Subgroups of free groups: a contribution to the Hanna Neumann conjecture[1]

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Résumé. On démontre que la conjecture forte de Hanna Neumann, sur le rang de l'intersection de sous-groupes de type fini d'un groupe libre, est vérifiée pour une large classe de sous-groupes, caractérisés par des propriétés géométriques. Un cas particulier de notre résultat entraîne que la conjecture est vraie pour les sous-groupes de type fini du groupe libre $F(A)$ (de base A) qui sont positivement engendrés, c'est-à dire engendrés par un ensemble fini de mots du monoïde libre de base A.

Abstract. We prove that the strengthened Hanna Neumann conjecture, on the rank of the intersection of finitely generated subgroups of a free group, holds for a large class of groups characterized by geometric properties. One particular case of our result implies that the conjecture holds for all positively finitely generated subgroups of the free group $F(A)$ (over the basis A), that is, for subgroups which admit a finite set of generators taken in the free monoid over A.

It has been known since Howson's 1954 paper [7] that the intersection of finitely generated subgroups of a free group is finitely generated. In fact, Howson gave an upper bound for the rank of the intersection $H \cap K$, in terms of the ranks of H and K.

In 1956, Hanna Neumann [13, 14] improved on Howson's bound, and formulated the following question. If H is a free group, we call $\widetilde{r}(H) = \max(\mathrm{rk}(H) - 1, 0)$ the *reduced rank* of H. Hanna Neumann showed that if H and K are finitely generated subgroups of a free group, then $\widetilde{r}(H \cap K) \leq 2\widetilde{r}(H)\widetilde{r}(K)$, and she asked if one might hope that

$$\widetilde{r}(H \cap K) \leq \widetilde{r}(H)\widetilde{r}(K).$$

[1]The authors gratefully acknowledge partial support from NSF grants number DMS-9970471 and DMS-0070701, and from 1999 INTAS grant number 1224 *Combinatorial and geometric theory of groups and semigroups and its applications to computer science.*

This question has become known as the **Hanna Neumann conjecture**. Hanna Neumann's proven bound was improved by Burns [2] in 1971. In 1991, Walter Neumann [15] formulated the following, stronger version of the question, which for convenience we refer to as the Strengthened Hanna Neumann Conjecture. Let H, K be finitely generated subgroups of a free group F and let $H\backslash F/K = \{HxK \mid x \in F\}$ be the set of double cosets. If $x \in F$, H^x denotes the conjugate $H^x = x^{-1}Hx$. Note that if $HxK = HyK$, then $\tilde{r}(H^x \cap K) = \tilde{r}(H^y \cap K)$.

Strengthened Hanna Neumann conjecture
If H and K are finitely generated subgroups of a free group F, then

$$\sum_{HgK \in H\backslash F/K} \tilde{r}(H^g \cap K) \leq \tilde{r}(H)\tilde{r}(K). \tag{1}$$

If H is a finitely generated subgroup of a free group F, we say that *SHN holds for H (with respect to F)* if (1) holds for every finitely generated subgroup K of F.

Walter Neumann [15] extended Burns' bound to the strengthened version of the conjecture, and Tardos [20] proved that SHN holds for all subgroups of rank 2. Tardos [21] further improved on Burns' bound for the strengthened conjecture in 1996. As far as we are aware, the best bound known to date is due to Dicks and Formanek [4], and it states that if H and K have rank at least 3, then

$$\sum_{HgK \in H\backslash F/K} \tilde{r}(H^g \cap K) \leq 2\tilde{r}(H)\tilde{r}(K) - 2(\tilde{r}(H) + \tilde{r}(K)) + 4,$$

proving in particular that SHN holds for subgroups of rank 3.

The aim of this paper is to prove that SHN holds for certain classes of subgroups H, characterized by geometric properties of the directed graph classically associated with a finitely generated subgroup of a free group. One particular case of our result implies that SHN holds for all positively finitely generated subgroups of the free group $F(A)$ (over the basis A), that is, for subgroups which admit a finite set of generators taken in the free monoid over A. This extends the well-known result that SHN holds for finite index subgroups (see [15]).

Our methods are elementary, and do not seem to relate in any obvious way with Dicks' alternative approach, whereby the problem is transformed into one on bipartite graphs [3]. Some of the results contained in this paper were announced in [12].

Note While this paper was under review, the authors came to know of the announcement of an alternate proof of SHN for positively finitely generated subgroups by Bilal Khan [10].

1 The graph-theoretical tools

It has become commonplace to use graph-theoretical tools to work with subgroups of a free group, and we will rely on this approach as well (Imrich [8, 9], Stallings [18], Gersten [5], Servatius [17], Nickolas [16], W. Neumann [15], Tardos [20], Dicks [3], etc.).

In this approach, one associates with each subgroup H of the free group $F(A)$ with basis A, a canonical *directed A-labeled graph* (that is, a directed graph whose edges are labeled with elements of A) denoted by $\Gamma(H)$.

Before we recall the definition of $\Gamma(H)$, let us make a few definitions concerning A-labeled graphs more precise. In a directed A-labeled graph, we consider paths which may traverse edges in either direction: the convention is that traversing an a-labeled edge ($a \in A$) in the backward direction carries the label a^{-1}. Thus every path has a label which is a word in the monoid $(A \sqcup A^{-1})^*$, freely generated by the disjoint union $A \sqcup A^{-1}$.

We say that a word $w \in (A \sqcup A^{-1})^*$ is *positive* if it contains no letter of A^{-1}, that is, if w lies in the free monoid A^*. A path is *positively labeled* if its label is a positive word.

A word in $(A \sqcup A^{-1})^*$ is said to be *(group-) reduced* if it does not contain any subword of the form aa^{-1} or $a^{-1}a$ ($a \in A$). The set of reduced words is in natural bijection with the free group $F(A)$. A path is said to be a *loop* if its initial and its final vertices coincide, and it is said to be *reduced* if its label is a reduced word. We say that a directed A-labeled graph is *connected* if given any pair of distinct vertices (u, v), there exists a path from u to v. Finally, such a graph is said to be *admissible* if it is connected and, for each $a \in A$ and each vertex v, there is at most one a-labeled edge starting at v, and at most one a-labeled edge into v.

The graph $\Gamma(H)$ can be constructed as follows: first we consider the graph $\Gamma_0(H)$, which represents $F(A)$ acting by right translations on the set of right H-cosets. The vertex set of $\Gamma_0(H)$ is the set $\{Hx \mid x \in F(A)\}$ and its edge set is $\{Hx \mid x \in F(A)\} \times A$: the pair (Hx, a) is an edge from Hx to Hxa, with label a. Next, $\Gamma(H)$ is the *H-core* of $\Gamma_0(H)$, that is, the least subgraph of $\Gamma_0(H)$ containing all the reduced loops based at the vertex H. We denote by $V(H)$ and $E(H)$ the sets of vertices and edges of $\Gamma(H)$.

By construction, $\Gamma(H)$ is admissible. In addition, it is well known [18] that $\Gamma(H)$ is finite if and only if H is finitely generated, that $\widetilde{r}(H) = |E(H)| - |V(H)|$, and that $\Gamma(H)$ uniquely characterizes the subgroup H. $\Gamma(H)$ can also be defined in terms of immersions on the bouquet of $|A|$ circles (Stallings [18]), or in terms of inverse automata (Margolis and Meakin [11]). Moreover, if H is given by a finite set of generators, $\Gamma(H)$ can be effectively constructed [18] (in fact in polynomial time [1]).

We say that a vertex v of a directed graph has *valence* k if v is adjacent to k edges in the underlying undirected graph. We call any vertex of valence less than 2 *extremal*, and any other vertex *internal*. For instance, in $\Gamma(H)$, there is at most one extremal vertex, and that is the vertex H. Iteratively removing extremal vertices in a directed A-labeled graph Γ yields the *core* of Γ, denoted by $c\Gamma$. If $c\Gamma$ is empty, we say that Γ is *contractible*: equivalently, Γ is a tree, or otherwise said, there is no reduced loop around any vertex of Γ.

In this framework, the intersection of subgroups of $F(A)$ can be represented in terms of pull-backs in the category of A-labeled graphs (i.e. the category of immersions into the bouquet of $|A|$ circles, Stallings [18]). More precisely, if H and K are subgroups of $F(A)$, we let $\Delta(H, K)$ be the directed A-labeled graph whose vertex set is $V(H) \times V(K)$, and whose a-labeled edges ($a \in A$) are of the form $((Hx, Ky), a)$, from vertex (Hx, Ky) to vertex (Hxa, Kya). It is well-known that $\Gamma(H \cap K)$ is isomorphic to the (H, K)-core of $\Delta(H, K)$. As $\Delta(H, K)$ may be disconnected, $\Gamma(H \cap K)$ is also isomorphic to the (H, K)-core of the connected component of $\Delta(H, K)$ containing the vertex (H, K).

Now proving the Hanna Neumann conjecture amounts to proving that the difference $|E| - |V|$ between the number of edges and the number of vertices in the connected component of the vertex (H, K) in $\Delta(H, K)$, is bounded above by $\tilde{r}(H)\tilde{r}(K) = (|E(H)| - |V(H)|)(|E(K)| - |V(K)|)$. In [15, 20] (see also [9]) it was also observed that

$$\sum_{HgK \in H \backslash F(A)/K} \tilde{r}(H^g \cap K) = \sum (|E| - |V|)$$

where the second sum runs over all the connected components of $\Delta(H, K)$ which are not contractible. In particular, the strengthened Hanna Neumann conjecture amounts to proving that $\sum(|E| - |V|) \leq \tilde{r}(H)\tilde{r}(K)$

Finally, let us note the following fact, whose verification is elementary.

Lemma 1.1 *Let H be a finitely generated subgroup of $F(A)$ and let $\varphi: F(A) \to F(B)$ be an injective morphism. If SHN holds for $\varphi(H)$ with respect to $F(B)$, then SHN holds for H with respect to $F(A)$.*

2 A result on the subgroups of a free group of rank two

In this section, we restrict our attention to a free group of rank two, and A_2 denotes the 2-letter alphabet $A_2 = \{a, b\}$.

For each vertex v of an admissible A_2-labeled graph Γ, let n_v be the number of edges out of v, and let $m_v = n_v - 1$. Then $m_v \in \{-1, 0, 1\}$, and we say that the vertex v is *positive* (resp. *zero*, *negative*) if m_v is equal to 1 (resp. 0, -1). In particular, if E and V are the sets of edges and of vertices of Γ, and if P and N are the sets of positive and negative vertices of V, then we have

$$|E| - |V| = \sum_{v \in V} m_v = |P| - |N|.$$

The aim of this section is to prove the following theorem.

Theorem 2.1 *Let H and K be finitely generated subgroups of $F(\{a, b\})$ and let n denote the number of negative vertices in $\Gamma(H)$. Then*

$$\sum_{HgK \in H \backslash F(A_2)/K} \tilde{r}(H^g \cap K) \leq (\tilde{r}(H) + n)\tilde{r}(K).$$

2.1 Basic paths

Let Δ be a directed A_2-labeled graph. A *basic* path in Δ is a positively labeled path starting in a positive vertex, ending in a negative vertex, and such that all its interior vertices (the vertices other than the origin and the end of the path) are zero vertices.

Lemma 2.2 *Let Δ be an admissible A_2-labeled graph. Two basic paths of Δ having a common zero vertex, coincide from that vertex on. In particular, there are at most two basic paths originating in a given positive vertex, and no vertex occurs twice within a basic path.*

Proof. This is elementary. $\square$

Lemma 2.3 *Let Δ be an admissible A_2-labeled graph, without extremal vertices. Every edge into a negative vertex is the last edge of a basic path.*

Proof. Let v_0 be a negative vertex and let $v_1 \xrightarrow{a_1} v_0$ be an edge into v_0. Since v_0 is negative, we have $v_1 \neq v_0$. Suppose we have constructed a positively labeled path $v_i \xrightarrow{a_i} v_{i-1} \cdots v_1 \xrightarrow{a_1} v_0$ $(i \geq 1)$ such that $v_0, v_1, \ldots, v_i$ are pairwise distinct and $v_1, \ldots, v_{i-1}$ are zero vertices.

If v_i is positive, we are done. Otherwise, v_i is a zero vertex and there is a positive edge into v_i, say $v_{i+1} \xrightarrow{a_{i+1}} v_i$ (since Δ has no extremal vertices).

If $v_{i+1} = v_j$ for some $1 \leq j \leq i$, then v_i and v_{j-1} must coincide, as they are the end vertex of the only edge out of the zero vertex v_j; and this contradicts the assumption that $v_0, v_1, \ldots, v_i$ are pairwise distinct. Thus, as long as the v_i $(i \geq 1)$ are all zero vertices, we are able to increase the length of our acceptable path. Since the graph Δ is finite, this process cannot go on forever and v_i is positive for some i, thus proving the lemma. $\square$

2.2 Matchings

We now come to a technically important *matching* property.

Proposition 2.4 *Let Δ be an admissible A_2-labeled graph without extremal vertices, and let $v_1, \ldots, v_t$ be the negative vertices of Δ. Then there exist pairwise distinct positive vertices $u_1, \ldots, u_t$ such that, for each $1 \leq i \leq t$, there exists a basic path from u_i to v_i.*

Proof. By Lemma 2.3, each negative vertex is the end vertex of two distinct basic paths, so we may consider $2t$ basic paths in Δ, with $2t$ pairwise distinct edges. Now Lemma 2.2 shows that these basic paths have no overlap, except maybe in their initial and final vertices. Thus, if Δ_0 is the subgraph of Δ consisting of these $2t$ paths, then every vertex in Δ_0 has valence 1 or 2.

It follows that each connected (undirected) component Θ of Δ_0 is topologically either a circle or a path. If Θ is a circle then it is the union of edge-disjoint basic paths of the form $w_i' \to w_i$ $(1 \leq i \leq r)$, $w_{i+1}' \to w_i$ $(1 \leq i < r)$ and $w_1' \to w_r$ with the w_i in $\{v_1, \ldots, v_t\}$; and if Θ is a path then it is the union of edge-disjoint basic paths of the form $w'_i \to w_i$ and $w'_{i+1} \to w_i$ $(1 \leq i \leq r)$ with the w_i in $\{v_1, \ldots, v_t\}$:

In either case, it is a simple matter to choose pairwise disjoint basic paths, connecting the negative vertices with some of the positive vertices as announced. □

Investigating how such matchings behave under the operation of a pull-back, we prove the following.

Proposition 2.5 *Let Γ_1 and Γ_2 be admissible A_2-labeled graphs, such that Γ_2 has no extremal vertex. Let Δ be the pull-back of Γ_1 and Γ_2, and let Γ be the union of the non-contractible connected components of Δ. Finally let P_1 and N_1 (resp. P_2 and N_2, P and N) be the sets of positive and negative vertices of Γ_1 (resp. Γ_2, Γ). Then we have*

$$|P| - |N| \leq |P_1|(|P_2| - |N_2|).$$

Proof. Let $\mathfrak{M}$ be a matching of the negative vertices of Γ_2 with some of its positive vertices, as per Proposition 2.4: $\mathfrak{M} = \{(u_1 \to v_1), \ldots, (u_t \to v_t)\}$ where the $u_i \to v_i$ are basic paths of Γ_2, the positive vertices u_i are pairwise distinct, the negative vertices v_i are pairwise distinct, and all the negative vertices of Γ_2 occur in $\mathfrak{M}$. In particular, $P_2 - N_2$ is equal to the number of positive vertices of Γ_2 which do not occur in $\mathfrak{M}$.

We now *lift* $\mathfrak{M}$ in Γ. Formally, for each basic path $u \xrightarrow{p} v$ in $\mathfrak{M}$ and for each positive vertex of the form (u', u) in Γ (i.e. u' is a positive vertex in Γ_1 and (u', u) lies in Γ), we consider the longest prefix p' of p which labels a path in Δ (or equivalently in Γ), starting in (u', u).

Since we are assuming that (u', u) is positive, there are both an a-labeled and a b-labeled edge out of (u', u) in Δ, so the prefix p' is non-empty. Let (z', z) be the vertex reached after following the path labeled p' and starting at vertex (u', u). Then (z', z) is a negative vertex of Γ. Indeed, if there is an edge out of (z', z), say, with label a, then there are a-labeled edges out of z' in Γ_1 and out of z in Γ_2. In particular, $z \neq v$ and $p'a$ must be a prefix of p, contradicting the choice of p'. So the path $(u', u) \longrightarrow (z', z)$ labeled p' is a basic path of Γ.

Let $\mathfrak{N}$ be the collection of basic paths of Γ thus constructed. By definition, their initial positive vertices are pairwise distinct (i.e. we construct exactly one path for each positive vertex of Γ with second coordinate in $\mathfrak{M}$). Let us verify that the negative vertices at the end of these basic paths are also pairwise distinct.

Let us assume that $(u', u) \xrightarrow{p_1} (z', z)$ and $(w', w) \xrightarrow{p_2} (z', z)$ are lifts of initial segments of elements of $\mathfrak{M}$ as above. In particular, $u \xrightarrow{p_1} z$ and $w \xrightarrow{p_2} z$ are initial segments of basic paths in $\mathfrak{M}$. As the elements of $\mathfrak{M}$ are pairwise disjoint and u, w are positive vertices of Γ_2, the two paths are equal in Γ_2, i.e., $p_1 = p_2$ and $u = w$. As Γ_1 is admissible, it follows that $u' = w'$ as well.

To conclude, we observe that $\mathfrak{N}$ establishes a bijection between a subset of P and a subset of N. If P' (resp. N') is the set of positive (resp. negative) vertices of Γ not in $\mathfrak{N}$, then $|P| - |N| = |P'| - |N'|$. Let also P_2' be the set of positive vertices of Γ_2 not in $\mathfrak{M}$: similarly, $|P_2'| = |P_2| - |N_2|$. Moreover, by construction of $\mathfrak{N}$, $P' \subseteq P_1 \times P_2'$. Thus

$$|P| - |N| = |P'| - |N'| \leq |P'| \leq |P_1||P_2'| = |P_1|(|P_2| - |N_2|),$$

which concludes our proof. □

2.3 Proof of Theorem 2.1

Let H, K be non-trivial finitely generated subgroups of $F(A_2)$. There exists an element $x \in F(A_2)$ such that $\Gamma(K^x)$ has no extremal vertex: it suffices to take for x the label of a path in $\Gamma(K)$ from vertex K to a vertex of the core $c\Gamma(K)$. Now we observe that if $Hg_1K, \ldots, Hg_rK$ is a list of the elements of $H\backslash F(A_2)/K$, then $Hg_1xK^x, \ldots, Hg_rxK^x$ is a list of the elements of $H\backslash F(A_2)/K^x$. Moreover, $H^{gx} \cap K^x = (H^g \cap K)^x$, so we get

$$\sum_{HgK^x \in H\backslash F(A_2)/K^x} \tilde{r}(H^g \cap K^x) = \sum_{HgK \in H\backslash F(A_2)/K} \tilde{r}(H^g \cap K).$$

Let $\Delta(H, K^x)$ be the pull-back of $\Gamma(H)$ and $\Gamma(K^x)$, and let Γ be the union of the non-contractible connected components of $\Delta(H, K^x)$. We let $P(H)$ and $N(H)$ (resp. $P(K^x)$ and $N(K^x)$, P and N) be the set of positive and negative vertices of $\Gamma(H)$ (resp. $\Gamma(K^x)$, Γ). As $\Gamma(K)$ has no extremal vertex, we may apply Proposition 2.5 to $\Gamma_1 = \Gamma(H)$ and $\Gamma_2 = \Gamma(K^x)$ to get

$$|P| - |N| \leq |P(H)|(|P(K^x)| - |N(K^x)|) = (\tilde{r}(H) + n)\tilde{r}(K^x) = (\tilde{r}(H) + n)\tilde{r}(K).$$

But $|P| - |N| = \sum_{HgK \in H\backslash F(A_2)/K} \tilde{r}(H^g \cap K)$. So we have proved the required inequality, namely

$$\sum_{HgK \in H\backslash F(A_2)/K} \tilde{r}(H^g \cap K) \leq (\tilde{r}(H) + n)\tilde{r}(K).$$

3 Applications

3.1 Subgroups of a free group of rank two

First we observe that Theorem 2.1 immediately implies the strengthened Hanna Neumann conjecture for an interesting class of subgroups of the free group of rank two.

Theorem 3.1 *Let H be a finitely generated subgroup of $F(\{a,b\})$. If $\Gamma(H)$ has no negative vertex, then SHN holds for H.*

Negative vertices are a special case of valence 2 vertices, where the two incident edges on the given vertex are oriented towards that vertex. Let us now call such a vertex a *sink*. There are three other types of valence 2 vertices: *sources*, where the two incident edges are outgoing edges; *ab-vertices*, with an incoming a-labeled edge and an outgoing b-labeled edge; and the dually defined *ba-vertices*.

Corollary 3.2 *Let H be a finitely generated subgroup of $F(\{a,b\})$. If the core of $\Gamma(H)$ has no sink (resp. no source, no ab-vertex, no ba-vertex), then SHN holds for H.*

Proof. We know that for some $x \in F(\{a,b\})$, $\Gamma(H^x)$ is isomorphic to the core of $\Gamma(H)$. Now by Lemma 1.1, it suffices to show that SHN holds for H^x. Thus we may assume without loss of generality that $\Gamma(H)$ is equal to $c\Gamma(H)$.

The statement relative to the no-sink case is exactly Theorem 3.1. In order to prove the no-source (resp. no ab-vertex, no ba-vertex) statement, we consider the automorphism φ of $F(\{a,b\})$ given by $\varphi(a) = a^{-1}$ and $\varphi(b) = b^{-1}$ (resp. $\varphi(a) = a$ and $\varphi(b) = b^{-1}$, $\varphi(a) = a^{-1}$ and $\varphi(b) = b$). It is immediate that the graph $\Gamma(\varphi(H))$ has no negative vertices. Thus SHN holds for $\varphi(H)$, and hence it does for H as well by Lemma 1.1. $\square$

Note that $\Gamma(H)$ may have a sink when its core does not: take for instance $H = \langle ab^{-1}ab^2a^{-1}\rangle$. So Corollary 3.2 is somewhat more general than Theorem 3.1, even in the no-sink case.

In the case where both $c\Gamma(H)$ and $c\Gamma(K)$ contain all four types of valence 2 vertices, Theorem 2.1 gives a bound on $\sum_{HgK\in H\backslash F(\{a,b\})/K} \tilde{r}(H^g \cap K)$ which may be better, in many cases, than the bound obtained by Dicks and Formanek (see the introduction), but it seems difficult to transcribe the geometric parameters of this bound (essentially the minimum of the numbers of sinks, sources, ab-vertices or ba-vertices in $c\Gamma(H)$ and $c\Gamma(K)$) into algebraic parameters.

3.2 Positively generated subgroups

In this section, A denotes any finite alphabet, possibly with more than two letters. We say that a subgroup H of a free group $F(A)$ is *positively generated* if it has a set of generators consisting of positive words.

As $\Gamma(H)$ is the union of the reduced loops around the vertex H in $\Gamma_0(H)$ (see Section 1), and as the labels of these loops are exactly the reduced words in H, it is not difficult to see that if $\{h_i\}_{i\in I}$ is a generating set of H, then $\Gamma(H)$ is the union of the reduced loops around the vertex H labeled by the h_i. In particular, if H is positively generated, then every vertex of $\Gamma(H)$ sits on a positively labeled loop around H.

Theorem 3.3 *Let H be a finitely, positively generated subgroup of $F(A)$. Then SHN holds for H.*

Proof. We already observed that every vertex of $\Gamma(H)$ is the origin of an edge. Thus, if A has 2 letters, then $\Gamma(H)$ has no negative vertex and SHN holds for H by Theorem 2.1.

In general, suppose that $A = \{a_1, \ldots, a_n\}$ with $n \geq 2$. For each $1 \leq i \leq n$, let $\varphi(a_i) = a^i b^i$. Then φ defines an injective morphism from $F(A)$ into $F(\{a, b\})$, which maps positive words to positive words. In particular, if H is a positively generated subgroup of $F(A)$, then $\varphi(H)$ is also positively generated. Therefore SHN holds for $\varphi(H)$, and hence SHN holds for H by Lemma 1.1. $\square$

We note that being positively generated is not an *algebraic* property of a subgroup, in the sense that it is not preserved under automorphisms of the free group. For instance, $H = \langle a \rangle$ is positively generated. However, if φ is the automorphism φ defined by $\varphi(a) = ab^{-1}$ and $\varphi(b) = b$, then $\varphi(H) = \langle ab^{-1} \rangle$ is not positively generated. It is nevertheless interesting to note the following characterization of positively generated subgroups of $F(A)$ [19].

Proposition 3.4 *Let H be a finitely generated subgroup of the free group $F(A)$. Then H is positively generated if and only if each vertex of $\Gamma(H)$ sits on a positively labeled loop around the vertex H, or equivalently, if and only if the directed graph $\Gamma(H)$ is strongly connected.*

As the proof of this characterization in [19] is constructive, classical results on the complexity of a depth-first search in a graph can be used to yield the following corollary.

Corollary 3.5 *Given a finitely generated subgroup H of $F(A)$ such that $\Gamma(H)$ has n vertices, one can decide whether H is positively generated in time $O(n \log n)$. If H is positively generated, one can exhibit a positive basis of H.*

It follows from a result of Héam [6, Prop. 14] that the finitely, positively generated subgroups are exactly the finitely generated subgroups which are the topological closure of a submonoid of A^* in the profinite topology of $F(A)$.

Note also that for a subgroup H of $F(\{a, b\})$, being positively generated is not equivalent to $\Gamma(H)$ having no sink. For example, let $H = \langle abab^{-1}, b^2a^{-1} \rangle$: H is not positively generated, yet $\Gamma(H)$ doesn't have negative vertices.

3.3 Concluding remarks

We remark that Theorem 3.3 extends the well-known result on the solution of the strengthened Hanna Neumann conjecture for finite index subgroups. Indeed, if H has finite index in $F(A)$, then $\Gamma(H)$ is a cover of the bouquet of A circles, that is, every vertex is the origin of an a-labeled edge for each letter $a \in A$ [18]. As $\Gamma(H)$ is admissible, this means that each letter $a \in A$ labels a permutation of the (finite) vertex set of $\Gamma(H)$. It follows that if $u \xrightarrow{a} v$ is an edge in $\Gamma(H)$, then some positive power of a labels a path from v to u, and hence $\Gamma(H)$ is strongly connected. In view of Proposition 3.4, it follows that finite index subgroups are always positively generated.

A simple way to extend our results is by using Lemma 1.1. For the purpose of this discussion, say that a subgroup H of $F(\{a,b\})$ is *favorable* if it satisfies the hypothesis of Corollary 3.2, that is, if $c\Gamma(H)$ avoids one of the four types of valence 2 vertices, namely sources, sinks, ab-vertices and ba-vertices. Then SHN holds for any subgroup of a free group which is the pre-image by an injective morphism of a favorable subgroup.

Unfortunately, we do not know how to characterize the finitely generated subgroups of free groups which fall in this category. In fact, at this point, we do not even have an example of a finitely generated subgroup H of a free group such that no injective image of H in $F(\{a,b\})$ is favorable.

For instance, we observe that the class of favorable subgroups of $F(\{a,b\})$ is not preserved by automorphisms. Indeed let $H = \langle abab^{-1}\rangle$ and let φ be defined on $F(\{a,b\})$ by $\varphi(a) = ab$ and $\varphi(b) = b$. Then φ is an automorphism of $F(\{a,b\})$ and $\varphi(H) = \langle abba\rangle$. Thus $\varphi(H)$ is positively generated even though H is not favorable.

It would be of interest to characterize the class of subgroups of a free group that admit an injective morphism onto a favorable subgroup.

Acknowledgements

The authors wish to thank the anonymous referee for his/her careful reading of the paper. Her/his useful suggestions led to the simplification of several proofs.

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