Graphs, intersections of subgroups of free groups and corank [1]

Pascal Weil (LaBRI, CNRS) [2]

The topic of this paper is the discussion of a combinatorial problem on certain edge-labeled graphs, motivated by a conjecture in the theory of free groups.

The group-theoretic question is the following. It is known that every subgroup of a free group $F(A)$ is free, and that the intersection of finitely generated subgroups is finitely generated (Howson, 1954 [4]). Hanna Neumann proved the following inequality,

$$\mathrm{rank}(H \cap K) - 1 \leq 2(\mathrm{rank}(H) - 1)(\mathrm{rank}(K) - 1),$$

and she conjectured (1956 [7]) that in fact,

$$\mathrm{rank}(H \cap K) - 1 \leq (\mathrm{rank}(H) - 1)(\mathrm{rank}(K) - 1).$$

Intermediate results appeared in the literature since, proving in particular that the conjecture holds when H or K has rank at most 3 (see below), but the general case is still open.

The main result which we discuss here is the proof that the conjecture holds when one of the subgroups is positively generated, that is, it admits a system of generators which can be written without using the inverses of the letters in A. This result is due to J. Meakin and the author and a complete proof can be found in [6]. We also refer to [6] for a more thorough bibliography. Another proof was found independently by B. Khan [3].

The best results known to date on H. Neumann's conjecture are briefly recalled in the first section. Then we make explicit the classical connection between the subgroups of free groups and labeled directed graphs and we discuss a graph-theoretic statement whose proof would imply H. Neumann's conjecture. In the third section we summarize the main arguments in the proof a particular case of the conjecture and its application to the positively generated case. We conclude with an open discussion on the exact scope of the partial result we obtained.

1. Partial results on H. Neumann's conjecture

If H is a finitely generated subgroup of a free group F, we let $\widetilde{r}(H) = \max(\mathrm{rank}(H) - 1, 0)$. Then H. Neumann's conjecture reads

$$\widetilde{r}(H \cap K) \leq \widetilde{r}(H)\widetilde{r}(K),$$

and the bound she proved reads $\widetilde{r}(H \cap K) \leq 2\widetilde{r}(H)\widetilde{r}(K)$. If $x \in F$, we let $H^x = x^{-1}Hx$ and we denote by $H\backslash F/K = \{HxK \mid x \in F\}$ the set of *double cosets*. Note that if $HxK = HyK$, then $\widetilde{r}(H^x \cap K) = \widetilde{r}(H^y \cap K)$. In 1991, Walter Neumann [8] formulated the following, so-called *strengthened Hanna Neumann conjecture*. It states that for each finitely generated subgroup H of F, we have

$$\text{for each finitely generated subgroup } K, \quad \sum_{HgK \in H\backslash F/K} \widetilde{r}(H^g \cap K) \leq \widetilde{r}(H)\widetilde{r}(K). \qquad [\mathbf{SHN}(H)]$$

The (geometric) reason for this new conjecture will become clearer when we discuss graph-theoretic aspects. The best results known to date, as far as we are aware, are that $\mathrm{SHN}(H)$ holds if H has rank 2 (Tardos [10]) or 3 (Dicks and Formanek [2]), and if both H and K have rank at least 3, then [2]

$$\textstyle\sum_{HgK \in H\backslash F/K} \widetilde{r}(H^g \cap K) \leq 2\widetilde{r}(H)\widetilde{r}(K) - 2(\widetilde{r}(H) + \widetilde{r}(K)) + 4.$$

[1]Partial support from 1999 INTAS grant number 1224 *Combinatorial and geometric theory of groups and semigroups and its applications to computer science* is gratefully acknowledged.

[2]LaBRI – Université Bordeaux-I and CNRS – 351 cours de la Libération – 33405 Talence Cedex – France. Pascal.Weil@labri.fr

2. Graphs and subgroups of free groups

If v is a vertex of a graph Γ (directed or not), the *v-core* of Γ is the least subgraph containing v and all the undirected loops of Γ around v. In particular, the v-core of Γ is connected and every vertex in it is adjacent to two different edges, except maybe v itself.

An *A-graph* is a triple $\Gamma = (V, E, \lambda)$ where (V, E) is a directed graph and $\lambda: E \to A$ is a *labeling function*. Γ is *admissible* if for each vertex $v \in V$ and each letter $a \in A$, at most one edge starting (resp. ending) in v has label a.

Finally, if $\Gamma = (V, E, \lambda)$ and $\Delta = (W, F, \lambda)$ are admissible A-graphs, the *fiber product* $\Gamma \times_A \Delta$ is the admissible A-graph with vertex set $V \times W$ and edges of the form $(e, f): (v, w) \xrightarrow{a} (v', w')$, for all edges $e: v \xrightarrow{a} v'$ in E and $f: w \xrightarrow{a} w'$ in F with the same label $a \in A$.

To describe the *admissible A-graph $\Gamma(H)$ associated with a subgroup H* of $F(A)$, we first consider the A-graph $\Gamma_0(H)$ with vertex set $F(A)/H$ and edge set $F(A)/H \times A$, with $(Hx, a): Hx \xrightarrow{a} Hxa$. ($\Gamma_0(H)$ represents the action of $F(A)$ by right translations on the set of right H-cosets.) Next we let $\Gamma(H) = (V(H), E(H), \lambda)$ be the H-core of $\Gamma_0(H)$. Then $\Gamma(H)$ is finite if and only if H is finitely generated, and $\tilde{r}(H) = |E(H)| - |V(H)| = \text{co-rank}(\Gamma(H)) - 1$. $\Gamma(H)$ can also be defined in terms of immersions on the bouquet of $|A|$ circles (Stallings [9]), or in terms of inverse automata (Margolis and Meakin [5]). Moreover, if H is given by a finite set of generators, $\Gamma(H)$ can be effectively constructed (see [9]) (in fact, in polynomial time [1]). Finally, if H and K are subgroups of $F(A)$, then $\Gamma(H \cap K)$ is isomorphic to the (H, K)-core of $\Gamma(H) \times_A \Gamma(K)$ [9].

Note also that if Γ is a (finite) connected admissible A-graph with a distinguished vertex v such that Γ is its own v-core, then there exists a unique (finitely generated) subgroup H of $F(A)$ such that $\Gamma(H)$ is isomorphic to Γ, in an isomorphism mapping v to vertex H.

Now if Γ is an admissible A-graph, we let $\mathcal{R}(\Gamma) = \sum(|E(\Gamma_i)| - |V(\Gamma_i)|)$, where the sum runs over all the connected components Γ_i of Γ which are not trees. If Γ is connected, then $\mathcal{R}(\Gamma) = \text{co-rank}(\Gamma) - 1$. It follows from the above discussion that H. Neumann's conjecture states that if Γ and Δ are connected admissible A-graphs and Θ is a connected component of $\Gamma \times_A \Delta$, then $\mathcal{R}(\Theta) \leq \mathcal{R}(\Gamma)\mathcal{R}(\Delta)$. Furthermore, one can verify [8, 10] that $\text{SHN}(H)$ holds if and only if, for each connected admissible A-graph Δ, $\mathcal{R}(\Gamma(H) \times_A \Delta) \leq \mathcal{R}(\Gamma(H))\mathcal{R}(\Delta)$. In fact, one can assume w.l.o.g. that every vertex of Δ is adjacent to at least two edges (we say that Δ is a *core graph*). Thus the strengthened conjecture reduces to showing that if Γ and Δ are connected admissible A-graphs and Δ is a core graph, then

$$\mathcal{R}(\Gamma \times_A \Delta) \leq \mathcal{R}(\Gamma)\mathcal{R}(\Delta). \qquad \textbf{[Graph version of SHN]}$$

3. A particular case

Here we restrict our attention to the case where $A = \{a, b\}$. For each vertex v of an admissible A-graph Γ, let n_v be the number of edges out of v, and let $m_v = n_v - 1$. Then $m_v \in \{-1, 0, 1\}$, and we say that the vertex v is *positive* (resp. *zero*, *negative*) if m_v is equal to 1 (resp. 0, -1). In particular, if $P(\Gamma)$ and $N(\Gamma)$ are the sets of positive and negative vertices in Γ, then we have

$$|E(\Gamma)| - |V(\Gamma)| = \textstyle\sum_{v \in V(\Gamma)} m_v = |P(\Gamma)| - |N(\Gamma)|.$$

We prove the following theorem.

Theorem 1 *Let Γ and Δ be connected admissible A-graphs. If Δ is a core graph and Γ has n negative vertices, then $\mathcal{R}(\Gamma \times_A \Delta) \leq (\mathcal{R}(\Gamma) + n)\mathcal{R}(\Delta)$.*

The main steps of the proof are as follows. We consider in Γ so-called *basic paths*, which are positively labeled paths starting in a positive vertex, ending in a negative vertex, and visiting only zero vertices inbetween. The first important lemma states that if Δ is a core graph and if $v_1, \ldots, v_t$

are the negative vertices of Δ, then there exist pairwise distinct positive vertices $u_1, \ldots, u_t$ and for each $1 \leq i \leq t$, a basic path from u_i to v_i. This collection of t pairwise edge-disjoint basic paths is called a *matching* for Δ.

Next we consider connected admissible A-graphs Γ and Δ such that Δ is a core graph, Θ the union of the connected components of $\Gamma \times_A \Delta$ which are not trees, $(u_1 \to v_1, \ldots, u_t \to v_t)$ a matching for Δ, and p_i the label of the basic path $u_i \to v_i$. For each positive vertex u' of Γ such that (u', u_i) is a (positive) vertex of Θ, we show that a non-empty prefix of p_i labels a basic path of Θ. Moreover, the negative vertices at the end of the basic paths obtained in this fashion are pairwise distinct when $i = 1, \ldots, t$ and u' runs over the positive vertices of Γ. Now the positive vertices (u', u) of Θ not involved in this partial matching must be such that $u \notin \{u_1, \ldots, u_t\}$. This is then used to show that $|P(\Theta)| - |N(\Theta)| \leq |P(\Gamma)|(|P(\Delta)| - |N(\Delta)|)$. The result follows since $\mathcal{R}(X) = |P(X)| - |N(X)|$ for $X = \Gamma, \Delta, \Theta$.

Corollary 2 *If H is a finitely generated subgroup of $F(\{a, b\})$ and $\Gamma(H)$ has n negative vertices, then $\sum_{HgK \in H\backslash F/K} \tilde{r}(H^g \cap K) \leq (\tilde{r}(H) + n)\tilde{r}(K)$.*

It is not difficult to show that if H is positively generated, then $\Gamma(H)$ has no negative vertices, so SHN(H) holds for these subgroups.

4. Conclusion

One can show that if $\varphi: F(B) \to F(C)$ is an injective morphism between free groups, H is a finitely generated subgroup of $F(B)$ and SHN($\varphi(H)$) holds, then SHN(H) holds as well. Since the morphism φ from $F(\{x_1, \ldots, x_n\})$ to $F(A)$ given by $\varphi(x_i) = a^i b^i$ for each i is injective and maps a positively generated subgroup to another, we now have proved the announced result.

Theorem 3 *If H is a positively generated subgroup of a free group, then SHN(H) holds.*

It is interesting to note that the property of being positively generated is not preserved by the automorphisms of the free group, and much less by the inverses of injective morphisms between free groups. It would be very useful, to delimit the scope of our result, to characterize those subgroups H of $F(A)$ for which there exists an injective endomorphism φ of $F(A)$ such that $\varphi(H)$ is positively generated, or at least, to give a non-trivial necessary condition for a subgroup to fall in this class. Indeed, SHN(H) holds for all these subgroups.

References

[1] J.-C. Birget, S. Margolis, J. Meakin, P. Weil. PSPACE-completeness of certain algorithmic problems on the subgroups of free groups, *Theoret. Comp. Sci.* **242** (2000) 247–281.

[2] W. Dicks and E. Formanek. The rank three case of the Hanna Neumann conjecture, to appear.

[3] B. Khan. The Hanna Neumann conjecture is true when one subgroup has a positive generating set, preprint.

[4] A.G. Howson. On the intersection of finitely generated free groups, *J. London Math. Soc.* **29** (1954) 428–434.

[5] S. Margolis, J. Meakin. Free inverse monoids and graph immersions, *Intern. J. of Algebra and Computation* **3** (1993) 79-100.

[6] J. Meakin, P. Weil. Subgroups of free groups: a contribution to the Hanna Neumann conjecture, submitted.

[7] H. Neumann. On intersections of finitely generated subgroups of free groups, *Publ. Math. Debrecen* **4** (1956) 186–189. Addendum, *Publ. Math. Debrecen* **5** (1958) 128.

[8] W.D. Neumann. On intersections of finitely generated subgroups of free groups, in L.G. Kovács (ed.) *Groups - Canberra 1989*, Lect. Notes in Math. **1456**, 161–170, Springer, 1990.

[9] J.R. Stallings. Topology of finite graphs, *Invent. Math.* **71** (1983) 551–565.

[10] G. Tardos. On the intersection of subgroups of a free group, *Invent. Math.* **108** (1992) 29–36.