Counting planar maps, coloured or uncoloured

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Abstract

We present recent results on the enumeration of q-coloured planar maps, where each monochromatic edge carries a weight ν. This is equivalent to weighting each map by its Tutte polynomial, or to solving the q-state Potts model on random planar maps. The associated generating function, obtained by Olivier Bernardi and the author, is differentially algebraic. That is, it satisfies a (non-linear) differential equation. The starting point of this result is a functional equation written by Tutte in 1971, which translates into enumerative terms a simple recursive description of planar maps. The proof follows and adapts Tutte's solution of *properly* q-coloured triangulations (1973-1984).

We put this work in perspective with the much better understood enumeration of families of *uncoloured* planar maps, for which the recursive approach almost systematically yields algebraic generating functions. In the past 15 years, these algebraicity properties have been explained combinatorially by illuminating bijections between maps and families of plane trees. We survey both approaches, recursive and bijective.

Comparing the coloured and uncoloured results raises the question of designing bijections for coloured maps. No complete bijective solution exists at the moment, but we present bijections for certain specialisations of the general problem. We also show that for these specialisations, Tutte's functional equation is much easier to solve that in the general case.

We conclude with some open questions.

1 Introduction

A planar map is a proper embedding in the sphere of a finite connected graph, defined up to continuous deformation. The enumeration of these objects has been a topic of constant interest for 50 years, starting with a series of papers by Tutte in the early 1960s; these papers were mostly based on recursive descriptions of maps (e.g. [103]). The last 15 years have witnessed a new burst of activity in this field, with the development of rich bijective approaches [98, 39], and their applications to the study of random maps of large size [78, 85]. In such enumerative problems, maps are usually *rooted* by orienting one edge. Figure 1 sets a first exercise in map enumeration.

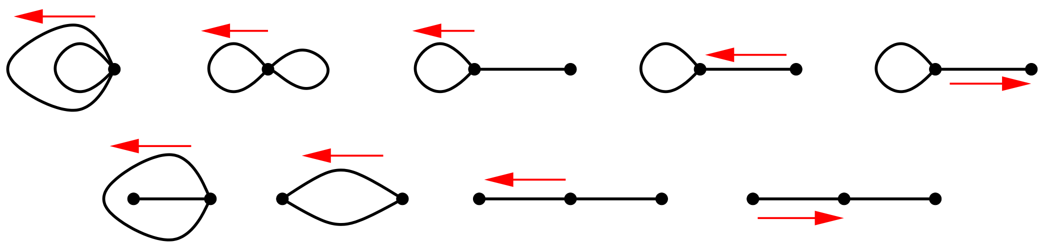

Figure 1: There are 9 rooted planar maps with two edges.

Planar maps are not only studied in combinatorics and probability, but also in theoretical physics. In this context, maps are considered as random surfaces, and constitute a model of 2-dimensional *quantum gravity*. For many years, maps were studied independently in combinatorics and in physics, and another approach for counting them, based on the evaluation of certain matrix integrals, was introduced in the 1970s in physics [42, 18], and much developed since then [55, 88]. More recently, a fruitful exchange started between the two communities. Some physicists have become masters in combinatorial methods [35, 37], while the matrix integral approach has been taken over by some probabilists [71].

From the physics point of view, it is natural to equip maps with additional structures, like particles, trees, spins, and more generally classical models of statistical physics. In combinatorics however, a huge majority of papers deal with the enumeration of bare maps. There has been some exceptions to this rule in the past few years, with combinatorial solutions of the Ising and hard-particle models on planar maps [34, 38, 39]. But there is also an earlier, and major, exception to this rule: Tutte's study of properly q-coloured triangulations (Figure 2).

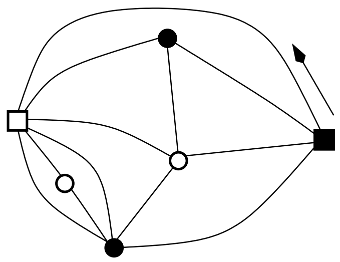

Figure 2: A (rooted) triangulation of the sphere, properly coloured with 4 colours.

This ten years long study (1973-1984) plays a central role in this paper. For a very long time, it remained an isolated *tour de force* with no counterpart for other families of planar maps or for more general colourings, probably because the corresponding series of papers [110, 108, 107, 109, 111, 112, 113, 114, 115, 116] looks quite formidable. Our main point here is to report on recent advances in the enumeration of (non-necessarily properly) q-coloured maps, in the steps of Tutte. In the associated generating function, every monochromatic edge is assigned a weight ν: the case $\nu = 0$ thus captures proper colourings. In physics terms, we are studying the q-state Potts model on planar maps. A third equivalent formulation is that we count planar maps weighted by their Tutte polynomial — a bivariate generalisation of the chromatic polynomial, introduced by Tutte, who called it the dichromatic polynomial. Since the Tutte polynomial has numerous interesting specialisations, giving for example the number of trees, forests, acyclic orientations, proper colourings of course, or the partition function of the Ising model, or the reliability and flow polynomials, we are covering several models at the same time.

We shall put this work in perspective with the (much better understood) enumeration of uncoloured maps, to which we devote Sections 3 and 4. We first present in Section 3 the robust recursive approach found in the early work of Tutte. It applies in a rather uniform way to many families of maps, and yields for their generating functions functional equations that we call *polynomial equations with one catalytic* *variable*. A typical example is (3.1). It is now understood that the solutions of these equations are always algebraic, that is, satisfy a polynomial equation. For instance, there are $2 \cdot 3^n \binom{2n}{n}/((n+1)(n+2))$ rooted planar maps with n edges, and their generating function, that is, the series

$$M(t) := \sum_{n \geq 0} \frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n} t^n,$$

satisfies

$$M(t) = 1 - 16t + 18tM(t) - 27t^2 M(t)^2.$$

Thus algebraicity is intimately connected with (uncoloured) planar maps. In Section 4, we present two more recent bijective approaches that relate maps to plane trees, which are algebraic objects *par excellence*. Not only do these bijections give a better understanding of algebraicity properties, but they also explain why many families of maps are counted by simple formulas.

In Section 5, we discuss the recursive approach for q-coloured maps. The corresponding functional equation (5.3) was written in 1971 by Tutte —who else?—, but was left untouched since then. It involves two "catalytic" variables, and it has been known for a long time that its solution is not algebraic. The key point of this section, due to Olivier Bernardi and the author, is the solution of this equation, in the form of a system of differential equations that defines the generating function of q-coloured maps. This series is thus *differentially algebraic*, like Tutte's solution of properly coloured triangulations. Halfway on the long path that leads to the solution stands an interesting intermediate result: when $q \neq 4$ is of the form $2 + 2\cos(j\pi/m)$, for integers j and m, the generating function of q-coloured planar maps is algebraic. This includes the values $q = 2$ and $q = 3$, for which we give explicit results. We also discuss certain specialisations for which the equation becomes easier to solve, like the enumeration of maps equipped with a bipolar orientation, or with a spanning tree.

Since we are still in the early days of the enumeration of coloured maps, it is not surprising that bijective approaches are at the moment one step behind. Still, a few bijections are available for some of the simpler specialisations mentioned above. They are presented in Section 6. We conclude with open questions, dealing with both uncoloured and coloured enumeration.

This survey is sometimes written in an informal style, especially when we describe bijections. Proofs are only given when they are new, or especially simple and illuminating. The reference list, although long, is certainly not exhaustive. In particular, the papers cited in this introduction are just examples illustrating our topic, and should be considered as pointers to the relevant literature. More references are given further in the paper. Two approaches that have been used to count maps are utterly absent from this paper: methods based on characters of the symmetric group and symmetric functions [68, 69], which do not exactly address the same range of problems, and the matrix integral approach, which is powerful [55], but is not always fully rigorous. The Potts model has been addressed via matrix integrals [51, 56, 123]. We refer to [15] for a description our current understanding of this work.

2 Definitions and notation

2.1 Planar maps

A *planar map* is a proper embedding of a connected planar graph in the oriented sphere, considered up to orientation preserving homeomorphism. Loops and multiple edges are allowed. The *faces* of a map are the connected components of its complement. The numbers of vertices, edges and faces of a planar map M, denoted by $\mathrm{v}(M)$, $\mathrm{e}(M)$ and $\mathrm{f}(M)$, are related by Euler's relation $\mathrm{v}(M) + \mathrm{f}(M) = \mathrm{e}(M) + 2$. The *degree* of a vertex or face is the number of edges incident to it, counted with multiplicity. A map is *m-valent* if all its vertices have degree m. A *corner* is a sector delimited by two consecutive edges around a vertex; hence a vertex or face of degree k defines k corners. The *dual* of a map M, denoted M^*, is the map obtained by placing a vertex of M^* in each face of M and an edge of M^* across each edge of M; see Figure 3.

For counting purposes it is convenient to consider *rooted* maps. A map is rooted by orienting an edge, called the *root-edge*. The origin of this edge is the *root-vertex*. The face that lies to the right of the root-edge is the *root-face*. In figures, we take the root-face as the infinite face (Figure 3). This explains why we often call the root-face the *outer* (or: *infinite*) face, and its degree the *outer degree*. The other faces are said to be *finite*. From now on, every map is *planar* and *rooted*. By convention, we include among rooted planar maps the *atomic* map m_0 having one vertex and no edge. The set of rooted planar maps is denoted $\mathcal{M}$.

A map is *separable* if it is atomic or can be obtained by gluing two non-atomic maps at a vertex. Observe that both maps with one edge are non-separable.

Figure 3: A rooted planar map and its dual (rooted at the dual edge).

2.2 Power series

Let A be a commutative ring and x an indeterminate. We denote by $A[x]$ (resp. $A[[x]]$) the ring of polynomials (resp. formal power series) in x with coefficients in A. If A is a field, then $A(x)$ denotes the field of rational functions in x, and $A((x))$ the field of Laurent series[1] in x. These notations are generalised to polynomials, fractions and series in several indeterminates. We denote by bars the reciprocals of variables: that is, $\bar{x} = 1/x$, so that $A[x, \bar{x}]$ is the ring of Laurent polynomials in x with coefficients in A. The coefficient of x^n in a Laurent series $F(x)$ is denoted

[1] A *Laurent series* is a series of the form $\sum_{n \geq n_0} a(n)x^n$, for some $n_0 \in \mathbf{Z}$.

by $[x^n]F(x)$. The *valuation* of a Laurent series $F(x)$ is the smallest d such that x^d occurs in $F(x)$ with a non-zero coefficient. If $F(x) = 0$, then the valuation is $+\infty$. If $F(x;t)$ is a power series in t with coefficients in $A((x))$, that is, a series of the form

$$F(x;t) = \sum_{n\geq 0, i\in\mathbf{Z}} f(i;n)x^i t^n,$$

where for all n, almost all coefficients $f(i;n)$ such that $i < 0$ are zero, then the *positive part* of $F(x;t)$ in x is the following series, which has coefficients in $xA[[x]]$:

$$[x^>]F(x;t) := \sum_{n\geq 0, i>0} f(i;n)x^i t^n.$$

We define similarly the non-negative part of $F(x;t)$ in x.

A power series $F(x_1,\ldots,x_k) \in \mathbb{K}[[x_1,\ldots,x_k]]$, where $\mathbb{K}$ is a field, is *algebraic* (over $\mathbb{K}(x_1,\ldots,x_k)$) if it satisfies a polynomial equation $P(x_1,\ldots,x_k,F(x_1,\ldots,x_k)) = 0$. The series $F(x_1,\ldots,x_k)$ is *D-finite* if for all $i \leq k$, it satisfies a (non-trivial) linear differential equation in x_i with coefficients in $\mathbb{K}[x_1,\ldots,x_k]$. We refer to [81, 82] for a study of these series. All algebraic series are D-finite. A series $F(x)$ is *differentially algebraic* if it satisfies a (non-necessarily linear) differential equation with coefficients in $\mathbb{K}[x]$.

2.3 The Potts model and the Tutte polynomial

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Let ν be an indeterminate, and take $q \in \mathbf{N}$. A *colouring* of the vertices of G in q colours is a map $c : V(G) \to \{1,\ldots,q\}$. An edge of G is *monochromatic* if its endpoints share the same colour. Every loop is thus monochromatic. The number of monochromatic edges is denoted by $m(c)$. The *partition function of the Potts model* on G counts colourings by the number of monochromatic edges:

$$\mathrm{P}_G(q,\nu) = \sum_{c:V(G)\to\{1,\ldots,q\}} \nu^{m(c)}.$$

The Potts model is a classical magnetism model in statistical physics, which includes (for $q = 2$) the famous Ising model (with no magnetic field) [120]. Of course, $\mathrm{P}_G(q,0)$ is the chromatic polynomial of G.

If G_1 and G_2 are disjoint graphs and $G = G_1 \cup G_2$, then clearly

$$\mathrm{P}_G(q,\nu) = \mathrm{P}_{G_1}(q,\nu)\,\mathrm{P}_{G_2}(q,\nu). \tag{2.1}$$

If G is obtained by attaching G_1 and G_2 at one vertex, then

$$\mathrm{P}_G(q,\nu) = \frac{1}{q}\,\mathrm{P}_{G_1}(q,\nu)\,\mathrm{P}_{G_2}(q,\nu). \tag{2.2}$$

The Potts partition function can be computed by induction on the number of edges. If G has no edge, then $\mathrm{P}_G(q,\nu) = q^{|V(G)|}$. Otherwise, let e be an edge of G. Denote by $G\backslash e$ the graph obtained by deleting e, and by G/e the graph obtained by contracting e (if e is a loop, then it is simply deleted). Then

$$\mathrm{P}_G(q,\nu) = \mathrm{P}_{G\backslash e}(q,\nu) + (\nu-1)\,\mathrm{P}_{G/e}(q,\nu). \tag{2.3}$$

Indeed, it is not hard to see that $\nu \, \mathrm{P}_{G/e}(q,\nu)$ counts colourings for which e is monochromatic, while $\mathrm{P}_{G\backslash e}(q,\nu) - \mathrm{P}_{G/e}(q,\nu)$ counts those for which e is bichromatic. One important consequence of this induction is that $\mathrm{P}_G(q,\nu)$ is always a *polynomial* in q and ν. We call it the *Potts polynomial* of G. Since it is a polynomial, we will no longer consider q as an integer, but as an indeterminate, and sometimes evaluate $\mathrm{P}_G(q,\nu)$ at real values q. We also observe that $\mathrm{P}_G(q,\nu)$ is a multiple of q: this explains why we will weight maps by $\mathrm{P}_G(q,\nu)/q$.

Up to a change of variables, the Potts polynomial is equivalent to another, maybe better known, invariant of graphs, namely the *Tutte polynomial* $\mathrm{T}_G(\mu,\nu)$ (see e.g. [19]):

$$\mathrm{T}_G(\mu,\nu) := \sum_{S\subseteq E(G)} (\mu-1)^{\mathrm{c}(S)-\mathrm{c}(G)}(\nu-1)^{\mathrm{e}(S)+\mathrm{c}(S)-\mathrm{v}(G)},$$

where the sum is over all spanning subgraphs of G (equivalently, over all subsets of edges) and v(.), e(.) and c(.) denote respectively the number of vertices, edges and connected components. For instance, the Tutte polynomial of a graph with no edge is 1. The equivalence with the Potts polynomial was established by Fortuin and Kasteleyn [62]:

$$\mathrm{P}_G(q,\nu) \;=\; \sum_{S\subseteq E(G)} q^{\mathrm{c}(S)}(\nu-1)^{\mathrm{e}(S)} \;=\; (\mu-1)^{\mathrm{c}(G)}(\nu-1)^{\mathrm{v}(G)}\, \mathrm{T}_G(\mu,\nu), \qquad (2.4)$$

for $q=(\mu-1)(\nu-1)$. In this paper, we work with P_G rather than T_G because we wish to assign real values to q (this is more natural than assigning real values to $(\mu-1)(\nu-1)$). However, one property looks more natural in terms of T_G: if G and G^* are dual connected planar graphs (that is, if G and G^* can be embedded as dual planar maps) then

$$\mathrm{T}_{G^*}(\mu,\nu) = \mathrm{T}_G(\nu,\mu).$$

Translating this identity in terms of Potts polynomials thanks to (2.4) gives:

$$\begin{aligned}
\mathrm{P}_{G^*}(q,\nu) &= q(\nu-1)^{\mathrm{v}(G^*)-1}\,\mathrm{T}_{G^*}(\mu,\nu)\\
&= q(\nu-1)^{\mathrm{v}(G^*)-1}\,\mathrm{T}_G(\nu,\mu)\\
&= \frac{(\nu-1)^{\mathrm{e}(G)}}{q^{\mathrm{v}(G)-1}}\,\mathrm{P}_G(q,\mu), \qquad (2.5)
\end{aligned}$$

where $\mu = 1+q/(\nu-1)$ and the last equality uses Euler's relation: $\mathrm{v}(G)+\mathrm{v}(G^*)-2 = \mathrm{e}(G)$.

3 Uncoloured planar maps: the recursive approach

In this section, we describe the first approach that was used to count maps: the recursive method. It is based on very simple combinatorial operations (like the deletion or contraction of an edge), which translate into non-trivial functional equations defining the generating functions. A recent theorem, generalising the so-called *quadratic method*, states that the solutions of all equations of this type are algebraic. Since the recursive method applies to many families of maps, numerous algebraicity results follow.

3.1 A functional equation for planar maps

Consider a rooted planar map, distinct from the atomic map. Delete the root-edge. If this edge is an isthmus, one obtains two connected components M_1 and M_2, and otherwise a single component M, which we can root in a canonical way (Figure 4). Conversely, starting from an ordered pair (M_1, M_2) of maps, there is a unique way to connect them by a new (root) edge. If one starts instead from a single map M, there are $d+1$ ways to add a root edge, where $d = \mathrm{df}(M)$ is the degree of the root-face of M (Figure 5).

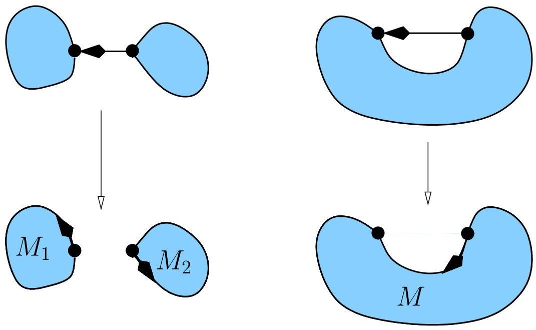

Figure 4: Deletion of the root-edge in a planar map.

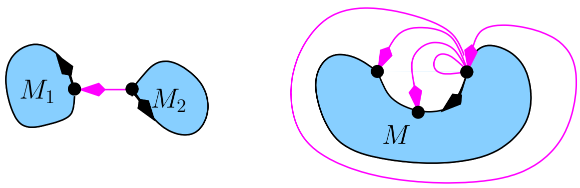

Figure 5: Reconstruction of a planar map.

Hence, to derive from this recursive description of planar maps a functional equation for their generating function, we need to take into account the degree of the root-face, by an additional variable y. Hence, let

$$M(t;y) = \sum_{M \in \mathcal{M}} t^{\mathrm{e}(M)} y^{\mathrm{df}(M)} = \sum_{d \geq 0} y^d M_d(t)$$

be the generating function of planar maps, counted by edges and outer-degree. The series $M_d(t)$ counts by edges maps with outer degree d. The recursive description of maps translates as follows:

$$\begin{aligned} M(t;y) &= 1 + y^2 t M(t;y)^2 + t \sum_{d \geq 0} M_d(t)(y + y^2 + \cdots + y^{d+1}) \\ &= 1 + y^2 t M(t;y)^2 + ty \frac{yM(t;y) - M(t;1)}{y-1}. \end{aligned} \tag{3.1}$$

Indeed, connecting two maps M_1 and M_2 by an edge produces a map of outer-degree $\mathrm{df}(M_1) + \mathrm{df}(M_2) + 2$, while the $d+1$ ways to add an edge to a map M such that $\mathrm{df}(M) = d$ produce $d+1$ maps of respective outer degree $1, 2, \ldots, d+1$, as can be seen on Figure 5. The term 1 records the atomic map.

The above equation was first written by Tutte in 1968 [105]. It is typical of the type of equation obtained in (recursive) map enumeration. More examples will be given in Section 3.2. One important feature in this equation is the divided difference

$$\frac{yM(t;y) - M(t;1)}{y-1},$$

which prevents us from simply setting $y = 1$ to solve for $M(t;1)$ first, and then for $M(t;y)$. The parameter $\mathrm{df}(M)$, and the corresponding variable y, are said to be *catalytic* for this equation — a terminology borrowed to Zeilberger [122].

Such equations do not only occur in connection with maps: they also arise in the enumeration of polyominoes [24, 59, 101], lattice walks [31, 3, 52, 76, 96], permutations [25, 28, 121]... The solution of these equations has naturally attracted some interest. The "guess and check" approach used in the early 1960s is now replaced by a general method, which we present below in Section 3.3. This method implies in particular that *the solution of any (well-founded) polynomial equation with one catalytic variable is algebraic*. It generalises the *quadratic method* developed by Brown [46] for equations of degree 2 that involve a single additional unknown series (like $M(t;1)$ in the equation above) and also the *kernel method* that applies to linear equations, and seems to have first appeared in Knuth's *Art of Computer Programming* [76, Section 2.2.1, Ex. 4] (see also [2, 31, 96]).

Contraction vs. deletion. Before we move to more examples, let us make a simple observation. Another natural way to decrease the edge number of a map is to contract the root-edge, rather than delete it (if this edge is a loop, one just erases it). When one tries to use this to count planar maps, one is lead to introduce the degree of the root-vertex as a catalytic parameter, and a corresponding variable x in the generating function. This yields the same equation as above:

$$M(t;x) = 1 + x^2 t M(t;x)^2 + t \sum_{d \geq 0} M_d(t)(x + x^2 + \cdots + x^{d+1}).$$

As illustrated by Figure 6, the term 1 records the atomic map, the second term corresponds to maps in which the root-edge is a loop, and the third term to the remaining cases. In particular, the sum $(x + x^2 + \cdots + x^{d+1})$ now describes how to

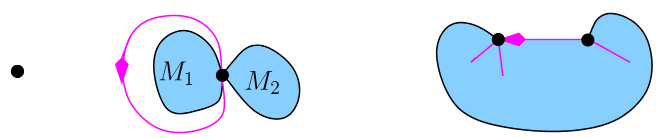

Figure 6: Contraction of the root-edge in a planar map.

distribute the adjacent edges when a new edge is inserted. Given that the contraction operation is the dual of the deletion operation, it is perfectly natural to obtain the same equation as before. The reason why we mention this alternative construction is that, when we establish below a functional equation for maps weighted by their Potts (or Tutte) polynomial, we will have to use simultaneously these two operations, as suggested by the recursive description (2.3) of the Potts polynomial. This will naturally result in equations with *two* catalytic variables x and y.

3.2 More functional equations

The recursive method is extremely robust. We illustrate this by a few examples. Two of them — maps with prescribed face degrees, and Eulerian maps with prescribed face degrees — actually cover infinitely many families of maps. Some of these examples also have a colouring flavour.

Maps with prescribed face degrees. Consider for instance the enumeration of *triangulations*, that is, maps in which all faces have degree 3. The recursive deletion of the root-edge gives maps in which all finite faces have degree 3, but the outer face may have any degree: these maps are called *near-triangulations*. We denote by $\mathcal{T}$ the set of near-triangulations. The deletion of the root-edge in a near triangulation gives either two near-triangulations, or a single one, *the outer degree of which is at least two* (Figure 7). In both cases, there is unique way to reconstruct the map we started from. Let $T(t;y) \equiv T(y)$ be the generating function of near-triangulations, counted by edges and by the outer degree:

$$T(t;y) = \sum_{M \in \mathcal{T}} t^{\mathrm{e}(M)} y^{\mathrm{df}(M)} = \sum_{d \geq 0} y^d T_d(t).$$

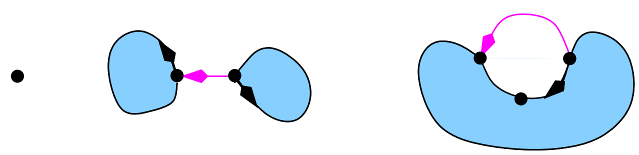

Figure 7: Deletion of the root-edge in a near-triangulation.

The above recursive description translates into

$$T(y) = 1 + ty^2T(y)^2 + t\,\frac{T(y) - T_0 - yT_1}{y}, \tag{3.2}$$

where $T_0 = 1$ counts the atomic map. We have again a divided difference, this time at $y = 0$. Its combinatorial interpretation ("it is forbidden to add an edge to a map of outer degree 0 or 1") differs from the interpretation of the divided difference occurring in (3.1) ("there are multiple ways to add an edge"). Still, both equations are of the same type and will be solved by the same method. Note that we have omitted the variable t in the notation $T(y)$, which we will do quite often in this paper, to avoid heavy notation and enhance the catalytic parameter(s).

Consider now *bipartite* planar maps, that is, maps that admit a proper 2-colouring (and then a unique one, if the root-vertex is coloured white). For planar maps, this is equivalent to saying that all faces have an even degree. Let $B(t;y) = \sum_{d\geq 0} B_d(t)y^d$ be the generating function of bipartite maps, counted by edges (variable t) and by half the outer degree (variable y). Then the deletion of the root-edge translates as follows (Figure 8):

$$\begin{aligned} B(y) &= 1 + tyB(y)^2 + t\sum_{d\geq 0} B_d(y + y^2 + \cdots + y^d) \\ &= 1 + tyB(y)^2 + ty\frac{B(y) - B(1)}{y-1}. \end{aligned} \tag{3.3}$$

This is again a quadratic equation with one catalytic variable, y.

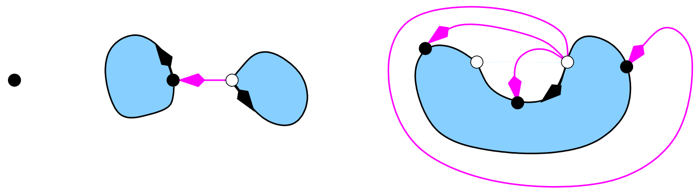

Figure 8: Deletion of the root-edge in a bipartite map.

More generally, it was shown by Bender and Canfield [6] that the recursive approach applies to any family of maps for which the face degrees belong to a given set D, provided D differs from a finite union of arithmetic progressions by a finite set. In all cases, the equation is quadratic, but may involve more than a single additional unknown function. For instance, when counting near-quadrangulations rather than near-triangulations, Eq. (3.2) is replaced by

$$Q(y) = 1 + ty^2Q(y)^2 + t\,\frac{Q(y) - Q_0 - yQ_1 - y^2Q_2}{y^2},$$

where Q_i counts near-quadrangulations of outer degree i. Bender and Canfield solved these equations using a theorem of Brown from which the quadratic method is derived, proving in particular that the resulting generating function is always algebraic. Their result only involves the edge number, but, when D is finite, it can be refined by keeping track of the vertex degree distribution [29].

Eulerian maps with prescribed face degrees. A planar map is Eulerian if all vertices have an even degree. Equivalently, its faces admit a proper 2-colouring (and a unique one, if the root-face is coloured white). Of course, Eulerian maps are the duals of bipartite maps, so that their generating function (by edges, and half-degree of the root-vertex) satisfies (3.3). But we wish to impose conditions on the face degrees of Eulerian maps (dually, on the vertex degrees of bipartite maps). This includes as a special case the enumeration of (non-necessarily Eulerian) maps with prescribed face degrees, discussed in the previous paragraph: indeed, if we require that all black faces of an Eulerian map have degree 2, each black face can be contracted into a single edge, leaving a standard map with prescribed (white) face degrees.

Generally speaking, it is difficult to count families of maps with conditions on the vertex degrees *and* on the face degrees (and being Eulerian is a condition on vertex degrees). However, it was shown in [29] that the enumeration of Eulerian maps such that all black faces have degree in $D_\bullet$ and all white faces have degree in $D_\circ$ can be addressed by the recursive method when $D_\bullet$ and $D_\circ$ are finite. This is also true when $D_\bullet = \{m\}$ and $D_\circ = m\mathbf{N}$ (such maps are called *m-constellations*).

Let us take the example of Eulerian near-triangulations. All finite faces have degree 3, while the infinite face, which is white by convention, has degree $3d$ for some $d \in \mathbf{N}$. In order to decompose these maps, we now delete all the edges that bound the black face adjacent to the root-edge (Figure 9). This leaves 1, 2 or 3 connected components, which are themselves Eulerian near-triangulations, and which we root in a canonical way. Let $E(z; y) \equiv E(y) = \sum_{d\geq 0} E_d(z)y^d$ be the generating function of Eulerian near-triangulations, counted by black faces (variable z) and by the outer degree, divided by 3 (variable y). The above decomposition gives:

$$E(y) = 1 + zyE(y)^3 + 2zE(y)(E(y) - E_0) + z(E(y) - E_0) + z\,\frac{E(y) - E_0 - yE_1}{y}.$$

This is a cubic equation with one catalytic variable, which is routinely solved by the method presented below in Section 3.3.

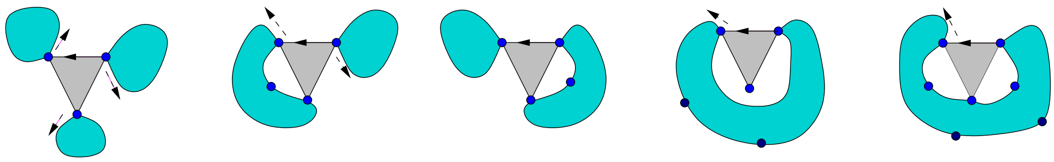

Figure 9: Decomposition of Eulerian near-triangulations.

The enumeration of Eulerian triangulations is often presented as a colouring problem [36, 54], for the following reason: a planar triangulation admits a proper 3-colouring of its vertices if and only if it is (properly) face-bicolourable, that is, Eulerian[2]. Moreover, fixing the colours of the endpoints of the root-edge determines completely the colouring. More generally, let us say that a q-colouring is *cyclic* if around any face, one meets either the colours $1, 2, \ldots q, 1, 2, \ldots, q$, in this order, or $q, q-1, \ldots, 1, q, q-1, \ldots, 1$. Then for $q \geq 3$, a planar map admits a cyclic q-colouring if and only if it is Eulerian and all its face degrees are multiples of q. In this case, it has exactly $2q$ cyclic colourings. The m-constellations defined above are of this type (with $m = q$).

Other families of maps. Beyond the two general enumeration problems we have just discussed, the recursive approach applies to many other families of planar maps:

[2]It is easy to see that the condition is necessary: around a face, in clockwise order, one meets either the colours 1, 2, 3 in this order, or 3, 2, 1, and all faces that are adjacent to a 123-face are of the 321-type. The converse is easily seen to hold by induction on the face number, using Figure 9.

loopless maps [8, 119], maps with higher connectivity [43, 47, 67], dissections of a regular polygon [44, 45, 103], triangulations with large vertex degrees [13], maps on surfaces of higher genus [5, 7, 65]... The resulting equations are often fruitfully combined with *composition equations* that relate the generating functions of two families of maps, for instance general planar maps and non-separable planar maps (see. e.g., [104, Eq. (6.3)] or [103, Eq. (2.5)]).

3.3 Equations with one catalytic variable and algebraicity theorems

In this section, we state a general theorem that implies that the solutions of all the functional equations we have written so far are algebraic. We then explain how to solve in practice these equations. The method extends the *quadratic method* that applies to quadratic equations with a unique additional unknown series [68, Section 2.9].

Let $\mathbb{K}$ be a field of characteristic 0, typically $\mathbf{Q}(s_1, \ldots, s_k)$ for some indeterminates $s_1, \ldots, s_k$. Let $F(y) \equiv F(t; y)$ be a power series in $\mathbb{K}(y)[[t]]$, that is, a series in t with rational coefficients in y. Assume that these coefficients have no pole at $y = 0$. The following divided difference (or discrete derivative) is then well-defined:

$$\Delta F(y) = \frac{F(y) - F(0)}{y}.$$

Note that

$$\lim_{y \to 0} \Delta F(y) = F'(0),$$

where the derivative is taken with respect to y. The operator $\Delta^{(i)}$ is obtained by applying i times Δ, so that:

$$\Delta^{(i)} F(y) = \frac{F(y) - F(0) - yF'(0) - \cdots - y^{i-1}/(i-1)!\, F^{(i-1)}(0)}{y^i}.$$

Now

$$\lim_{y \to 0} \Delta^{(i)} F(y) = \frac{F^{(i)}(0)}{i!}.$$

Assume $F(t; y)$ satisfies a functional equation of the form

$$F(y) \equiv F(t; y) = F_0(y) + t\, Q\Big(F(y), \Delta F(y), \Delta^{(2)} F(y), \ldots, \Delta^{(k)} F(y), t; y\Big), \quad (3.4)$$

where $F_0(y) \in \mathbb{K}(y)$ and $Q(y_0, y_1, \ldots, y_k, t; y)$ is a polynomial in the $k + 2$ indeterminates $y_0, y_1, \ldots, y_k, t$, and a rational function in the last indeterminate y, having coefficients in $\mathbb{K}$. This equation thus involves, in addition to $F(y)$ itself, k additional unknown series, namely $F^{(i)}(0)$ for $0 \le i < k$.

Theorem 3.1 ([29, 15]) *Under the above assumptions, the series $F(t; y)$ is algebraic over $\mathbb{K}(t, y)$.*

In practice, one proceeds as follows to obtain an algebraic system of equations defining the k unknown series $F^{(i)}(0)$. An example will be detailed further down. Write (3.4) in the form

$$P(F(y), F(0), \ldots, F^{(k-1)}(0), t; y) = 0, \quad (3.5)$$

for some polynomial $P(y_0, y_1, \ldots, y_k, t; y)$, and consider the following equation in Y:

$$\frac{\partial P}{\partial y_0}(F(Y), F(0), \ldots, F^{(k-1)}(0), t; Y) = 0. \tag{3.6}$$

On explicit examples, it is usually easy to see that this equation admits k solutions $Y_0, \ldots, Y_{k-1}$ in the ring of Puiseux series in t with a non-negative valuation (a Puiseux series is a power series in a fractional power of t, for instance a series in $\sqrt{t}$). By differentiating (3.5) with respect to y, it then follows that

$$\frac{\partial P}{\partial y}(F(Y), F(0), \ldots, F^{(k-1)}(0), t; Y) = 0. \tag{3.7}$$

Hence the following system of $3k$ algebraic equations holds: for $i = 0, \ldots, k-1$,

$$\begin{aligned} P(F(Y_i), F(0), \ldots, F^{(k-1)}(0), t; Y_i) &= 0, \\ \frac{\partial P}{\partial y_0}(F(Y_i), F(0), \ldots, F^{(k-1)}(0), t; Y_i) &= 0, \\ \frac{\partial P}{\partial y}(F(Y_i), F(0), \ldots, F^{(k-1)}(0), t; Y_i) &= 0. \end{aligned} \tag{3.8}$$

This system involves $3k$ unknown series, namely Y_i, $F(Y_i)$, and $F^{(i)}(0)$ for $0 \leq i < k$. The fact that the series $F^{(i)}(0)$ are derivatives of F plays no particular role. Observe that the above system consists of k times the *same* triple of equations, so that elimination in this system is not obvious [29] (and will often end up being very heavy). When $k = 1$, however, obtaining a solution takes three lines in Maple. Consider for instance the equation (3.2) we have obtained for near-triangulations. Eq. (3.6) reads in this case

$$Y = t + 2tY^3T(Y),$$

and it is clear that it has a unique solution, which is a formal power series in t with constant term 0. Indeed, the coefficient of t^n in Y can be determined inductively in terms of the coefficients of T. Then (3.7) reads

$$T(Y) = 1 + 3tY^2T(Y)^2 - tT_1.$$

These two equations, combined with the original equation (3.2) taken at $y = Y$, form a system of three polynomial equations involving $Y, T(Y)$ and T_1, from which Y and $T(Y)$ are readily eliminated by taking resultants. This leaves a polynomial equation for the unknown series T_1, which counts near-triangulations of outer degree 1:

$$T_1 = t^2 - 27t^5 + 30\,t^3T_1 + t(1 - 96\,t^3){T_1}^2 + 64\,t^5{T_1}^3.$$

One can actually go further and obtain simple expressions for the coefficients of T_1. The above equation admits rational parametrisations, for instance

$$t^3 = X(1-2X)(1-4X), \quad t\,T_1 = \frac{X(1-6X)}{1-4X},$$

and the Lagrange inversion formula yields the number of near-triangulations of outer degree 1 having $3n+2$ edges (hence $n+2$ vertices) as

$$\frac{2 \cdot 4^n (3n)!!}{n!!(n+2)!},$$

where $n!! = n(n-2)(n-4)\cdots(n - 2\lfloor\frac{n-1}{2}\rfloor)$. The existence of such simple formulas will be discussed further, in connection with bijective approaches (Section 4).

Algebraicity results. The general algebraicity result for solutions of polynomial equations with one catalytic variable (Theorem 3.1), combined with the wide applicability of the recursive method, implies that many families of planar maps have an algebraic generating function. In the following theorem, the term *generating function* refers to the generating function by vertices, faces and edges (of course, one of these statistics is redundant, by Euler's formula).

Theorem 3.2 ([6, 29]) *For any set $D \subset \mathbf{N}$ that differs from a finite union of arithmetic progressions by a finite set, the generating function of maps such that all faces have their degree in D is algebraic. If D is finite, this holds has well for the refined generating function that keeps track of the number of i-valent faces, for all $i \in D$.*

For any finite sets $D_\circ$ and $D_\bullet$ in $\mathbf{N}$, the generating function of face-bicoloured maps such that all white (resp. black) faces have their degree in $D_\circ$ (resp. $D_\bullet$) is algebraic. This holds as well for the generating function that keeps track of the number of i-valent white and black faces, for all $i \in \mathbf{N}$.

Finally, the generating function of face-bicoloured planar maps such that all black faces have degree m, and all white faces have their degree in $m\mathbf{N}$, is algebraic.

Where is the quadratic method? To finish this section, let us briefly sketch why the above procedure for solving equations with one catalytic variable generalises the quadratic method. The first two equations of (3.8) show that $y_0 = F(Y_i)$ is a double root of $P(y_0, F(0), \ldots, F^{k-1}(0), t; Y_i)$. Hence $y = Y_i$ cancels the discriminant of $P(y_0, F(0), \ldots, F^{k-1}(0), t; y)$, taken with respect to y_0. When P has degree 2 in y_0, it is easy to see that the third equation of (3.8) means that Y_i is actually a double root of the discriminant [29, Section 3.2]: this is the heart of the quadratic method, described in [68, Section 2.9]. That each series Y_i is a multiple root of the discriminant actually holds for equations of higher degree, but this is far from obvious [29, Section 6].

4 Uncoloured planar maps: bijections

So far, we have emphasised the fact that many families of planar maps have an algebraic generating function. It turns out that many of them are also counted by remarkably simple numbers, which have a strong flavour of tree enumeration. Both observations raise a natural question: is it possible to explain the algebraicity and/or the numbers more combinatorially, via bijections that would relate maps to trees?

We present in this section two bijections between planar maps and some families of trees that allow one to determine very elegantly the number of planar maps having n edges. The first bijection also explains combinatorially why the associated generating function is algebraic. The second one has other virtues, as it allows to record the distances of vertices to the root-vertex. This property has proved extremely useful is the study of random maps of large size and their scaling limit [49, 78, 80, 86, 85]. Both bijections could probably qualify as *Proofs from The Book* [1]. Both are robust enough to be generalised to many other families of maps, as discussed in Section 4.2.

4.1 Two proofs from The Book?

Both types of bijections involve families of maps with bounded vertex degrees (or, dually, bounded face degrees). So let us first recall why planar maps are equivalent to planar 4-valent maps (or dually, to quadrangulations).

Take a planar map, and create a vertex in the middle of each edge, called *e-vertex* to distinguish it from the vertices of the original map. Then, turning inside each face, join by an edge each pair of consecutive e-vertices (Figure 10). The e-vertices, together with these new edges, form a 4-valent map. Root this map in a canonical way. This construction is a bijection between rooted planar maps with n edges and rooted 4-valent maps with n vertices.

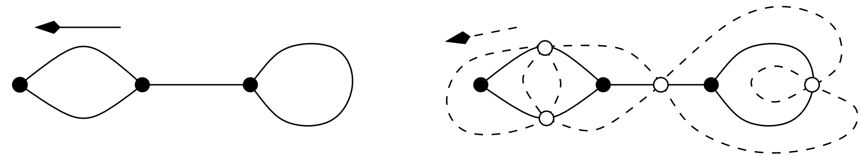

Figure 10: A planar map with n edges and the corresponding 4-valent planar map with n vertices (dashed lines).

Four-valent maps and blossoming trees. The first bijection, due to Schaeffer [98], transforms 4-valent maps into *blossoming trees*. A blossoming tree is a (plane) binary tree, rooted at a leaf, such that every inner node carries, in addition to its two children, a *flower* (Figure 11). There are three possible positions for each flower. If the tree has n inner nodes, it has n flowers and $n+2$ leaves. Flowers and leaves are called *half-edges*.

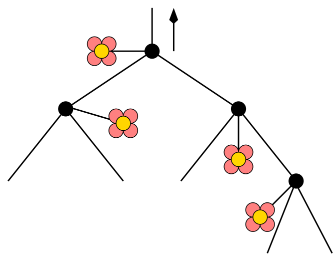

Figure 11: A blossoming tree with $n = 4$ inner nodes.

One obtains a blossoming tree by *opening* certain edges of a 4-valent map (Figure 12). Take a 4-valent map M. First, cut the root-edge into two half-edges, that become leaves: the first of them will be the root of the final tree. Then, start walking around the infinite face in counterclockwise order, beginning with the root edge. Each time a non-separating[3] edge has just been visited, cut it into two half-edges: the first one becomes a flower, and the second one, a leaf. Proceed until all edges

[3] An edge is *separating* if its deletion disconnects the map, *non-separating* otherwise; a map is a tree if and only if all its edges are separating.

are separating edges; this may require to turn several times around the map. The final result is a blossoming tree, denoted $\Psi(M)$.

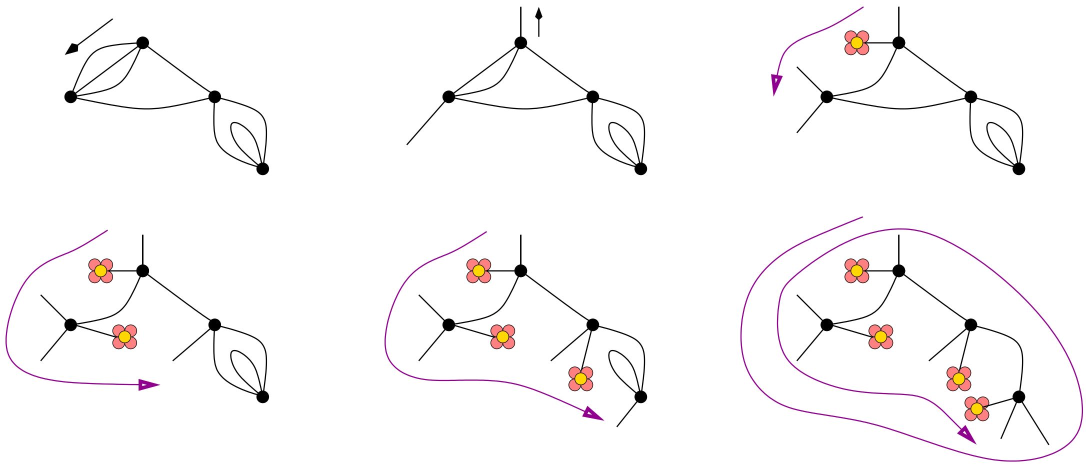

Figure 12: Opening the edges of a 4-valent map gives a blossoming tree.

Conversely, one can construct a map by matching leaves and flowers in a blossoming tree T as follows (Figure 13, left). Starting from the root, walk around the infinite face of T in counterclockwise order. Each time a flower is immediately followed by a leaf in the cyclic sequence of half-edges, merge them into an edge in counterclockwise direction; this creates a new finite face that encloses no unmatched half-edges. Stop when all flowers have been matched. At this point, exactly two leaves remain unmatched (because there are n flowers and $n+2$ leaves). Observe that the same two leaves remain unmatched if one starts walking around the tree from another position than the root. The tree T is said to be *balanced* if one of the unmatched leaves is the root leaf. In this case, match it to the other unmatched leaf to form the root-edge of the map $\Phi(T)$. Again, the complete procedure may require to turn several times around the tree. We discuss further down what can be done if T is *not* balanced.

Proposition 4.1 ([98]) *The map Ψ is a bijection between 4-valent planar maps with n vertices and balanced blossoming trees with n inner nodes. Its reverse bijection is Φ.*

With this bijection, it is easy to justify combinatorially the algebraicity of the generating function of 4-valent maps.

Corollary 4.2 *The generating function of 4-valent planar maps, counted by vertices, is*

$$M(t) = T(t) - tT(t)^3$$

where $T(t)$, the generating function of blossoming trees (counted by inner nodes), satisfies

$$T(t) = 1 + 3tT(t)^2.$$

Proof By decomposing blossoming trees into two subtrees and a flower, it should be clear that their generating function $T(t)$ satisfies the above equation. Via the bijection Ψ, counting maps boils down to counting balanced blossoming trees. Their generating function is $T(t) - U(t)$, where $U(t)$ counts *unbalanced* blossoming trees. Consider such a tree, and look at the flower that matches the root leaf (Figure 13). This flower is attached to an inner node. Delete this node: this leaves, beyond the flower, a 3-tuple of blossoming trees. This shows that $U(t) = tT(t)^3$. $\square$

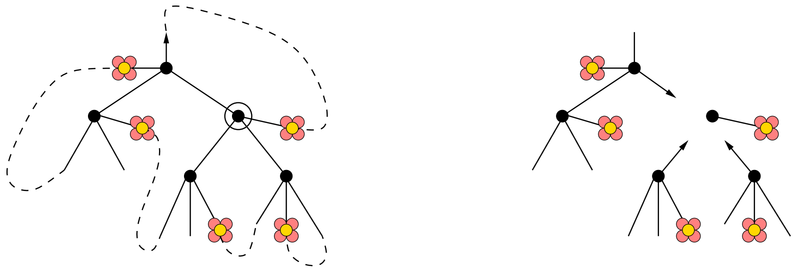

Figure 13: An unbalanced blossoming tree gives rise to three blossoming trees.

This bijection was originally designed [98] to explain combinatorially the simple formulas that occur in the enumeration of maps in which all vertices have an even degree — like 4-valent maps.

Corollary 4.3 *The number of* 4*-valent rooted planar maps with* n *vertices is*

$$\frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n}.$$

Proof We will prove that the above formula counts balanced blossoming trees of size n. Clearly, the total number of blossoming trees of this size is

$$t_n = \frac{3^n}{n+1} \binom{2n}{n}$$

(because binary trees are counted by the Catalan numbers $\binom{2n}{n}/(n+1)$). Marking a blossoming tree at one of its two unmatched leaves is equivalent, up to a re-rooting of the tree, to marking a balanced blossoming tree at one of its $n+2$ leaves. This shows that $2t_n = (n+2)b_n$, where b_n counts balanced blossoming trees, and the result follows. $\square$

A more general construction. A variant $\overline{\Phi}$ of the above bijection sends pairs (T, ϵ) formed of a (non-necessarily balanced) blossoming tree T and of a sign $\epsilon \in \{+, -\}$ onto rooted 4-valent maps with a distinguished face. This construction works as follows. In the tree T, one matches flowers and leaves as described above. The two unmatched leaves are then used to form the root edge, the orientation of which is chosen according to the sign ϵ. This gives a 4-valent rooted map. One then marks the face of this map located to the right of the half-edge where T is rooted. For example, the two maps associated with the (unbalanced) tree of Figure 13 are shown in Figure 14.

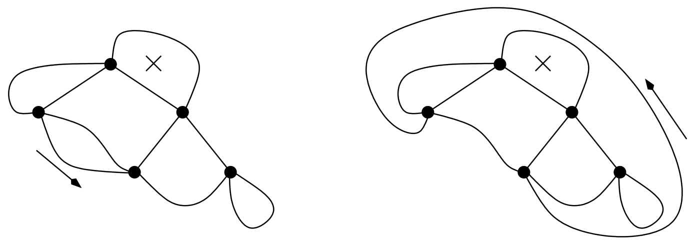

Figure 14: The two maps associated to the tree of Figure 13 via the map $\overline{\Phi}$.

This construction is bijective. Since a 4-valent map with n vertices has $n+2$ faces, it proves that the number m_n of such maps satisfies $(n+2)m_n = 2 \cdot 3^n \binom{2n}{n}/(n+1)$.

The bijection Φ described earlier can actually be seen as a specialisation of $\overline{\Phi}$: If T is balanced, and one chooses to orient the root edge of the map in such a way it starts with the root half-edge of the tree, the map M one obtains satisfies $\Psi(M) = T$. The distinguished face is in this case the root-face, and is thus canonical.

Quadrangulations and labelled trees. The second bijection starts from the duals of 4-valent maps, that is, from quadrangulations. It transforms them into *well labelled trees*. A *labelled tree* is a rooted plane tree with labelled vertices, such that:

– the labels belong to $\{1, 2, 3, \ldots\}$,
– the smallest label that occurs is 1,
– the labels of two adjacent vertices differ by $0, \pm 1$.

The tree is *well labelled* if, in addition, the root vertex has label 1.

This bijection was first found by Cori & Vauquelin in 1981 [50], but the simple description we give here was only discovered later by Schaeffer [49, 99]. As above, there are two versions of this bijection: the most general one sends rooted quadrangulations with n faces and a distinguished (or: *pointed*) vertex v_0 onto pairs (T, ϵ) formed of a labelled tree with n edges T and of a sign $\epsilon \in \{+, -\}$. Equivalently, it sends rooted quadrangulations with a pointed vertex v_0 *such that the root edge is oriented away from* v_0 (in a sense that will be explained below) to labelled trees. The other bijection is a restriction, which sends rooted quadrangulations (pointed canonically at their root-vertex) onto well labelled trees.

So let us describe directly the more general bijection $\overline{\Lambda}$. Take a rooted quadrangulation Q with a pointed vertex v_0, such that the root-edge is oriented away from v_0. By this, we mean that the starting point of the root-edge is closer to v_0 than the endpoint, in terms of the graph distance[4]. Label all vertices by their distance to v_0. The labels of two neighbours differ by ± 1. If the starting point of the root-edge has label ℓ, then the endpoint has label $\ell + 1$. The labelling results in two types of faces: when walking inside a face with the edges on the left, one sees either a cyclic sequence of labels of the form $\ell, \ell+1, \ell, \ell+1$, or a sequence of the form $\ell, \ell+1, \ell+2, \ell+1$. In the former case, create an edge in the face joining the two corners labelled $\ell + 1$. In the latter one, create an edge from the "first" corner labelled $\ell + 1$ (in the order described above) to the corner labelled $\ell + 2$. See Figure 15 for an example. The set

[4] The fact that Q is a quadrangulation, and hence a bipartite map, prevents two neighbour vertices to be at the same distance from v_0.

of edges created in this way forms a tree, which we root at the edge created in the outer face of Q, oriented away from the endpoint of the root-edge of Q (Figure 16). This tree, $\overline{\Lambda}(Q)$, contains all vertices of Q, except the marked one.

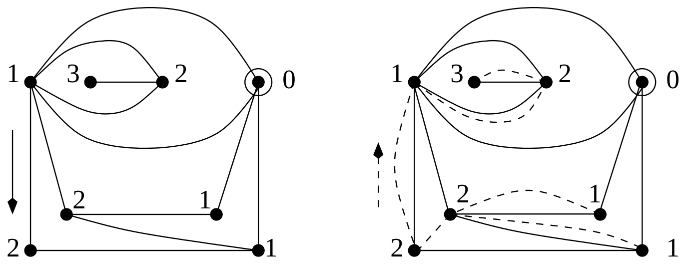

Figure 15: From a rooted quadrangulation with a pointed vertex to a labelled tree (in dashed lines).

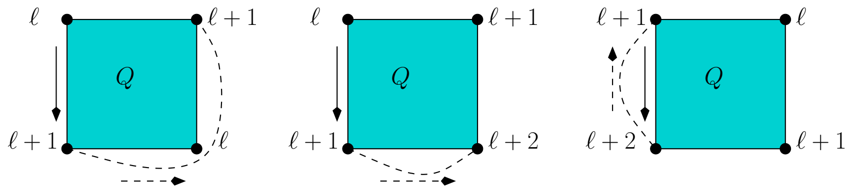

Figure 16: How to root the tree $\overline{\Lambda}(Q)$. The only vertices that are shown are those of the root-face of Q.

The reverse bijection $\overline{V}$ works as follows (see Figure 17 for an example). Start from a labelled tree T. Create a new vertex v_0, away from the tree. Then, visit the corners of the tree in counterclockwise order. From each corner labelled ℓ, send an edge to the next corner labelled $\ell - 1$ (or to v_0 if $\ell = 1$). This set of edges forms a quadrangulation Q. Each face of Q contains an edge of T. Choose the root-edge of Q in the face containing the root-edge of T, according to the rules of Figure 16.

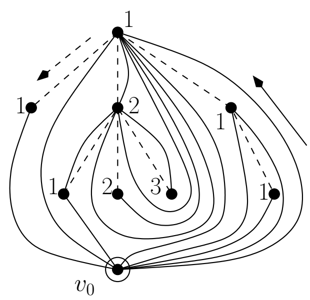

Figure 17: From a labelled tree (dashed lines) to a rooted quadrangulation with a pointed vertex.

Proposition 4.4 *The map $\overline{\Lambda}$ sends bijectively pointed rooted quadrangulations with*

n faces such that the root-edge is oriented away from the pointed vertex to labelled trees with n edges. The reverse bijection is $\overline{V}$.

When specialised to rooted quadrangulations pointed canonically at their root-vertex, $\overline{\Lambda}$ induces a bijection Λ between rooted quadrangulations and well labelled trees.

Let us now discuss the consequences of this bijection, in terms of algebraicity and in terms of closed formulas. It is possible to use Λ to prove that the generating function of rooted quadrangulations is algebraic [50], and satisfies the system of Corollary 4.2, but this is not as simple as the proof of Corollary 4.2 given above. What *is* simple is to use $\overline{\Lambda}$ to count quadrangulations, and hence recover Corollary 4.3.

This alternative proof works as follows. First, observe that there are $3^n C_n$ labelled trees with n edges, where $C_n = \binom{2n}{n}/(n+1)$ counts rooted plane trees with n edges. Indeed, $3^n C_n$ is clearly the number of trees labelled 0 at the root vertex, such that the labels are in $\mathbf{Z}$ and differ by at most 1 along edges. If ℓ_0 denotes the smallest label of such a tree, adding $1-\ell_0$ to all labels gives a labelled tree, and this transformation is reversible. Now, the above proposition implies that $3^n C_n = (n+2)q_n/2$, where q_n is the number of quadrangulations with n faces. Indeed, there are $n+2$ ways to point a vertex in such a quadrangulation, and half of these pointings are such that the root-edge is oriented away from the pointed vertex.

4.2 More bijections

Even though it is difficult to invent bijections, the two bijections presented above have now been adapted to many other map families, including the two general families described in Section 3.2: maps with prescribed face degrees (or, dually, prescribed vertex degrees), and Eulerian maps with prescribed face degrees (dually, bipartite maps with prescribed vertex degrees).

On the "blossoming" side, Schaeffer's bijection [98] was originally designed, not only for 4-valent maps, but for maps with prescribed vertex degrees, *provided these degrees are even*. The case of general degrees was solved (bijectively) a few years later by a trio of theoretical physicists, Bouttier, Di Francesco and Guitter [35]. The equations they obtain differ from those obtained by Bender & Canfield via the recursive method [6]. See [29, Section 10] for the correspondence between the two solutions. The case of *bipartite* maps with prescribed vertex degrees was then solved by Schaeffer and the author [34]. The special case of m-constellations was solved earlier in [33].

On the "labelled" side, the extensions of the Cori-Vauquelin-Schaeffer bijection (which applied to quadrangulations) to maps with prescribed face degrees, and to *Eulerian* maps with prescribed face degrees, came in a single paper, again due to Bouttier *et al.* [37].

Other bijections of the blossoming type exist for certain families of maps that are constrained, for instance, by higher connectivity conditions, by forbidding loops, or for dissections of polygons [63, 94, 95]. On the labelled side, there exist bijections for non-separable maps [53, 74], for d-angulations with girth d [17], and for maps of higher genus [48]. But the trees are then replaced by more complicated objects, namely one-face maps of higher genus.

All these bijections shed a much better light on planar maps, by revealing their hidden tree-like structure. As already mentioned, they often preserve important statistics, like distances to the root-vertex. In terms of proving algebraicity results, two restrictions should be mentioned:

– when the degrees are not bounded, it takes a bit of algebra to derive, from the system of equations that describes the structure of trees, polynomial equations satisfied by their generating functions,

– these bijections usually establish the algebraicity of the generating function of maps that are *doubly marked* (like rooted maps with a distinguished vertex, or with a distinguished face). The argument used to prove Corollary 4.2 has in general no simple counterpart.

5 Coloured planar maps: the recursive approach

We have now reviewed two combinatorial approaches (one recursive, one bijective) for the enumeration of families of planar uncoloured maps. We now move to the central topic of this paper, namely the enumeration of coloured planar maps, and compare both types of problems.

A first simple observation is that algebraicity will no longer be the rule. Indeed, it has been known for a long time [91] that the generating function of planar maps, weighted by their number of spanning trees (which is the specialisation $\mu = \nu = 1$ of the Tutte polynomial [102]) is:

$$\sum_{M \in \mathcal{M}} t^{\mathrm{e}(M)}\, \mathrm{T}_M(1,1) = \sum_{n \geq 0} \frac{1}{(n+1)(n+2)} \binom{2n}{n} \binom{2n+2}{n+1} t^n.$$

The asymptotic behaviour of the n^{th} coefficient, being $\kappa\, 16^n n^{-3}$, prevents this series from being algebraic [60]. The transcendence of this series implies that it cannot be described by a polynomial equation with one catalytic variable (Theorem 3.1). However, it is not difficult to write an equation with *two* catalytic variables for maps weighted by their Tutte (or Potts) polynomial. This equation is based on the recursive description (2.3). We present this equation in Section 5.1, and another one, for triangulations, in Section 5.2.

The whole point is now to solve equations with two catalytic variables. Much progress has been made in the past few years on the *linear* case. The equations for coloured maps are not linear, but they become linear (or quasi-linear, in a sense that will be explained) for certain special cases, like the enumeration of maps equipped with a spanning tree or a bipolar orientation. Sections 5.3 and 5.4 are devoted to these two simpler problems. They show how the *kernel method*, which was originally designed to solve linear equations with one catalytic variable [2, 31, 96], can be extended to equations with two catalytic variables. Sections 5.3 and 5.4 actually present two variants of this extension.

We then return to the general case. Following the complicated approach used by Tutte to count properly coloured triangulations [116], we obtain two kinds of results:

– when $q \neq 4$ is of the form $2 + 2\cos j\pi/m$, for integers j and m, the generating function of q-coloured maps satisfies also an equation with a *single* catalytic variable, and is thus algebraic. Explicit results are given for $q = 2$ and $q = 3$;

– in general, the generating function of q-coloured maps satisfies a non-linear differential equation.

These results, due to Olivier Bernardi and the author [15, 14], are presented without proof in Sections 5.5 and 5.6. We do not make explicit the differential equation satisfied by the generating function of q-coloured maps, but give an (explicit) *system* of differential equations, which we hope to simplify in a near future.

5.1 A functional equation for coloured planar maps

Let $\mathcal{M}$ be the set of rooted maps. For M in $\mathcal{M}$, recall that $\mathrm{dv}(M)$ and $\mathrm{df}(M)$ denote respectively the degrees of the root-vertex and root-face of M. We define the *Potts generating function* of planar maps by:

$$M(x,y) \equiv M(q,\nu,t,w;x,y) = \frac{1}{q}\sum_{M\in\mathcal{M}} t^{\mathrm{e}(M)} w^{\mathrm{v}(M)-1} x^{\mathrm{dv}(M)} y^{\mathrm{df}(M)}\, \mathrm{P}_M(q,\nu). \quad (5.1)$$

Since there is a finite number of maps with a given number of edges, and $\mathrm{P}_M(q,\nu)$ is a multiple of q, the generating function $M(x,y)$ is a power series in t with coefficients in $\mathbf{Q}[q,\nu,w,x,y]$. Keeping track of the number of vertices allows us to go back and forth between the Tutte and Potts polynomial, thanks to (2.4).

Proposition 5.1 *The Potts generating function of planar maps satisfies:*

$$\begin{aligned} M(x,y) \;=\; & 1 + xywt\,(qy + (\nu-1)(y-1))\, M(x,y)M(1,y) \\ & + xyt(x\nu-1)M(x,y)M(x,1) \\ & + xywt(\nu-1)\frac{xM(x,y)-M(1,y)}{x-1} + xyt\frac{yM(x,y)-M(x,1)}{y-1}. \end{aligned} \quad (5.2)$$

Observe that (5.2) characterises $M(x,y)$ entirely as a series in $\mathbf{Q}[q,\nu,w,x,y][[t]]$ (think of extracting recursively the coefficient of t^n in this equation). Note also that when $\nu=1$, then $\mathrm{P}_M(q,\nu)=q^{\mathrm{v}(M)}$, so that we are essentially counting planar maps by edges, vertices, and by the root-degrees dv and df. The variable x is no longer catalytic: it can be set to 1 in the functional equation, which becomes an equation for $M(1,y)$ with a single catalytic variable y.

Proof This equation is not difficult to establish using the recursive definition of the Potts polynomial (2.3) in terms of deletion and contraction of edges. Of course, one chooses to delete or contract the root-edge of the map. Let us sketch the proof to see where each term of the equation comes from. Equation (2.3) gives

$$M(x,y) = 1 + M_{\backslash}(x,y) + (\nu-1)M_{/}(x,y),$$

where the term 1 is the contribution of the atomic map m_0,

$$M_{\backslash}(x,y) = \frac{1}{q}\sum_{M\in\mathcal{M}\backslash\{m_0\}} t^{\mathrm{e}(M)} w^{\mathrm{v}(M)-1} x^{\mathrm{dv}(M)} y^{\mathrm{df}(M)}\, \mathrm{P}_{M\backslash e}(q,\nu),$$

and

$$M_{/}(x,y) = \frac{1}{q}\sum_{M\in\mathcal{M}\backslash\{m_0\}} t^{\mathrm{e}(M)} w^{\mathrm{v}(M)-1} x^{\mathrm{dv}(M)} y^{\mathrm{df}(M)}\, \mathrm{P}_{M/e}(q,\nu),$$

where $M\backslash e$ and M/e denote respectively the maps obtained from M by deleting and contracting the root-edge e.

A. The series $M_\backslash$. We consider the partition $\mathcal{M} \setminus \{m_0\} = \mathcal{M}_1 \uplus \mathcal{M}_2 \uplus \mathcal{M}_3$, where $\mathcal{M}_1$ (resp. $\mathcal{M}_2$, $\mathcal{M}_3$) is the subset of maps in $\mathcal{M} \setminus \{m_0\}$ such that the root-edge is an isthmus (resp. a loop, resp. neither an isthmus nor a loop). We denote by $M^{(i)}(x,y)$, for $1 \le i \le 3$, the contribution of $\mathcal{M}_i$ to the generating function $M_\backslash(x,y)$, so that

$$M_\backslash(x,y) = M^{(1)}(x,y) + M^{(2)}(x,y) + M^{(3)}(x,y).$$

• Contribution of $\mathcal{M}_1$. Deleting the root-edge of a map in $\mathcal{M}_1$ leaves two maps M_1 and M_2, as illustrated in Figure 4 (left). The Potts polynomial of this pair can be determined using (2.1). One thus obtains

$$M^{(1)}(x,y) = qxy^2tw\, M(1,y)M(x,y),$$

as the degree of the root-vertex of M_1 does not contribute to the degree of the root-vertex of the final map.

• Contribution of $\mathcal{M}_2$. Deleting the root-edge of a map in $\mathcal{M}_2$ leaves two maps M_1 and M_2 attached by their root vertex, as illustrated in Figure 6. The Potts polynomial of this pair can be determined using (2.2). One thus obtains

$$M^{(2)}(x,y) = x^2yt\, M(x,1)M(x,y),$$

as the degree of the root-face of M_1 does not contribute to the degree of the root-face of the final map.

• Contribution of $\mathcal{M}_3$. Deleting the root-edge of a map in $\mathcal{M}_3$ leaves a single map M. If the outer degree of M is d, there are $d+1$ ways to add a new (root-)edge to M, as illustrated in Figure 5 (right). However, a number of these additions create a loop, and thus their generating function must be subtracted. One thus obtains

$$M^{(3)}(x,y) = xt \sum_{d\ge 0} M_d(q,\nu,t,w;x)(y + y^2 + \cdots + y^{d+1}) - xyt\, M(x,1)M(x,y),$$

where $M_d(q,\nu,w,t;x)$ is the coefficient of y^d in $M(x,y)$. This gives

$$M^{(3)}(x,y) = xyt\frac{yM(x,y) - M(x,1)}{y-1} - xyt\, M(x,1)M(x,y),$$

and finally

$$M_\backslash(x,y) = qxy^2tw\, M(1,y)M(x,y) + x(x-1)yt\, M(x,1)M(x,y) + xyt\,\frac{yM(x,y) - M(x,1)}{y-1}.$$

B. The series $M_/$. The study of this series is of course very similar to the previous one, by duality. One finds:

$$M_/(x,y) = x^2yt\, M(x,1)M(x,y) + xy(y-1)tw\, M(1,y)M(x,y) + xytw\,\frac{xM(x,y) - M(1,y)}{x-1}.$$

Adding the series 1, $M_\backslash$ and $(\nu-1)M_/$ gives the functional equation. $\square$

Remark. Equation (5.2) is equivalent to an equation written by Tutte in 1971:

$$\tilde{M}(x,y) = 1 + xyw(y\mu - 1)\tilde{M}(x,y)\tilde{M}(1,y) \ + \ xyz(x\nu - 1)\tilde{M}(x,y)\tilde{M}(x,1) \\ + xyw\left(\frac{x\tilde{M}(x,y) - \tilde{M}(1,y)}{x-1}\right) + xyz\left(\frac{y\tilde{M}(x,y) - \tilde{M}(x,1)}{y-1}\right), \quad (5.3)$$

where $\tilde{M}(x,y)$ counts maps weighted by their Tutte polynomial [106]:

$$\tilde{M}(x,y) \equiv \tilde{M}(\mu,\nu,w,z;x,y) = \sum_{M\in\mathcal{M}} w^{\mathrm{v}(M)-1} z^{\mathrm{f}(M)-1} x^{\mathrm{dv}(M)} y^{\mathrm{df}(M)}\, \mathrm{T}_M(\mu,\nu).$$

We call the above series *the Tutte generating function of planar maps.* The relation (2.4) between the Tutte and Potts polynomials and Euler's relation ($\mathrm{v}(M) + \mathrm{f}(M) - 2 = \mathrm{e}(M)$) give

$$M(q,\nu,t,w;x,y) = \tilde{M}\left(1 + \frac{q}{\nu-1}, \nu, (\nu-1)tw, t; x, y\right),$$

from which (5.2) easily follows.

5.2 More functional equations

In a similar fashion, one can write a functional equation for coloured non-separable planar maps [84]. It is equivalent to (5.2) via a simple composition argument [15, Section 14]. Writing equations for coloured maps with prescribed face degrees is harder, as the contraction of the root-edge changes the degree of the finite face located to the left of the root-edge. This is however not a serious problem if one counts *proper* colourings of triangulations (the faces of degree 2 that occur can be "smashed" into a single edge), and in 1971, Tutte came up with the following equation, the solution of which kept him busy during the following decade:

$$T(x,y) = xy^2q(q-1) + \frac{xz}{yq}T(1,y)T(x,y) \\ + xz\frac{T(x,y) - y^2T_2(x)}{y} - x^2yz\frac{T(x,y) - T(1,y)}{x-1} \quad (5.4)$$

where $T_2(x) = [y^2]T(x,y)$. The series $T(x,y)$ defined by this equation is

$$T(x,y) = \sum_T z^{\mathrm{f}(T)-1} x^{\mathrm{dv}(T)} y^{\mathrm{df}(T)}\, \mathrm{P}_T(q,0), \quad (5.5)$$

where the sum runs over all non-separable near-triangulations (maps in which all finite faces have degree 3). Note that the number of edges and the number of vertices of T can be obtained from $\mathrm{f}(T)$ and $\mathrm{df}(T)$, using

$$\mathrm{v}(T) + \mathrm{f}(T) = 2 + \mathrm{e}(T) \quad \text{and} \quad 2\,\mathrm{e}(T) = 3(\mathrm{f}(T) - 1) + \mathrm{df}(T). \quad (5.6)$$

There seems to be no straightforward extension to the Potts generating function[5], and it is not until recently that an equation was obtained for the Potts generating

[5] Although another type of functional equation is given in [56, Sec. 3] for the Potts generating function of cubic maps, using matrix integrals.

function $Q(x,y)$ of *quasi-triangulations* [15]. We refer to that paper for the precise definition of this class of maps, which is not so important here. What *is* important is that the series $Q(0,y)$ is the Potts generating function of near-triangulations:

$$Q(0,y) \equiv Q(q,\nu,t,z;0,y) = \frac{1}{q}\sum_{T\in\mathcal{T}} t^{\mathrm{e}(T)} z^{\mathrm{f}(T)-1} y^{\mathrm{df}(T)}\,\mathrm{P}_T(q,\nu).$$

Proposition 5.2 *The Potts generating function of quasi-triangulations satisfies*

$$\begin{aligned} Q(x,y) = 1 + zt\,\frac{Q(x,y)-1-yQ_1(x)}{y} + xzt(Q(x,y)-1) + xyztQ_1(x)Q(x,y) \\ + yzt(\nu-1)Q(x,y)(2xQ_1(x)+Q_2(x)) + y^2t\left(q+\frac{\nu-1}{1-xzt\nu}\right)Q(0,y)Q(x,y) \\ + \frac{yt(\nu-1)}{1-xzt\nu}\,\frac{Q(x,y)-Q(0,y)}{x} \end{aligned} \tag{5.7}$$

where $Q_1(x) = [y]Q(x,y)$ *and* $Q_2(x) = [y^2]Q(x,y) = \dfrac{(1-2xzt\nu)}{zt\nu}Q_1(x)$.

Tutte's equation (5.4) for non-separable, properly coloured near-triangulations can be recovered from this proposition [15, Section 14]. In Section 5.4, we will use the following (equivalent) equation for the generating function of quasi-triangulations, weighted by their Tutte polynomial:

$$\begin{aligned} \tilde{Q}(x,y) = 1 + tz\,\frac{\tilde{Q}(x,y)-1-y\tilde{Q}_1(x)}{y} + xtz\left(\tilde{Q}(x,y)-1\right) + xytz\tilde{Q}_1(x)\tilde{Q}(x,y) \\ + tzy(\nu-1)\tilde{Q}(x,y)\left(2x\tilde{Q}_1(x)+\tilde{Q}_2(x)\right) \\ + y^2t\left(\mu+\frac{tx\nu\, z}{1-x\nu\, tz}\right)\tilde{Q}(0,y)\tilde{Q}(x,y) + \frac{yt}{1-x\nu\, tz}\,\frac{\tilde{Q}(x,y)-\tilde{Q}(0,y)}{x}, \end{aligned} \tag{5.8}$$

where $\tilde{Q}_1(x) = [y]\tilde{Q}(x,y)$ and $\tilde{Q}_2(x) = [y^2]\tilde{Q}(x,y) = \dfrac{(1-2xzt\nu)}{zt\nu}\tilde{Q}_1(x)$. We call the specialisation $\tilde{Q}(0,y)$ *the Tutte generating function of near-triangulations:*

$$\tilde{Q}(0,y) \equiv \tilde{Q}(q,\nu,t,z;0,y) = \sum_{T\in\mathcal{T}} t^{\mathrm{e}(T)} z^{\mathrm{f}(T)-1} y^{\mathrm{df}(T)}\,\mathrm{T}_T(\mu,\nu).$$

5.3 A linear case: bipolar orientations of maps

Let G be a connected graph with a root-edge (s,t). A *bipolar orientation* of G is an acyclic orientation of the edges of G such that s is the single source and t the single sink. Such orientations exist if and only if G is non-separable. It is known [70, 77] that the number of bipolar orientations of G is:

$$(-1)^{\mathrm{v}(G)}\frac{\partial\,\mathrm{P}_G}{\partial q}(1,0).$$

This number is also called the *chromatic invariant* of G [19, p. 355]. This expression implies that the generating function of (non-atomic) planar maps equipped with a bipolar orientation, counted by edges (t), non-root vertices (w), degree of the root-vertex (x) and of the root-face (y) is

$$B(t,w;x,y) = -\frac{\partial}{\partial q}\left(qM(q,0,t,-w;x,y)-q\right)\big|_{q=1} = -\frac{\partial M}{\partial q}(1,0,t,-w;x,y),$$

where $M(x,y)$ is the Potts generating function of planar maps, defined by (5.1) (we have used $M(1,0,t,w;x,y)=1$). By differentiation, it is easy to derive from (5.2) an equation with two catalytic variables satisfied by $B(x,y)$. Using again $M(1,0,t,w;x,y)=1$, this equation is found to be *linear*:

$$\left(1+\frac{xytw}{1-x}+\frac{xyt}{1-y}\right)B(x,y) = xy^2wt + \frac{x^2ywt}{1-x}B(1,y) + \frac{xy^2t}{1-y}B(x,1). \tag{5.9}$$

Similarly, using (5.6), one finds that the generating function of planar near-triangulations equipped with a bipolar orientation, counted by non-root faces (z), degree of the root-vertex (x) and of the root-face (y) is

$$B^{\triangleleft}(z;x,y) = -\frac{\partial T}{\partial q}(0,iz;x,iy),$$

where $i^2=-1$ and $T(q,z;x,y)$ is Tutte's generating function for coloured non-separable near-triangulations, defined by (5.5). Given that $T(1,z;x,y)=0$, the equation satisfied by $B^{\triangleleft}$ is again linear:

$$\left(1-\frac{xz}{y}-\frac{zx^2y}{x-1}\right)B^{\triangleleft}(x,y) = xy^2 - xzyB_2^{\triangleleft}(x) - \frac{zx^2yB^{\triangleleft}(1,y)}{x-1} \tag{5.10}$$

with $B_2^{\triangleleft}(x) = [y^2]B^{\triangleleft}(x,y)$.

In the past few years, much progress has been made in the solution of linear equations with two catalytic variables [27, 26, 32, 30, 89, 90]. It is now understood that a certain group of rational transformations, which leaves invariant the *kernel* of the equation (the coefficient of $B(x,y)$) plays an important role. In particular, when this group is finite, the equation can often be solved in an elementary way, using what is sometimes called the *algebraic version* of the kernel method [27, 30]. This is the case for both (5.9) and (5.10). We detail the solution of (5.10), and explain how to adapt it to solve (5.9).

Proposition 5.3 *The number of bipolar orientations of near-triangulations having $m+1$ vertices is*

$$\frac{(3m)!}{(4m^2-1)m!^2(m+1)!}.$$

The number of bipolar orientations of near-triangulations having $m+1$ vertices and a root-face of degree j is

$$\frac{j(j-1)(3m-j-1)!}{m!(m+1)!(m-j+1)!}. \tag{5.11}$$

For $m\geq 2$, the number of bipolar orientations of near-triangulations having $m+1$ vertices, a root-vertex of degree i and a root-face of degree j is

$$\frac{(i-1)(j-1)(2m-j-2)!(3m-i-j-1)!}{(m-1)!m!(m-j+1)!(2m-i-j+1)!}\left((2j+i-6)m+i+3j-j^2-ij\right).$$

The corresponding generating functions in 1, 2 and 3 variables are D-finite.

The last two formulas are due to Tutte [110, Eqs. (32) and (34)]. He also derived them from the functional equation (5.10), but his proof involved a lot of guessing, while ours is constructive. The first formula in Proposition 5.3 seems to be new.

Proof It will prove convenient to set $x = 1/(1-u)$. After multiplying (5.10) by $(x-1)/x^2/y$, the equation we want to solve reads:

$$u\bar{y}\left(1-u-z(y\bar{u}+\bar{y})\right)B^{\triangleleft}\left(\frac{1}{1-u},y\right) = uy - R(u) - S(y), \qquad (5.12)$$

with $\bar{u} = 1/u$, $\bar{y} = 1/y$, $R(u) = zuB_2^{\triangleleft}\left(\frac{1}{1-u}\right)$ and $S(y) = zB^{\triangleleft}(1,y)$. Let $K(u,y) = 1-u-z(y\bar{u}+\bar{y})$ be the *kernel* of this equation. This kernel is invariant by the transformations:

$$\Phi : (u,y) \mapsto (yz\bar{u}, y) \quad \text{and} \quad \Psi : (u,y) \mapsto (u, u\bar{y}).$$

Both transformations are involutions, and, by applying them iteratively to (u,y), one obtains 6 pairs (u',y') on which $K(\cdot,\cdot)$ takes the same value:

$$(u,y)\xrightarrow{\Phi}(yz\bar{u},y)\xrightarrow{\Psi}(yz\bar{u},z\bar{u})\xrightarrow{\Phi}(z\bar{y},z\bar{u})\xrightarrow{\Psi}(z\bar{y},u\bar{y})\xrightarrow{\Phi}(u,u\bar{y})\xrightarrow{\Psi}(u,y). \qquad (5.13)$$

For each such pair (u',y'), the corresponding specialisation of (5.12) reads

$$(u'/y')\,K(u,y)B^{\triangleleft}(1/(1-u'),y') = u'y' - R(u') - S(y').$$

We form the alternating sum of the 6 equations of this form obtained from the pairs (5.13). The series $R(\cdot)$ and $S(\cdot)$ cancel out, and, after dividing by $K(u,y)$, we obtain:

$$\begin{aligned} u\bar{y}B^{\triangleleft}\left(\frac{1}{1-u},y\right) - z\bar{u}B^{\triangleleft}\left(\frac{1}{1-yz\bar{u}},y\right) + yB^{\triangleleft}\left(\frac{1}{1-yz\bar{u}},z\bar{u}\right) \\ - u\bar{y}B^{\triangleleft}\left(\frac{1}{1-z\bar{y}},z\bar{u}\right) + z\bar{u}B^{\triangleleft}\left(\frac{1}{1-z\bar{y}},u\bar{y}\right) - yB^{\triangleleft}\left(\frac{1}{1-u},u\bar{y}\right) \\ = \frac{uy - y^2z\bar{u} + yz^2\bar{u}^2 - z^2\bar{y}\bar{u} + zu\bar{y}^2 - u^2\bar{y}}{1-u-z(y\bar{u}+\bar{y})}. \end{aligned}$$

The above identity holds in the ring of formal power series in z with coefficients in $\mathbf{Q}(u,y)$, which we consider as a sub-ring of Laurent series in u and y. Recall that $B^{\triangleleft}(x,y)$ has coefficients in $xy^2\mathbf{Q}[x,y]$. Hence, in the left-hand side of this identity, the terms with positive exponents in u and y are exactly those of $u\bar{y}B^{\triangleleft}\left(\frac{1}{1-u},y\right)$. It follows that the latter series is the positive part (in u and y) of the rational function

$$R(z;u,y) := \frac{uy - y^2z\bar{u} + yz^2\bar{u}^2 - z^2\bar{y}\bar{u} + zu\bar{y}^2 - u^2\bar{y}}{1-u-z(y\bar{u}+\bar{y})}.$$

It remains to perform a coefficient extraction. One first finds:

$$\frac{1}{1-u-z(\bar{y}+y\bar{u})} = \sum_{n\geq 0}\sum_{a\geq -n}\sum_{b=-n}^{n} z^n u^a y^b \binom{n}{\frac{b+n}{2}}\binom{\frac{b+n}{2}+n+a}{n}$$

where the sum is restricted to triples (n, a, b) such that $n+b$ is even. An elementary calculation then yields the expansion of $R(z; u, y)$, and finally

$$B^{\triangleleft}\left(z; \frac{1}{1-u}, y\right) = \sum_{n\geq 0}\sum_{i\geq 0}\sum_{j=2}^{n+2} z^n u^i y^j \frac{(i+1)(j-1)(i+j)\left(\frac{3n+j}{2}+i-1\right)!}{\left(\frac{n-j}{2}+1\right)!\left(\frac{n+j}{2}+i+1\right)!\left(\frac{n+j}{2}\right)!}, \quad (5.14)$$

where the sum is restricted to triples (n, i, j) such that $n+j$ is even. In particular, the case $u = 0$ shows that the number of bipolar orientations of near-triangulations having n finite faces and a root-face of degree j is

$$\frac{(j-1)j\left(\frac{3n+j}{2}-1\right)!}{\left(\frac{n-j}{2}+1\right)!\left(\frac{n+j}{2}+1\right)!\left(\frac{n+j}{2}\right)!},$$

which coincides with (5.11), given that the number of vertices of such maps is $1 + (n+j)/2$.

The first formula of the proposition is then obtained by summing over j. The third one is easily verified using (5.14). However, it can also be *derived* from (5.14) if one prefers a constructive proof. One proceeds as follows. First, observe that if a rational function $R(u)$ is of the form $P(1/(1-u))$, for some Laurent polynomial P, then $P(x)$ coincides with the expansion of $R(1-\bar{x})$ as a Laurent series in $\bar{x}$. In particular, if $P(x) \in x\mathbf{Q}[x]$, then $P(x)$ is the positive part in x of the expansion of $R(1-\bar{x})$ in $\bar{x}$. The coefficient of $z^n y^j$ in $B^{\triangleleft}(z; x, y)$ is precisely in $x\mathbf{Q}[x]$, so that we can apply this extraction procedure to the right-hand side of (5.14). We first express the coefficient of $z^n y^j$ as a rational function of u, using

$$\sum_{i\geq 0} u^i \binom{a+b+i}{a} = \frac{1}{u^b(1-u)^{a+1}} - \sum_{j=0}^{b-1} \frac{1}{u^{b-j}}\binom{a+j}{a}.$$

Then, we set $u = 1-\bar{x}$, expand in $\bar{x}$ this rational function, and extract the positive part in x. For the above series, this gives:

$$\begin{aligned}
[x^>]\sum_{i\geq 0} u^i \binom{a+b+i}{a} &= [x^>]\frac{x^{a+1}}{(1-\bar{x})^b} - \sum_{j=0}^{b-1}\frac{1}{(1-\bar{x})^{b-j}}\binom{a+j}{a} \\
&= [x^>]\frac{x^{a+1}}{(1-\bar{x})^b} \\
&= \sum_{k=0}^{a} x^{a+1-k}\binom{k+b-1}{k}.
\end{aligned}$$

Combining these two ingredients yields the third formula of the proposition.

Finally, the form of these three formulas, together with the closure properties of D-finite series [81, 82], imply that the associated generating functions are D-finite. □

The same method allows us to solve the linear equation (5.9) obtained for bipolar orientations of general maps. One sets $x = 1+u$ and $y = 1+v$. It is also convenient to write

$$B(x, y) = xy^2tw + x^2y^2t^2w\, G(x, y).$$

The equation satisfied by G reads

$$uv\,(1 - t(1+\bar u)(1+\bar v)(u+vw))\,G(1+u,1+v) = \\ uv - tu(1+u)G(1+u,1) - twv(1+v)G(1,1+v).$$

The relevant transformations Φ and Ψ are now

$$\Phi : (u,v) \mapsto (\bar u w v, v) \quad \text{and} \quad \Psi : (u,v) \mapsto (u, u\bar v \bar w).$$

Again, they generate a group of order 6. One finally obtains that $G(1+u,1+v)$ is the non-negative part (in u and v) of the following rational function:

$$\frac{(1-\bar u \bar v)(u\bar v - w\bar u)(\bar u v - \bar v \bar w)}{1 - t(1+\bar u)(1+\bar v)(u+vw)}.$$

A coefficient extraction, combined with Lemma 6 of [26], yields the following results.

Proposition 5.4 *For $1 \leq m < n$, the number of bipolar orientations of planar maps having n edges and $m+1$ vertices is*

$$\frac{2}{(n-1)n^2}\binom{n}{m-1}\binom{n}{m}\binom{n}{m+1}.$$

For $1 \leq m < n$ and $2 \leq j \leq m+1$, the number of bipolar orientations of planar maps having n edges, $m+1$ vertices and a root-face of degree j is

$$\frac{j(j-1)}{(n-1)n^2}\binom{n}{m}\binom{n}{m+1}\binom{n-j-1}{m-j+1}.$$

For $n \geq 3$, $1 \leq m < n$, $2 \leq i \leq n-m+1$ and $2 \leq j \leq m+1$, the number of bipolar orientations of planar maps having n edges, $m+1$ vertices, a root-vertex of degree i and a root-face of degree j is

$$\frac{(i-1)(j-1)}{(n-1)n}\binom{n}{m}\left[\binom{n-j-1}{n-m-2}\binom{n-i-1}{m-2} - \binom{n-j-1}{n-m-1}\binom{n-i-1}{m-1}\right].$$

The associated generating functions are D-finite.

The solution we have sketched is very close to [26, Section 2]. Eq. (5.9) was also solved independently by Baxter [4], but his solution involved some guessing, while the one we have presented here is constructive.

5.4 A quasi-linear case: spanning trees

When $\mu = \nu = 1$, the Tutte polynomial $\mathrm{T}_G(\mu,\nu)$ gives the number of spanning trees of G. The equations (5.3) and (5.8) that define the Tutte generating functions of our main two families of planar maps turn out to be much easier to solve in this case.

Consider first general planar maps, and Tutte's equation (5.3). We replace w by wt and z by zt so that t keeps track of the edge number. When $\mu = \nu = 1$, the equation reads:

$$\left(1 - \frac{x^2ywt}{x-1} - \frac{xy^2zt}{y-1} - xyzt(x-1)\tilde{M}(x,1) - xywt(y-1)\tilde{M}(1,y)\right)\tilde{M}(x,y) = \\ 1 - \frac{xyzt}{y-1}\tilde{M}(x,1) - \frac{xywt}{x-1}\tilde{M}(1,y). \quad (5.15)$$

Observe that, up to a factor $(x-1)(y-1)$, the *same* linear combination of $\tilde{M}(x,1)$ and $\tilde{M}(1,y)$ appears on the right- and left-hand sides. This property was observed, but not fully exploited, by Tutte [117]. Bernardi [10] showed that it allows us to solve (5.15) using the standard kernel method usually applied to *linear* equations with two catalytic variables [26]. We thus obtain a new proof of the following result, due to Mullin [91]. Using Mullin's terminology, we say that a map equipped with a distinguished spanning tree is *tree-rooted*.

Proposition 5.5 *The number of tree-rooted planar maps with n edges is*

$$\frac{(2n)!(2n+2)!}{n!(n+1)!^2(n+2)!}.$$

The number of tree-rooted planar maps with $i+1$ vertices and $j+1$ faces is

$$\frac{(2i+2j)!}{i!(i+1)!j!(j+1)!}.$$

The associated generating functions are D-finite.

Proof Set

$$S(u,v) \equiv S(w,z,t;u,v) = \frac{1}{(1-ut)(1-vt)}\tilde{M}\left(w,z,t^2;\frac{1}{1-ut},\frac{1}{1-vt}\right).$$

Eq. (5.15) can be rewritten as

$$\left(1 - t(u+v+w\bar{u}+z\bar{v})\right)S(u,v) = \left(1 - uvt^2S(u,v)\right)\left(1 - tz\bar{v}S(u,0) - tw\bar{u}S(0,v)\right). \quad (5.16)$$

Observe that the (Laurent) polynomial $(1 - t(u+v+w\bar{u}+z\bar{v}))$ is invariant by the transformation $u \mapsto w\bar{u}$. Seen as a polynomial in v, it has two roots. Exactly one of them, denoted $V \equiv V(w,z,t;u)$ is a formal power series in t with coefficients in $\mathbf{Q}[w,z,u,\bar{u}]$, satisfying

$$V = t\left(z + (u+w\bar{u})V + V^2\right). \quad (5.17)$$

In (5.16), specialise v to V. The left-hand side vanishes, and hence the right-hand side vanishes as well. Since its first factor is not zero, there holds

$$tzuS(u,0) + twVS(0,V) = uV.$$

Now replace u by $w\bar{u}$ in (5.16), and specialise again v to V. This gives

$$tzw\bar{u}S(w\bar{u},0) + twVS(0,V) = w\bar{u}V.$$

By taking the difference of the last two equations, we obtain

$$tzuS(u,0) - tzw\bar{u}S(w\bar{u},0) = (u - w\bar{u})V. \tag{5.18}$$

Since $S(u,0)$ is a series in t with coefficients in $\mathbf{Q}[w,z,u]$, this equation implies that $tzuS(u,0)$ is the positive part in u of $(u - w\bar{u})V$. The number $\mathrm{TR}(i,j)$ of tree-rooted planar maps having $i+1$ vertices and $j+1$ faces is the coefficient of $w^i z^j t^{i+j}$ in $\tilde{M}(w,z,t;1,1)$, that is, the coefficient of $w^i z^j t^{2i+2j}$ in $S(u,0)$, or the coefficient of $w^i z^{j+1} t^{2i+2j+1} u$ in $tzuS(u,0)$. The Lagrange inversion formula, applied to (5.17), yields

$$[w^i z^j t^n u^{n+1-2i-2j}]V = \frac{(n-1)!}{i!(j-1)!j!(n+1-i-2j)!}. \tag{5.19}$$

Hence

$$\begin{aligned} \mathrm{TR}(i,j) &= [w^i z^{j+1} t^{2i+2j+1} u]\,(tzuS(u,0)) \\ &= [w^i z^{j+1} t^{2i+2j+1} u^0]V - [w^{i-1} z^{j+1} t^{2i+2j+1} u^2]V \qquad \text{(by (5.18))}, \end{aligned}$$

which, thanks to (5.19), gives the second result of the proposition. The first one follows by summing over all pairs (i,j) such that $i+j=n$. Alternatively, one can apply the Lagrange inversion formula to the equation satisfied by V when $w=z=1$, which is $V = t(1+uV)(1+\bar{u}V)$. □

Similarly, we can derive from the functional equation (5.8) defining the Tutte-generating function of quasi-triangulations the number of tree-rooted near-triangulations having a fixed outer degree and number of vertices. This result is also due to Mullin [91], but the proof is new.

Proposition 5.6 *The number of tree-rooted near-triangulations having $i+1$ vertices and a root-face of degree d is*

$$\frac{d}{(i+1)(4i-d)}\binom{3i-d}{i}\binom{4i-d}{i}.$$

The associated generating function is D-finite.

Proof We specialise to $\mu = \nu = z = 1$ the equation (5.8) that defines the Tutte-generating function of quasi-triangulations. We then replace $\tilde{Q}_2(x)$ by its expression in terms of $\tilde{Q}_1$. Again, the same linear combination of $\tilde{Q}_1(x)$ and $\tilde{Q}(0,y)$ occurs in the right- and left-hand sides, and the equation can be rewritten as

$$\begin{aligned} &\left(1 - t\bar{y} - xy(1-tx) - \frac{ty}{x(1-tx)}\right)\tilde{Q}(x,y) = \\ &\qquad \left(1 - t\bar{y} - tR_1(x) - \frac{ty}{x(1-tx)}\tilde{Q}(0,y)\right)\left(1 - xy\tilde{Q}(x,y)\right) \end{aligned} \tag{5.20}$$

where $R_1(x) = x + \tilde{Q}_1(x)$. Let us denote $u := x(1-xt)$. Equivalently, we introduce a new indeterminate u and set

$$x = X(u) := \frac{1-\sqrt{1-4ut}}{2t}.$$

The (Laurent) polynomial $(1 - t\bar{y} - uy - t\bar{u}y)$ occurring in the left-hand side of (5.20) is invariant by the transformation $y \mapsto tu\bar{y}/(t + u^2)$. As a polynomial in u, it has two roots. One of them is a power series in t with constant term 0, satisfying

$$U = t\,\frac{y + U\bar{y}}{1 - Uy}. \tag{5.21}$$

In (5.20), specialise x to $X(U)$. The left-hand side vanishes, leaving

$$tUR_1(X(U)) + ty\tilde{Q}(0, y) = U(1 - t\bar{y}).$$

If we first replace y by $t\bar{y}U/(t + U^2) = t/(1 - t\bar{y})$ in (5.20) before specialising x to $X(U)$, we obtain instead

$$tUR_1(X(U)) + \frac{t^2}{1 - t\bar{y}}\,\tilde{Q}\left(0, \frac{t}{1 - t\bar{y}}\right) = t\bar{y}U.$$

By taking the difference of the last two equations, one finds:

$$ty\tilde{Q}(0, y) - \frac{t^2}{1 - t\bar{y}}\,\tilde{Q}\left(0, \frac{t}{1 - t\bar{y}}\right) \quad = \quad U(1 - 2t\bar{y}).$$

Since $\tilde{Q}(0, y)$ is a series in t with coefficients in $\mathbf{Q}[y]$, this equation implies that $ty\tilde{Q}(0, y)$ is the positive part in y of $U(1 - 2t\bar{y})$. The Lagrange inversion formula, applied to (5.21), gives:

$$[t^n y^{3i-n+2}]U = \frac{1}{n}\binom{n}{i+1}\binom{n+i-1}{i}.$$

This yields

$$\begin{aligned}
[t^n y^{3i-n}]\tilde{Q}(0, y) \quad &= \quad \frac{1}{n+1}\binom{n+1}{i+1}\binom{n+i}{i} - \frac{2}{n}\binom{n}{i+1}\binom{n+i-1}{i} \\
&= \quad \frac{3i - n}{(i+1)(n+i)}\binom{n}{i}\binom{n+i}{i},
\end{aligned}$$

which is equivalent to the proposition, as a near-triangulation with n edges and outer degree $3i - n$ has $i + 1$ vertices. □

5.5 When q is a Beraha number: Algebraicity

We now report on more difficult results obtained recently by Olivier Bernardi and the author [15] by following and adapting Tutte's enumeration of properly q-coloured triangulations [116].

For certain values of q, it is possible to derive from the equation with two catalytic variables defining $M(x, y)$ an equation with a single catalytic variable (namely, y) satisfied by $M(1, y)$. For instance, one can derive from the case $q = 1$ of (5.2) that $M(y) \equiv M(1, y)$ satisfies

$$M(y) = 1 + y^2 t\nu w M(y)^2 + \nu t y\,\frac{yM(y) - M(1)}{y - 1}.$$

This is only a moderately exciting result, as the latter equation is just the standard functional equation (3.1) obtained by deleting recursively the root-edge in planar maps.

But let us be persistent. When $q = 2$, one can derive from (5.2) that $M(y) \equiv M(1, y)$ satisfies a polynomial equation with one catalytic variable, involving two additional unknown series, namely $M(1)$ and $M'(1)$. This equation is rather big (see [15, Section 12]), and we do not write it here. No combinatorial way to derive it is known at the moment. When $\nu = 0$, the series $M'(1)$ disappears, and one recovers the standard equation (3.3) obtained by deleting recursively the root-edge in bipartite planar maps.

This construction works as soon as $q \neq 0, 4$ is of the form $2 + 2\cos(j\pi/m)$, for integers j and m. These numbers generalise *Beraha's numbers* (obtained for $j = 2$), which occur frequently in connection with chromatic properties of planar graphs [9, 58, 72, 73, 87, 97]. They include the three integer values $q = 1, 2, 3$. Given that the solutions of polynomial equations with one catalytic variable are always algebraic (Theorem 3.1), the following algebraicity result holds [15].

Theorem 5.7 *Let $q \neq 0, 4$ be of the form $2 + 2\cos j\pi/m$ for two integers j and m. Then the series $M(q, \nu, t, w; x, y)$, defined by* (5.2)*, is algebraic over $\mathbf{Q}(q, \nu, t, w, x, y)$.*

A similar method works for quasi-triangulations.

Theorem 5.8 *Let $q \neq 0, 4$ be of the form $2 + 2\cos j\pi/m$ for two integers j and m. Then the series $Q(q, \nu, t, z; x, y)$, defined by* (5.7)*, is algebraic over $\mathbf{Q}(q, \nu, t, z, x, y)$.*

For the two integer values $q = 2$ (the Ising model) and $q = 3$ (the 3-state Potts model), we have applied the procedure described in Section 3.3 to obtain explicit algebraic equations satisfied by $M(q, \nu, t, 1; 1, 1)$ and $Q(q, \nu, t, 1; 1, 1)$. However, when $q = 3$, we could only solve the case $\nu = 0$ (corresponding to proper colourings). The final equations are remarkably simple. We give them here for general planar maps. For triangulations, these equations are not new: the Ising model on triangulations was already solved by several other methods (including bijective ones, see Section 6.3 for details), and properly 3-coloured triangulations are just Eulerian triangulations, as discussed in Section 3.2. With the help of Bruno Salvy, we have also conjectured an algebraic equation of degree 11 for the generating function of properly 3-coloured cubic maps (maps in which all vertices have degree 3). By the duality relation (2.5), this corresponds to the series $Q(q, \nu, t, 1; 1, 1)$ taken at $q = 3$, $\nu = -2$.

Theorem 5.9 *The Potts generating function of planar maps $M(2, \nu, t, w; x, y)$, defined by* (5.1) *and taken at $q = 2$, is algebraic. The specialisation $M(2, \nu, t, w; 1, 1)$ has degree* 8 *over $\mathbf{Q}(\nu, t, w)$.*

When $w = 1$, the degree decreases to 6*, and the equation admits a rational parametrisation. Let $S \equiv S(t)$ be the unique power series in t with constant term* 0 *satisfying*

$$S = t\,\frac{\left(1 + 3\nu S - 3\nu S^2 - \nu^2 S^3\right)^2}{1 - 2S + 2\nu^2 S^3 - \nu^2 S^4}.$$

Then

$$M(2,\nu,t,1;1,1) = \frac{1+3\,\nu\,S-3\,\nu\,S^2-\nu^2S^3}{\left(1-2\,S+2\,\nu^2S^3-\nu^2S^4\right)^2}\times$$
$$\left(\nu^3S^6+2\,\nu^2(1-\nu)S^5+\nu\,(1-6\,\nu)S^4-\nu\,(1-5\,\nu)S^3+(1+2\,\nu)S^2-(3+\nu)S+1\right).$$

Theorem 5.10 *The Potts generating function of planar maps $M(3,\nu,t,w;x,y)$, defined by (5.1) and taken at $q=3$, is algebraic.*

The specialisation $M(3,0,t,1;1,1)$ that counts properly *three-coloured planar maps by edges, has degree 4 over $\mathbf{Q}(t)$, and admits a rational parametrisation. Let $S\equiv S(t)$ be the unique power series in t with constant term 0 satisfying*

$$t=\frac{S(1-2\,S^3)}{\left(1+2S\right)^3}.$$

Then

$$M(3,0,t,1;1,1) = \frac{(1+2\,S)\left(1-2\,S^2-4\,S^3-4\,S^4\right)}{\left(1-2\,S^3\right)^2}.$$

5.6 The general case: differential equations

The culminating, and final point in Tutte's study of properly coloured triangulations was a non-linear differential equation satisfied by their generating function. For the more complicated problem of counting maps weighted by their Potts polynomial, we have come with a *system* of differential equations that defines the corresponding generating function [14]. One compact way to write this system is as follows.

Theorem 5.11 *Let $\beta=\nu-1$ and*

$$\Delta(t,v)=(q\nu+\beta^2)-q(\nu+1)v+(\beta t(q-4)(wq+\beta)+q)v^2.$$

There exists a unique triple $(A(t,v),B(t,v),C(t,v))$ of polynomials in v with coefficients in $\mathbf{Q}[q,\nu,w][[t]]$, having degree $4,2$ and 2 respectively in v, such that

$$\begin{aligned} A(0,v) &= (1-v)^2, & A(t,0) &= 1,\\ B(0,v) &= 1-v, & C(t,0) &= w(q+2\beta)-1-\nu, \end{aligned}$$

and

$$\frac{1}{C(t,v)}\frac{\partial}{\partial v}\left(\frac{v^4C(t,v)^2}{A(t,v)\Delta(t,v)^2}\right)=\frac{v^2}{B(t,v)}\frac{\partial}{\partial t}\left(\frac{B(t,v)^2}{A(t,v)\Delta(t,v)^2}\right). \tag{5.22}$$

Let $A_i(t)$ (resp. $B_i(t)$) denote the coefficient of v^i in $A(t,v)$ (resp. $B(t,v)$). Then the Potts generating function of planar maps, $M(1,1)\equiv M(q,\nu,t,w;1,1)$, defined by (5.1), is related to A and B by

$$\begin{aligned}12\,t^2w\left(q\nu+\beta^2\right)M(1,1)-A_2(t)+2\,B_2(t)-8\,t\,(w(q+2\beta)-\nu-1)\,B_1(t)+B_1(t)^2\\ =4\,t\left(1-3\,(\beta+2)^2t+(6\,(\beta+2)(q+2\,\beta)t+q+3\,\beta)\,w-3\,t(q+2\,\beta)^2w^2\right).\end{aligned}$$

Comments
1. Let us write

$$A(t,v)=\sum_{j=0}^{4}A_j(t)v^i,\quad B(t,v)=\sum_{j=0}^{2}B_j(t)v^i,\quad C(t,v)=\sum_{j=0}^{2}C_j(t)v^i.$$

The differential equation (5.22) then translates into a system of 9 differential equations (with respect to t) relating the 11 series A_j, B_j, C_j. However, $A_0(t)=A(t,0)$ and $C_0(t)=C(t,0)$ are given explicitly as initial conditions, so that there are really as many unknown series in t as differential equations. Observe moreover that no derivative of the series $C_j(t)$ arise in the system. This is why we only need initial conditions for the series $A_j(t)$ and $B_j(t)$. They are prescribed by the values of $A(0,v)$ and $B(0,v)$.
2. The form of the above result is very close to Tutte's solution of properly coloured planar triangulations, which can be stated as in Theorem 5.12 below. However, Tutte's case is simpler, as it boils down to only 4 differential equations. This explains why Tutte could derive from his system a *single* differential equation for the generating function of properly coloured triangulations. More precisely, it follows from the theorem below that, if $t=z^2$ and $H\equiv H(t)=t^2T_2(1)$,

$$2q^2(1-q)t+(qt+10H-6tH')H''+q(4-q)(20H-18tH'+9t^2H'')=0. \quad (5.23)$$

So far, we have not been able to derive from Theorem 5.11 a single differential equation for coloured planar maps.

Theorem 5.12 *Let*

$$\Delta(v)=v+4-q.$$

There exists a unique pair $(A(z,v),B(z,v))$ of polynomials in v with coefficients in $\mathbf{Q}[q][[z]]$, having degree 3 and 1 respectively in v, such that

$$\begin{array}{ll} A(0,v)=1+v/4, & A(z,0)=1,\\ B(0,v)=1, & \end{array}$$

and

$$-\frac{4z}{v}\frac{\partial}{\partial v}\left(\frac{v^3}{A(z,v)}\right)=\frac{1}{B(z,v)\Delta(v)}\frac{\partial}{\partial z}\left(\frac{B(z,v)^2}{A(z,v)}\right).$$

Let $A_i(z)$ (resp. $B_i(z)$) denote the coefficient of v^i in $A(z,v)$ (resp. $B(z,v)$). Then the face generating function of properly q-coloured planar near-triangulations having outer-degree 2, denoted $T_2(q,z;1)$ and defined by (5.4), is related to A and B by

$$\begin{aligned} 20\,z^4(q-4)T_2(q,z;1)/q-2\,B_1(z)^2-\left(96\,z^2-24\,z^2q+1\right)B_1(z)+2\,A_2(z)\\ -2\,z^2\left(10-q+432\,z^2-216\,z^2q+27\,z^2q^2\right)=0.\end{aligned}$$

6 Some bijections for coloured planar maps

Certain specialisations of the Potts generating function of planar maps can be determined using a purely bijective approach.

6.1 Bipolar orientations of maps

The numbers that arise in the enumeration of bipolar orientations of planar maps (Proposition 5.4) are known to count other families of objects: *Baxter permutations* (not the same Baxter as in [4]!), pairs of *twin trees*, and certain configurations of *non-intersecting lattice paths.* Several bijections have been established recently between bipolar orientations and these families [20, 57, 64]. Let us mention, however, that the only family that is simple to enumerate is that of non-intersecting lattice paths (via the Lindström-Gessel-Viennot theorem). Hence bijections with this family are the only ones that really provide a self-contained proof of Proposition 5.4.

Regarding bipolar orientations of triangulations (Proposition 5.3), we are currently working on certain bijections with *Young tableaux* of height at most 3, in collaboration with Nicolas Bonichon and Éric Fusy.

6.2 Spanning trees

It is not hard to count in a bijective manner tree-rooted maps with $i+1$ vertices and $j+1$ faces (Proposition 5.5). The construction below, which is usually attributed to Lehman and Walsh [118], is actually not far from Mullin's original proof [91]. Starting from a tree-rooted map (M, T), one walks around the tree T in counterclockwise order, starting from the root-edge of M (Figure 18, left), and:

– when an edge e of T is met, one walks along this edge, and writes a when e is met for the first time, $\bar{a}$ otherwise;

– when an edge e not in T is met, one crosses the edge, and writes b when e is met for the first time, $\bar{b}$ otherwise.

This gives a shuffle of two Dyck words[6] u and v, one of length $2i$ on the alphabet $\{a, \bar{a}\}$ (since there are i edges in T), one of length $2j$ on the alphabet $\{b, \bar{b}\}$ (since there are j edges not in the tree). The number of such shuffles is

$$\binom{2i+2j}{2i} C_i C_j,$$

where $C_i = \binom{2i}{i}/(i+1)$ is the number of Dyck words of length $2i$. The construction is easily seen to be bijective, and this gives the second result of Proposition 5.5.

As already explained, the first result of Proposition 5.5 follows by summing over all i, j such that $i + j = n$. A *direct* bijective proof was only obtained in 2007 by Bernardi [12]. It transforms a tree-rooted map into a pair formed of a plane tree and a non-crossing partition. See [16] for a recent extension to maps of higher genus.

Mullin's original construction [91] decouples the tree-rooted map (M, T) into two objects (Figure 18, right):

– a plane tree with j edges, which is the dual of T and corresponds to the Dyck word v on $\{b, \bar{b}\}$ described above,

[6] A Dyck word on the alphabet $\{a, \bar{a}\}$ is a word that contains as many occurrences of a and $\bar{a}$, and such that every prefix contains at least as many a's as $\bar{a}$'s. A shuffle of two Dyck words can be seen as a walk in the first quadrant of $\mathbf{Z}^2$, starting and ending at the origin.

– a plane tree T', which consists of T and of $2j$ half-edges; this tree can be seen as the Dyck word u shuffled with the word c^{2j}.

The vertex degree distribution of M coincides with the degree distribution of T', and Mullin used this property to count tree-rooted maps with prescribed vertex degrees (or dually, with prescribed face degrees, since a map and its dual have the same number of spanning trees). Indeed, it is easy to count trees with a prescribed degree distribution [100, Thm. 5.3.10]. In particular, the number of plane trees with root-degree d, such that n_k non-root vertices have degree k, for $k \geq 1$, and carrying in addition $2j$ half-edges, is

$$T'(d, j, n_1, n_2, \ldots) := \frac{d(2j - 1 + \sum_k n_k)!}{(2j)! \prod_k n_k!},$$

so that the number of tree-rooted maps (M, T) in which the root-vertex has degree d and n_k non-root vertices have degree k, for $k \geq 1$, is

$$\frac{1}{j+1}\binom{2j}{j} T'(d, j, n_1, n_2, \ldots) = \frac{d(2j - 1 + \sum_k n_k)!}{j!(j+1)! \prod_k n_k!},$$

where $j = \mathrm{e}(M) - \mathrm{v}(M) + 1 = (d + \sum_k (k-2) n_k)/2$ is the *excess* of M (and also the number of faces, minus 1). In particular, the number of tree-rooted maps (M, T) having a root-vertex of degree d and $2i - d$ non-root vertices of degree 3 is

$$\frac{d(4i - d - 1)!}{i!(i+1)!(2i - d)!},$$

since such maps have excess i. This is the dual statement of Proposition 5.6.

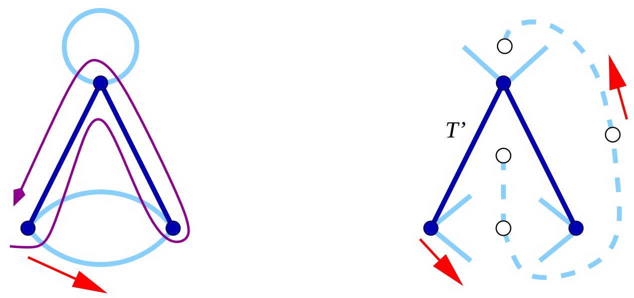

Figure 18: Left: The tour of a tree-rooted map gives an encoding by a shuffle of Dyck words, here $bbaa\bar{b}\bar{b}\bar{a}b\bar{b}\bar{a}$. Right: Alternatively, one can decouple a tree-rooted map into the dual plane tree (dashed lines) and a plane tree T' carrying half-edges.

6.3 The Ising model ($q = 2$)

As observed in [34], a simple transformation relates the Potts generating function of maps at $q = 2$ to the enumeration of bipartite maps by vertex degrees.

Proposition 6.1 *Let $B(t, v, w; x)$ be the generating function of planar bipartite maps, counted by edges (t), non-root vertices of degree 2 (variable v), non-root vertices of* *degree $\neq 2$ (variable w), and degree of the root-vertex (x). Let $M(q, \nu, t, w; x, y)$ be the Potts generating function of planar maps, defined by* (5.1)*. Then*

$$M\left(2, tv, \frac{t}{1-t^2v^2}, w; x, 1\right) = B(t, v+w, w; x).$$

This identity can be refined by keeping track of the number of non-root vertices of each degree and colour. Let

$$\overline{M}(\nu, t, x_1, x_2, \ldots, y_1, y_2, \ldots; x) = \sum_M \nu^{m(M)} t^{\mathrm{e}(M)} x^{\mathrm{dv}(M)} \prod_{i\geq 1} x_i^{\mathrm{v}_i^\circ(M)} y_i^{\mathrm{v}_i^\bullet(M)},$$

where the sum runs over all 2-coloured maps M rooted at a black vertex, $m(M)$ is the number of monochromatic edges in M, and $\mathrm{v}_i^\circ(M)$ (resp. $\mathrm{v}_i^\bullet(M)$) is the number of non-root white (resp. black) vertices of degree i. Let

$$\overline{B}(t, x_1, x_2, \ldots, y_1, y_2, \ldots; x) = \sum_M t^{\mathrm{e}(M)} x^{\mathrm{dv}(M)} \prod_{i\geq 1} x_i^{\mathrm{v}_i^\circ(M)} y_i^{\mathrm{v}_i^\bullet(M)},$$

where the sum runs over all bipartite maps, properly bicoloured in such a way the root-vertex is black. Then

$$\overline{M}\left(tv, \frac{t}{1-t^2v^2}, x_1, x_2, \ldots, y_1, y_2, \ldots; x\right) = \overline{B}(t, x_1, v+x_2, x_3, \ldots, y_1, v+y_2, y_3, \ldots; x).$$

Proof We establish directly the second identity, which implies the first one by specialising each x_i and y_i to w. Take a 2-coloured planar map M, rooted at a black vertex. On each edge, add a (possibly empty) sequence of square vertices of degree 2, in such a way the resulting map is properly bicoloured. An example is shown on Figure 19. Every monochromatic edge receives an odd number of square vertices, while every dichromatic edge receives an even number of these vertices. Each addition of a vertex of degree 2 also results in the addition of an edge. Since M can be recovered from the bipartite map by erasing all square vertices, the identity follows. $\square$

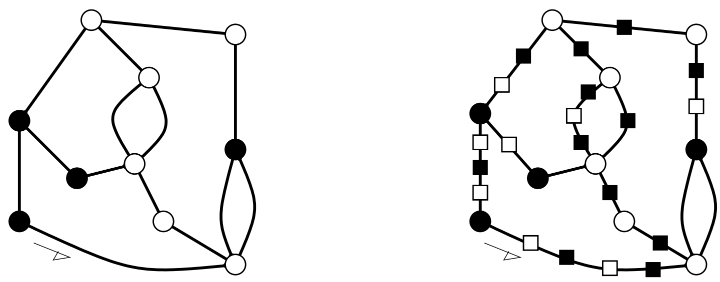

Figure 19: A 2-coloured map and one of the associated bipartite maps.

Recall from Sections 3 and 4 that the enumeration of bipartite maps with prescribed vertex degrees can be addressed via the recursive method (and equations with one catalytic variable) [29], bijections with blossoming trees [34], or bijections with labelled trees [37]. In particular, the last two approaches explain bijectively[7] the algebraicity of the associated generating function, at least when the vertex degrees are bounded.

7 Final comments and questions

We conclude with a number of questions raised by this survey. The first type of question asks what problems have an algebraic solution. Of course, all methods (recursive, bijective, or via matrix integrals...) are welcome to answer them. We then go on with a list of problems that have been solved by a recursive approach, but are still waiting for a purely bijective proof. We also mention questions dealing with asymptotic properties of maps.

7.1 Algebraicity

Theorem 3.2 states several algebraicity results for maps with prescribed face degrees. By comparing the results dealing with general maps to those dealing with Eulerian maps, it appears that our understanding of Eulerian maps with unbounded degrees is probably still incomplete.

Question 7.1 *Let $D_\bullet$ and $D_\circ$ be two subsets of $\mathbf{N}$. Under what conditions on these sets is the generating function of Eulerian maps such that all black (resp. white) faces have their degree in $D_\bullet$ (resp. $D_\circ$) algebraic?*

This question can in principle be addressed via the equations of [34, 37]. Algebraicity is known to hold when $D_\bullet$ and $D_\circ$ are finite, and when $D_\bullet = \{m\}$ and $D_\circ = m\mathbf{N}$. A natural sub-case that could be addressed first is the following[8].

Question 7.2 *Is the generating function of Eulerian planar maps in which all face degrees are multiples of m algebraic?*

Recall that for $m \geq 3$, these are the maps that admit a cyclic m-colouring (Section 3.2). Algebraicity has already been proved when $m = 2$, that is, for maps that are both Eulerian and bipartite [83, 93].

Eulerian maps are required to have even vertex-degrees. But one could think of other restrictions than parity.

Question 7.3 *Under what condition is the generating function of maps in which both the vertex degrees and the face degrees are constrained algebraic?*

This question seems of course very hard to address. A positive answer is known in at least one case: the generating function of triangulations in which all vertices have degree at least d is algebraic for all d [13, 66].

[7] up to minor restrictions imposed at the root of the map

[8] Having raised the question, the author has started to explore it... and come with a positive answer [22]. Algebraicity can be proved either via the equations of [34], or via a bijection with $(m + 1)$-constellations.

7.2 Bijections

Our first question may seem surprising at first sight.

Question 7.4 *Design bijections between families of trees and families of planar maps with unbounded degrees.*

Indeed, it seems that all bijections that can be used to count families of maps with unbounded degrees use a detour via maps with bounded degrees. The simplest example, presented in Section 4, is that of general planar maps: we have first shown that they are in bijection with 4-valent maps (or, dually, quadrangulations), before describing two types of bijections between 4-valent maps and trees. We could actually content ourselves with this situation: after all, isn't a combination of two beautiful bijections twice as beautiful as a single bijection? But there exist problems with an algebraic solution, dealing with maps with unbounded degrees, that have not been solved by a direct bijection so far, like the Ising model on general planar maps (Theorem 5.9), or the hard-particle model on general planar maps [29]. Discovering such bijections could also give an algebra-free proof of the fact that maps in which all degrees are multiples of m are algebraic; the bijection of [35] gives indeed a proof, but requires a bit of algebra. Moreover, this could be a purely bijective way to address the questions raised above on the algebraicity of Eulerian maps in which all face degrees are multiples of m.

Question 7.5 *Design bijections for q-coloured maps.*

This can take several directions:

- find bijections for the special values of q (like $q = 3$) that are known to yield algebraic generating functions (Theorems 5.7 and 5.8);
- find bijections for specialisations of the Potts generating function of maps, like those presented in Section 6 for spanning trees and bipolar orientations;
- finally, one would dream of designing bijections that would establish directly differential equations for coloured maps, starting with the (relatively simple?) case of triangulations (see (5.23)). The author is currently working on an interesting construction of Bouttier *et al.* [39], which allows to count spanning forests on maps and to derive certain differential equations in a simpler way than the recursive approach [23].

Finally, we have discussed in Section 4 two families of bijections, but a third one could exist, as suggested by Bernardi's beautiful construction for loopless triangulations [11].

Question 7.6 *Is the bijection of [11] the tip of some iceberg?*

7.3 Asymptotics of maps

Question 7.7 *What is the asymptotic number of properly q-coloured planar maps having n edges?*

This question has been studied by Odlyzko and Richmond [92] for triangulations, starting from the differential equation (5.23). For $q \in [15/11, 4] \cup [5, \infty)$, they proved that the number of properly q-coloured triangulations with n faces is of the form $\kappa \mu^n n^{-5/2}$. The exponent $-5/2$ is typical in the enumeration of (uncoloured) planar maps.

The asymptotic behaviour of the number of n-edge q-coloured planar maps has been worked out in [15] for $q = 2$ and $q = 3$, using the explicit results of Theorems 5.9 and 5.10. Again, the exponent is $-5/2$. The same question can be asked when a parameter $\nu \neq 0$ weights monochromatic edges. For $q = 2$, the exponent is still $-5/2$, except at the critical value $\nu = (3 + \sqrt{5})/2$, where it becomes $-7/3$. See [75, 21] for similar results on maps of fixed vertex degree.

The proofs of these results use the solutions of the difficult functional equations (5.2) and (5.4). It would be extremely interesting to be able to understand the asymptotic behaviour of these numbers (or the singular behaviour of the associated series) *directly from these equations*. At the moment, we do not known how to do this, even in the case of one catalytic variable.

Question 7.8 *Develop a "singularity analysis" [61] for equations with catalytic variables.*

Finally, the asymptotic geometry of random uncoloured maps has attracted a lot of attention in the past few years [40, 41, 49], and a limit object, *the Brownian map*, has been identified [78, 85, 86]. Similar questions can be addressed for maps equipped with an additional structure.

Question 7.9 *Is there a scaling limit for maps equipped with a spanning tree? a spanning forest? for properly q-coloured maps?*

The final section of [79] suggests a partial, and conjectural answer to this question for maps equipped with certain statistical physics models, including the Ising model. Of course, the first point is to determine how the average distance between two vertices of these maps scales.

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References

[1] M. Aigner and G. M. Ziegler. *Proofs from The Book*. Springer-Verlag, Berlin, fourth edition, 2010.

[2] C. Banderier, M. Bousquet-Mélou, A. Denise, P. Flajolet, D. Gardy, and D. Gouyou-Beauchamps. Generating functions for generating trees. *Discrete Math.*, 246(1-3):29–55, 2002.

[3] C. Banderier and P. Flajolet. Basic analytic combinatorics of directed lattice paths. *Theoret. Comput. Sci.*, 281(1-2):37–80, 2002.

[4] R. J. Baxter. Dichromatic polynomials and Potts models summed over rooted maps. *Ann. Comb.*, 5(1):17–36, 2001.

[5] E. A. Bender and E. R. Canfield. The asymptotic number of rooted maps on a surface. *J. Combin. Theory Ser. A*, 43(2):244–257, 1986.

[6] E. A. Bender and E. R. Canfield. The number of degree-restricted rooted maps on the sphere. *SIAM J. Discrete Math.*, 7(1):9–15, 1994.

[7] E. A. Bender, E. R. Canfield, and L. B. Richmond. The asymptotic number of rooted maps on a surface. II. Enumeration by vertices and faces. *J. Combin. Theory Ser. A*, 63(2):318–329, 1993.

[8] E. A. Bender and N. C. Wormald. The number of loopless planar maps. *Discrete Math.*, 54(2):235–237, 1985.

[9] S. Beraha, J. Kahane, and N. J. Weiss. Limits of chromatic zeros of some families of maps. *J. Combin. Theory Ser. B*, 28(1):52–65, 1980.

[10] O. Bernardi. Comptage des cartes planaires boisées. Personal communication.

[11] O. Bernardi. Bijective counting of Kreweras walks and loopless triangulations. *J. Combin. Theory Ser. A*, 114(5):931–956, 2007.

[12] O. Bernardi. Bijective counting of tree-rooted maps and shuffles of parenthesis systems. *Electron. J. Combin.*, 14(1):Research Paper 9, 36 pp. (elect.), 2007.

[13] O. Bernardi. On triangulations with high vertex degree. *Ann. Comb.*, 12(1):17–44, 2008.

[14] O. Bernardi and M. Bousquet-Mélou. Counting colored planar maps: differential equations, In preparation.

[15] O. Bernardi and M. Bousquet-Mélou. Counting colored planar maps: algebraicity results. *J. Combin. Theorey Ser. B*, to appear. Arxiv:0909:1695.

[16] O. Bernardi and G. Chapuy. A bijection for covered maps, or a shortcut between Harer-Zagier's and Jackson's formulas. Arxiv:1001.1592, 2010.

[17] O. Bernardi and É. Fusy. A bijection for triangulations, quadrangulations, pentagulations, etc. Arxiv:1007.1292, 2010.

[18] D. Bessis, C. Itzykson, and J. B. Zuber. Quantum field theory techniques in graphical enumeration. *Adv. in Appl. Math.*, 1(2):109–157, 1980.

[19] B. Bollobás. *Modern graph theory*, volume 184 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998.

[20] N. Bonichon, M. Bousquet-Mélou, and É. Fusy. Baxter permutations and plane bipolar orientations. *Sém. Lothar. Combin.*, 61A:Art. B61Ah, 29 pp. (electronic), 2009.

[21] D. V. Boulatov and V. A. Kazakov. The Ising model on a random planar lattice: the structure of the phase transition and the exact critical exponents. *Phys. Lett. B*, 186(3-4):379–384, 1987.

[22] M. Bousquet-Mélou. Algebraicity results for planar Eulerian maps. In preparation.

[23] M. Bousquet-Mélou. Spanning forests on planar maps. In preparation.

[24] M. Bousquet-Mélou. A method for the enumeration of various classes of column-convex polygons. *Discrete Math.*, 154(1-3):1–25, 1996.

[25] M. Bousquet-Mélou. Multi-statistic enumeration of two-stack sortable permutations. *Electron. J. Combin.*, 5(1):Research Paper 21, 12 pp., 1998.

[26] M. Bousquet-Mélou. Four classes of pattern-avoiding permutations under one roof: generating trees with two labels. *Electronic J. Combinatorics*, 9(2):Research Paper 19, 2003.

[27] M. Bousquet-Mélou. Walks in the quarter plane: Kreweras' algebraic model. *Ann. Appl. Probab.*, 15(2):1451–1491, 2005.

[28] M. Bousquet-Mélou and S. Butler. Forest-like permutations. *Ann. Comb.*, 11:335–354, 2007.

[29] M. Bousquet-Mélou and A. Jehanne. Polynomial equations with one catalytic variable, algebraic series and map enumeration. *J. Combin. Theory Ser. B*, 96:623–672, 2006.

[30] M. Bousquet-Mélou and M. Mishna. Walks with small steps in the quarter plane. *Contemp. Math.*, 520:1–40, 2010. ArXiv:0810.4387.

[31] M. Bousquet-Mélou and M. Petkovšek. Linear recurrences with constant coefficients: the multivariate case. *Discrete Math.*, 225(1-3):51–75, 2000.

[32] M. Bousquet-Mélou and M. Petkovšek. Walks confined in a quadrant are not always D-finite. *Theoret. Comput. Sci.*, 307(2):257–276, 2003.

[33] M. Bousquet-Mélou and G. Schaeffer. Enumeration of planar constellations. *Adv. in Appl. Math.*, 24(4):337–368, 2000.

[34] M. Bousquet-Mélou and G. Schaeffer. The degree distribution of bipartite planar maps: applications to the Ising model. In K. Eriksson and S. Linusson, editors, *Formal Power Series and Algebraic Combinatorics*, pages 312–323, Vadstena, Sweden, 2003. ArXiv math.CO/0211070.

[35] J. Bouttier, P. Di Francesco, and E. Guitter. Census of planar maps: from the one-matrix model solution to a combinatorial proof. *Nuclear Phys. B*, 645(3):477–499, 2002. ArXiv:cond-mat/0207682.

[36] J. Bouttier, P. Di Francesco, and E. Guitter. Counting colored random triangulations. *Nucl. Phys. B*, 641:519–532, 2002.

[37] J. Bouttier, P. Di Francesco, and E. Guitter. Planar maps as labeled mobiles. *Electron. J. Combin.*, 11(1):Research Paper 69, 27 pp. (electronic), 2004.

[38] J. Bouttier, P. Di Francesco, and E. Guitter. Combinatorics of bicubic maps with hard particles. *J. Phys. A*, 38(21):4529–4559, 2005.

[39] J. Bouttier, P. Di Francesco, and E. Guitter. Blocked edges on Eulerian maps and mobiles: application to spanning trees, hard particles and the Ising model. *J. Phys. A*, 40(27):7411–7440, 2007.

[40] J. Bouttier and E. Guitter. Statistics in geodesics in large quadrangulations. *J. Phys. A*, 41(14):145001, 30, 2008.

[41] J. Bouttier and E. Guitter. The three-point function of planar quadrangulations. *J. Stat. Mech. Theory Exp.*, 7:P07020, 39, 2008.

[42] E. Brézin, C. Itzykson, G. Parisi, and J. B. Zuber. Planar diagrams. *Comm. Math. Phys.*, 59(1):35–51, 1978.

[43] W. G. Brown. Enumeration of non-separable planar maps. *Canad. J. Math.*, 15:526–545, 1963.

[44] W. G. Brown. Enumeration of triangulations of the disk. *Proc. London Math. Soc. (3)*, 14:746–768, 1964.

[45] W. G. Brown. Enumeration of quadrangular dissections of the disk. *Canad. J. Math.*, 17:302–317, 1965.

[46] W. G. Brown. On the existence of square roots in certain rings of power series. *Math. Ann.*, 158:82–89, 1965.

[47] W. G. Brown and W. T. Tutte. On the enumeration of rooted non-separable planar maps. *Canad. J. Math.*, 16:572–577, 1964.

[48] G. Chapuy, M. Marcus, and G. Schaeffer. A bijection for rooted maps on orientable surfaces. *SIAM J. Discrete Math.*, 23(3):1587–1611, 2009.

[49] P. Chassaing and G. Schaeffer. Random planar lattices and integrated super-Brownian excursion. *Probab. Theory Related Fields*, 128(2):161–212, 2004.

[50] R. Cori and B. Vauquelin. Planar maps are well labeled trees. *Canad. J. Math*, 33:1023–1042, 1981.

[51] J.-M. Daul. q-States Potts model on a random planar lattice. ArXiv:hep-th/9502014.

[52] A. de Mier and M. Noy. A solution to the tennis ball problem. *Theoret. Comput. Sci.*, 346(2-3):254–264, 2005.

[53] A. Del Lungo, F. Del Ristoro, and J.-G. Penaud. Left ternary trees and non-separable rooted planar maps. *Theoret. Comput. Sci.*, 233(1-2):201–215, 2000.

[54] P. Di Francesco, B. Eynard, and E. Guitter. Coloring random triangulations. *Nuclear Phys. B*, 516(3):543–587, 1998.

[55] P. Di Francesco, P. Ginsparg, and J. Zinn-Justin. 2D gravity and random matrices. *Phys. Rep.*, 254(1-2), 1995. 133 pp.

[56] B. Eynard and G. Bonnet. The Potts-q random matrix model: loop equations, critical exponents, and rational case. *Phys. Lett. B*, 463(2-4):273–279, 1999.

[57] S. Felsner, É. Fusy, M. Noy, and D. Orden. Bijections for Baxter families and related objects. *J. Combin. Theory Ser. A*, 118(3):993–1020, 2011.

[58] P. Fendley and V. Krushkal. Tutte chromatic identities from the Temperley-Lieb algebra. *Geom. Topol.*, 13(2):709–741, 2009.

[59] S. Feretić and D. Svrtan. On the number of column-convex polyominoes with given perimeter and number of columns. In Barlotti, Delest, and Pinzani, editors, *Proceedings of the 5th Conference on Formal Power Series and Algebraic Combinatorics (Florence, Italy)*, pages 201–214, 1993.

[60] P. Flajolet. Analytic models and ambiguity of context-free languages. *Theoret. Comput. Sci.*, 49(2-3):283–309, 1987.

[61] P. Flajolet and A. Odlyzko. Singularity analysis of generating functions. *SIAM J. Discrete Math.*, 3(2):216–240, 1990.

[62] C. M. Fortuin and P. W. Kasteleyn. On the random cluster model: I. Introduction and relation to other models. *Physica*, 57:536–564, 1972.

[63] É. Fusy, D. Poulalhon, and G. Schaeffer. Dissections and trees, with applications to optimal mesh encoding and to random sampling. *ACM Trans. Algorithms*, 4(2), 2008. Art. 19.

[64] É. Fusy, D. Poulalhon, and G. Schaeffer. Bijective counting of plane bipolar orientations and Schnyder woods. *European J. Combin.*, 30(7):1646–1658, 2009.

[65] Z. Gao. The number of degree restricted maps on general surfaces. *Discrete Math.*, 123(1-3):47–63, 1993.

[66] Z. Gao and N. C. Wormald. Enumeration of rooted cubic planar maps. *Ann. Comb.*, 6(3-4):313–325, 2002.

[67] Z. J. Gao, I. M. Wanless, and N. C. Wormald. Counting 5-connected planar triangulations. *J. Graph Theory*, 38(1):18–35, 2001.

[68] I. P. Goulden and D. M. Jackson. Transitive factorizations in the symmetric group, and combinatorial aspects of singularity theory. *European J. Combin.*, 21(8):1001–1016, 2000.

[69] I. P. Goulden and D. M. Jackson. The KP hierarchy, branched covers, and triangulations. *Adv. Math.*, 219(3):932–951, 2008.

[70] C. Greene and T. Zaslavsky. On the interpretation of Whitney numbers through arrangements of hyperplanes, zonotopes, non-Radon partitions, and orientations of graphs. *Trans. Amer. Math. Soc.*, 280(1):97–126, 1983.

[71] A. Guionnet and É. Maurel-Segala. Combinatorial aspects of matrix models. *ALEA Lat. Am. J. Probab. Math. Stat.*, 1:241–279, 2006.

[72] J. L. Jacobsen, J.-F. Richard, and J. Salas. Complex-temperature phase diagram of Potts and RSOS models. *Nuclear Phys. B*, 743(3):153–206, 2006.

[73] J. L. Jacobsen and J. Salas. Transfer matrices and partition-function zeros for antiferromagnetic Potts models. II. Extended results for square-lattice chromatic polynomial. *J. Statist. Phys.*, 104(3-4):701–723, 2001.

[74] B. Jacquard and G. Schaeffer. A bijective census of nonseparable planar maps. *J. Combin. Theory Ser. A*, 83(1):1–20, 1998.

[75] V. A. Kazakov. Ising model on a dynamical planar random lattice: exact solution. *Phys. Lett. A*, 119(3):140–144, 1986.

[76] D. E. Knuth. *The art of computer programming. Vol. 1: Fundamental algorithms*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont, 1968.

[77] B. Lass. Orientations acycliques et le polynôme chromatique. *European J. Combin.*, 22(8):1101–1123, 2001.

[78] J.-F. Le Gall. The topological structure of scaling limits of large planar maps. *Invent. Math.*, 169(3):621–670, 2007.

[79] J.-F. Le Gall and G. Miermont. Scaling limits of random planar maps with large faces. *Ann. Probab.*, 39(1):1–69, 2011.

[80] J.-F. Le Gall and F. Paulin. Scaling limits of bipartite planar maps are homeomorphic to the 2-sphere. *Geom. Funct. Anal.*, 18(3):893–918, 2008.

[81] L. Lipshitz. The diagonal of a D-finite power series is D-finite. *J. Algebra*, 113(2):373–378, 1988.

[82] L. Lipshitz. D-finite power series. *J. Algebra*, 122:353–373, 1989.

[83] V. A. Liskovets and T. R. S. Walsh. Enumeration of Eulerian and unicursal planar maps. *Discrete Math.*, 282(1-3):209–221, 2004.

[84] Y. Liu. Chromatic sum equations for rooted planar maps. In *Proceedings of the fifteenth Southeastern conference on combinatorics, graph theory and computing (Baton Rouge, La., 1984)*, volume 45, pages 275–280, 1984.

[85] J.-F. Marckert and G. Miermont. Invariance principles for random bipartite planar maps. *Ann. Probab.*, 35(5):1642–1705, 2007.

[86] J.-F. Marckert and A. Mokkadem. Limit of normalized quadrangulations: the Brownian map. *Ann. Probab.*, 34(6):2144–2202, 2006.

[87] P. P. Martin. The Potts model and the Beraha numbers. *J. Phys. A*, 20(6):L399–L403, 1987.

[88] M. L. Mehta. A method of integration over matrix variables. *Comm. Math. Phys.*, 79(3):327–340, 1981.

[89] M. Mishna. Classifying lattice walks restricted to the quarter plane. *J. Combin. Theory Ser. A*, 116(2):460–477, 2009. ArXiv:math/0611651.

[90] M. Mishna and A. Rechnitzer. Two non-holonomic lattice walks in the quarter plane. *Theoret. Comput. Sci.*, 410(38-40):3616–3630, 2009. ArXiv:math/0701800.

[91] R. C. Mullin. On the enumeration of tree-rooted maps. *Canad. J. Math.*, 19:174–183, 1967.

[92] A. M. Odlyzko and L. B. Richmond. A differential equation arising in chromatic sum theory. In *Proceedings of the fourteenth Southeastern conference on combinatorics, graph theory and computing (Boca Raton, Fla., 1983)*, volume 40, pages 263–275, 1983.

[93] D. Poulalhon and G. Schaeffer. A note on bipartite Eulerian planar maps. Available at `http://www.lix.polytechnique.fr/∼schaeffe`.

[94] D. Poulalhon and G. Schaeffer. A bijection for triangulations of a polygon with interior points and multiple edges. *Theoret. Comput. Sci.*, 307(2):385–401, 2003.

[95] D. Poulalhon and G. Schaeffer. Optimal coding and sampling of triangulations. *Algorithmica*, 46(3-4):505–527, 2006.

[96] H. Prodinger. The kernel method: a collection of examples. *Sém. Lothar. Combin.*, 50:Art. B50f, 19 pp. (electronic), 2003/04.

[97] H. Saleur. Zeroes of chromatic polynomials: a new approach to Beraha conjecture using quantum groups. *Comm. Math. Phys.*, 132(3):657–679, 1990.

[98] G. Schaeffer. Bijective census and random generation of Eulerian planar maps with prescribed vertex degrees. *Electron. J. Combin.*, 4(1):Research Paper 20, 14 pp. (electronic), 1997.

[99] G. Schaeffer. *Conjugaison d'arbres et cartes combinatoires aléatoires*. PhD thesis, Université Bordeaux 1, France, 1998.

[100] R. P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999.

[101] H. N. V. Temperley. Combinatorial problems suggested by the statistical mechanics of domains and of rubber-like molecules. *Phys. Rev. (2)*, 103:1–16, 1956.

[102] W. T. Tutte. A contribution to the theory of chromatic polynomials. *Canadian J. Math.*, 6:80–91, 1954.

[103] W. T. Tutte. A census of planar triangulations. *Canad. J. Math.*, 14:21–38, 1962.

[104] W. T. Tutte. A census of planar maps. *Canad. J. Math.*, 15:249–271, 1963.

[105] W. T. Tutte. On the enumeration of planar maps. *Bull. Amer. Math. Soc.*, 74:64–74, 1968.

[106] W. T. Tutte. Dichromatic sums for rooted planar maps. *Proc. Sympos. Pure Math.*, 19:235–245, 1971.

[107] W. T. Tutte. Chromatic sums for rooted planar triangulations. II. The case $\lambda = \tau + 1$. *Canad. J. Math.*, 25:657–671, 1973.

[108] W. T. Tutte. Chromatic sums for rooted planar triangulations. III. The case $\lambda = 3$. *Canad. J. Math.*, 25:780–790, 1973.

[109] W. T. Tutte. Chromatic sums for rooted planar triangulations. IV. The case $\lambda = \infty$. *Canad. J. Math.*, 25:929–940, 1973.

[110] W. T. Tutte. Chromatic sums for rooted planar triangulations: the cases $\lambda = 1$ and $\lambda = 2$. *Canad. J. Math.*, 25:426–447, 1973.

[111] W. T. Tutte. Chromatic sums for rooted planar triangulations. V. Special equations. *Canad. J. Math.*, 26:893–907, 1974.

[112] W. T. Tutte. On a pair of functional equations of combinatorial interest. *Aequationes Math.*, 17(2-3):121–140, 1978.

[113] W. T. Tutte. Chromatic solutions. *Canad. J. Math.*, 34(3):741–758, 1982.

[114] W. T. Tutte. Chromatic solutions. II. *Canad. J. Math.*, 34(4):952–960, 1982.

[115] W. T. Tutte. Map-colourings and differential equations. In *Progress in graph theory (Waterloo, Ont., 1982)*, pages 477–485. Academic Press, Toronto, ON, 1984.

[116] W. T. Tutte. Chromatic sums revisited. *Aequationes Math.*, 50(1-2):95–134, 1995.

[117] W. T. Tutte. Dichromatic sums revisited. *J. Combin. Theory Ser. B*, 66(2):161–167, 1996.

[118] T. Walsh and A. B. Lehman. Counting rooted maps by genus. II. *J. Combinatorial Theory Ser. B*, 13:122–141, 1972.

[119] T. R. S. Walsh and A. B. Lehman. Counting rooted maps by genus. III: Nonseparable maps. *J. Combinatorial Theory Ser. B*, 18:222–259, 1975.

[120] D. J. A. Welsh and C. Merino. The Potts model and the Tutte polynomial. *J. Math. Phys.*, 41(3):1127–1152, 2000.

[121] D. Zeilberger. A proof of Julian West's conjecture that the number of two-stack-sortable permutations of length n is $2(3n)!/((n+1)!(2n+1)!)$. *Discrete Math.*, 102(1):85–93, 1992.

[122] D. Zeilberger. The umbral transfer-matrix method: I. Foundations. *J. Comb. Theory, Ser. A*, 91:451–463, 2000.

[123] P. Zinn-Justin. The dilute Potts model on random surfaces. *J. Statist. Phys.*, 98(1-2):j10–264, 2000.

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