which are not systems of greatest congruences over some pseudovariety.

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