

A note on the continuous extensions of injective morphisms between free groups to relatively free profinite groups¹

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Abstract Let \mathbf{V} be a pseudovariety of finite groups such that free groups are residually \mathbf{V} , and let $\varphi: F(A) \rightarrow F(B)$ be an injective morphism between finitely generated free groups. We characterize the situations where the continuous extension $\hat{\varphi}$ of φ between the pro- \mathbf{V} completions of $F(A)$ and $F(B)$ is also injective. In particular, if \mathbf{V} is extension-closed, this is the case if and only if $\varphi(F(A))$ and its pro- \mathbf{V} closure in $F(B)$ have the same rank. We examine a number of situations where the injectivity of $\hat{\varphi}$ can be asserted, or at least decided, and we draw a few corollaries.

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In this paper, we are interested in the pro- \mathbf{V} topologies on finitely generated free groups, where \mathbf{V} is a *pseudovariety of groups* (a class of finite groups closed under taking subgroups, quotients and finite direct products). These topologies were introduced in the 1950s by Hall. When \mathbf{V} is the class of all finite groups, the finite index subgroups are exactly the open subgroups, and Hall proved [6] that every finitely generated subgroup is closed. More recent papers (Ribes and Zalesskiĭ [9], Margolis, Sapir, Weil [7], Weil [12]) focused on the problem of effectively computing the pro- \mathbf{V}

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closure $Cl_{\mathbf{V}}(H)$ of a given finitely generated subgroup H of a free group. It is known for instance that if \mathbf{V} is extension-closed, then $Cl_{\mathbf{V}}(H)$ has finite rank, at most equal to the rank of H [9]. In general, a finite rank subgroup may have an infinite rank closure (e.g. if \mathbf{V} is the pseudovariety of finite abelian groups), or it may be the case that if H is a finite rank subgroup, then its closure always has finite rank, possibly greater than the rank of H (e.g. if \mathbf{V} is the pseudovariety of finite nilpotent groups [7]). It is interesting to note that deciding whether a given subgroup is pro- \mathbf{V} -closed is equivalent to deciding an extension property for a certain set of partial isomorphisms of a finite set [7]. If \mathbf{V} consists of all finite p -groups, for some fixed prime p , the closure of a given finite rank subgroup can be effectively computed [9], in polynomial time [7]. On the other hand, it is not known whether the pro-solvable closure of a finite rank subgroup is effectively computable; a positive solution for this difficult open question would have interesting consequences in finite monoid theory [7] and in computational complexity (Straubing, Thérien [10]).

It is also known that finite rank closed subgroups are free factors of clopen subgroups, and that the converse holds if \mathbf{V} is extension-closed [9]. Moreover, again in the extension-closed case, if H is a finite rank pro- \mathbf{V} -closed subgroup of a free group F , then the pro- \mathbf{V} topology of H coincides with the topology it inherits from F . The central result in this paper (Theorem 1.1) characterizes the situations where this property of coincidence of topologies holds: it is equivalent to another extension property, namely to the fact that a certain injective morphism between two free groups F and F' admits an injective continuous extension between the pro- \mathbf{V} completions of F and F' . It turns out that in the extension-closed case, this is equivalent to the fact that H and its pro- \mathbf{V} closure have equal rank.

After the proof of the main result, we list a number of immediate consequences: for instance, it follows from our result that if \mathbf{V} is extension-closed, the continuous extension of an injective endomorphism of the free group of rank 2 is always injective. In the last section, we illustrate our result by considering a simple example of an injective morphism $\varphi: F \rightarrow F'$ between finitely generated free groups whose continuous extension $\hat{\varphi}$ to the pro- p completions is not injective, and we exhibit a sequence $(t_n)_n$ of elements of F whose limit points are non-trivial elements of $\ker \hat{\varphi}$.

1 Injective extendability

If A is an alphabet (that is, a finite non-empty set), then $F(A)$ denotes the free group on A . Let \mathbf{V} be a pseudovariety of finite groups: the pro- \mathbf{V} topology on a group G is the least topology which makes every morphism from G into an element of \mathbf{V} continuous. A basis of neighborhoods of 1 in this topology is given by the finite-index normal subgroups K of G such that $G/K \in \mathbf{V}$.

The pro- \mathbf{V} topology on G is Hausdorff if and only if G is residually \mathbf{V} . In that case, the pro- \mathbf{V} topology on G can be defined by an ultrametric distance function. This situation arises in particular if G is a free group and \mathbf{V} is a non-trivial extension-closed pseudovariety. In the sequel, we consider only pseudovarieties \mathbf{V} such that free groups are residually \mathbf{V} .

If G is a group, we denote by \hat{G} the pro- \mathbf{V} completion of G : it is compact and totally disconnected. If $G = F(A)$, we write $\hat{F}_{\mathbf{V}}(A)$ for \hat{G} ; this is also the free pro- \mathbf{V} group on A . If $H \subseteq G$, we write $Cl_{\mathbf{V}}(H)$ (or simply $Cl(H)$) for the closure of H in G . If $H \subseteq \hat{G}$, we write \overline{H} for the closure of H in \hat{G} . In particular, if $H \subseteq G$, $\overline{H} = Cl(H)$ and, if G is residually \mathbf{V} , then $Cl(H) = \overline{H} \cap G$.

We note that every morphism $\varphi: F \rightarrow F'$ between free groups is uniformly continuous when both groups are equipped with their respective pro- \mathbf{V} topologies. In particular, φ admits a (uniquely defined) continuous extension between the pro- \mathbf{V} completions, written $\hat{\varphi}: \hat{F} \rightarrow \hat{F}'$.

For a justification of these assertions, we refer the readers, for instance, to [7]. We now consider injective morphisms, and we state our main result.

Theorem 1.1 *Let $\varphi: F(A) \rightarrow F(B)$ be an injective morphism and let $H = \varphi(F(A))$. Let \mathbf{V} be a pseudovariety of groups such that free groups are residually \mathbf{V} . The following conditions are equivalent:*

- *The continuous extension of φ , $\hat{\varphi}: \hat{F}_{\mathbf{V}}(A) \rightarrow \hat{F}_{\mathbf{V}}(B)$ is one-to-one.*
- *The pro- \mathbf{V} topology on H coincides with the topology on H induced by the pro- \mathbf{V} topology on $F(B)$.*

If, in addition, \mathbf{V} is extension-closed, these properties are equivalent to:

- *H and $Cl(H)$ have the same rank.*

The proof of Theorem 1.1 follows directly from Propositions 1.4, 1.6 and 1.8 below.

1.1 Comparing the pro- \mathbf{V} topologies on a subgroup

We will use the following elementary remark.

Lemma 1.2 *Let $\varphi: F(A) \rightarrow F(B)$ be a morphism between finitely generated free groups, let $H = \varphi(F(A))$ be the range of φ and let $\hat{\varphi}: \hat{F}_{\mathbf{V}}(A) \rightarrow \hat{F}_{\mathbf{V}}(B)$ be the continuous extension of φ between the pro- \mathbf{V} completions of $F(A)$ and $F(B)$. Then the range of $\hat{\varphi}$ is \overline{H} .*

Proof. By continuity, we have $\hat{\varphi}(\hat{F}_{\mathbf{V}}(A)) = \hat{\varphi}(\overline{F(A)}) \subseteq \overline{\varphi(F(A))} = \overline{H}$. Of course, we also have $H \subseteq \hat{\varphi}(\hat{F}_{\mathbf{V}}(A))$. Finally, as $\hat{F}_{\mathbf{V}}(A)$ is compact and $\hat{\varphi}$ is continuous, the group $\hat{\varphi}(\hat{F}_{\mathbf{V}}(A))$ is closed, so $\hat{\varphi}(\hat{F}_{\mathbf{V}}(A)) = \overline{H}$. \square

Next we consider the following property of a finitely generated subgroup H of a free group $F(B)$:

[Property Coinc(\mathbf{V})] The pro- \mathbf{V} topology on H coincides with the topology on H induced by the pro- \mathbf{V} topology on $F(B)$.

This property translates as follows.

Lemma 1.3 *Let H be a finitely generated subgroup of a free group $F(B)$, let $\iota: H \rightarrow F(B)$ be the natural injection of H into $F(B)$, and let $\hat{\iota}: \hat{H} \rightarrow \hat{F}_{\mathbf{V}}(B)$ be the continuous extension of ι between the pro- \mathbf{V} completions of H and $F(B)$. Then H has Property Coinc(\mathbf{V}) if and only if $\hat{\iota}$ is injective. In particular, \overline{H} is homeomorphic to \hat{H} .*

Proof. This is immediate once we observe that $\hat{\iota}$ has range \overline{H} (by Lemma 1.2). \square

We are now ready to prove the first equivalence in Theorem 1.1.

Proposition 1.4 *Let \mathbf{V} be a pseudovariety of groups such that free groups are residually \mathbf{V} . Let φ , $\hat{\varphi}$ and H be as in the statement of Theorem 1.1. Then $\hat{\varphi}$ is injective if and only if H has Property Coinc(\mathbf{V}).*

Proof. Let $\psi: F(A) \rightarrow H$ be the restriction of φ to an isomorphism between $F(A)$ and H , and let $\hat{\psi}: \hat{F}_{\mathbf{V}}(A) \rightarrow \hat{H}$ be the continuous extension of ψ . As ψ is an isomorphism, $\hat{\psi}$ is a homeomorphism.

Let $\iota: H \rightarrow F(B)$ and $\hat{\iota}: \hat{H} \rightarrow \hat{F}_{\mathbf{V}}(B)$ be as in Lemma 1.3. We observe that $\varphi = \iota \circ \psi$, so that $\hat{\varphi} = \hat{\iota} \circ \hat{\psi}$. As $\hat{\psi}$ is a homeomorphism, $\hat{\varphi}$ is one-to-one if and only if $\hat{\iota}$ is. By Lemma 1.3, this is equivalent to H having Property Coinc(\mathbf{V}), as we wanted to prove. \square

1.2 The extension-closed case

The following sufficient condition for a finitely generated subgroup to have Property Coinc(\mathbf{V}) was proved in [7, Proposition 2.17]:

Proposition 1.5 *If \mathbf{V} is extension-closed, then every finitely generated, closed subgroup of $F(A)$ has Property Coinc(\mathbf{V}).*

We will see that this sufficient condition is not necessary (Proposition 1.7 below). However, we can immediately use this property to prove one half of the remaining equivalence.

Proposition 1.6 *Let \mathbf{V} be a non-trivial extension-closed pseudovariety of groups. Let $\varphi: F(A) \rightarrow F(B)$ be a morphism between free groups, let $H = \varphi(F(A))$, and let $\hat{\varphi}$ be the continuous extension of φ between the pro- \mathbf{V} completions of $F(A)$ and $F(B)$. If φ and $\hat{\varphi}$ are injective, then H and $Cl(H)$ have equal ranks.*

Proof. By Lemma 1.2, the range of $\hat{\varphi}$ is \overline{H} . Since $\hat{\varphi}$ is injective between two compact spaces, it is a homeomorphism onto its image, so \overline{H} is homeomorphic to the free pro- \mathbf{V} group of rank $|A| = \text{rank}(H)$.

On the other hand, we know from Proposition 1.5 that $Cl(H)$ has Property Coinc(\mathbf{V}). Applying Lemma 1.3 to $Cl(H)$, we find that $\overline{Cl(H)}$ is homeomorphic to the free pro- \mathbf{V} group of rank $\text{rank}(Cl(H))$.

But $\overline{H} = \overline{Cl(H)}$, so we have proved that $\text{rank}(H) = \text{rank}(Cl(H))$. \square

Before we prove the reverse implication, we show the following result, which does not require the hypothesis that \mathbf{V} is extension-closed.

Proposition 1.7 *Let H be a finitely generated subgroup of the free group $F(A)$. If H is dense in the pro- \mathbf{V} topology of $F(A)$ and if $\text{rank}(H) = \text{rank}(F(A))$, then H has Property Coinc(\mathbf{V}).*

Proof. As H and $F(A)$ have the same rank, we may consider an injective endomorphism ψ of $F(A)$ with range H . Let $\hat{\psi}: \hat{F}_{\mathbf{V}}(A) \rightarrow \hat{F}_{\mathbf{V}}(A)$ be the continuous extension of ψ . By Lemma 1.2, $\hat{\psi}$ is an onto endomorphism of $\hat{F}_{\mathbf{V}}(A)$. But every onto continuous endomorphism of a finitely generated profinite group is injective [4, Prop. 15.3]. So $\hat{\psi}$ is injective: by Proposition 1.4, this implies that H has Property Coinc(\mathbf{V}). \square

We can now give the last element in the proof of Theorem 1.1.

Proposition 1.8 *Let H be a finitely generated subgroup of the free group $F(A)$. If \mathbf{V} is extension-closed and $\text{rank}(H) = \text{rank}(Cl(H))$, then H has Property Coinc(\mathbf{V}).*

Proof. By Proposition 1.5, the pro- \mathbf{V} topology on $Cl(H)$ coincides with the topology on $Cl(H)$ induced by the pro- \mathbf{V} topology on $F(A)$. Therefore H is dense in the pro- \mathbf{V} topology on $Cl(H)$. Now Proposition 1.7 implies that the pro- \mathbf{V} topology on H coincides with the topology on H induced by the pro- \mathbf{V} topology on $Cl(H)$, and this concludes the proof. \square

2 Corollaries

The following collection of remarks is immediately deduced from Theorem 1.1. Throughout this section, \mathbf{V} denotes a pseudovariety of groups such that free groups are residually \mathbf{V} , $\varphi: F(A) \rightarrow F(B)$ is an injective morphism between free groups, $\hat{\varphi}: \hat{F}_{\mathbf{V}}(A) \rightarrow \hat{F}_{\mathbf{V}}(B)$ is the continuous extension of φ between the pro- \mathbf{V} completions of $F(A)$ and $F(B)$, and $H = \varphi(F(A))$.

Corollary 2.1 *Whether $\hat{\varphi}$ is injective depends only on H , not on φ .*

If $\mathbf{V} = \mathbf{G}$, the pseudovariety of all finite groups, the pro- \mathbf{V} completion of a group is called its profinite completion. It is well-known that for the pro- \mathbf{G} topology, every finitely generated subgroup of the free group is closed [6]. As a result, we have:

Corollary 2.2 *Every injective morphism between free groups of finite rank admits an injective continuous extension to the profinite completions of these groups.*

Let p be a prime number and let \mathbf{G}_p be the pseudovariety of finite p -groups. The pro- \mathbf{G}_p completion of a group is called its pro- p completion. It is shown in [9] that if p is a prime number, one can effectively compute the pro- p closure of a finitely generated subgroup of the free group (see [7] for a polynomial time algorithm). It follows that:

Corollary 2.3 *Given a prime number p , one can decide whether the continuous extension of φ to the pro- p completions is injective.*

Let \mathbf{G}_{sol} be the pseudovariety of finite solvable groups; the pro- \mathbf{G}_{sol} completion of a group is called its pro-solvable completion. It is also shown in [12] that one can compute the rank of the pro-solvable closure of a finite index subgroup:

Corollary 2.4 *If H has finite index, one can decide whether the continuous extension of φ to the pro-solvable completions is injective.*

For the general case however, we do not know whether one can effectively compute the rank of the pro-solvable closure of a given finitely generated subgroup (see the conclusion of [7] or [12] for a discussion). In particular, we do not know whether the injectivity of the continuous extension of φ to the pro-solvable completions is decidable.

In [7], the pro- \mathbf{V} topology is considered also when \mathbf{V} is the pseudovariety \mathbf{G}_{nil} of finite nilpotent groups, a pseudovariety which is not extension-closed. An example is given of a finitely generated subgroup which is closed in that topology yet does not have Property Coinc(\mathbf{V}) [7, Example 1.10]. This shows that the extension-closed assumption in Proposition 1.8 cannot be dispensed with. However, we also know [7] that the pro-nilpotent completion of the free group (its pro- \mathbf{G}_{nil} completion) is a subdirect product of its pro- p completions (p prime). Therefore we have:

Corollary 2.5 *The continuous extension of φ to the pro-nilpotent completions is injective if and only if, for each prime p , the continuous extension of φ to the pro- p completions is injective.*

In view of Theorem 1.1, deciding the injectivity of the pro-nilpotent extension of φ is equivalent to deciding whether every p -closure of H has the same rank as H . But there are only finitely many subgroups of $F(A)$ of the form $Cl_p(H)$, and they are effectively computable [7, 12]. It follows that:

Corollary 2.6 *It is decidable whether the extension of φ to the pro-nilpotent completions is injective.*

Returning to extension-closed pseudovarieties, it is shown in [9] that $\text{rank}(Cl(H)) \leq \text{rank}(H)$. As free groups and free pro- \mathbf{V} groups of rank 1 are commutative, it follows that if H has rank 1 or 2, then $\text{rank}(H) = \text{rank}(Cl(H))$. This translates into the following result.

Corollary 2.7 *If \mathbf{V} is extension-closed and φ is defined on the free group of rank 1 or 2, then $\hat{\varphi}$ is injective.*

This last result leads to the following consequences. Let \mathbf{G} be the pseudovariety of all finite groups, B be a finite alphabet, and $u \in \hat{F}_{\mathbf{G}}(B)$. We say that a finite group G satisfies the pseudo-identity $u = 1$ if, for every

continuous morphism $\psi: \hat{F}_{\mathbf{G}}(B) \rightarrow G$, we have $\psi(u) = 1$. The class of finite groups which satisfy a given set of pseudo-identities is a pseudovariety, and every pseudovariety can be defined in this fashion (Reiterman's theorem, see [1]).

Let $A = \{a, b\}$, B be a finite alphabet, $\varphi: F(A) \rightarrow F(B)$ be an injective morphism (that is, $\varphi(a)\varphi(b) \neq \varphi(b)\varphi(a)$) and $\hat{\varphi}: \hat{F}_{\mathbf{G}}(A) \rightarrow \hat{F}_{\mathbf{G}}(B)$ be the continuous extension of φ to the free profinite groups over A and B .

Corollary 2.8 *Let \mathbf{V} be a non-trivial extension-closed pseudovariety and let $(u_i)_{i \in I}$ be a collection of elements of $\hat{F}_{\mathbf{G}}(A)$. If \mathbf{V} satisfies the pseudo-identities $\hat{\varphi}(u_i) = 1$, then \mathbf{V} satisfies the pseudo-identities $u_i = 1$.*

To build from this result, let us observe that, if p is a prime number, it is immediate that a finite group is a p -group if and only if every one of its cyclic subgroup is a p -group. That is equivalent to saying that \mathbf{G}_p is defined by a set of one-variable pseudo-identities. In fact, it is even the case that there exists a single element $u_p \in \hat{F}_{\mathbf{G}}(a)$ such that a finite group is a p -group if and only if it satisfies the pseudo-identity $u_p = 1$ (u_p is the limit in the profinite topology of the sequence $a^{p^{n!}}$, denoted $u_p = a^{p^\omega}$ [2, Example 2.6(1)]).

It is also known that a finite group is nilpotent (resp. solvable) if and only if each of its 2-generated subgroups is nilpotent (resp. solvable), so that the pseudovarieties \mathbf{G}_{nil} and \mathbf{G}_{sol} are both defined by a set of 2-variable pseudo-identities. In the nilpotent case, this is a result of Neumann and Taylor [8] and in the solvable case, it was proved by Thompson [11], see also Flavell [3]. In fact, it is known that there exists an element $u_{nil}(a, b)$ (resp. $u_{sol}(a, b)$) of $\hat{F}_{\mathbf{G}}(A)$ such that the single 2-variable pseudo-identity $u_{nil}(a, b) = 1$ defines exactly \mathbf{G}_{nil} , Almeida [2, Example 2.7(1)] (resp. $u_{sol}(a, b) = 1$ defines exactly \mathbf{G}_{sol} , Bandman *et al.* [5]).

Thus, Corollary 2.8 implies the following.

Corollary 2.9 *Let B be a finite alphabet, and let $x, y \in F(B)$ such that $xy \neq yx$. Let \mathbf{V} be an extension-closed pseudovariety of groups.*

- *If \mathbf{V} satisfies the pseudo-identity $x^{p^\omega} = 1$, then $\mathbf{V} = \mathbf{G}_p$.*
- *If \mathbf{V} satisfies the pseudo-identity $u_{nil}(x, y) = 1$, then $\mathbf{V} = \mathbf{G}_p$ for some prime p .*
- *If \mathbf{V} satisfies the pseudo-identity $u_{sol}(x, y) = 1$, then $\mathbf{V} = \mathbf{G}_{sol}$.*

This can be rewritten in the, perhaps more readable, following form (without actually using the subtle results of the existence of a *single* pseudo-identity defining \mathbf{G}_p , \mathbf{G}_{nil} or \mathbf{G}_{sol}).

Corollary 2.10 *Let B be a finite alphabet, and let $x, y \in F(B)$ such that $xy \neq yx$. Let \mathbf{V} be an extension-closed pseudovariety of groups.*

- *If for every morphism $\psi: F(B) \rightarrow G$ into an element $G \in \mathbf{V}$, $\psi(x)$ has exponent a power of p (for some fixed prime p), then $\mathbf{V} = \mathbf{G}_p$.*
- *If for every morphism $\psi: F(B) \rightarrow G$ into an element $G \in \mathbf{V}$, $\psi(x)$ and $\psi(y)$ generate a nilpotent subgroup of G , then $\mathbf{V} = \mathbf{G}_p$ for some prime p .*
- *If for every morphism $\psi: F(B) \rightarrow G$ into an element $G \in \mathbf{V}$, $\psi(x)$ and $\psi(y)$ generate a solvable subgroup of G , then $\mathbf{V} = \mathbf{G}_{sol}$.*

3 An example

We now give an explicit example of an injective morphism φ between finitely generated free groups whose continuous extension $\hat{\varphi}$ between the corresponding free pro- p groups is not injective, and we exhibit a sequence of words $(t_n)_n$ such that $\lim \varphi(t_n) = 1$, yet 1 is not a limit point of $(t_n)_n$: thus the limit points of $(t_n)_n$ are non-trivial elements of $\ker \hat{\varphi}$. Put differently, this means that p -groups ultimately satisfy $\varphi(t_n) = 1$, yet there exists a p -group that does not satisfy any of the identities $t_n = 1$.

Let q be a fixed odd prime, let $B = \{x, y\}$, and let H be the kernel of the morphism from $F(B)$ into the additive group $\mathbb{Z}/q\mathbb{Z}$ which maps letters x and y to 1. Then H has rank $q + 1$ and if $A = \{a_0, a_1, \dots, a_q\}$, then H is the range of the injective morphism $\varphi: F(A) \rightarrow F(B)$ given by

$$\varphi(a_i) = x^i y x^{-(i+1)}, i = 0, \dots, q - 2 \quad \varphi(a_{q-1}) = x^{q-1} y \quad \varphi(a_q) = x^q.$$

As $F(B)/H$ is a q -group, H is closed in the pro- q topology: by Theorem 1.1, the continuous extension of φ to the pro- q completions of $F(A)$ and $F(B)$ is injective. On the other hand, one can show that for every other prime number p , H is dense in the pro- p topology (see [7, Section 3.1]), and by Theorem 1.1 again, the continuous extension of φ to the pro- p completions is not injective.

Let $u_0 = x$, $v_0 = y$, and for each $n \geq 0$, $u_{n+1} = u_n v_n$ and $v_{n+1} = v_n u_n$. We now fix a prime number $p \neq q$. It is well-known that the identities

$u_n = v_n$ are ultimately verified by every finite p -group (Engel identities, [8]). This means that the sequence $(u_n v_n^{-1})_n$ converges to 1 in the pro- p topology. It is also easily verified that, for each $n \geq 0$, the word $u_n v_n^{-1}$ is reduced and lies in H . Thus there exists a (unique) word $t_n \in F(A)$ such that $\varphi(t_n) = u_n v_n^{-1}$ for each $n \geq 0$.

Let S be the q -dimensional vector space over the p -element field \mathbb{F}_p with basis e_0, \dots, e_{q-1} . Let π be the projection of $F(A)$ onto S defined by $\pi(a_i) = e_i$ for $i = 0, \dots, q-1$ and $\pi(a_q) = 0$. We prove that $\pi(t_n)$ is never 0 in S , so that the additive group S (an abelian p -group) does not satisfy $t_n = 1$.

We consider the morphism $\pi \circ \varphi^{-1}$ from H to S . For each $n \geq 0$, let $s_n = \pi(t_n) = \pi \circ \varphi^{-1}(u_n v_n^{-1})$. Then

$$s_{n+1} = \pi \circ \varphi^{-1}(u_{n+1} v_{n+1}^{-1}) = \pi \circ \varphi^{-1}(u_n v_n u_n^{-1} v_n^{-1}).$$

Since H is normal, $u_n v_n u_n^{-1} v_n^{-1} \in H$ and we have

$$s_{n+1} = \pi \circ \varphi^{-1}(u_n v_n u_n^{-1} v_n^{-1}) + \pi \circ \varphi^{-1}(u_n v_n^{-1}) = s_n - \pi \circ \varphi^{-1}(u_n u_n v_n^{-1} u_n^{-1}).$$

Let σ be the linear isomorphism of S given by $\sigma(e_i) = e_{i+1}$ where the indices i and $i+1$ are taken modulo q . We leave it to the reader to verify that, for each $i = 0, \dots, q-1$, we have

$$\pi \circ \varphi^{-1}(x \varphi(a_i) x^{-1}) = \pi \circ \varphi^{-1}(y \varphi(a_i) y^{-1}) = \sigma \circ \pi(a_i).$$

It follows that for each $g \in F(A)$, we have

$$\pi \circ \varphi^{-1}(x \varphi(g) x^{-1}) = \pi \circ \varphi^{-1}(y \varphi(g) y^{-1}) = \sigma \circ \pi(g).$$

Now u_n is a positive word in x and y of length 2^n , so we have

$$s_{n+1} = s_n - \sigma^{2^n}(s_n) = (\text{id} - \sigma^{2^n})(s_n).$$

We observe that the linear transformation σ of V has order q and a one-dimensional eigenspace associated to the eigenvalue 1 (generated by $w = e_0 + \dots + e_{q-1}$) and the supplementary hyperplane W of equation $x_0 + \dots + x_{q-1} = 0$ is stable under σ . Now we use the fact that $\sigma^q = \text{id}$ and q is odd: for each integer $n \geq 0$, σ is equal to a power of σ^{2^n} . As 1 is not an eigenvalue of the restriction $\sigma|_W$, it cannot be an eigenvalue of any 2^n -th power of $\sigma|_W$. It follows that the linear transformation $\text{id} - \sigma^{2^n}$ of W is invertible for all $n \geq 0$. As $s_0 \neq 0$, we find that $s_n = \pi(t_n) \neq 0$ for each n , as announced.

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