Profinite methods in semigroup theory

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A highlight of the Conference held at the University of Nebraska-Lincoln in May 2000 was the announcement by Rhodes of a result [84] which was to constitute the last step in the proof of the decidability of the complexity of finite semigroups and automata (the *complexity problem*). This problem has been open for more than thirty years and has inspired many active branches of semigroup theory.

We must state straightaway that about a year after the Conference and while a first version of this paper was already in press, Rhodes and Steinberg discovered a mistake in an earlier paper by Almeida and Weil [28]. Unfortunately, this error invalidated an important link in the chain of results which was to settle the complexity problem. Thus this problem is still open... More details about this mistake and its repercussions will be given in the rest of this introduction and in the relevant parts of this survey (Sections 5.3 and 7).

Rhodes' result fits in a strategy which aimed at solving the complexity problem by a sequence of results, often very interesting in their own right. In particular, Rhodes' paper [84] actually states and proves a more specific result, which seemed then to complete the whole construction.

As we mentioned above, there is now a gap in this construction but it is interesting nonetheless to consider the methods which participate in it, as all of them have had a profound impact on the development of semigroup theory. A number of techniques and notions have come to play a role in the study of the complexity problem, which did not exist in the 1960s when the problem was stated, notably the notion of pseudovariety of finite semigroups in the 1970s, the introduction of category-theoretic tools in semigroup theory in the 1980s, and the introduction of profinite methods.

This survey concentrates on the latter topic, which is especially interesting in the context outlined above since profinite semigroup theory provides the framework in which to formulate the precise questions which Rhodes answered, and makes it possible to understand the importance of the recent paper by McCammond [61], another essential ingredient in the study of the decidability of complexity. Our paper aims to speak to both specialists and non-specialists. For the latter it offers an outlook on semigroup theory which will allow them to appreciate the way in which the study of relatively free profinite semigroups has enriched and modified finite semigroup theory, and has provided useful tools for the study of the complexity problem. The end of the paper, somewhat more technical, is intended to help the specialists of pseudovariety theory who were not directly involved in the most recent results in that direction, to get a general view of the present situation.

The part of semigroup theory we are concerned with here is the theory of finite semigroups, and especially the classification of finite semigroups in pseudovarieties. The introduction of profinite methods in the mainstream of semigroup theory is rather recent. Reiterman's seminal theorem was published in 1982, and Almeida's work developing the theory of implicit operations in the specific framework of semigroup theory started in the mid 1980s. Important results started flowing rapidly,

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and there are at present a number of surveys and basic references, some dated [3, 117] and some still very useful, notably Almeida's book [5] but also [25, 20]. However the pace of activity and the wealth of deep results have been such in the late 1990s that it is no doubt useful to stop for a moment and give a new survey.

To put it very briefly, the general idea is that in order to study a class of finite semigroups $\mathbf{V}$, it may be sufficient to study a projective limit of elements of $\mathbf{V}$. In general, the resulting semigroup is uncountably infinite, but it is naturally equipped with a compact topology, so that its study retains a certain flavor of finite semigroup theory. Moreover, with respect to $\mathbf{V}$, this semigroup behaves somewhat like the free object of a variety (in Birkhoff's sense).

As we will see, the applications of this approach concern mostly the study of the lattice $\mathcal{P}\mathbf{s}$ of pseudovarieties of finite semigroups: either the determination of specific sublattices (e.g. the lattice of the semidirectly closed aperiodic pseudovarieties; or the lattice of sub-pseudovarieties of certain interesting pseudovarieties), or the study of certain structuring operations on $\mathcal{P}\mathbf{s}$, including many decomposition and decidability problems such as the complexity problem.

There are also many important connections with the theory of formal languages, via Eilenberg's variety theorem which relates pseudovarieties of finite semigroups and certain classes of recognizable languages closed under natural operations. In fact, the connections with language theory are two-way, as languages and automata provide very useful tools in certain proofs.

After we present the foundations of the theory in Section 1, we give a few structural results in Section 2, both to have instructive examples and because some of the structure theorems we discuss served as an inspiration in the elaboration of more general results. In Section 3, we survey the contribution of profinite methods to the pointlike problem, which gives us an opportunity to study in detail a type of compactness argument often used in the theory. Sections 4 and 5 are devoted to the operation of the Mal'cev product and the semidirect product on the lattice $\mathcal{P}\mathbf{s}$, or rather to the applications of the theory of relatively free profinite semigroups and of pseudo-identities to this study. The precise definition of the complexity of a finite semigroup can be found in Section 5. The case of products with pseudovarieties of groups, which gives rise to a number of specific problems and specific connections with the vast domain of profinite group theory, is reviewed in Section 6.

Finally, we concentrate more at length in Section 7 on the recent and very subtle developments with respect to the semidirect product operation on $\mathcal{P}\mathbf{s}$, and especially on the notions of hyper-decidable, reducible and tame pseudovarieties and their application to the decidability of iterated semidirect products. These notions are crucial to the understanding of the statement and the proof of the recent results of McCammond and Rhodes, and of the reasons why these results do not, as it was hoped, complete the proof of the decidability of the complexity of a finite semigroup. The last section proposes, in lieu of a conclusion, a brief discussion of some extensions of the results presented herein.

To simplify the exposition, we have chosen to express most of the results in terms of pseudovarieties of semigroups. Analogous results also hold for pseudovarieties of monoids, and in dealing with examples, we sometimes skip from semigroup land to monoid land at short notice. This should not trouble the readers. Interesting extensions also exist in the realm of ordered semigroups and monoids, with important applications: these are briefly reviewed in the last section.

1 Pseudovarieties and pseudo-identities

The general references on pseudovarieties are the books of Eilenberg [45] and Pin [67], and on pseudo-identities, the book of Almeida [5].

1.1 Varieties and pseudovarieties

Recall that a *variety* is a class of algebras (here, for the most part, semigroups or monoids) closed under taking sub-algebras, homomorphic images (or quotients) and direct products. We assume that the reader is familiar with the basic concepts underlying the statement of Birkhoff's theorem on varieties, notably the concept of free object of a variety over a set [38]. We denote by $F_A(\mathcal{V})$ the free object of the variety $\mathcal{V}$ on the alphabet A. Here we call *alphabet* any non-empty set.

Birkhoff's theorem states that varieties are exactly the so-called *equational classes* of algebras, that is, the classes which can be defined by a set of *identities*. In the case of semigroups, this means that each variety is described by a set of formal equalities $u = v$ where u and v are elements of a free semigroup. If Σ is a set of identities, we denote by $[\Sigma]$ the variety defined by these identities. For instance, $[xy = yx]$ is the variety of commutative semigroups. In order to handle varieties of monoids, we also allow the use of the empty word. For instance, if n is a fixed integer, $[x^n = 1]$ is the variety of monoids consisting of the groups of exponent n. The classification of semigroups in varieties gives rise to classical problems, like the decidability of the word problem in some $F_A(\mathcal{V})$ (do two words represent the same element?), the identity problem (does an identity follow from a given finite set of identities?), the finite basis problem (can the variety $\mathcal{V}$ be defined by a finite set of identities?), etc.

Apart from the trivial variety, no variety consists only of finite algebras, because of the possibility of taking arbitrary direct products. The analogue of varieties for the classification of finite algebras, is the notion of pseudovariety: a *pseudovariety* is a class of finite algebras closed under taking sub-algebras, homomorphic images and finite direct products. In particular, when we use the word pseudovariety we always mean, but do not always write, a class of finite algebras.

Equational classes of finite algebras (i.e. the finite elements of varieties) are clearly pseudovarieties. To distinguish varieties and pseudovarieties, we denote by $[\![\Sigma]\!]$ the class of finite algebras satisfying the identities in Σ. For instance $\mathbf{Com} = [\![xy = yx]\!]$ is the pseudovariety of (finite) commutative semigroups, and $\mathbf{B} = [\![x^2 = x]\!]$ is the pseudovariety of (finite) idempotent semigroups.

However, most interesting pseudovarieties are not equational. What's more, they often satisfy no non-trivial identity. Important examples are: $\mathbf{S}$, the pseudovariety of all (finite) semigroups; $\mathbf{M}$, the pseudovariety of all monoids; $\mathbf{A}$, the pseudovariety of *aperiodic* semigroups (in which every subgroup is trivial); $\mathbf{N}$, the pseudovariety of *nilpotent* semigroups (in which every long enough product is 0); $\mathbf{J}$, the pseudovariety of $\mathcal{J}$*-trivial* semigroups (in which every principal 2-sided ideal has a single generator). Groups also form a pseudovariety of monoids: every submonoid of a finite group is again a group, in contrast with the situation prevailing in arbitrary cardinality. This pseudovariety is written $\mathbf{G}$. We denote by $\mathbf{G}_p$, $\mathbf{G}_{sol}$ and $\mathbf{G}_{nil}$ the pseudovarieties of p-groups, solvable groups and nilpotent groups.

None of these pseudovarieties satisfies a non-trivial identity. Indeed, if u and v are distinct words in a free semigroup (or monoid), there exists a morphism into an element of $\mathbf{A}$, $\mathbf{N}$, $\mathbf{J}$, $\mathbf{G}$, $\mathbf{G}_p$, etc, which maps u and v to distinct elements; which is another way of saying that the free semigroup (or monoid) is residually $\mathbf{A}$, $\mathbf{N}$, $\mathbf{J}$, $\mathbf{G}$, $\mathbf{G}_p$, etc. For $\mathbf{N}$, it suffices to take the quotient of the free semigroup by the ideal of all words of length greater than that of u and v. The natural projection onto that semigroup maps u and v onto themselves, and hence separates them. Thus the free semigroup is residually $\mathbf{N}$, and also residually $\mathbf{V}$ for each pseudovariety $\mathbf{V}$ containing $\mathbf{N}$ (e.g. $\mathbf{A}$, $\mathbf{J}$).

For groups, the classical argument is the following: consider the set P of the prefixes of u and v, that is, the (possibly empty) words w such that u or v can be written in the form ww'. For each letter a, let $\mu(a)$ be the partial one-to-one function from P to P which maps $w \in P$ to wa if $wa \in P$.

Next, let $\sigma(a)$ be a permutation of P extending $\mu(a)$. The mapping $a \mapsto \sigma(a)$ induces a morphism σ from the free semigroup into the permutation group of P. It is easily verified that the image of the empty word $1 \in P$ by $\sigma(u)$ (resp. $\sigma(v)$) is u (resp. v) itself. Thus $\sigma(u) \neq \sigma(v)$, completing the proof that the free group is residually $\mathbf{G}$. For p-groups, the argument is more complex, and can be found for instance in [45, vol. B].

Eilenberg and Schützenberger first proposed, in the 1970s, a framework to allow the description of pseudovarieties by identities. A semigroup S is said to *ultimately satisfy* a sequence of identities $(u_n = v_n)_{n \geq 0}$ if it satisfies almost all these identities (all but a finite number of them). It is rapidly seen that the class of finite semigroups which ultimately satisfy a given sequence of identities is a pseudovariety. Eilenberg and Schützenberger proved that the converse also holds [45, vol. B] (see also [67]).

Theorem 1.1 *A class of finite algebras is a pseudovariety if and only if it is the class of algebras which ultimately satisfy a given sequence of identities.*

For instance, $\mathbf{G}$ is ultimately defined by the sequence $(x^{n!} = 1)_n$, since every finite group G satisfies $x^{n!} = 1$ for all $n \geq |G|$. Similarly, $\mathbf{G}_p$ is ultimately defined by the sequence $(x^{p^n})_n$.

It is also well-known [45, 67] that $\mathbf{A}$ is ultimately defined by $(x^{n+1} = x^n)_n$; $\mathbf{J}$ is ultimately defined by the sequence alternating the identities $x^{n+1} = x^n$ and $(xy)^n = (yx)^n$; and $\mathbf{N}$ is ultimately defined by the sequence alternating the identities $x^n y = x^n$ and $yx^n = x^n$, which we can shorten to $x^n = 0$.

1.2 Relatively free profinite semigroups

It is not surprising that this idea of ultimate definition of a pseudovariety by a sequence of identities, akin to a limit process, can be better expressed in a topological framework. The notions we present now, which lead to Reiterman's theorem on pseudovarieties and pseudo-identities (1982) do exactly that.

Several points of view can be adopted to handle limit processes for sequences of identities, that is, of words. Let A be a finite alphabet and let $\mathbf{V}$ be a pseudovariety of semigroups. We may first associate with $\mathbf{V}$ a distance $d_{\mathbf{V}}$ on the free semigroup A^+, defined as follows.

If $u, v \in A^+$, we let $r_{\mathbf{V}}(u, v)$ be the least cardinality of an element $V \in \mathbf{V}$ such that there exists a morphism $\varphi: A^+ \to V$ satisfying $\varphi(u) \neq \varphi(v)$. Next we let $d_{\mathbf{V}}(u, v) = 2^{-r_{\mathbf{V}}(u,v)}$. If u and v cannot be separated by a morphism into an element of $\mathbf{V}$, we let $d_{\mathbf{V}}(u, v) = 0$.

It is easily verified that $d_{\mathbf{V}}$ is an écart (or pseudometric): $d_{\mathbf{V}}(u, u) = 0$, $d_{\mathbf{V}}(u, v) = d_{\mathbf{V}}(v, u) \geq 0$ and $d_{\mathbf{V}}$ satisfies the triangle inequality. More than that, $d_{\mathbf{V}}$ is ultrametric, that is,

$$d_{\mathbf{V}}(u, w) \leq \max(d_{\mathbf{V}}(u, v), d_{\mathbf{V}}(v, w)),$$

and precompact: for each $\varepsilon > 0$, there exists a finite set of words $(u_i)_{1 \leq i \leq n}$ such that every word sits within distance ε of one of the u_i. For $d_{\mathbf{V}}$ to be a distance, we need to have also that $d_{\mathbf{V}}(u, v) > 0$ if $u \neq v$. That is, $d_{\mathbf{V}}$ is a distance if and only if $\mathbf{V}$ satisfies no non-trivial identity on alphabet A.

The écart $d_{\mathbf{V}}$ defines a uniform structure on A^+, and through it a topology, which is Hausdorff if and only if $\mathbf{V}$ satisfies no non-trivial identity on alphabet A (see [79] for a discussion of which uniformities may arise in this fashion). Multiplication is continuous for this topology. For instance, if $\mathbf{V} = \mathbf{G}$ and u is a word, then the sequence $u^{n!}$ $d_{\mathbf{V}}$-converges to the empty word 1.

Next we consider the completion of the uniform space $(A^+, d_{\mathbf{V}})$, which we denote by $\hat{F}_A(\mathbf{V})$. In the case where $(A^+, d_{\mathbf{V}})$ is not Hausdorff, this means first taking the quotient of A^+ by the relation given by $u \sim v$ if $d_{\mathbf{V}}(u, v) = 0$, a congruence, and then taking the usual completion of the resulting space, see [5, 25]. Since $d_{\mathbf{V}}$ is ultrametric and precompact, $\hat{F}_A(\mathbf{V})$ is a compact and totally disconnected topological semigroup. We write $\imath$ the natural mapping from A^+ into $\hat{F}_A(\mathbf{V})$: $\imath$ is a morphism, which is one-to-one if and only if $\mathbf{V}$ satisfies no non-trivial equality. In fact, one can show that $\imath(A^+)$ is the free object on A in the variety generated by the elements of $\mathbf{V}$. For this reason, we slightly abuse notation and write $F_A(\mathbf{V})$ for $\imath(A^+)$.

The same topological semigroup can also be described as the *pro-*$\mathbf{V}$ *completion* of the free semigroup. Recall that a topological semigroup S is said to be *profinite* if it is the projective limit of a collection of finite semigroups (which are implicitly assumed to be equipped with the discrete topology), see for instance [25]. This is equivalent to being compact and totally disconnected, or compact and zero-dimensional. The *profinite completion* of a semigroup S is the projective limit of its finite quotients. Similarly, a semigroup is *pro-*$\mathbf{V}$ if it is a projective limit of elements of $\mathbf{V}$, or equivalently, if it is profinite and its finite continuous homomorphic images are in $\mathbf{V}$. The *pro-*$\mathbf{V}$ *completion* of S is the projective limit of its quotients in $\mathbf{V}$.

One can show that $\hat{F}_A(\mathbf{V})$ is the pro-$\mathbf{V}$ completion of A^+, and that it is the *free pro-*$\mathbf{V}$ *semigroup* on A in the sense that it is characterized, up to homeomorphism, by the following universal property: if S is a pro-$\mathbf{V}$ semigroup and $\varphi\colon A \to S$ is a mapping (resp. $\varphi\colon A^+ \to S$ is a morphism), then there is a unique continuous morphism $\hat{\varphi}\colon \hat{F}_A(\mathbf{V}) \to S$ such that $\varphi = \hat{\varphi} \circ \imath$.

$$\begin{tikzcd} A^+ \arrow[r, "\imath"] \arrow[dr, "\varphi"'] & \hat{F}_A(\mathbf{V}) \arrow[d, "\hat{\varphi}"] \\ & S \end{tikzcd}$$

Observe that if $\mathbf{V}$ and $\mathbf{W}$ are pseudovarieties and $\mathbf{V} \subseteq \mathbf{W}$, then every pro-$\mathbf{V}$ semigroup is pro-$\mathbf{W}$ as well. Then it follows from the universal property above that the identity function on A induces a continuous onto morphism from $\hat{F}_A(\mathbf{W})$ onto $\hat{F}_A(\mathbf{V})$, termed the *natural projection.*

The free pro-$\mathbf{S}$ semigroup is simply called the *free profinite semigroup*, written $\widehat{A^+}$, and if $\mathbf{V}$ is a pseudovariety, the natural projection from $\widehat{A^+}$ onto $\hat{F}_A(\mathbf{V})$ is denoted by $\mathrm{pr}_{\mathbf{V}}$. Again, if $\varphi\colon A^+ \to S$ is a morphism and $S \in \mathbf{V}$, then φ induces a continuous morphism $\hat{\varphi}\colon \widehat{A^+} \to S$, which can be factored through $\hat{F}_A(\mathbf{V})$ via $\mathrm{pr}_{\mathbf{V}}$.

$$\begin{tikzcd} A^+ \arrow[r, "\imath"] \arrow[d, "\varphi"'] & \widehat{A^+} \arrow[dl, "\hat{\varphi}"'] \arrow[d, "\mathrm{pr}_{\mathbf{V}}"] \\ S & \hat{F}_A(\mathbf{V}) \arrow[l] \end{tikzcd}$$

1.3 Implicit operations

Another point of view which is frequently adopted in the literature, notably in Reiterman's paper [82] and in Almeida's work [5], is that of implicit operations.

Let A be an n-letter alphabet. The *A-ary*, or *n-ary implicit operations over the pseudovariety* $\mathbf{V}$ are the the A-ary operations defined on all the elements of $\mathbf{V}$, which can be added to the semigroup signature without losing morphisms. For instance, taking the inverse of an element is a unary operation, which is well defined on each finite group, and for each (monoid) morphism $\varphi\colon G \to H$ between finite groups, $\varphi(g^{-1}) = \varphi(g)^{-1}$ for all $g \in G$. Thus, the operation $g \mapsto g^{-1}$ is a unary implicit operation over the pseudovariety (of monoids) $\mathbf{G}$. Clearly, the same operation is not well defined on arbitrary finite monoids.

More formally, an A-ary implicit operation x is a collection $x = (x_S)_S$, indexed by the elements of $\mathbf{V}$, such that for each $S \in \mathbf{V}$, x_S is a mapping from S^A into S and for each morphism $\varphi\colon S \to T$ between elements of $\mathbf{V}$, $\varphi \circ x_S = x_T \circ \varphi^A$.

$$\begin{tikzcd} S^A \arrow[r, "\varphi^A"] \arrow[d, "x_S"'] & T^A \arrow[d, "x_T"] \\ S \arrow[r, "\varphi"] & \mathrm{T} \end{tikzcd}$$

The set of all A-ary implicit operations over $\mathbf{V}$ is denoted by $\bar{\Omega}_A\mathbf{V}$, or $\bar{\Omega}_n\mathbf{V}$. This set constitutes a semigroup for the pointwise multiplication, namely the operation $(xy)_S = x_S y_S$. It is equipped with a topology, defined by the distance d, whose definition will look familiar: if x, y are distinct elements of $\bar{\Omega}_A\mathbf{V}$, $r(x, y)$ is the least cardinality of an element $S \in \mathbf{V}$ such that $x_S \neq y_S$, and $d(x, y) = 2^{-r(u,v)}$. For this topology, $\bar{\Omega}_A\mathbf{V}$ is a topological semigroup, and one can verify that it is naturally homeomorphic with $\hat{F}_A(\mathbf{V})$, see for instance [25].

The analogue of the subsemigroup A^+ (in fact, $\imath(A^+)$) in $\bar{\Omega}_A\mathbf{V}$ is the set of so-called explicit operations. Each word in A^+ can be seen as a monomial in $|A|$ non-commuting variables, and as such, it defines an A-ary operation on every semigroup (in $\mathbf{V}$). For instance, if $w = abacb$, then the corresponding A-ary operation on S is given by $(s_a, s_b, s_c) \mapsto s_a s_b s_a s_c s_b$. Each such operation is called an *explicit operation*. Note that the product of explicit operations in $\bar{\Omega}_A\mathbf{V}$ corresponds to the concatenation of the words defining those operations.

The set of explicit operations is written $\Omega_A\mathbf{V}$ and it is dense in $(\bar{\Omega}_A\mathbf{V}, d)$, justifying the notation. The homeomorphism result mentioned above states precisely that $\bar{\Omega}_A\mathbf{V}$ is a pro-$\mathbf{V}$ semigroup, the continuous morphism from $\hat{F}_A(\mathbf{V})$ into $\bar{\Omega}_A(\mathbf{V})$ induced by mapping each letter $a \in A$ on the explicit operation $a \in \Omega_A\mathbf{V}$ is a homeomorphism, and the image of $F_A(\mathbf{V})$ under this homeomorphism is $\Omega_A\mathbf{V}$.

1.4 Some elements of free profinite semigroups

The structural properties of the semigroups $\hat{F}_A(\mathbf{V})$ (or $\bar{\Omega}_A\mathbf{V}$) are difficult to apprehend in general. We will see specific examples of such semigroups in Section 2. Here we discuss the properties of certain elements in free profinite semigroups.

As we saw in Section 1.3, each element $x \in \hat{F}_A(\mathbf{V})$ can be seen as an A-ary operation defined on each finite semigroup $S \in \mathbf{V}$. But in general, this operation is not *computable*, in the sense that there does not exist an algorithm which, given an element S of $\mathbf{V}$, produces a table for the operation x_S. Indeed, $\hat{F}_A(\mathbf{V})$ is uncountable in general, while there can be only countably many algorithms.

The simplest class of elements of $\hat{F}_A(\mathbf{V})$ is formed by the elements of $\imath(A^+) = F_A(\mathbf{V})$, i.e. the explicit operations. These are evidently computable.

It is a general property of compact semigroups that if $x \in \hat{F}_A(\mathbf{V})$, the closed subsemigroup $\overline{\langle x \rangle}$ generated by x contains a single idempotent, written x^ω. This idempotent sits in the minimal ideal of $\overline{\langle x \rangle}$, a group, and we denote by $x^{\omega-1}$ the inverse in that group of $x^\omega x = x^{\omega+1}$.

The notation x^ω is classically used in finite semigroup theory [67]: if x is an element of a finite semigroup S, then x^ω denotes the only idempotent power of x and one shows that $x^\omega = x^{n!}$ for each $n \geq |S|$. These two uses of the ω-power notation are not in contradiction: one verifies easily that if S is an element of $\mathbf{V}$, φ is a continuous morphism from $\hat{F}_A(\mathbf{V})$ into S and $x \in \hat{F}_A(\mathbf{V})$, then $\varphi(x^\omega) = \varphi(x)^\omega$. In particular, if x is computable, then so is x^ω.

This notation allows the compact description of a number of important pseudovarieties. For instance, finite groups are exactly the finite monoids in which $x^\omega = 1$ for each element x. Since in a finite semigroup, x^ω is the ultimate value of the sequence $x^{n!}$, this observation can be seen as shorthand for the ultimate definition of $\mathbf{G}$ by a sequence of identities as in Section 1.1. Getting a little ahead of ourselves, we express this characterization by writing $\mathbf{G} = [\![x^\omega = 1]\!]$.

Similarly, we have

$$\mathbf{A} = [\![x^{\omega+1} = x^\omega]\!], \quad \mathbf{J} = [\![x^{\omega+1} = x^\omega, (xy)^\omega = (yx)^\omega]\!], \quad \mathbf{N} = [\![x^\omega = 0]\!].$$

Let κ be the signature composed of the semigroup multiplication and the unary operation $x \mapsto x^{\omega-1}$. The above examples show that this signature constitutes a sufficient vocabulary to describe many important pseudovarieties (see Sections 2 and 7 for a discussion of questions raised by this approach).

However, not all pseudovarieties can be described this way. For instance, one can show that the pseudovarieties of p-groups, nilpotent groups or solvable groups cannot be defined by a set of identities in the signature κ. However, they can be described by using other elements of the free profinite monoid, with a somewhat more complicated definition.

If p is a prime number and $x \in \widehat{A^*}$, then the sequence $(x^{p^{n!}})_n$ is convergent, and its limit is written x^{p^ω} [10]. (The sequence $(x^{p^n})_n$ does not converge, contrary to the author's assertion in [117].) Then one can verify that $\mathbf{G}_p$ is described by the equality $x^{p^\omega} = 1$ in the following sense: let S be a finite monoid, let $s \in S$, let $\varphi\colon \widehat{A^*} \to S$ be a continuous morphism, and let $x \in \widehat{A^*}$ be such that $\varphi(x) = s$; finally, let $s^{p^\omega} = \varphi(x^{p^\omega})$. One can verify that s^{p^ω} is in fact independent of the choice of φ and of the element $x \in \varphi^{-1}(s)$. Moreover, a finite monoid S is a p-group if, for each $s \in S$, we have $s^{p^\omega} = 1$.

Similarly, for $x, y \in \widehat{A^*}$, let us write $[x, y] = x^{\omega-1}y^{\omega-1}xy$. Then we let $z_1 = [x, y]$ and $z_{n+1} = [z_n, y]$. The sequence $(z_{n!})_n$ converges, and if we write $[x,_\omega y]$ for its limit, then the pseudovariety described by $[x,_\omega y] = 1$ is that of the finite Engel groups, that is $\mathbf{G}_{nil}$ [10]. A conjectural description of the pseudovariety of solvable groups in the same spirit is briefly discussed in [10, Ex. 2.7].

1.5 Pseudo-identities: Reiterman's theorem

The pseudovarieties discussed in Section 1.4 all admit a description by formal equalities between elements of free profinite semigroups, much in the spirit of the equational description of varieties. This is a general fact, formalized by Reiterman's theorem.

By definition, a *pseudo-identity* is a formal equality of the form $u = v$, where u and v are elements of $\widehat{A^+}$ for some finite alphabet A. We say that a finite semigroup S *satisfies the pseudo-identity* $u = v$ if for any continuous morphism $\varphi\colon \widehat{A^+} \to S$, we have $\varphi(u) = \varphi(v)$. One verifies easily that if Σ is a set of pseudo-identities, the class $[\![\Sigma]\!]$ of all finite semigroups which satisfy the elements of Σ is a pseudovariety. In 1982, Reiterman proved the following theorem [82].

Theorem 1.2 *A class of finite semigroups* $\mathbf{V}$ *is a pseudovariety if and only if there exists a set of pseudo-identities* Σ *such that* $\mathbf{V} = [\![\Sigma]\!]$.

In fact, Reiterman's theorem is a little more precise, in that it describes the sub-pseudovarieties of a pseudovariety $\mathbf{V}$ by pseudo-identities which are formal equalities between elements of free pro-$\mathbf{V}$ semigroups, see [5].

Section 1.4 above provides a number of examples of sets of pseudo-identities describing classical pseudovarieties. For all these pseudovarieties, we had already seen descriptions according to Eilenberg and Schützenberger's Theorem (Theorem 1.1 above), by means of sequences of identities. One can verify that if $\mathbf{V}$ is ultimately defined by a sequence of identities $(u_n = v_n)_n$, then $\mathbf{V}$ is defined by the limit points of the sequence $(u_n = v_n)_n$, that is, by the pseudo-identities of the form $u = v$ such that there exist $1 \leq n_1 < n_2 < \ldots$ with $u = \lim_k u_{n_k}$ and $v = \lim_k v_{n_k}$. However, the usage of pseudo-identities provides a better grasp on the decidability, and even the algorithmic complexity, of the membership problem in the pseudovariety.

The description of pseudovarieties by means of sets of pseudo-identities raises the same questions that are usually considered for varieties, such as the existence of finite basis of pseudo-identities, or the deduction of a pseudo-identity from a given set of pseudo-identities. Almeida gave a characterization of the closed sets of pseudo-identities, i.e. the sets of pseudo-identities Σ for which there exists a pseudovariety $\mathbf{V}$ such that Σ is the class of all the pseudo-identities satisfied by all the elements of $\mathbf{V}$ [5]. We note that Reiterman's proof of Theorem 1.2 relies on the compactness properties of the free profinite semigroup, and hence indirectly on Tikhonov's theorem and the axiom of choice. Higgins gave an algebraic proof of the same theorem, which does not rest on the usage of that axiom [53].

1.6 On the applications of relatively free profinite semigroups

In a sense, the free pro-$\mathbf{V}$ semigroup $\hat{F}_A(\mathbf{V})$ *contains all the information about the A-generated elements of* $\mathbf{V}$, since it is the projective limit of these finite semigroups. Thus, any information we can get on the structure of $\hat{F}_A(\mathbf{V})$ can be translated into information on the A-generated elements of $\mathbf{V}$.

One important result concretely connecting the free pro-$\mathbf{V}$ semigroups and the elements of $\mathbf{V}$ concerns the *variety of languages* associated with $\mathbf{V}$, that is, the class of recognizable languages accepted by a semigroup in $\mathbf{V}$ [45, 67]. Almeida showed that if $\mathcal{B}$ is a basis of clopen sets for the topology of $\hat{F}_A(\mathbf{V})$, then the languages of the form $B \cap A^+$ $(B \in \mathcal{B})$ form a basis for the Boolean algebra of the languages of A^+ recognized by a semigroup in $\mathbf{V}$ [5].

Most of the applications of the study of the free pro-$\mathbf{V}$ semigroups however deal with the structure of the lattice of sub-pseudovarieties of $\mathbf{V}$ and the decomposition of pseudovarieties by product operations like the Mal'cev or the semidirect product, the topic of Sections 4 and beyond. Before we can go on with these questions, we give a quick description of the structural properties of a few relatively free profinite semigroups.

2 Calculation of certain relatively free profinite semigroups

In this section, we describe the algebraic and topological structure of $\hat{F}_A(\mathbf{V})$ for certain pseudovarieties $\mathbf{V}$. Proofs and details can be found, for the most part, in [5].

Say a pseudovariety $\mathbf{V}$ is *locally finite* if for each finite alphabet A, there is a bound on the cardinality of the A-generated elements of $\mathbf{V}$. This is equivalent to saying that the free object on A in the variety generated by $\mathbf{V}$ is finite. In that case, topology does not play a useful role in the description of the finitely generated relatively free pro-$\mathbf{V}$ semigroups. Indeed, for each finite alphabet A, $\hat{F}_A(\mathbf{V}) = F_A(\mathbf{V})$ is finite. It is well-known that every pseudovariety generated by a finite collection of semigroups is locally finite. An important example of such a pseudovariety is provided by *semilattices*, $\mathbf{Sl} = [\![xy = yx,\ x^2 = x]\!]$, which is generated by the multiplicative semigroup $\{0, 1\}$. Another example of a locally finite pseudovariety, for which the word problem is well-known [54] and which is not generated by a finite set of semigroups, is that of idempotent semigroups, $\mathbf{B} = [\![x^2 = x]\!]$.

Going back to semilattices, it is also well-known that the free semilattice over alphabet A $F_A(\mathbf{Sl})$ is isomorphic to the semigroup of non-empty subsets of A under union. In this case, the natural morphism from the free semigroup onto the free semilattice is called the *(alphabetic) content morphism*, written c, and it maps a word u to the set $c(u)$ of letters occurring in u. We also denote by c the continuous extension of the content morphism from $\widehat{A^+}$ onto $F_A(\mathbf{Sl})$, so that we can talk of the alphabetic content of an arbitrary element of $\widehat{A^+}$. In fact, if $\mathbf{V}$ is any pseudovariety containing $\mathbf{Sl}$, we again call content morphism the natural projection from $\hat{F}_A(\mathbf{V})$ onto $F_A(\mathbf{Sl})$, so that the following diagram commutes, and the elements of $\hat{F}_A(\mathbf{V})$ also have a well-defined content.

$$\begin{tikzcd}
\widehat{A^+} \arrow[r, "\mathrm{pr}_{\mathbf{V}}"] \arrow[dr, "c"'] & \hat{F}_A(\mathbf{V}) \arrow[d, "c"] \\
 & F_A(\mathbf{Sl})
\end{tikzcd}$$

It is important to note that it is because a pseudovariety $\mathbf{V}$ contains $\mathbf{Sl}$ that it makes sense to talk of the content of an element of $\hat{F}_A(\mathbf{V})$. If we consider the pseudovariety $\mathbf{G}$, which does not contain $\mathbf{Sl}$, and if $a \in A$, there is no natural notion of alphabetic content of the element $a \in \hat{F}_A(\mathbf{G})$. Indeed, for any letter $b \in A$, we have $a = ab^\omega = \lim_n ab^{n!}$ in $\hat{F}_A(\mathbf{G})$ so that we would have to add the letter b to the content of a...

The idea illustrated by the case of semilattices, that distinctions operated between elements of a free pro-$\mathbf{V}$ semigroup (such as the content) can *ipso facto* be operated between elements of free pro-$\mathbf{W}$ semigroups if $\mathbf{V} \subseteq \mathbf{W}$, is again illustrated by the study of the free pro-nilpotent semigroups.

Recall that $\mathbf{N} = [\![x^\omega = 0]\!]$ is the class of semigroups in which long enough products are equal to zero (i.e., an absorbing element). Since $\hat{F}_A(\mathbf{N})$ is the completion of $(A^+, d_{\mathbf{N}})$, every one of its elements is the limit of a sequence $(x_n)_n$ of words. It is not difficult to show that if the length of the x_n is uniformly bounded, then the convergence of the sequence implies that it is ultimately constant. On the other hand, if there is no bound on the length of the x_n, convergence implies that $\lim_n |x_n| = \infty$, so that in every finite nilpotent semigroup $S \in \mathbf{N}$, x_n is eventually 0. With these observations, one can verify that $\hat{F}_A(\mathbf{N}) = A^+ \cup \{0\}$ where 0 is an absorbing element, and is the limit of all the sequences of words whose length tends to infinity. Thus, if $\mathbf{V}$ is a pseudovariety containing $\mathbf{N}$, the natural projection establishes a distinction between the elements of $\imath(A^+)$ in $\hat{F}_A(\mathbf{V})$ and the others, that is, between the explicit and the non-explicit operations: the non-explicit operations are exactly those whose image in the natural projection onto $\hat{F}_A(\mathbf{N})$ is zero. Again, looking at pseudovarieties which do not contain $\mathbf{N}$, this distinction is not founded. Whether we look in $\hat{F}_A(\mathbf{G})$ or in $\hat{F}_A(\mathbf{Sl})$, the element a can be considered as explicit as well as non-explicit, that is, it is the image (in the natural projection) both of explicit and non-explicit elements of $\widehat{A^+}$. Pushing this idea further, Almeida shows that if $\mathbf{V}$ is a pseudovariety containing $\mathbf{N}$, then an element of $\hat{F}_A(\mathbf{V})$ is non-explicit if and only if it can be written in the form $xy^\omega z$ [5].

Similarly, the consideration of the pseudovariety $\mathbf{K} = [\![x^\omega y = x^\omega]\!]$ allows us to talk of the prefixes of length n or even ω of an element of a free pro-$\mathbf{V}$ semigroup: one shows that $\hat{F}_A(\mathbf{K})$ is homeomorphic to the semigroup $A^+ \cup A^{\mathbb{N}}$, where a sequence of words $(u_n)_n$ converges to an infinite word u if for each $k \geq 1$, the sequence of the length k prefixes of the u_n is eventually constant, equal to the length k prefix of u. Thus, if $\mathbf{V}$ is a pseudovariety containing $\mathbf{K}$ (and hence, containing $\mathbf{N}$), then the image $\pi(x)$ of an element $x \in \hat{F}_A(\mathbf{V})$ under the natural projection onto $\hat{F}_A(\mathbf{K})$ is

x itself if x is explicit, or an infinite word which can be considered as the length ω prefix of x otherwise: in that case, $\pi(x)$ is the infinite word $\pi(x) = a_1a_2\cdots$ such that if $(x_n)_n$ is a sequence of words converging to x in $\hat{F}_A(\mathbf{V})$ and $k \geq 1$, then almost all the x_n begin with $a_1a_2\cdots a_k$. The natural projection on the free pro-$\mathbf{D}$ semigroup, where $\mathbf{D} = [\![yx^\omega = x^\omega]\!]$ is the right-left dual of $\mathbf{K}$, allows the definition of the length ω suffix of a non-explicit implicit operation over a sufficiently large pseudovariety.

There exists a vast literature on the free pro-$\mathbf{G}$ monoids, and more generally on the free pro-$\mathbf{H}$ monoids when $\mathbf{H}$ is a pseudovariety of groups: it is not difficult to see that $\hat{F}_A(\mathbf{G})$ is in fact the free profinite group, that is, the profinite completion of the free group and not only of the free monoid. The study of profinite groups arises in certain areas of number theory [47, 93, 44]. In Section 6 we give a short discussion of certain geometric properties of the free pro-$\mathbf{H}$ groups and their Cayley graphs, which play an interesting part in the computation of products involving $\mathbf{H}$. Finally, let us mention that factorization results extending the reduced factorization which is classical in (discrete) free groups, can be found in [24].

Factorization results are also the main feature of the study of free pro-$\mathbf{DS}$ semigroups. The pseudovariety $\mathbf{DS}$ is the class of the finite semigroups in which every regular element sits in a group, or equivalently the class of finite semigroups in which every $\mathcal{D}$-class is a subsemigroup [5, 67]. In spite of this technical definition, $\mathbf{DS}$ is an important pseudovariety, in part because it is the non-regular analogue of the much studied class of completely regular pseudovarieties. Part of its interest stems also from the fact that we have a factorization theorem for the elements of the free pro-$\mathbf{DS}$ semigroups, due to Almeida and Azevedo [12, 5].

This factorization theorem, which we will presently discuss, is at the root of most of the complete structural descriptions of relatively free profinite semigroups known to date, and of many join decomposition results.

In short, Almeida and Azevedo proved that $\hat{F}_A(\mathbf{DS})$ is generated algebraically by the letters of A and by its regular elements. Moreover, each regular $\mathcal{D}$-class of $\hat{F}_A(\mathbf{DS})$ is characterized by its content, that is, by the common image of its elements under the content morphism. In other words, if $x \in \widehat{A^+}$, then x admits a factorization in the form $x = u_0x_1u_1\cdots x_ku_k$ where the u_i are words (explicit operations) and the x_i are such that $\mathrm{pr}_{\mathbf{DS}}(x_i)$ is regular; and if x and y are regular elements of $\hat{F}_A(\mathbf{DS})$, then $x\,\mathcal{D}\,y$ if and only if $c(x) = c(y)$. The proof of this theorem, which relies on elegant considerations on the number of repeated occurrences of a subword in the terms of a sequence of words converging in $\hat{F}_A(\mathbf{DS})$, can be found in [5, 25].

This factorization theorem is rather vague, as factorization theorems go: it claims no uniqueness of the decomposition and some of the factors, namely the regular ones, are incompletely characterized. But with these limitations, this theorem has led to a number of remarkable results.

The most striking, and inspiring, application of this factorization result is the elucidation of the structure of the free pro-$\mathbf{J}$ semigroups. Note that if $\mathbf{V}$ is a sub-pseudovariety of $\mathbf{DS}$, it follows from the existence of the natural projection from $\hat{F}_A(\mathbf{DS})$ onto $\hat{F}_A(\mathbf{V})$ that $\hat{F}_A(\mathbf{V})$ too is generated by the letters in A and by its regular elements. But in a $\mathcal{J}$-trivial semigroup, every $\mathcal{D}$-class is a singleton, and every regular $\mathcal{D}$-class consists of a single idempotent. Thus $\hat{F}_A(\mathbf{J})$ has exactly as many idempotents as there are non-empty subsets of A, and it is generated by A and these $2^{|A|}-1$ idempotents. Moreover, Almeida showed that the resulting factorization can be put in a normal form which provides a uniqueness statement: every element $x \in \hat{F}_A(\mathbf{J})$ can be written in a unique fashion in the form $x = u_0x_1u_1\cdots x_ku_k$, where the x_i are idempotent elements of $\hat{F}_A(\mathbf{J})$, the u_i are words, possibly empty, the first (resp. last) letter of a word in this factorization does not occur in the content of the preceding (resp. subsequent) idempotent factor, and if a word factor u_i is empty $(0 < i < k)$, then $c(x_i)$ and $c(x_{i+1})$ are $\subseteq$-incomparable [5].

This unique factorization theorem for $\hat{F}_A(\mathbf{J})$ has a number of interesting consequences. For instance, if we observe that an idempotent of content $B \subseteq A$ is equal to the ω-power of any word of content B, we see that $\hat{F}_A(\mathbf{J})$ is finitely generated as a κ-algebra, and hence countable in contrast with most relatively free profinite semigroups. In fact, Almeida showed that the variety of κ-algebras generated by $\mathbf{J}$ has a nice finite basis of identities, $\hat{F}_A(\mathbf{J})$ is the free A-generated κ-algebra in that variety, and it has a decidable word problem [5]. This result, long isolated, was to serve as the inspiration for some of the constructions reported in Section 7.2.

The structure of $\hat{F}_A(\mathbf{J})$ was also used to study certain sub-pseudovarieties of $\mathbf{J}$, motivated by language-theoretic considerations such as the class of $\mathcal{J}$-trivial semigroups which are also locally semilattices (Selmi [94]). Finally, the factorization theorem for pro-$\mathbf{J}$ semigroups was the model for another complete description: that of the free pro-$\mathbf{R}$ semigroups, where $\mathbf{R} = [\![(xy)^\omega x = (xy)^\omega]\!]$ is the pseudovariety of $\mathcal{R}$-trivial semigroups. There the description of the elements uses certain arity ω products, and can be expressed in terms of ordinal words (Almeida and Weil [27]).

Intermediate pseudovarieties between $\mathbf{J}$ and $\mathbf{DS}$ were also considered, with less precise factorization theorems, see in particular Azevedo's work on $\mathbf{DO}$ (the semigroups in which every regular $\mathcal{D}$-class is the direct product of a group and a rectangular band) [35], Almeida [6] and Trotter and Weil [115] on $\mathbf{DA}$ (the aperiodic semigroups in $\mathbf{DS}$), Costa [40] on many subpseudovarieties of $\mathbf{DO}$, etc.

Even with such partial knowledge of the structure of the free pro-$\mathbf{DS}$ semigroups, it is possible to derive from the factorization theorem for the elements of $\hat{F}_A(\mathbf{DS})$ sufficient information to compute certain joins of sub-pseudovarieties of $\mathbf{DS}$ which had until then resisted calculation, in particular the joins $\mathbf{G} \vee \mathbf{Com} = \mathbf{ZE}$ computed by Almeida [2, 5] in answer to a question posed by Rhodes [87], and $\mathbf{R} \vee \mathbf{L}$ (Almeida and Azevedo [11, 5]). See also the computation of $\mathbf{G} \vee (\mathbf{J} \cap \mathbf{Ecom}) = \mathbf{DG} \cap \mathbf{Ecom}$ (Almeida and Weil [24, 25]). The latter computation uses also the reduced factorization statement for the elements of the free profinite group mentioned earlier, and it can be relativized to a pseudovariety of groups $\mathbf{H}$, provided $\mathbf{H}$ satisfies certain geometric properties [24, 25].

3 Pointlike subsets

Let S be a finite semigroup and let $\mathbf{V}$ be a pseudovariety. A subset X of S is called a $\mathbf{V}$-*pointlike* if for each relational morphism $\tau\colon S \to T$ with $T \in \mathbf{V}$, there exists an element $t \in T$ such that $X \subseteq \tau^{-1}(t)$. Recall that a *relational morphism* τ as above is a mapping from S into the non-empty subsets of T, such that for each $s, s' \in S$, we have $\tau(s)\tau(s') \subseteq \tau(ss')$. Every relational morphism can be factored in the form $\tau = \psi \circ \varphi^{-1}$ where φ and ψ are morphisms into S and T respectively and φ is onto.

The question whether, given S and $\mathbf{V}$, one can decide which subsets of S are $\mathbf{V}$-pointlikes was the focus of a fair amount of the semigroup theory literature in the 1980s and the 1990s. Deep results give positive answers to this problem when $\mathbf{V}$ is the pseudovariety of groups (Ash [32]) or of aperiodic semigroups (Henckell [49]), see [20, 50]. Even though this decidability question is not as central as it was, having been subsumed by more general questions which will be discussed in Section 7, the methods developed to tackle it have been an inspiration for the techniques which are the topic of this paper.

In addition, the pointlike question provides a relatively simple framework to present a compactness argument which is one of the central mechanisms of the application of profinite semigroup theory to finite semigroup theory. It is illustrated in the proof of the following theorem [77].

Theorem 3.1 *Let $\varphi: A^+ \to S$ be a morphism onto a finite semigroup and let $\hat\varphi: \widehat{A^+} \to S$ be its continuous extension. Let $\mathbf{V}$ be a pseudovariety of semigroups. A subset X of S is a $\mathbf{V}$-pointlike if and only if there exists $x \in \hat F_A(\mathbf{V})$ such that $X \subseteq \hat\varphi \circ \mathrm{pr}_{\mathbf{V}}^{-1}(x)$.*

$$\begin{tikzcd}
A^+ \arrow[r, "\imath"] \arrow[dr, "\varphi"'] & \widehat{A^+} \arrow[r, "\mathrm{pr}_{\mathbf{V}}"] \arrow[d, "\hat\varphi"] & \hat F_A(\mathbf{V}) \\
& S \arrow[ur, "\varphi_{\mathbf{V}}"'] &
\end{tikzcd}$$

First we introduce a useful piece of notation. With φ, S and $\mathbf{V}$ as in the above statement, we denote by $\varphi_{\mathbf{V}}$ the composite $\varphi_{\mathbf{V}} = \mathrm{pr}_{\mathbf{V}} \circ \hat\varphi^{-1}$ and we call it the *natural relational morphism from S into $\hat F_A(\mathbf{V})$*.

Let us assume that $X \subseteq \hat\varphi \circ \mathrm{pr}_{\mathbf{V}}^{-1}(x) = \varphi_{\mathbf{V}}^{-1}(x)$ for some element $x \in \hat F_A(\mathbf{V})$, and let τ be a relational morphism from S into a semigroup $T \in \mathbf{V}$. Using the fact that $\hat F_A(\mathbf{V})$ is free pro-$\mathbf{V}$, one verifies that there exists a continuous morphism $\psi: \hat F_A(\mathbf{V}) \to T$ such that $\tau(s) \subseteq \psi \circ \varphi_{\mathbf{V}}(s)$ for each $s \in S$. It follows that $X \subseteq \tau^{-1}(\psi(x))$.

The converse is more interesting. Let $S_1, \ldots, S_k$ be the finite collection of the subsets of S of the form $\varphi_{\mathbf{V}}^{-1}(x)$ $(x \in \hat F_A(\mathbf{V}))$. For each $1 \leq i \leq k$, let $X_i = \{x \in \hat F_A(\mathbf{V}) \mid \varphi_{\mathbf{V}}^{-1}(x) = S_i\}$ and let $\sim_i$ be the syntactic congruence of X_i; that is, the least congruence on $\hat F_A(\mathbf{V})$ such that X_i is a union of classes (see [45, vol. A]). One can show easily that each X_i is closed. As the X_i form a partition of $\hat F_A(\mathbf{V})$, each of them is in fact clopen. Now a compactness argument due to Hunter [55] implies that each $\sim_i$ is a clopen congruence (a clopen subset of $\hat F_A(\mathbf{V}) \times \hat F_A(\mathbf{V})$), so that the quotient morphism $\pi_i: \hat F_A(\mathbf{V}) \to V_i = \hat F_A(\mathbf{V})/\!\sim_i$ is a continuous morphism onto a finite semigroup $V_i \in \mathbf{V}$. Finally, let $V = \prod_1^k V_i$ and $\pi: \hat F_A(\mathbf{V}) \to V$ be given by $\pi(x) = (\pi_i(x))_{1 \leq i \leq k}$. If a subset $X \subseteq S$ is a $\mathbf{V}$-pointlike, then there exists an element $v = (v_i)_i \in V$ such that $X \subseteq (\pi \circ \varphi_{\mathbf{V}})^{-1}(v) = \varphi_{\mathbf{V}}^{-1}(\pi^{-1}(v))$. Now let $x, y \in \pi^{-1}(v)$, and let i be such that $x \in X_i$. Since $\pi_i(x) = v_i = \pi_i(y)$, that is, $x \sim_i y$, we also have $y \in X_i$. Thus $\varphi_{\mathbf{V}}^{-1}(x) = \varphi_{\mathbf{V}}^{-1}(y) = S_i$, and finally $X \subseteq \varphi_{\mathbf{V}}^{-1}(x)$, which concludes the proof. $\square$

We will see several instances of similar statements, where the consideration of the collection of all the relational morphisms from S into an element of $\mathbf{V}$ can be replaced by that of a single relational morphism into a free pro-$\mathbf{V}$ semigroup.

4 Mal'cev products

Let $\mathbf{V}$ and $\mathbf{W}$ be pseudovarieties of semigroups. By definition, $\mathbf{W} ⓜ \mathbf{V}$ is the class of all finite semigroups S for which there exists a relational morphism $\tau: S \to V$ into an element $V \in \mathbf{V}$ such that $\tau^{-1}(e) \in \mathbf{W}$ for each idempotent e of V. It is a routine verification that $\mathbf{W} ⓜ \mathbf{V}$ is a pseudovariety.

Mal'cev products are a natural consideration in group theory. Indeed, if $\mathbf{V}$ and $\mathbf{W}$ are pseudovarieties of groups, a finite group G is in $\mathbf{W} ⓜ \mathbf{V}$ if it an extension of a group in $\mathbf{W}$ by one in $\mathbf{V}$. As it turns out, this operation on pseudovarieties is also very interesting for semigroups and monoids, and the computation of Mal'cev products poses challenging problems.

Important pseudovarieties have Mal'cev product decompositions. For instance, we have $\mathbf{J} = \mathbf{N} ⓜ \mathbf{Sl}$ and $\mathbf{DA} = \mathbf{LI} ⓜ \mathbf{Sl}$, where $\mathbf{LI} = [\![x^\omega y x^\omega = x^\omega]\!]$ is the pseudovariety of locally trivial semigroups. On the other hand, neither $\mathbf{S}$ nor $\mathbf{A}$ is Mal'cev reducible [64]. The following decompositions will be discussed again in Section 6: if $\mathbf{V}$ is a pseudovariety of semigroups, $\mathbf{EV}$ denotes the pseudovariety of semigroups in which the subsemigroup generated by the idempotents is in $\mathbf{V}$, and we have

$$\mathbf{ESl} = \mathbf{Sl} \textcircled{m} \mathbf{G} \qquad \text{and} \qquad \mathbf{EJ} = \mathbf{J} \textcircled{m} \mathbf{G}.$$

The following theorem, due to Pin and Weil [77], provides a theoretical characterization of the elements of $\mathbf{W} \textcircled{m} \mathbf{V}$. Its proof relies on a compactness argument very similar to that in the proof of Theorem 3.1.

Theorem 4.1 *Let* $\mathbf{V}$ *and* $\mathbf{W}$ *be pseudovarieties of semigroups, let* S *be a finite semigroup, let* $\varphi\colon A^+ \to S$ *be an onto morphism, and let* $\varphi_{\mathbf{V}}$ *be the natural relational morphism from* S *into* $\hat{F}_A(\mathbf{V})$*. Then* $S \in \mathbf{W} \textcircled{m} \mathbf{V}$ *if and only if* $\varphi_{\mathbf{V}}^{-1}(e) \in \mathbf{W}$ *for each idempotent* e *of* $\hat{F}_A(\mathbf{V})$*.*

An immediate consequence of this result, which does not follow immediately from the definition of the Mal'cev product, describes the behavior of this product under certain intersections [77].

Corollary 4.2 *Let* $(\mathbf{W}_i)_{i\in I}$ *and* $\mathbf{V}$ *be pseudovarieties of semigroups. Then*

$$\left(\bigcap_{i\in I} \mathbf{W}_i\right) \textcircled{m} \mathbf{V} = \bigcap_{i\in I} (\mathbf{W}_i \textcircled{m} \mathbf{V}).$$

According to Theorem 4.1, in order to determine whether a finite semigroup S, given together with an onto morphism $\varphi\colon A^+ \to S$, belongs to a pseudovariety $\mathbf{W} \textcircled{m} \mathbf{V}$, it suffices to decide whether the subsemigroups of S of the form $\varphi_{\mathbf{V}}^{-1}(e)$ lie in $\mathbf{W}$, when e is an idempotent element of $\hat{F}_A(\mathbf{V})$. In particular, $\mathbf{W} \textcircled{m} \mathbf{V}$ is decidable if $\mathbf{W}$ is decidable and the finite collections of the $\varphi_{\mathbf{V}}^{-1}(e)$ $(e = e^2 \in \hat{F}_A(\mathbf{V}))$ can be effectively computed.

This is the case notably in the case where $\mathbf{V}$ is locally finite and, $\pi\colon A^* \to F_A(\mathbf{V})$ being the natural projection, the rational languages of the form $\pi^{-1}(e)$ $(e = e^2 \in F_A(\mathbf{V}))$ can be effectively computed [118]. Indeed, if we can find a rational expression for each such language, then we can compute the image of that language under a morphism φ from A^+ onto S. Thus, for instance, we have the following result.

Proposition 4.3 *If* $\mathbf{W}$ *is a decidable pseudovariety, then* $\mathbf{W} \textcircled{m} \mathbf{Sl}$ *and* $\mathbf{W} \textcircled{m} \mathbf{B}$ *are decidable.*

Another very interesting situation occurs when $\mathbf{V}$ is a pseudovariety of groups, in which case the collection of the $\varphi_{\mathbf{V}}^{-1}(e)$ consists of a single subsemigroup, namely $\varphi_{\mathbf{V}}^{-1}(1)$. This particular situation is discussed in Section 6 below.

Theorem 4.1 can also be used to describe a defining set of pseudo-identities for $\mathbf{W} \textcircled{m} \mathbf{V}$, given one for $\mathbf{W}$ [77, Thm 4.1].

Theorem 4.4 *Let* $\mathbf{V}$ *and* $\mathbf{W}$ *be pseudovarieties of semigroups, and let* Σ *be a defining set of pseudo-identities of* $\mathbf{W}$*. Then* $\mathbf{W} \textcircled{m} \mathbf{V}$ *is defined by the pseudo-identities of the form* $x(\vec{z}) = y(\vec{z})$ *obtained from the elements* $x = y$ *of* Σ *as follows:* $\vec{z} = (z_a)_{a\in A}$ *is a vector of elements of* $\widehat{B^+}$*, indexed by the alphabet* A *of variables occurring in the pseudo-identity* $x = y$*,* $\mathbf{V}$ *satisfies* $z_a = z_b = z_a^2$ *for all* $a, b \in A$*,* B *is a finite alphabet, and* $x(\vec{z})$ *(resp.* $y(\vec{z})$*) is the image of* x *(resp.* y*) in the continuous morphism from* $\widehat{A^+}$ *into* $\widehat{B^+}$ *which maps letter* a *to* z_a*.*

This general statement sometimes allows exact computations, depending on our capability to solve the equations $\mathbf{V} \models z_a = z_b = z_a^2$ in $\widehat{B^+}$. For instance, using the fact that nilpotent semigroups have a single idempotent, namely 0, and that every implicit operation which is not explicit is of the form $xy^\omega z$ (see Section 2) and is equal to 0 in $\hat{F}_A(\mathbf{N})$, we can use Theorem 4.4 to compute defining pseudo-identities for products of the form $\mathbf{W} ⓜ \mathbf{N}$, given defining pseudo-identities for $\mathbf{W}$, and to show that if $\mathbf{W}$ is decidable, then so is $\mathbf{W} ⓜ \mathbf{N}$. We obtain the following general result [77].

Proposition 4.5 *Let $\mathbf{W}$ be a pseudovariety of semigroups. If $\mathbf{W}$ is decidable (resp. is finitely based, has finite rank), then the same holds for $\mathbf{W} ⓜ \mathbf{N}$.*

We have $\mathbf{W} ⓜ \mathbf{J} = (\mathbf{W} ⓜ \mathbf{N}) ⓜ \mathbf{Sl}$ and this pseudovariety is decidable if $\mathbf{W}$ is.

It is a good exercise to compare the result of the application of Theorem 4.4 to the product $\mathbf{N} ⓜ \mathbf{Sl} = \mathbf{J}$ to the pseudo-identities defining $\mathbf{J}$, given in Section 1.4.

5 Semidirect products

Much attention has been paid to the semidirect product operation in the lattice $\mathcal{P}\mathbf{s}$, in part due to the work on the complexity problem mentioned in the introduction. Deciding complexity is in fact deciding the membership problem in certain iterated semidirect products, see below.

If S and T are semigroups, a left action of T on S is given by a morphism λ from T in the endomorphisms of S, written $\lambda(t)(s) = t \cdot s$. Such an action determines a semidirect product $S * T$, which consists of the set $S \times T$ and the semigroup law $(s,t)(s',t') = (s + t \cdot s', tt')$. For clarity, the semigroup operation of T is written multiplicatively and the operation of S additively.

This operation is then extended to pseudovarieties: if $\mathbf{V}$ and $\mathbf{W}$ are pseudovarieties of semigroups, $\mathbf{W} * \mathbf{V}$ is defined to be the pseudovariety generated by the $S * T$ ($S \in \mathbf{W}$, $T \in \mathbf{V}$), where the action of T on S is left unitary: if T is a monoid, we require that $1_T \cdot s = s$ for each $s \in S$. In the case where $\mathbf{V}$ is a pseudovariety of monoids, we also require the action to be right unitary, that is, $t \cdot 0_S = 0_S$ for each $t \in T$. It is well-known that this operation on pseudovarieties is associative [45, vol. B].

The wreath product of S and T, written $S \circ T$, is the particular semidirect product $S^{T^1} * T$, where S^{T^1} denotes the semigroup of mappings from $T^1 = T \cup \{1\}$ into S, and the action of T on S^{T^1} is given by $(t \cdot f)(u) = f(ut)$ for each $t \in T$, $f \in S^{T^1}$ and $u \in T^1$.

The semidirect product is obviously a natural tool for obtaining composition and decomposition results. It is particularly relevant in the context of the theory of finite semigroups and monoids since it was proved that every such object divides a sufficiently long semidirect product of "simple" monoids: this is the theorem of Krohn and Rhodes, proved in the early 1960s, see [45, vol. B].

Theorem 5.1 *Every finite semigroup S divides a wreath product of simple groups dividing S and copies of the 3-element monoid $U_2 = \{1, a, b\}$ where $ax = a$ and $bx = b$ for each x.*

As the wreath product is associative, a wreath product of groups is a group, and a wreath product of copies of U_2 is aperiodic, it follows that every finite semigroup sits in some iterated semidirect product of the pseudovarieties $\mathbf{A}$ and $\mathbf{G}$. More precisely, let $\mathbf{C}_0 = \mathbf{A}$ and for each $n \geq 0$, let $\mathbf{C}_{n+1} = \mathbf{A} * \mathbf{G} * \mathbf{C}_n$. The least n such that a given semigroup S lies in $\mathbf{C}_n$ is called the *(group) complexity* of S. This complexity measures the least number of groups needed in a semidirect decomposition of S up to division. Complexity 0 consists of the aperiodic semigroups, a clearly decidable pseudovariety. It has been known for a long time [85] that the hierarchy of the $\mathbf{C}_n$ is infinite, that is, $\mathbf{C}_n$ is properly contained in $\mathbf{C}_{n+1}$ for each n. The decidability of the group complexity of a finite semigroup (the *complexity problem*) has been an open problem for more than thirty years, and much of the modern theory of finite semigroups was developed in relation with the study of this problem. As we already mentioned, it briefly seemed as if a solution to this problem followed from a paper submitted by Rhodes [84], but the discovery of a mistake in an earlier paper [28] brought the problem back to its *open* status.

At any rate, much work has gone in the past twenty years into the computation of semidirect products of pseudovarieties, apart from the case of products involving the pseudovarieties **A** and **G**. In this section, we present the contributions of profinite methods to this effort.

5.1 The free object approach

It is well-known in group theory that if G is a group and N is a normal subgroup, then G can be embedded in the wreath product $N \circ G/N$. A similar construction led Almeida and Weil to the following result [26].

Theorem 5.2 *Let* $\mathbf{V}$ *and* $\mathbf{W}$ *be pseudovarieties of semigroups and let* A *be a finite alphabet. The action of* $\hat{F}_A(\mathbf{V})$ *on the set* $\hat{F}_A(\mathbf{V})^1 \times A$ *given by* $x \cdot (y, a) = (xy, a)$ *is continuous and naturally extends to a continuous action of* $\hat{F}_A(\mathbf{V})$ *on* $\hat{F}_{\hat{F}_A(\mathbf{V})^1\times A}(\mathbf{W})$. *Moreover, letting* $\theta(a) = ((1, a), a)$ $(a \in A)$ *defines a continuous one-to-one morphism from* $\hat{F}_A(\mathbf{W} * \mathbf{V})$ *into the resulting semidirect product* $\hat{F}_{\hat{F}_A(\mathbf{V})^1\times A}(\mathbf{W}) * \hat{F}_A(\mathbf{V})$.

Note that in this statement, we are considering the free pro-**W** semigroup over a set of generators, $\hat{F}_A(\mathbf{V})^1 \times A$, which is usually not finite, but always a profinite set (i.e. a compact and totally disconnected topological space, but also one that can be seen as the projective limit of a collection of finite sets). Free pro-**W** semigroups over profinite sets of generators can be defined in a natural way, to make this statement completely rigorous. We refer the reader to [26] for the details.

A comparable result for varieties of groups can be found in H. Neumann's [66]. Theorem 5.2 was first proved by Almeida [4] in the case where **V** and **W** have finite free objects over each finite alphabet. The spirit behind it, at least in Almeida's finitary case, is very close to the ideas at work in the language-theoretic connection between semidirect products and sequential functions, and in Straubing's wreath product principle [105, 67].

Example 5.3 The rather dry statement of Theorem 5.2 may become clearer after looking at a simple example. Let us consider a pseudovariety of semigroups $\mathbf{V}$ and the morphism $\theta\colon \hat{F}_A(\mathbf{N} * \mathbf{V}) \longrightarrow \hat{F}_{\hat{F}_A(\mathbf{V})^1\times A}(\mathbf{N}) * \hat{F}_A(\mathbf{V})$ in Theorem 5.2. Since $\hat{F}_B(\mathbf{N}) = B^+ \cup \{0\}$ (even if B is a profinite set), the first component of $\theta(a_1 \cdots a_n)$ $(a_1, \ldots, a_n \in A)$ is

$$(1, a_1)(a_1, a_2)\cdots(a_1 \cdots a_{n-1}, a_n)$$

where the first coordinates of each pair are taken in $\hat{F}_A(\mathbf{V})^1$ and the product of the pairs is taken in $\hat{F}_{\hat{F}_A(\mathbf{V})^1\times A}(\mathbf{N})$. This observation shows that if $x \in \hat{F}_A(\mathbf{N} * \mathbf{V})$ is not explicit, then the first component of $\theta(x)$ is 0. Since the second component of $\theta(x)$ is the value of x in $\hat{F}_A(\mathbf{V})$, it is not difficult to verify that $\hat{F}_A(\mathbf{N} * \mathbf{V})$ is isomorphic to the closed subsemigroup of $\hat{F}_A(\mathbf{N}) \times \hat{F}_A(\mathbf{V})$ generated by the (a, a) $(a \in A)$, and hence that $\mathbf{N} * \mathbf{V} = \mathbf{N} \vee \mathbf{V}$ [26].

The same reasoning shows that $\mathbf{K} * \mathbf{V} = \mathbf{K} \vee \mathbf{V}$ [116]. □

Let us now consider the products of the form $\mathbf{Sl} * \mathbf{V}$, where $\mathbf{V}$ is a pseudovariety of semigroups. In that case, Theorem 5.2 can be stated in terms of the geometry of the *profinite Cayley graphs* of $\hat{F}_A(\mathbf{V})$. This topological graph, written $\Gamma_A(\mathbf{V})$, has vertex set the elements of $\hat{F}_A(\mathbf{V})^1$ and its edges represent right multiplication by the letters of A: thus the set $\hat{F}_A(\mathbf{V})^1 \times A$ is identified with the set of edges of $\Gamma_A(\mathbf{V})$. The graph $\Gamma_A(\mathbf{V})$ is naturally equipped with a topology, inherited from $\hat{F}_A(\mathbf{V})$, which makes it a profinite graph, i.e. a compact and totally disconnected topological graph. Now, if B is a profinite set, $\hat{F}_B(\mathbf{Sl})$ is easily seen to be the semigroup of the closed non-empty subsets of B under union [26]. So $\hat{F}_{\hat{F}_A(\mathbf{V})^1\times A}(\mathbf{Sl})$ is the set of closed sets of edges of $\Gamma_A(\mathbf{V})$.

If $u = a_1 \cdots a_n$ is a finite word on alphabet A, then $\theta(u) = (P, u)$ where the second component is the value of u in $\hat{F}_A(\mathbf{V})$ and P is the set of edges of the path labeled u and starting at vertex 1 in $\Gamma_A(\mathbf{V})$. In general (i.e., taking limits), if $x \in \hat{F}_A(\mathbf{Sl} * \mathbf{V})$, then $\theta(x) = (P, [x]_\mathbf{V})$ where $[x]_\mathbf{V}$ is the value of x in $\mathbf{V}$ and P is a set of edges of $\Gamma_A(\mathbf{V})$ which can be seen as a path from 1 to $[x]_\mathbf{V}$. The characterization of the sets P of edges of $\Gamma_A(\mathbf{V})$ which may arise as the first coordinate of $\theta(x)$ for some $x \in \hat{F}_A(\mathbf{Sl} * \mathbf{V})$ given in [26] contains a mistake, which was fixed by Rhodes and Steinberg [90]: we say that a subgraph X of a profinite graph Γ is a *support* if for each continuous graph morphism from Γ onto a finite graph, the image of X is the set of edges of a path. Then we have the following statement [26, 90].

Theorem 5.4 *Let $\mathbf{V}$ be a pseudovariety of semigroups and let A be a finite alphabet. Then $\hat{F}_A(\mathbf{Sl} * \mathbf{V})$ is homeomorphic to the semigroup consisting of all pairs (P, x), where $x \in \hat{F}_A(\mathbf{V})$, P is a support in $\Gamma_A(\mathbf{V})$ containing 1 and x, and the product is given by*

$$(P, x)(Q, y) = (P \cup xQ, xy).$$

Example 5.5 Let us consider the product $\mathbf{V} = \mathbf{Sl} * \mathbf{N}$ [26]. Since there is only one non-explicit element in $\hat{F}_A(\mathbf{N})$, the non-explicit elements of the free pro-$\mathbf{V}$ semigroups are characterized by an infinite word u on the finite alphabet A, which labels a path in the Cayley graph $\Gamma_A(\mathbf{N})$ from 1 to 0, and by a number of loop edges around vertex 0. In other words, they are characterized by their length ω prefix and their "alphabetical content at infinity". This can be formalized properly to prove that $\mathbf{Sl} * \mathbf{N} = [\![x^\omega y^2 = x^\omega y;\ x^\omega yz = x^\omega zy]\!]$.

Let us now consider the product $\mathbf{W} = \mathbf{Sl} * \mathbf{K}$. The non-explicit vertices of $\Gamma_A(\mathbf{K})$ are all sinks, in the sense that every edge out of them is a loop. This can be used to verify that the free pro-$\mathbf{W}$ semigroup on A is homeomorphic to $\hat{F}_A(\mathbf{Sl} * \mathbf{N})$, and hence that $\mathbf{Sl} * \mathbf{K} = \mathbf{Sl} * \mathbf{N}$ [26].

These results can be extended to all products of the form $\mathbf{V} * \mathbf{N}$ and $\mathbf{V} * \mathbf{K}$ where $\mathbf{V}$ is a pseudovariety of semigroups generated by monoids (Steinberg [95]). □

The case of products of the form $\mathbf{V} * \mathbf{H}$, where $\mathbf{H}$ is a pseudovariety of groups is discussed in Section 6 below.

Theorem 5.4 was also used by Costa to study the structure of the free pro-$\mathbf{LSl}$ semigroup, since it is well-known that $\mathbf{LSl} = \mathbf{Sl} * \mathbf{D}$. The free pro-$\mathbf{D}$ semigroups are algebraically rather simple (see Section 2), but the geometry of their profinite Cayley graphs turns out to be very challenging. Costa was able to show, for instance, that $\hat{F}_A(\mathbf{LSl})$ is uncountably high and uncountably wide, that is, it has uncountable ascending chains and uncountable antichains for the $\mathcal{J}$-order [39]. Costa's results are among the few structure theorems for free pro-$\mathbf{V}$ semigroups when $\mathbf{V}$ is not contained in $\mathbf{DS}$.

5.2 The categorical approach

Another very successful approach in handling semidirect product is based on the use of categories and semigroupoids. Pioneering work in this direction can be found in the papers of Margolis and Pin [63], Straubing [105] or Thérien and Weiss [112] but the proper concepts were finally set by Tilson [113].

Here, we consider categories as algebraic objects generalizing monoids: a *category* is a graph C equipped with an associative partial product operation on its edge set. Two edges can be multiplied if and only if they are consecutive, and if $x \in C(u,v)$ (i.e. x is an edge of C from vertex u to vertex v) and $y \in C(v,w)$, then $xy \in C(u,w)$. Traditionally, category theorists talk of objects instead of vertices and of morphisms instead of edges. Associativity is required wherever it makes sense. Finally, one requires that for each vertex u, the semigroup $C(u,u)$ has a local identity, that is, there exists an edge $1_u \in C(u,u)$ such that $1_u x = x$ (resp. $x 1_u = x$) for each $x \in C(u,v)$ (resp. $x \in C(v,u)$) and for each vertex v. Clearly, each $C(u,u)$ is a monoid, called a *local monoid* of C, and each monoid can be seen as a one-vertex category. If the local identity condition is not required, we say that C is a *semigroupoid*: semigroupoids are to categories as semigroups are to monoids.

The notions of morphism, division and direct product in semigroup theory naturally extend to semigroupoids, and one may consider pseudovarieties of categories or semigroupoids. We refer the reader to [113] and [28] for the precise definitions. The major contribution in Tilson's [113] lies in the definition of a semigroupoid canonically associated with a relational morphism between semigroups $\tau: S \to T$, written D_τ and called the *derived semigroupoid of* τ, and in the following theorem.

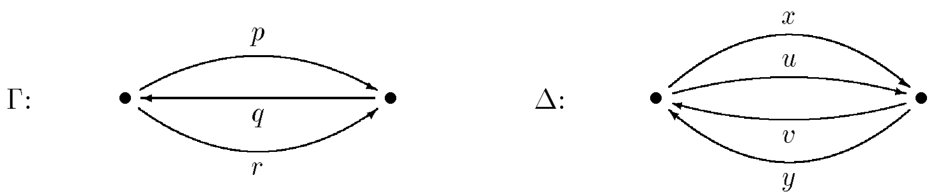
Theorem 5.6 *Let S be a finite semigroup and let $\mathbf{V}$ and $\mathbf{W}$ be pseudovarieties of semigroups. Then $S \in \mathbf{W} * \mathbf{V}$ if and only if there exists a relational morphism $\tau: S \to T$ such that $T \in \mathbf{V}$ and D_τ divides an element of $\mathbf{W}$.*

The advent of Tilson's theorem made it clear to semigroup theorists that the study of finite semigroups and monoids now must include the study of pseudovarieties of categories and semigroupoids. However, some more work is involved in order to turn Theorem 5.6 into a technique to compute semidirect products, as in Theorem 5.7 below.

If $\mathbf{V}$ is a pseudovariety of semigroups, then the class of finite semigroupoids which divide a semigroup in $\mathbf{V}$ forms a pseudovariety of semigroupoids, called the *global of* $\mathbf{V}$ and written $\mathbf{gV}$. In view of Theorem 5.6, globals of pseudovarieties of semigroups are especially important to describe semidirect products.

A version of Reiterman's theorem exists for pseudovarieties of categories and semigroupoids, developed by Almeida and Weil [28], and independently by Jones [56]: every pseudovariety of semigroupoids is defined by a set of pseudo-identities, that is, by pairs of coterminal edges of finitely generated free profinite semigroupoids. To be more precise: if Γ is a finite graph, we let Γ^+ be the free semigroupoid over Γ, that is, the semigroupoid with the same vertex set as Γ and with edges the non-empty paths in Γ. The free profinite semigroupoid over Γ, denoted by $\widehat{\Gamma^+}$, is constructed in analogy with the free profinite semigroup, either by taking the completion of Γ^+ for a well-chosen metric, or by taking the projective limit of all the morphisms from Γ^+ onto a finite semigroupoid. For the details, see Almeida and Weil [28], or the work of Teixeira [108] and Steinberg [97].

For instance, Thérien and Weiss showed that $\mathbf{gCom}$ is defined by the (pseudo-)identity $(\Gamma; pqr = rqp)$ where Γ is the graph shown below [112]; and Knast's theorem [59] can be rephrased in the language of pseudo-identities, to state that $\mathbf{gJ}$ is defined by the pseudo-identity $(\Delta; (xy)^\omega(uv)^\omega = (xy)^\omega xv(uv)^\omega)$, where Δ is the following graph:



Before we give, in Theorem 5.7 below, a precise statement on using such pseudo-identities in conjunction with Theorem 5.6 to compute semidirect products, let us briefly digress on the subject of the pseudovarieties of the form $\mathbf{gV}$ (when $\mathbf{V}$ is a pseudovariety of semigroups), which are a crucial ingredient in semidirect product computations.

In general, it is quite difficult to find pseudo-identities for $\mathbf{gV}$ even when $\mathbf{V}$ is seemingly well-known, as examplified by the non-trivial proofs of the results on $\mathbf{gJ}$ and $\mathbf{gCom}$ given above. Both these pseudovarieties have vertex rank 2, i.e., they have a defining set of pseudo-identities involving only graphs with at most 2 vertices. This means that a semigroupoid is in $\mathbf{gJ}$ (resp. $\mathbf{gCom}$) if and only if its subsemigroupoids with 2 vertices are. In [97], Steinberg gives several other examples of globals which have vertex rank 2, 3 or 4, notably the globals of right zero semigroups, right simple semigroups, completely simple semigroups, $\mathbf{N}$ and $\mathbf{N} \vee \mathbf{G}$. Almeida and Azevedo [13] compute the globals of the pseudovarieties of commutative monoids, and show specific examples of globals of decidable pseudovarieties with infinite vertex rank.

Globals of vertex rank 1 are particularly interesting. If $\mathbf{V}$ is a pseudovariety of semigroups, the class of finite semigroupoids C which are *locally in* $\mathbf{V}$, that is such that every local semigroup $C(u,u)$ is in $\mathbf{V}$, forms a pseudovariety written $\ell\mathbf{V}$. It is not difficult to verify that $\mathbf{V}$ is the class of semigroups in a pseudovariety of semigroupoids $\mathbf{W}$ if and only if $\mathbf{gV} \subseteq \mathbf{W} \subseteq \ell\mathbf{V}$. If $\mathbf{gV} = \ell\mathbf{V}$, we say that $\mathbf{V}$ is *local*. Local pseudovarieties are exactly those whose global has vertex rank 1. Many examples of local pseudovarieties are known, such as all pseudovarieties of monoids consisting only of groups [113], $\mathbf{Sl}$ (a rephrasing of a result of Brzozowski and Simon [37] found also in [45, vol. B]), $\mathbf{DS}$ [58] and $\mathbf{DA}$ [6].

If $\mathbf{V}$ is a local pseudovariety of semigroups, then $\mathbf{V}$ is decidable if and only if $\mathbf{gV}$ is. Of course, such results as those reported above on $\mathbf{gJ}$ and $\mathbf{gCom}$ show that these pseudovarieties are decidable. Almeida and Azevedo have shown that if $\mathbf{V}$ is a decidable pseudovariety of commutative monoids, then $\mathbf{gV}$ is decidable [13]. Decidability is also preserved by the operator $\mathbf{V} \mapsto \mathbf{gV}$ when $\mathbf{V}$ contains a certain 5-element semigroup [105, 113]. Steinberg [97] and Teixeira [108] completed earlier work by Thérien [111], Jones and Pustejovsky [57] and Rey [83] to prove that $\mathbf{g}(\mathbf{W} * \mathbf{V}) = \mathbf{gW} * \mathbf{gV}$ for an appropriate definition of the semidirect product of pseudovarieties of semigroupoids, see also Steinberg and Tilson's [104].

5.3 Pseudo-identities for semidirect products

Let us now go back to the description of a defining set of pseudo-identities for the semidirect product of pseudovarieties of semigroups or monoids. We need the following definition. Let Γ be a finite graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$ (always assumed to be disjoint sets), and let S be a semigroup. An *S-labeling* of Γ is a mapping δ from $V(\Gamma) \cup E(\Gamma)$ into S^1. It is said to be *consistent* if $\delta(i)\delta(x) = \delta(j)$ whenever $i, j \in V(\Gamma)$ and $x \in \Gamma(i,j)$.

Note that an $\widehat{A^+}$-labeling δ of Γ induces naturally a continuous morphism from the free profinite semigroupoid $\widehat{\Gamma^+}$ into $\widehat{A^+}$, also denoted δ. If $i, j \in V(\Gamma)$ and $u, v \in \widehat{\Gamma^+}(i,j)$, that is, if $(\Gamma; u = v)$ is a pseudo-identity for semigroupoids, we denote by $\delta(\Gamma; u = v)$ the semigroup pseudo-identity $\delta(i)\delta(u) = \delta(i)\delta(v)$. The following theorem was proved by Almeida and Weil [28].

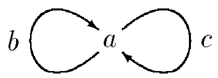
Theorem 5.7 *Let* $\mathbf{V}$ *and* $\mathbf{W}$ *be pseudovarieties of semigroups such that* $\mathbf{gW}$ *has finite vertex rank* r*, and let* Σ *be a defining set of pseudo-identities for* $\mathbf{gW}$*, all written on graphs with at most* r *vertices*[2]*. Then* $\mathbf{W} * \mathbf{V}$ *is defined by the pseudo-identities of the form* $\delta(\Gamma; u = v)$*, where* $(\Gamma; u = v)$ *is an element of* Σ *and* δ *is an* $\widehat{A^+}$*-labeling of* Γ *such that* $\mathrm{pr}_{\mathbf{V}} \circ \delta$ *is consistent.*

$$\begin{tikzcd}
 & \Gamma \arrow[dl, "\delta"'] \arrow[dr, "\mathrm{pr}_{\mathbf{V}} \circ \delta"] & \\
\widehat{A^*} \arrow[rr, "\mathrm{pr}_{\mathbf{V}}"] & & \hat{F}^1_A(\mathbf{V})
\end{tikzcd}$$

Applying Theorem 5.7 to determine membership of a finite semigroup S in a given product $\mathbf{W} * \mathbf{V}$ requires not only knowing a sufficiently simple set of pseudo-identities defining $\mathbf{gW}$, but it also requires understanding the consistent $\hat{F}_A(\mathbf{V})$-labelings of finite graphs, that is, solving certain systems of equations in free pro-$\mathbf{V}$ semigroups. The problem thus posed is discussed in Section 7 below. However, if $\mathbf{V}$ is sufficiently simple, or if the graphs underlying the pseudo-identities defining $\mathbf{gW}$ are small enough, it is already possible to use Theorem 5.7 to perform exact computations. Let us work out a simple example (see [28] for many more).

Example 5.8 We compute the pseudovariety $\mathbf{G} * \mathbf{V}$ when $\mathbf{V}$ is a pseudovariety of monoids. As $\mathbf{G}$ is local, we may take for the set of pseudo-identities Σ defining $\mathbf{gG}$ the pseudo-identities $(\Gamma; b^\omega c = cb^\omega = c)$ with Γ the one-vertex graph below, expressing that in every local monoid, each idempotent is the local identity.

Γ:



Let δ be an $\widehat{A^*}$-labeling of Γ mapping a to x, b to y and c to z. Then $\mathrm{pr}_{\mathbf{V}} \circ \delta$ is consistent if $\mathbf{V}$ satisfies $xy = xz = x$. The resulting pseudo-identities $\delta(\Gamma; b^\omega c = cb^\omega = c)$ are $xy^\omega z = xzy^\omega = xz$, that is,

$$(*) \qquad \mathbf{G} * \mathbf{V} = [\![xy^\omega z = xzy^\omega = xz \mid \mathbf{V} \text{ satisfies } xy = xz = x]\!].$$

Since we are dealing with monoids, and since $\mathbf{V}$ satisfies $x = x1$, we get in particular the pseudo-identities of the form $xy^\omega = x$ where $\mathbf{V}$ satisfies $xy = x$. But these pseudo-identities imply those occurring in $(*)$ above, so

$$(**) \qquad \mathbf{G} * \mathbf{V} = [\![xy^\omega = x \mid \mathbf{V} \text{ satisfies } xy = x]\!].$$

If $\mathbf{V} = \mathbf{J}$, we can be more precise. Using the description of the elements of $\hat{F}_A(\mathbf{J})$ and $\hat{F}_A(\mathbf{DS})$ and the easy verification that $\mathbf{G} * \mathbf{J} \subseteq \mathbf{DS}$ (see Section 2), we see that if $\mathbf{J}$ satisfies $xy = x$, then x has a factorization $x = u_0 x_1 u_1 \cdots u_{n-1} x_n$ such that the u_i are words, the x_i are regular in $\mathbf{DS}$, and idempotent in $\mathbf{J}$, and the content of y is contained in the content of x_n. This leads to the following simplification of $(**)$:

$$\mathbf{G} * \mathbf{J} = [\![x^\omega y^\omega = x^\omega \mid c(y) \subseteq c(x)]\!] = [\![(yx)^\omega y^\omega = (yx)^\omega]\!].$$

□

[2] In [28], it is claimed that the result in Theorem 5.7 holds for all choices of $\mathbf{W}$ and Σ ([28, Theorem 5.3]). But Rhodes and Steinberg recently found a mistake in the proof of an earlier proposition ([28, Proposition 3.6]), which invalidates this more general result. However, the proof of [28, Theorem 5.3] still establishes the result stated in this survey.

The results of this section were used also to study the sublattice of $\mathcal{P}\mathbf{s}$ formed by the pseudovarieties closed under semidirect product, see the work of Almeida, Azevedo and Teixeira [5, 108, 109, 110].

Finally, let us mention also, for the sake of completeness, that the methods discussed above have allowed a fruitful revisitation of Straubing's results on the decidability of the semidirect products of the form $\mathbf{V} * \mathbf{D}$ and of the comparison of $\mathbf{V} * \mathbf{D}$ with the pseudovariety $\mathbf{LV}$ (Delay theorem [105]). The central point of the argument for the membership in these pseudovarieties is the following: if S is a semigroup, the *Cauchy category of* S, written S_E is the category with vertices the idempotents of S and with edge sets $S_E(e,f) = eSf$. The product of consecutive edges is as in S. One can show that $S \in \mathbf{V} * \mathbf{D}$ if and only if $S_E \in \mathbf{gV}$ while $S \in \mathbf{LV}$ if and only if $S_E \in \ell\mathbf{V}$ [113]. For a discussion of these issues and extensions in the spirit of this paper, see Almeida and Weil [28], Steinberg [103] and Almeida, Azevedo and Teixeira [14].

6 Products with a pseudovariety of groups

Before we proceed with the applications of Theorem 5.7, let us discuss the products of the form $\mathbf{V} \textcircled{m} \mathbf{H}$ and $\mathbf{V} * \mathbf{H}$, when $\mathbf{H}$ is a pseudovariety of groups. The results seen so far have especially interesting expressions in that particular case, and new connections with profinite topologies have arisen in this context.

As we saw in Section 4, if $\mathbf{V}$ is a pseudovariety, S is a finite semigroup and $\varphi\colon A^+ \to S$ is an onto morphism, then $S \in \mathbf{V} \textcircled{m} \mathbf{G}$ if and only if the subsemigroup $\varphi_{\mathbf{G}}^{-1}(1)$ of S is in $\mathbf{V}$. Thus, if $\mathbf{V}$ is decidable, deciding membership in $\mathbf{V} \textcircled{m} \mathbf{G}$ depends on whether we can compute this subsemigroup.

It is not difficult to verify that this subsemigroup, called the ($\mathbf{G}$-)*kernel* of S and written $\mathbf{K}_{\mathbf{G}}(S)$, does not depend on the choice of the onto morphism φ. More precisely, one can verify [50] that it is the intersection of all the $\tau^{-1}(1)$, where τ is a relational morphism from S into a finite group. It is with this last definition that the kernel of a monoid was first introduced by Rhodes and Tilson in the 1970s [86].

Proving that the kernel of a finite semigroup is computable, and even efficiently computable, concentrated a fair amount of the research efforts in algebraic semigroup theory throughout the 1980s and the early 1990s, until the problem was solved by Ash [32].

Tilson noticed early on (see [50] for a survey) that $\mathbf{K}_{\mathbf{G}}(S)$ contains the idempotents of S, and that if $s \in \mathbf{K}_{\mathbf{G}}(S)$ and $yxy = y$ in S, then xsy and ysx lie in $\mathbf{K}_{\mathbf{G}}(S)$ as well. We say that $\mathbf{K}_{\mathbf{G}}(S)$ is *closed under weak conjugation*. Rhodes conjectured (the so-called *type II conjecture*) that this observation suffices to describe all the elements of the kernel, that is, $\mathbf{K}_{\mathbf{G}}(S)$ is the least subsemigroup of S closed under weak conjugation. This easily implies the decidability of the kernel. Ash's deep result [32] confirmed this conjecture. As we will discuss later (see Section 7), Ash's result implies in fact stronger properties of the pseudovariety $\mathbf{G}$.

Theorem 6.1 *Let S be a finite semigroup. Then $\mathbf{K}_{\mathbf{G}}(S)$ is the least subsemigroup of S closed under weak conjugation.*

The following proposition is easily verified.

Proposition 6.2 *Let $\mathbf{V}$ be a pseudovariety of semigroups. If $\mathbf{V}$ is decidable, then so is $\mathbf{V} \textcircled{m} \mathbf{G}$. Moreover $\mathbf{V} * \mathbf{G} \subseteq \mathbf{V} \textcircled{m} \mathbf{G}$ and if $\mathbf{V}$ is local, then $\mathbf{V} * \mathbf{G} = \mathbf{V} \textcircled{m} \mathbf{G}$.*

For instance, we have $\mathbf{Sl} \textcircled{m} \mathbf{G} = \mathbf{Sl} * \mathbf{G}$. This provides the setting for a famous result proved by Ash as a consequence of a partial solution of Rhodes' type II conjecture [31],

$$\mathbf{Sl} \textcircled{m} \mathbf{G} = \mathbf{Sl} * \mathbf{G} = \mathbf{ESl} = \mathbf{Inv},$$

where **Inv** denotes the pseudovariety generated by the inverse semigroups. It is interesting to note in this context that the construction of the free pro-$\mathbf{Sl} * \mathbf{G}$ semigroup, as in Theorem 5.4, is reminiscent of the description of the free inverse monoid due to Munn [60] (see Margolis and Meakin [62] for the more general construction of the relatively free E-unitary inverse monoid with maximal group image in a pseudovariety of groups $\mathbf{H}$). In Munn's construction, the free inverse monoid over A is identified with the set of pairs of the form (P, x) where x lies in the free group over A, $FG(A)$, and P is a connected subgraph of the Cayley graph of $FG(A)$ containing 1 and x. The product is as in Theorem 5.4. The precise relationship between Munn's construction, that in Theorem 5.4 and the free profinite inverse monoid is discussed by Almeida and Weil in [26] (see also Steinberg [100]).

We may have the equality $\mathbf{V} ⓜ \mathbf{G} = \mathbf{V} * \mathbf{G}$ even when $\mathbf{V}$ is not local. This is the case in particular when $\mathbf{V} = \mathbf{J}$, as was finally proved by Henckell and Rhodes [51, 50] in a famous result:

$$\mathbf{J} ⓜ \mathbf{G} = \mathbf{J} * \mathbf{G} = \mathbf{EJ} = \mathbf{BG} = \mathbf{PG},$$

where $\mathbf{BG}$ denotes the pseudovariety of semigroups in which no distinct idempotents are $\mathcal{R}$- or $\mathcal{L}$-equivalent, and $\mathbf{PG}$ denotes the pseudovariety generated by the power sets of finite groups (see [70] for a survey). We note that the equation $\mathbf{V} * \mathbf{G} = \mathbf{EV}$ has many solutions besides $\mathbf{V} = \mathbf{Sl}$ and $\mathbf{V} = \mathbf{J}$, which are discussed in Almeida and Escada's [17].

Ash's proof of the type II conjecture did not make explicit use of profinite objects or topology. However, before Rhodes' conjecture was proved, two related conjectures were proposed by Pin [68, 69] and Pin and Reutenauer [75], whose statements were entirely concerned with the properties of the pro-$\mathbf{G}$ topology of the free semigroup and the free group and whose confirmation would prove Rhodes' conjecture. We refer the reader to [50] for a detailed account of the history of partial results on these conjectures, and to [50] and [102] for a discussion of the relation between these conjectures.

To be more precise, Pin showed that an element s of a finite semigroup S lies in $\mathbf{K}_{\mathbf{G}}(S)$ if and only if the empty word lies in the topological closure of an explicitly computable recognizable language L_s, with respect to the pro-$\mathbf{G}$ topology on the free monoid [69]. The first conjecture of Pin then proposed an algorithm to compute the topological closure of a given recognizable language in the pro-$\mathbf{G}$ topology on the free monoid.

The second conjecture, of Pin and Reutenauer, had to do with a simple algorithm to compute the topological closure of a rational subset of a free group with respect to the pro-$\mathbf{G}$ topology. Its technical statement, now a theorem, centers on the difficult part of the justification of this algorithm.

Theorem 6.3 *Let $FG(A)$ be a finitely generated free group, and let $H_1, \ldots, H_n$ be finitely generated subgroups of $FG(A)$. Then the product $H_1 H_2 \cdots H_n$ is a closed subset of $FG(A)$.*

It is especially interesting to note that this second conjecture, with its appealingly simple statement, was proved directly by Ribes and Zalesskiĭ, independently of Ash's work and at approximatively the same time [88]. Their methods were seemingly very different[3], drawing on the theory of profinite graphs and groups and exploiting the geometric properties of the profinite Cayley graph $\Gamma_A(\mathbf{G})$: even though, seen as a discrete graph, it is simply an uncountable disjoint union of copies of the Cayley graph of the free group $FG(A)$, its topology makes it possible to uniquely define "geodesics" between any two vertices, and gives the whole graph tree-like properties.

[3] As it turns out, Theorem 6.3 cannot be deduced directly from the confirmation of Rhodes' conjecture, but it can be deduced from an intermediary result in Ash's proof of that conjecture, see Steinberg [102] and Delgado [43] for a discussion of this issue.

More recently, Herwig and Lascar gave a new proof of Theorem 6.3 based on model-theoretic arguments [52]. Coulbois then extended that proof and showed that the collection of groups in which the product of finitely generated subgroups is closed in the profinite topology, is closed under free product [41, 42], illuminating the fact that this property of groups can be seen as a strengthening of the LERF property.

The geometric ideas mentioned above gave rise to a number of developments, notably concerning the relativization of Rhodes' conjecture to other pseudovarieties of groups. Following work by Gildenhuys and Ribes [48], Almeida and Weil characterized the pseudovarieties of groups $\mathbf{H}$ such that the profinite Cayley graph of $\hat{F}_A(\mathbf{H})$ is a profinite tree, which they termed *arborescent* pseudovarieties: they are exactly the pseudovarieties $\mathbf{H}$ such that if $G/N \in \mathbf{H}$ and $N \in \mathbf{H} \cap \mathbf{Ab}$, then $G \in \mathbf{H}$ [24, 25]. In particular, every extension-closed pseudovariety is arborescent. Moreover the same authors gave a reduced factorization statement for the finitely generated free pro-$\mathbf{H}$ groups when $\mathbf{H}$ is arborescent, and used it to compute certain joins of pseudovarieties of monoids.

Many of the definitions and results given above can be relativized to a pseudovariety of groups $\mathbf{H}$ instead of $\mathbf{G}$. For instance, the $\mathbf{H}$-*kernel* of a semigroup S, written $\mathbf{K_H}(S)$, is defined by $\mathbf{K_H}(S) = \varphi_{\mathbf{H}}^{-1}(1)$. Again, this subsemigroup of S (which contains $\mathbf{K_G}(S)$) is the intersection of all the $\tau^{-1}(1)$ where τ is a relational morphism from S into a group in $\mathbf{H}$ and, if $\mathbf{V}$ is a pseudovariety, then $S \in \mathbf{V} \textcircled{m} \mathbf{H}$ if and only if $\mathbf{K_H}(S) \in \mathbf{V}$ [50].

In particular, the decidability of $\mathbf{V} \textcircled{m} \mathbf{H}$ depends on the decidability of $\mathbf{V}$ and the computability of $\mathbf{H}$-kernels. To summarize recent work on this question, we will simply indicate here that Ribes and Zalesskiĭ proved an analogue of Theorem 6.3 for finitely generated closed subgroups with respect to the pro-$\mathbf{H}$ topology when $\mathbf{H}$ is extension-closed [89], and that Steinberg extended that result for the product of 2 closed subgroups when $\mathbf{H}$ is only supposed to be arborescent [102].

Ribes and Zalesskiĭ showed also that one can compute the pro-$\mathbf{G}_p$ closure of a finitely generated subgroup of the free group, and combined these results to show that the $\mathbf{G}_p$-kernel of a finite monoid is effectively computable [89]. Thus, a product of the form $\mathbf{V} \textcircled{m} \mathbf{G}_p$ is decidable if $\mathbf{V}$ is (see also [119] and [101]).

Regarding the pro-$\mathbf{G}_{nil}$ topology, Margolis, Sapir and Weil showed that we can here too compute the pro-$\mathbf{G}_{nil}$ closure of a finitely generated subgroup of the free group, and that the $\mathbf{G}_{nil}$-kernel of a finite inverse monoid is computable [65]. Steinberg then showed that the regular elements of the $\mathbf{G}_{nil}$-kernel of a finite monoid are computable [100], so that the products of the form $\mathbf{V} \textcircled{m} \mathbf{G}_{nil}$ are decidable when $\mathbf{V}$ is a decidable pseudovariety consisting only of regular semigroups. Note that $\mathbf{G}_p$ is extension-closed whereas $\mathbf{G}_{nil}$ is not even arborescent. A particularly challenging open problem is to find out whether the $\mathbf{G}_{sol}$-kernel of a finite monoid is computable. A positive answer would also have interesting consequences in the theory of descriptive complexity, see [107].

The equality $\mathbf{V} \textcircled{m} \mathbf{H} = \mathbf{V} * \mathbf{H}$ still holds for any pseudovariety of groups $\mathbf{H}$ when $\mathbf{V}$ is local, for instance for $\mathbf{V} = \mathbf{Sl}$. The equality of $\mathbf{J} \textcircled{m} \mathbf{H}$ and $\mathbf{J} * \mathbf{H}$ is less robust: it holds when $\mathbf{H}$ is arborescent (Steinberg [102]), but Steinberg showed that $\mathbf{J} * \mathbf{H} \neq \mathbf{J} \textcircled{m} \mathbf{H}$ when $\mathbf{H}$ consists of abelian groups, or when the elements of $\mathbf{H}$ satisfy $x^n = 1$ for some fixed integer n [101]. The connections between $\mathbf{J} * \mathbf{H}$, $\mathbf{J} \textcircled{m} \mathbf{H}$, $\mathbf{BH}$, $\mathbf{PH}$ and the elements of $\mathbf{EJ}$ with subgroups in $\mathbf{H}$, are discussed in Steinberg's [98].

7 Hyperdecidability and tameness

We now go back to Theorem 5.7, and to what knowledge is required on pseudovarieties $\mathbf{V}$ and $\mathbf{W}$ (or at least, is sufficient) to be able to use that theorem to effectively decide membership in $\mathbf{W} * \mathbf{V}$. The material in this section is drawn from the work of Almeida and Steinberg [21, 20].

7.1 Hyperdecidable pseudovarieties

Let $\mathbf{V}$ and $\mathbf{W}$ be pseudovarieties of semigroups, let S be a finite semigroup, let $\varphi\colon A^+ \to S$ be an onto morphism, and let $\varphi_{\mathbf{V}} = \mathrm{pr}_{\mathbf{V}} \circ \hat{\varphi}^{-1}$ be the natural relational morphism from S into $\hat{F}_A(\mathbf{V})$. By Theorem 5.7, S lies in $\mathbf{W} * \mathbf{V}$ if and only if for each $(\Gamma; u = v)$ in Σ, a basis of pseudo-identities for $\mathbf{gW}$ with finite vertex rank:

if δ is an $\widehat{A^+}$-labeling of Γ such that $\mathrm{pr}_{\mathbf{V}} \circ \delta$ is consistent, and if γ is the S-labeling $\gamma = \hat{\varphi} \circ \delta$,

then $\gamma(i)\gamma(u) = \gamma(i)\gamma(v)$ (where i is the initial vertex of u and v).

$$\begin{tikzcd}
\Gamma \arrow[r, "\delta"] \arrow[d, "\gamma"'] & \widehat{A^+}^1 \arrow[d, "\mathrm{pr}_{\mathbf{V}}"] \arrow[dl, "\hat{\varphi}"'] \\
S^1 \arrow[r, "\varphi_{\mathbf{V}}"] & \hat{F}_A^1(\mathbf{V})
\end{tikzcd}$$

Note that $\mathrm{pr}_{\mathbf{V}} \circ \delta \subseteq \mathrm{pr}_{\mathbf{V}} \circ \hat{\varphi}^{-1} \circ \hat{\varphi} \circ \delta = \mathrm{pr}_{\mathbf{V}} \circ \hat{\varphi}^{-1} \circ \gamma = \varphi_{\mathbf{V}} \circ \gamma$. In particular, if $x \in \Gamma$, then $\mathrm{pr}_{\mathbf{V}} \circ \delta(x) \in \varphi_{\mathbf{V}}(\gamma(x))$: we say that $\mathrm{pr}_{\mathbf{V}} \circ \delta$ is $\varphi_{\mathbf{V}}$-*associated with* γ. Then it is an elementary verification that S lies in $\mathbf{W} * \mathbf{V}$ if and only if for each $(\Gamma; u = v)$ in Σ and for each S-labeling γ of Γ,

if there exists a consistent $\hat{F}_A(\mathbf{V})$-labeling of Γ which is $\varphi_{\mathbf{V}}$-associated with γ,

then $\gamma(i)\gamma(u) = \gamma(i)\gamma(v)$ (where i is the initial vertex of u and v).

$$\begin{tikzcd}
 & \Gamma \arrow[dl, "\gamma"'] \arrow[dr, "\delta"] & \\
S^1 \arrow[rr, "\varphi_{\mathbf{V}}"] & & \hat{F}_A^1(\mathbf{V})
\end{tikzcd}$$

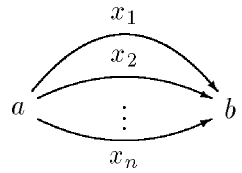
If an S-labeling γ satisfies the condition in the **if** statement above, we say that γ is $\mathbf{V}$-*inevitable* (this terminology goes back to Ash's paper [32]).

The condition of $\mathbf{V}$-inevitability for the labeling γ has an interesting expression in terms of systems of equations in $\hat{F}_A(\mathbf{V})$ with rational constraints. More precisely, we consider the following system of equations associated with Γ: the indeterminates are the Y_i ($i \in V(\Gamma)$) and Y_x ($x \in E(\Gamma)$), and the equations are of the form $Y_iY_x = Y_j$, one for each edge $x \in \Gamma(i,j)$.

Let us denote by $\overline{L}$ the closure of a language $L \subseteq A^+$ in $\hat{F}_A(\mathbf{V})$. For each $s \in S$, let L_s be the language $L_s = \varphi^{-1}(s)$. Then one can observe (see for instance [29]) that the S-labeling γ of Γ is $\mathbf{V}$-inevitable if and only if the above system of equations has a solution in $\hat{F}_A(\mathbf{V})$ such that, for each $i \in V(\Gamma)$ (resp. $x \in E(\Gamma)$), $Y_i \in \overline{L_{\gamma(i)}}$ (resp. $Y_x \in \overline{L_{\gamma(x)}}$). We denote by $\mathrm{Eq}(\Gamma, \gamma)$ this system of equations with constraints. In [16], Almeida and Delgado examine this type of systems in the case of pseudovarieties of groups.

The next step is to abstract out the pseudovariety $\mathbf{W}$ (or rather $\mathbf{gW}$) in the analysis, that is, to express a relevant condition on $\mathbf{V}$ without making reference to the basis of pseudo-identities Σ: we say that $\mathbf{V}$ is *hyperdecidable* if, given a finite semigroup S, a finite graph Γ and an S-labeling of Γ, one can decide whether that labeling is $\mathbf{V}$-inevitable, that is, one can decide whether $\mathrm{Eq}(\Gamma, \gamma)$ has a solution in $\hat{F}_A(\mathbf{V})$. This concept was first introduced by Almeida [8] with a definition in terms of the collection of the relational morphisms from S into the elements of $\mathbf{V}$, equivalent to that given here [9].

Hyperdecidability is a strong property for a pseudovariety $\mathbf{V}$, since it implies not only the decidability of the membership problem in $\mathbf{V}$, but also the decidability of the $\mathbf{V}$-pointlikes [8]. The latter decidability property is sometimes called *strong decidability* in the literature. To see how it is implied by hyperdecidability, it suffices to consider two-vertex, n-edge graphs as follows.



If S is a finite monoid and γ is an S-labeling of Γ mapping vertex a to 1, then γ is $\mathbf{V}$-inevitable if and only if the set $\{\gamma(b), \gamma(x_1), \ldots, \gamma(x_n)\}$ is $\mathbf{V}$-pointlike.

With this notion in hand, Almeida was able to prove the following theorem [8].

Theorem 7.1 *Let* $\mathbf{V}$ *and* $\mathbf{W}$ *be pseudovarieties of semigroups. If* $\mathbf{gW}$ *is decidable and has vertex rank a given integer* n*, if* $\mathbf{V}$ *is hyperdecidable, then* $\mathbf{W} * \mathbf{V}$ *is decidable.*

This theorem naturally led to the search for hyperdecidable pseudovarieties. In [8], Almeida observes that locally finite pseudovarieties are hyperdecidable provided they are *order-computable*, that is, they are decidable and there exists a computable function f such that the free object on n generators has at most $f(n)$ elements. In the same paper, Almeida also discusses hyperdecidability for pseudovarieties of groups, in relation with certain elements of Ash's proof of the Rhodes conjecture [32]. In particular, he shows how Ash's results imply that $\mathbf{G}$ is hyperdecidable. Almeida and Zeitoun [29] showed that $\mathbf{J}$ is hyperdecidable, and Steinberg extended this result to $\mathbf{J} \vee \mathbf{G}$ [99]. In fact, in [100], Steinberg furthers the discussion of group pseudovarieties, and proves in particular that $\mathbf{G}_p$ and $\mathbf{J} \vee \mathbf{G}_p$ are hyperdecidable. Almeida and Silva show a weak form of hyperdecidability for the pseudovariety $\mathbf{R}$ of $\mathcal{R}$-trivial semigroups [19], and Almeida and Trotter show that orthogroups form a hyperdecidable pseudovariety [22]. Let us note finally that if $\mathbf{V}$ is order-computable and $\mathbf{W}$ is hyperdecidable, then $\mathbf{W} \vee \mathbf{V}$ is hyperdecidable (Almeida [8]) and so is $\mathbf{W} * \mathbf{V}$ (Almeida and Silva [18]). More examples can be found in Almeida and Steinberg's [21].

7.2 Weakly σ-reducible pseudovarieties

It is difficult to show that a pseudovariety is hyperdecidable. A summary way of expressing the difficulty at hand is to say that, in general, $\hat{F}_A(\mathbf{V})$ is too complex, too "big", for us to be able to effectively solve the systems of the form $\mathrm{Eq}(\Gamma, \gamma)$ in it.

If σ is a set of implicit operations (that is, a set of elements of finitely generated free profinite semigroups) containing the semigroup multiplication (the explicit operation ab over the 2-letter alphabet $\{a, b\}$), we can look at each semigroup as a σ-algebra. We call such a set σ an *implicit signature*, and for each finite alphabet A, we may consider the free σ-algebra over A in the variety of σ-algebras generated by the elements of $\mathbf{V}$, which we write $F^\sigma_A(\mathbf{V})$.

If σ consists only in the semigroup multiplication, or more generally, if σ is a set of explicit operations, we get $F^\sigma_A(\mathbf{V}) = F_A(\mathbf{V})$, the semigroup of explicit operations. At the other extreme, if σ contains all A-ary implicit operations, then $F^\sigma_A(\mathbf{V}) = \hat{F}_A(\mathbf{V})$. In any case, $F^\sigma_A(\mathbf{V})$ is the sub-σ-algebra of $\hat{F}_A(\mathbf{V})$ generated by A.

The situation is most interesting when we consider intermediate signatures, say when σ contains non-explicit operations, yet is finite, or finitely describable; for instance, the implicit signature $\kappa = \{ab, a^{\omega-1}\}$, which we already encountered in Section 1.4. Then $F^\sigma_A(\mathbf{V})$ is countable and we can discuss such questions as the word problem in that σ-algebra, the identity problem, etc. If the elements of σ are computable, the setting is appropriate for the discussion of decidability questions.

The idea followed by Almeida and Steinberg [21] is to require more than hyperdecidability of the pseudovariety $\mathbf{V}$, in order to make solving systems of equations of the form $\mathrm{Eq}(\Gamma, \gamma)$ more effective: in fact, to make it sufficient to solve them in $F_A^\sigma(\mathbf{V})$ rather than in $\hat{F}_A(\mathbf{V})$ for a well-chosen implicit signature σ, in particular one for which solving such systems can be done effectively.

If S is a finite semigroup, Γ is a finite graph and γ is an S-labeling of Γ, recall that γ is $\mathbf{V}$-inevitable if there exists a $\hat{F}_A(\mathbf{V})$-labeling which is consistent and $\varphi_{\mathbf{V}}$-associated with γ, that is, if the system of equations with constraints $\mathrm{Eq}(\Gamma, \gamma)$ has a solution in $\hat{F}_A(\mathbf{V})$.

We say that $\mathbf{V}$ is *weakly σ-reducible* if the $\mathbf{V}$-inevitability of γ can be decided by solving $\mathrm{Eq}(\Gamma, \gamma)$ in $F_A^\sigma(\mathbf{V})$ rather than in the larger $\hat{F}_A(\mathbf{V})$. More precisely, $\mathbf{V}$ is weakly σ-reducible if and only if, for all S, Γ and γ as above, if $\mathrm{Eq}(\Gamma, \gamma)$ has a solution in $\hat{F}_A(\mathbf{V})$, then it has one in $F_A^\sigma(\mathbf{V})$. In other words, if there exists a $\hat{F}_A(\mathbf{V})$-labeling of Γ which is consistent and $\varphi_{\mathbf{V}}$-associated with γ, then there exists a $F_A^\sigma(\mathbf{V})$-labeling with the same properties.

$$\begin{tikzcd}
& \Gamma \arrow[dl, "\gamma"'] \arrow[d, "\delta"] & \\
S \arrow[r, "\varphi_{\mathbf{V}} \cap F_A^\sigma(\mathbf{V})"] \arrow[rr, bend right, "\varphi_{\mathbf{V}}"'] & F_A^\sigma(\mathbf{V}) \arrow[r, "\imath"] & \hat{F}_A(\mathbf{V})
\end{tikzcd}$$

It is interesting to note that this concept was already present implicitly in earlier work, such as Ash's [32], Almeida's [9], Delgado's [43] and Steinberg's [96].

We say that an implicit signature σ is *highly computable* if it is recursively enumerable and consists of computable operations, say, if there is an algorithm to enumerate algorithms computing the elements of σ. Almeida and Steinberg proved the following result [21].

Theorem 7.2 *Let σ be a highly computable implicit signature and let $\mathbf{V}$ be a pseudovariety. If $\mathbf{V}$ is weakly σ-reducible, if for each finite alphabet A, $F_A^\sigma(\mathbf{V})$ has a decidable word problem, and if for each rational language $L \subseteq A^*$, there is an algorithm to decide membership in $\overline{L}$ for the elements of $F_A^\sigma(\mathbf{V})$, then $\mathbf{V}$ is hyperdecidable.*

Many examples of weakly κ-reducible pseudovarieties can be found in [21] (see also [20]), notably those discussed at the end of Section 7.1, namely $\mathbf{J}$, $\mathbf{G}$, $\mathbf{G}_p$, $\mathbf{J} \vee \mathbf{G}$ and $\mathbf{J} \vee \mathbf{G}_p$.

7.3 σ-reducible pseudovarieties

Weak σ-reducibility is useful to prove hyperdecidability, and hence to handle semidirect products, yet it is not enough if we want to decide membership in iterated semidirect products — which is what is needed to decide the group complexity of a finite semigroup. Almeida and Steinberg introduced even stronger properties of a pseudovariety in the hope of solving this problem [21].

The first of these notions is a subtle shift from weak σ-reducibility, motivated by effectivity considerations. Let S be a finite semigroup and let $\varphi\colon A^+ \to S$ be an onto morphism. We consider a variant of the natural relational morphism $\varphi_{\mathbf{V}} = \mathrm{pr}_{\mathbf{V}} \circ \hat{\varphi}^{-1}$ from S into $\hat{F}_A(\mathbf{V})$ as follows: $\varphi_{\sigma,\mathbf{V}}$ is the relational morphism from S into $F_A^\sigma(\mathbf{V})$ whose graph is the sub-σ-algebra of $S \times F_A^\sigma(\mathbf{V})$ generated by the pairs $(\varphi(a), a)$ $(a \in A)$. It is immediate that for each $s \in S$, $\varphi_{\sigma,\mathbf{V}}(s)$ is contained in $\varphi_{\mathbf{V}}(s) \cap F_A^\sigma(\mathbf{V})$.

The definition of a σ-reducible pseudovariety is the same as that of a weakly σ-reducible one, only replacing $\varphi_{\mathbf{V}}$ by $\varphi_{\sigma,\mathbf{V}}$: we say that $\mathbf{V}$ is *σ-reducible* if for each finite monoid S, each finite graph Γ and each S-labeling γ of Γ, there exists a consistent $\hat{F}_A(\mathbf{V})$-labeling of Γ which is $\varphi_{\mathbf{V}}$-associated with γ if and only if there exists a consistent $F_A^\sigma(\mathbf{V})$-labeling which is $\varphi_{\sigma,\mathbf{V}}$-associated with γ.

$$\begin{tikzcd} & \Gamma \arrow[dl, "\gamma"'] \arrow[dr, "\delta"] & \\ S^1 \arrow[rr, "\varphi_{\sigma,\mathbf{V}}"] & & F_A^\sigma(\mathbf{V})^1 \end{tikzcd}$$

As $\varphi_{\sigma,\mathbf{V}}$ is contained in $\varphi_{\mathbf{V}}$, it follows easily that every σ-reducible pseudovariety is also weakly σ-reducible.

A first use of this definition is to simplify the defining set of identities for $\mathbf{W} * \mathbf{V}$ given by Theorem 5.7, when $\mathbf{V}$ is σ-reducible.

Theorem 7.3 *Let $\mathbf{V}$ and $\mathbf{W}$ be pseudovarieties of semigroups, let σ be an implicit signature such that $\mathbf{V}$ is σ-reducible, and let Σ be a defining set of pseudo-identities for $\mathbf{gW}$ with finite vertex rank.* [4] *Then $\mathbf{W} * \mathbf{V}$ is defined by the pseudo-identities of the form $\delta(\Gamma; u = v)$, where $(\Gamma; u = v)$ is an element of Σ, δ is an $F_A^\sigma(\mathbf{S})$-labeling of Γ and $\mathrm{pr}_{\mathbf{V}} \circ \delta$ is consistent.*

The difference between σ-reducibility and weak σ-reducibility, or rather between the restriction of $\varphi_{\mathbf{V}}$ to $F_A^\sigma(\mathbf{V})$ and $\varphi_{\sigma,\mathbf{V}}$ may appear slight. Let us consider however the very simple example where $\mathbf{V} = \mathbf{G}$ and σ consists only of the binary semigroup operation and the nullary choice of the identity, so that $F_A^\sigma(\mathbf{M}) = F_A^\sigma(\mathbf{G}) = A^*$, the free monoid. We know that an element $s \in S$ lies in the kernel $\mathbf{K_G}(S)$ if and only if $1 \in \varphi_{\mathbf{G}}(s)$. On the other hand, $1 \in \varphi_{\sigma,\mathbf{G}}(s)$ if and only if there exists a word $u = a_1a_2\cdots a_n$ $(n \geq 0)$ such that $s = \varphi(u)$ and $\mathrm{pr}_{\mathbf{G}}(u) = 1$. The latter is possible only if u is the empty word, and hence $s = 1$. Thus $\varphi_{\sigma,\mathbf{G}}^{-1}(1) = \{1\}$, whereas $\varphi_{\mathbf{G}}^{-1}(1) = \mathbf{K_G}(S)$.

Examples of σ-reducible pseudovarieties are briefly discussed in Section 7.5 below. For now, let us note the following necessary condition for a pseudovariety to be σ-reducible. A pseudovariety $\mathbf{V}$ is said to be *σ-equational* if it can be defined by a set of pseudo-identities involving elements of the free σ-algebras $F_A^\sigma(\mathbf{S})$, that is, pseudo-identities built up from the letters and the operations in σ. Almeida and Steinberg [21] show the following result.

Proposition 7.4 *Let $\mathbf{V}$ be a pseudovariety and let σ be an implicit signature. If $\mathbf{V}$ is σ-reducible, then $\mathbf{V}$ is σ-equational.*

It is immediate that $\mathbf{G}$ is κ-equational, but one can verify that $\mathbf{G}_p$ is not κ-equational. Therefore $\mathbf{G}_p$ is not κ-reducible, even though it is weakly κ-reducible. Almeida and Volkov showed that $\overline{\mathbf{Ab}}$ (the class of semigroups in which every subgroup is abelian) is not κ-equational, and hence it is not κ-reducible [23]. Note that $\mathbf{G}$ is also trivially η-equational, where $\eta = \{ab, a^\omega\}$; however $\mathbf{G}$ is not η-reducible [20], proving that the converse of Proposition 7.4 does not hold.

[4] The original statement of this theorem in [21] does not mention the assumption of finite vertex rank for $\mathbf{gW}$. However, the proof relies on Theorem 5.7 which imposes this hypothesis. When writing [21], Almeida and Steinberg were not aware of the mistake in [28] and used the incorrect assumptions of [28, Theorem 5.3], see the discussion around Theorem 5.7 above.

7.4 Tame pseudovarieties

These definitions and results finally led Almeida and Steinberg to the following concept [20, 21]: a pseudovariety $\mathbf{V}$ is *tame* if it behaves nicely, in the sense of the properties discussed above, with respect to a reasonable implicit signature σ. To be more precise, $\mathbf{V}$ is said to be tame if $\mathbf{V}$ is recursively enumerable and there exists a highly computable implicit signature σ such that $\mathbf{V}$ is σ-reducible and $F_A^\sigma(\mathbf{V})$ has a decidable word problem for each finite alphabet A. In that case, we sometimes say that $\mathbf{V}$ is *σ-tame.*

It is probably clear to the reader that being tame is a very strong property of a pseudovariety. In particular, it implies hyperdecidability [21, Thm 4.7].

Proposition 7.5 *Every tame pseudovariety is hyperdecidable.*

Let us briefly sketch the proof of this result, as it is an opportunity to discuss decidability issues and effectivity vs. efficiency. The object of the proof is to show that, given a finite graph Γ, a morphism $\varphi: A^+ \to S$ onto a finite semigroup S and an S-labeling γ of Γ, one can decide whether γ is $\mathbf{V}$-inevitable. The proof relies on two semi-algorithms. The first of these semi-algorithms runs through the onto morphisms $\psi: A^+ \to V$ with $V \in \mathbf{V}$ (using the fact that $\mathbf{V}$ is recursively enumerable); for each such semigroup, the algorithm considers the finitely many consistent V-labelings of Γ, and verifies whether these labelings are $\psi \circ \varphi^{-1}$-associated with γ. Now one can show that there exists a consistent $\hat{F}_A(\mathbf{V})$-labeling of Γ $\varphi_{\mathbf{V}}$-associated with γ if and only if for each onto morphism $\psi: A^+ \to V$ with $V \in \mathbf{V}$, there exists a consistent V-labeling of Γ $\psi \circ \varphi^{-1}$-associated with γ (the proof is based on a compactness argument, comparable with the proofs of Theorems 3.1 and 4.1). Thus, if γ is not $\mathbf{V}$-inevitable, then this semi-algorithm will eventually exhibit a counter-example to $\mathbf{V}$-inevitability, that is, it will produce an element V in $\mathbf{V}$ such that there is no consistent V-labeling of Γ which can be associated with γ.

The second semi-algorithm runs through the possible $F_A^\sigma(\mathbf{S})$-labelings δ of Γ (using the fact that σ is countable), such that $\mathrm{pr}_{\mathbf{V}} \circ \delta$ is consistent (using the fact that the word problem is decidable in $F_A^\sigma(\mathbf{V})$). For each such labeling, the semi-algorithm verifies whether $\mathrm{pr}_{\mathbf{V}} \circ \delta$ is contained in $\varphi_{\sigma,\mathbf{V}} \circ \gamma$. This verification can be performed in finite time, again using the decidability of the word problem in $F_A^\sigma(\mathbf{V})$. Since $\mathbf{V}$ is σ-reducible, this semi-algorithm will, if γ is $\mathbf{V}$-inevitable, eventually exhibit a consistent $F_A^\sigma(\mathbf{V})$-labeling of Γ $\varphi_{\sigma,\mathbf{V}}$-associated with γ.

Even though this argument is sufficient to establish the decidability of the $\mathbf{V}$-inevitability of a labeling of a finite graph, it is immediately apparent that the decision algorithm thus described is not computationally satisfying: this is a theoretical algorithm, in the sense that it is totally impractical. As is argued in [21, 20], it is likely that practical algorithms, for specific values of $\mathbf{V}$, will have to make use of the properties of $\mathbf{V}$.

At this point though, the usefulness of the notion of a tame pseudovariety is being questioned, in the wake of the discovery of the mistake in [28] and the subsequent strengthening of the hypothesis necessary for Theorem 5.7 to hold.

When writing [21], Almeida and Steinberg were unaware of the mistake in [28], and on the basis of an erroneous statement, they derived the following very promising statement: if $\mathbf{V}_1, \ldots, \mathbf{V}_n$ are recursively enumerable pseudovarieties, if $\mathbf{gV}_1$ is recursively definable (i.e. it is defined by a recursively enumerable list of computable pseudo-identities), and if $\mathbf{V}_2, \ldots, \mathbf{V}_n$ are tame, then the pseudovariety $\mathbf{V}_1 * \mathbf{V}_2 * \cdots * \mathbf{V}_n$ is decidable ([21, Thm 5.10], see also [20]).

If this statement was correct, it would be a major asset in the study of the semidirect product, and a crucial element of a solution of the complexity problem (see the next section). Unfortunately, the restrictive hypothesis which we now must assume to use Theorem 5.7 and its consequences, has ruined the proof of Almeida and Steinberg's statement: at this point we cannot draw any conclusion on the decidability of the semidirect product of n pseudovarieties ($n \geq 3$) from the assumption that these pseudovarieties are tame, even if their locals have finite vertex rank...

7.5 Examples

A number of important pseudovarieties are known to be κ-tame. For instance, the pseudovarieties $\mathbf{N}$, $\mathbf{K}$, $\mathbf{D}$, $\mathbf{LI}$, respectively of nilpotent, reverse definite, definite and locally trivial semigroups, are κ-tame (Almeida and Zeitoun [30], following Steinberg [96] and Almeida [7]). The pseudovariety $\mathbf{CS}$ of completely simple semigroups is also κ-tame (Almeida [9]).

The case of the pseudovariety $\mathbf{G}$, which is tame, is discussed by Almeida and Steinberg [21]. That this pseudovariety is recursively enumerable and κ-definable is trivial. The word problem in $F_A^\kappa(\mathbf{G})$ is well-known to be decidable, as this κ-algebra is the free group over A. The only non-trivial point is that $\mathbf{G}$ is κ-reducible: it is shown in [21, Thm 4.9] how the κ-reducibility of $\mathbf{G}$ can be derived from Ash's solution of the type II conjecture [32, Thm 2.1].

It is clear that $\mathbf{A}$, the pseudovariety of aperiodic semigroups, is κ-definable. McCammond showed that the word problem in $F_A^\kappa(\mathbf{A})$ is decidable for every finite alphabet A [61]. In [84], Rhodes shows that $\mathbf{A}$ is also κ-reducible. Thus $\mathbf{A}$ is tame. In view of the discussion at the end of Section 7.4, this result looked like it was the final step to the proof of the decidability of every iterated semidirect product involving the pseudovarieties $\mathbf{A}$ and $\mathbf{G}$, and hence clinched the complexity problem after more than thirty years of work! Unfortunately, the gap discovered by Rhodes and Steinberg in [28] has cancelled this last result.

Steinberg showed that every join of $\mathbf{J}$ with a κ-reducible pseudovariety of completely regular semigroups is κ-reducible [99]. In particular, $\mathbf{J} \vee \mathbf{G}$ is tame. This generalizes results of Almeida and Zeitoun on hyperdecidability [29], and of Steinberg on weak κ-reducibility [96]. Note also that the pseudovariety of orthogroups is κ-reducible (Almeida and Trotter [22]).

Not all tame pseudovarieties are κ-tame. As we saw in Section 7.3, the pseudovariety $\mathbf{G}_p$ is not κ-reducible, so that it is not κ-tame, even though it is weakly κ-reducible and the κ-algebra $F_A^\kappa(\mathbf{G}_p)$ has decidable word problem, being isomorphic to the free group on A. Nevertheless, Almeida showed that $\mathbf{G}_p$ is σ-tame for a well-crafted highly computable signature σ [10].

8 In lieu of conclusion

The join operation

One particularly important operation on the lattice $\mathcal{P}\mathbf{s}$ is hardly mentioned in this survey, namely the join operation. In fact, a number of early results obtained by profinite methods dealt with the computation of joins such as $\mathbf{R} \vee \mathbf{L}$ (Almeida and Azevedo [11]) and $\mathbf{G} \vee \mathbf{Com}$ (Almeida [2]). Both these joins turn out to have a nice, finite basis of pseudo-identities. It is known however that there exist decidable, finitely based pseudovarieties whose join is not decidable (Albert, Baldinger and Rhodes [1]). The pseudovariety $\mathbf{J} \vee \mathbf{G}$ has concentrated a lot of interest throughout the years, with a sequence of papers studying fragments of that join (e.g. $\mathbf{G} \vee (\mathbf{J} \cap \mathbf{ESl})$, which again has a finite basis of pseudo-identities [24]) or offering increasingly precise results: $\mathbf{J} \vee \mathbf{G}$ is decidable (Almeida, Azevedo and Zeitoun [15] and Steinberg [96]), it does not have a finite basis of pseudo-identities (Trotter and Volkov [114]), and it is κ-reducible (Steinberg [99]). On the joins $\mathbf{G} \vee \mathbf{Com}$ and $\mathbf{G} \vee (\mathbf{J} \cap \mathbf{ESl})$, see also Auinger [33, 34].

Elements of a general method for handling joins have started emerging (Steinberg [95, 96, 99]). Extensions are explored by Rhodes and Steinberg [90], applying the concept of semigroup expansion first introduced by Birget and Rhodes [36] to profinite semigroups. More specifically, the authors of that paper extend to the profinite realm the general expansion techniques invented by Elston [46]. The resulting theorems are quite general, and constitute a common generalization of the techniques presented above for the Mal'cev product and the semidirect product, which can also be applied to discuss joins.

The two-sided product

The two-sided semidirect product of semigroups and pseudovarieties was introduced by Rhodes and Tilson [91] as an alternative to the use of the semidirect product: this more powerful product yields right-left symmetrical decomposition results which are useful in certain cases [91, 92]. An analogue of the derived semigroupoid of a relational morphism, called the kernel, was defined by the same authors, and the results in Section 5 have natural counterparts for two-sided products, as was verified by Almeida and Weil [26, 28].

Ordered semigroups

Interest has emerged recently for classes of finite ordered semigroups and monoids (that is, where the order relation is compatible with the product operation), due to the connections with formal language theory first established by Pin [71, 72]. The basis for the use of profinite methods (pseudovarieties, pseudo-identities, etc.) were verified by Pin and Weil [76]. A Reiterman-like theorem describes the pseudovarieties of ordered semigroups by means of pseudo-identities of the form $u \leq v$, with $u, v \in \widehat{A^+}$ for some finite alphabet A.

For instance, the class of finite ordered monoids in which the unit is the maximum element forms the pseudovariety $\mathbf{J}^+ = [\![x \leq 1]\!]$. This pseudovariety gives rise to interesting developments: its localized version, $\mathbf{LJ}^+ = [\![x^\omega y x^\omega \leq x^\omega]\!]$, has ties with language theory, see below; and a 1985 result of Straubing and Thérien [106] shows that every monoid of $\mathbf{J}$ is the homomorphic image of one in $\mathbf{J}^+$ (formally: of a monoid which can be equipped with an order relation in which the unit is the maximum element). This means that $\mathbf{J}$, which can be seen as a pseudovariety of ordered monoids as well, is the join of $\mathbf{J}^+$ and of its dual $\mathbf{J}^- = [\![1 \leq x]\!]$, see [72].

Regarding Mal'cev products, Pin and Weil considered Mal'cev decomposition of pseudovarieties of ordered semigroups in the same paper which forms the basis of Section 4 above [77]. They consider pseudovarieties of ordered semigroups (resp. monoids) of the form $\mathbf{W} Ⓜ \mathbf{V}$ where $\mathbf{W}$ is a pseudovariety of ordered semigroups and $\mathbf{V}$ is a pseudovariety of unordered semigroups (resp. monoids).

For the semidirect product, the main techniques described in Section 5 can be adapted to the ordered case without major difficulty (Pin, Pinguet and Weil [80, 74] and [81] for some applications to language theory).

The products of the form $\mathbf{LJ}^+ Ⓜ \mathbf{V}$ have been studied in particular, because of deep connections with the concatenation product of recognizable languages (Pin and Weil [78]). The special case of products $\mathbf{LJ}^+ Ⓜ \mathbf{H}$ where $\mathbf{H}$ is a pseudovariety of groups, and of the class of monoids in that product, namely the pseudovariety $\mathbf{J}^+ Ⓜ \mathbf{H}$ is especially interesting, for its connections with both logical questions (see [98]) and concatenation hierarchies [73, 78]. The situation here is a little different from that described in Section 6, because $\mathbf{J}^+$ is a local pseudovariety [74], in contrast with its unordered counterpart $\mathbf{J}$. Some of Steinberg's results reported in Section 6 were proved directly in the more general setting of pseudovarieties of ordered semigroups [98].

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