

**Pattern avoiding permutations,
interval orders
and
involutions with no neighbour nesting**

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joint work with...

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2. University of Reykjavík

I. L'histoire

Pattern avoiding permutations

Classical pattern avoidance

Consider $\mathcal{S}_n(2-3-1)$; the collection of all permutations $\pi \in \mathcal{S}_n$ such that there do not exist indices $1 \leq i < j < k \leq n$ with $\pi_k < \pi_i < \pi_j$.

$$\mathcal{S}_n(2-3-1) = \mathcal{S}_n \left(\begin{array}{|c|c|c|} \hline & \bullet & \\ \hline \bullet & & \\ \hline & & \bullet \\ \hline \end{array} \right)$$

Generalized pattern avoidance

Consider $\mathcal{S}_n(23-1)$; the collection of all permutations $\pi \in \mathcal{S}_n$ such that there do not exist indices $1 \leq i < i+1 < k \leq n$ with $\pi_k < \pi_i < \pi_{i+1}$.

$$\mathcal{S}_n(23-1) = \mathcal{S}_n \left(\begin{array}{|c|c|c|} \hline & \bullet & \\ \hline \bullet & & \\ \hline & & \bullet \\ \hline \end{array} \right)$$

The ‘—’ represents an elastic length.

- E. Babson and E. Steingrímsson, Generalized permutation patterns and a classification of the Mahonian statistics, *Sém. Lothar. Combin.* **44** (2000) Art. B44b, 18 pp.

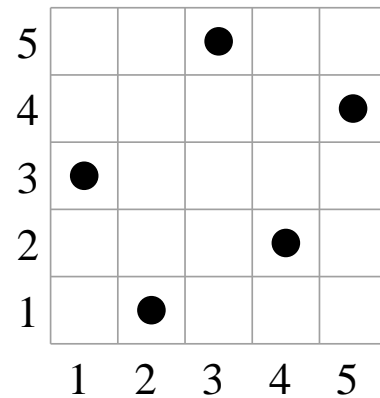
A new class of pattern avoiding permutations

What about

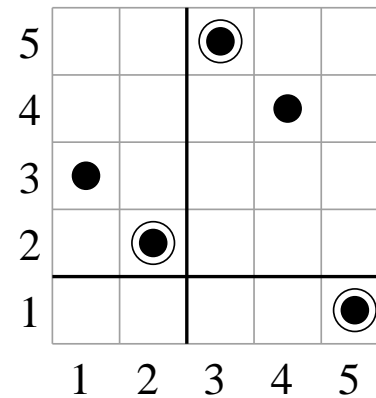
$$\mathcal{R}_n = \mathcal{S}_n \left(\begin{array}{|c|c|c|} \hline & \bullet & \\ \hline \bullet & & \\ \hline & & \bullet \\ \hline \end{array} \right) ?$$

A permutation is in \mathcal{R}_n if there do not exist indices i and k satisfying $1 \leq i < i + 1 < k \leq n$ and such that

$$\pi_k + 1 = \pi_i < \pi_{i+1}.$$



31524 $\in \mathcal{R}_5$



32541 $\notin \mathcal{R}_5$

Enumeration

Some known results:

- $\mathcal{S}_n \left(\begin{array}{|c|c|c|} \hline & \bullet & \\ \hline \bullet & & \\ \hline & & \bullet \\ \hline \end{array} \right) \Rightarrow C_n, \quad \text{Catalan numbers}$

- $\mathcal{S}_n \left(\begin{array}{|c|c|c|} \hline & \bullet & \\ \hline \bullet & & \\ \hline & & \bullet \\ \hline \end{array} \right) \Rightarrow B_n, \quad \text{Bell numbers}$

What about

- $\mathcal{R}_n = \mathcal{S}_n \left(\begin{array}{|c|c|c|} \hline & \bullet & \\ \hline \bullet & & \\ \hline & & \bullet \\ \hline \end{array} \right) \Rightarrow 1, 1, 2, 5, 15, 53, 217, 1014...?$

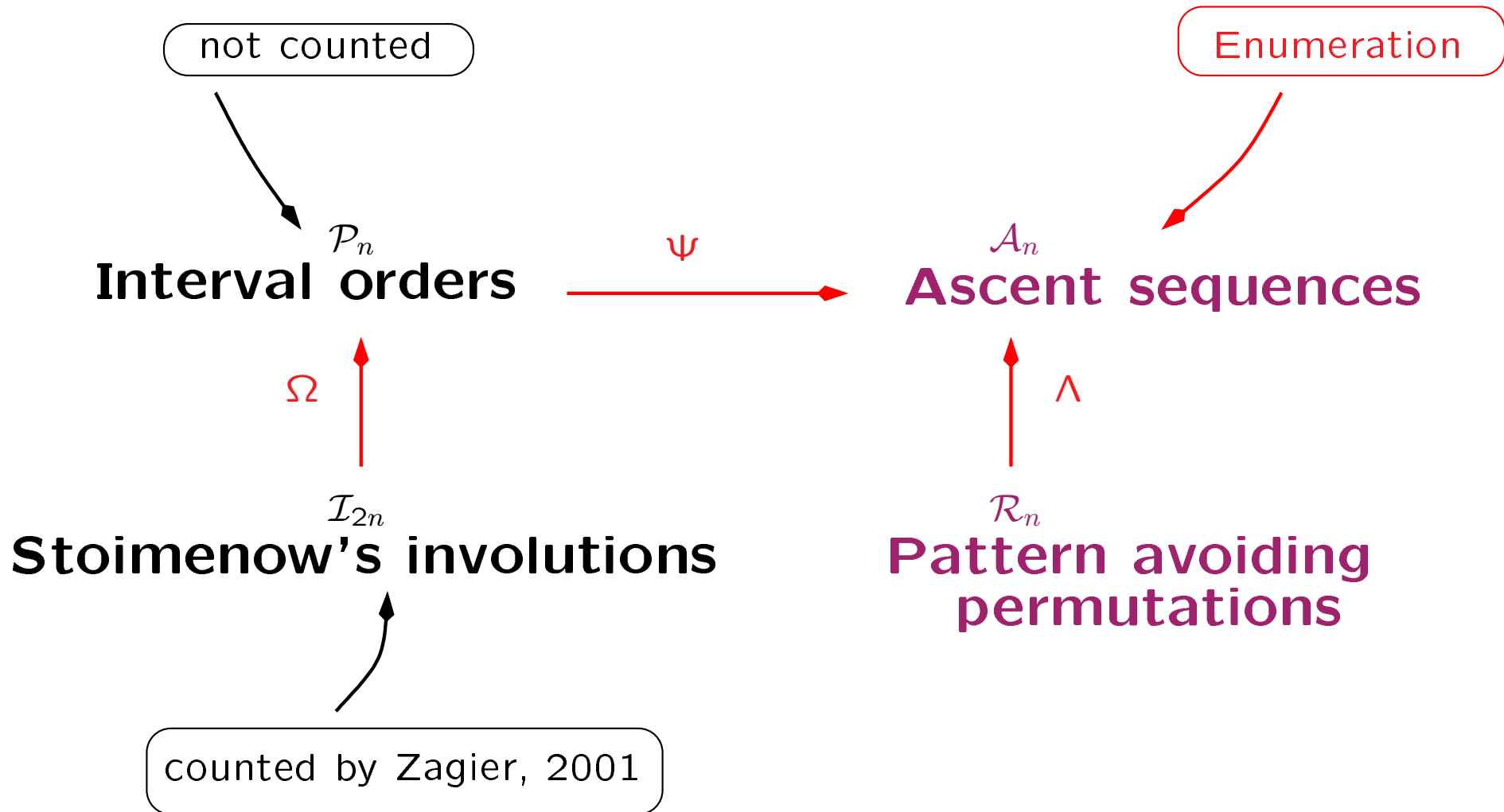
In the On-Line Encyclopedia of Integer Sequences: A022493

- Unlabelled interval orders on n points

- Linearized chord diagrams of degree n , introduced by [Stoimenow, 1998](#)

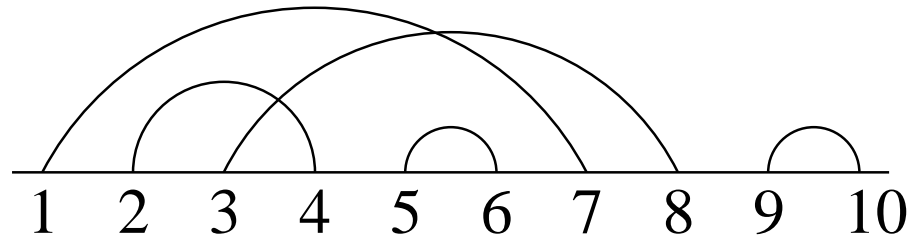
(here: involutions with no neighbour nesting)

In this talk



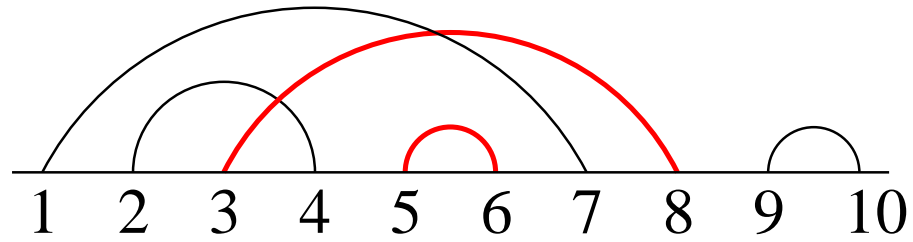
II. Involutions with no neighbour nestings and interval orders

Involutions with no neighbour nestings



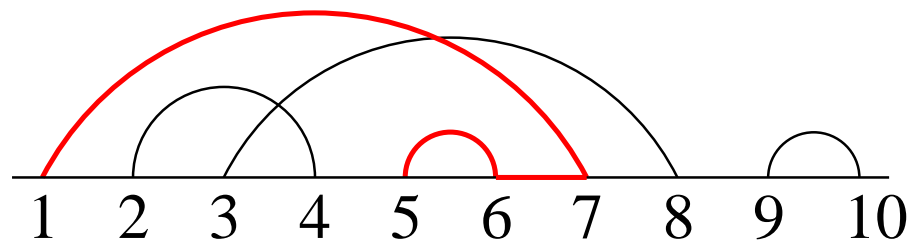
A (fixed-point free) involution

Involutions with no neighbour nestings



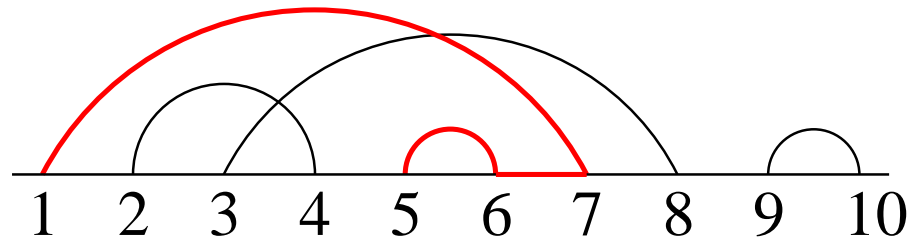
A **nesting** between two arches

Involutions with no neighbour nestings

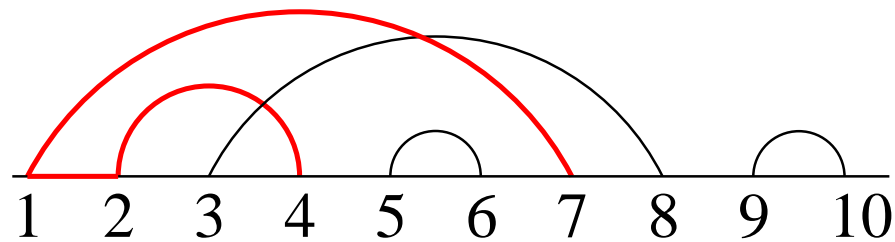


A neighbour nesting

Involutions with no neighbour nestings

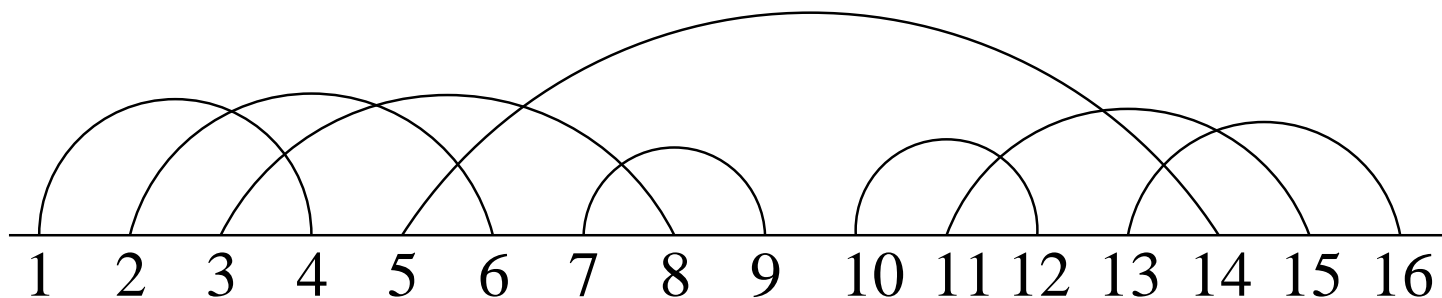


A neighbour nesting



Another one

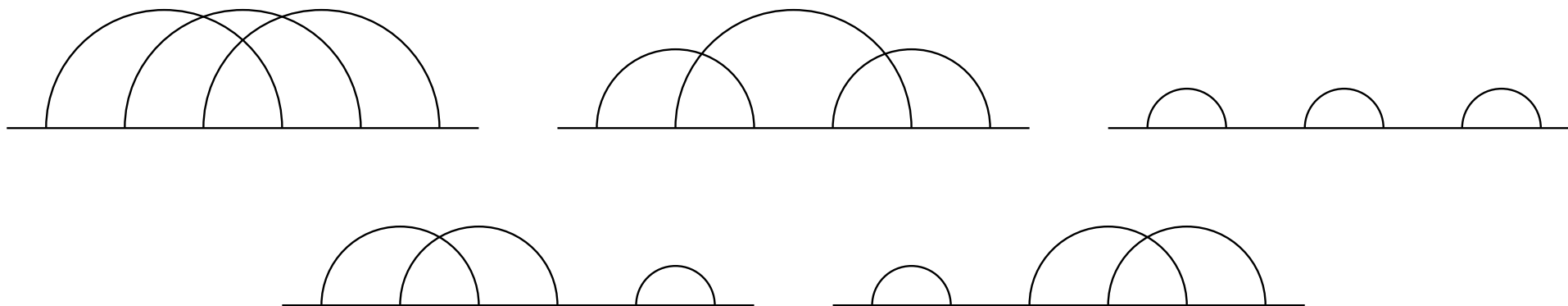
Involutions with no neighbour nestings



[Stoimenow 98]: bounds on the dimension of the vector space of Vassiliev invariants of degree n

Equivalently: all descents cross the diagonal

Example: $n = 3$

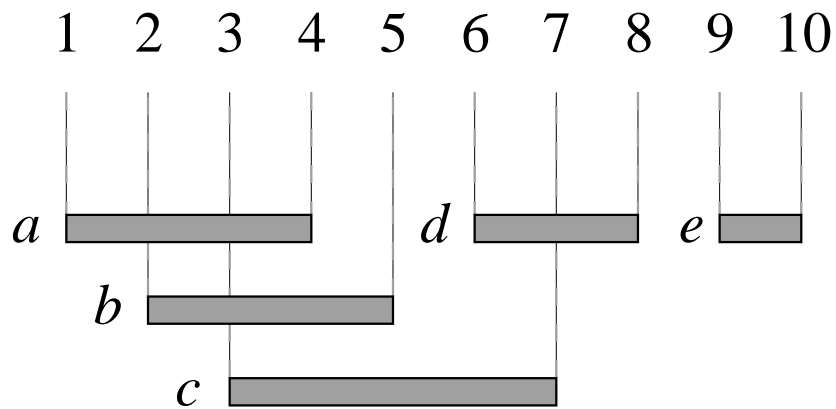


Interval orders

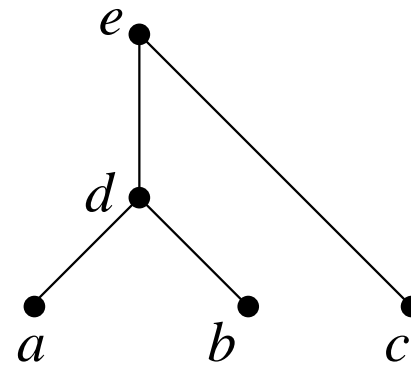
Definition: The relative order of a collection of intervals of the real line under the order relation:

$$[x, y] < [z, t] \iff y < z.$$

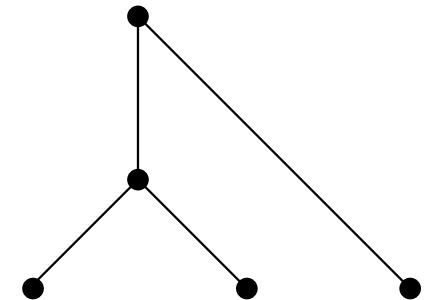
Example



Intervals



(Labelled) Poset



Unlabelled Poset

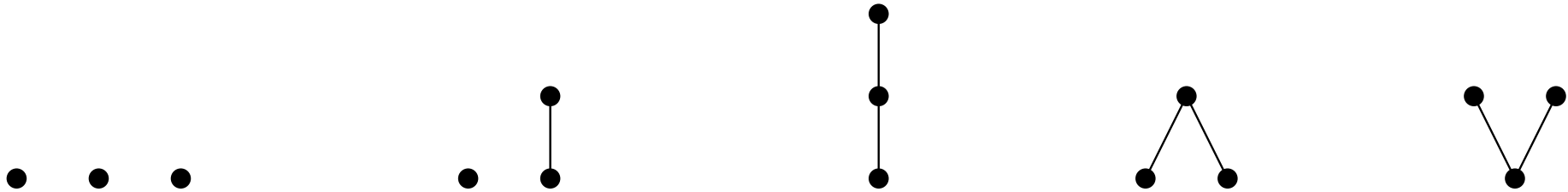
(Hasse diagram)

Remarks

- Different collections of intervals may give the same order:



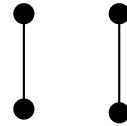
- Example: $n = 3$



- For the number of unlabelled posets, only 16 terms are known!
(18 in the labelled case)

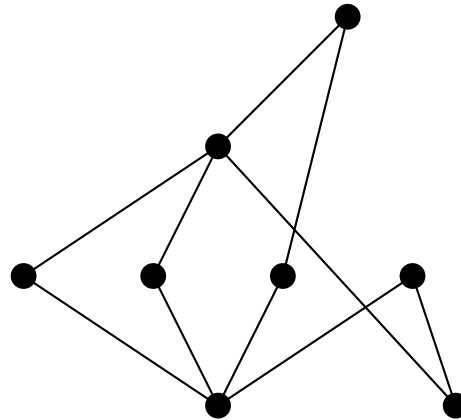
Interval orders: two characterizations

- **First characterization:** The poset P has no induced $(2 + 2)$ sub-poset



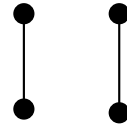
[Fishburn 70]

Example:



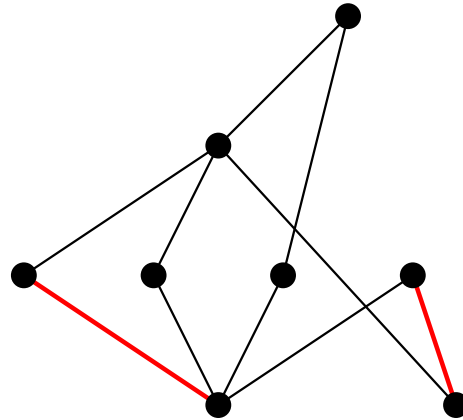
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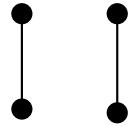
[Fishburn 70]

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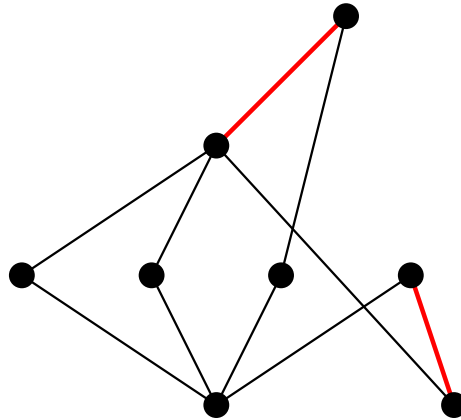
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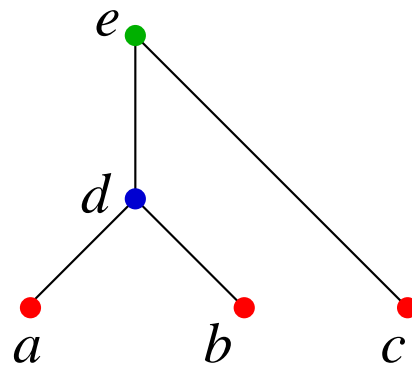
Interval orders: two characterizations

- Second characterization [Fishburn 70]. The predecessor sets (or: down-sets)

$$D(x) = \{y \in P : y < x\}, \quad \text{for } x \in P$$

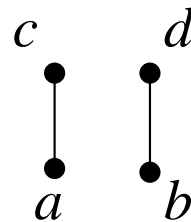
can be totally ordered by inclusion.

Example:



$$\emptyset = D(a) = D(b) = D(c) \subset D(d) = \{a, b\} \subset D(e) = \{a, b, c, d\}$$

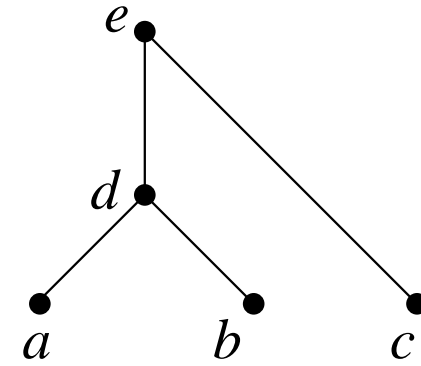
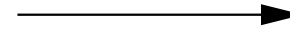
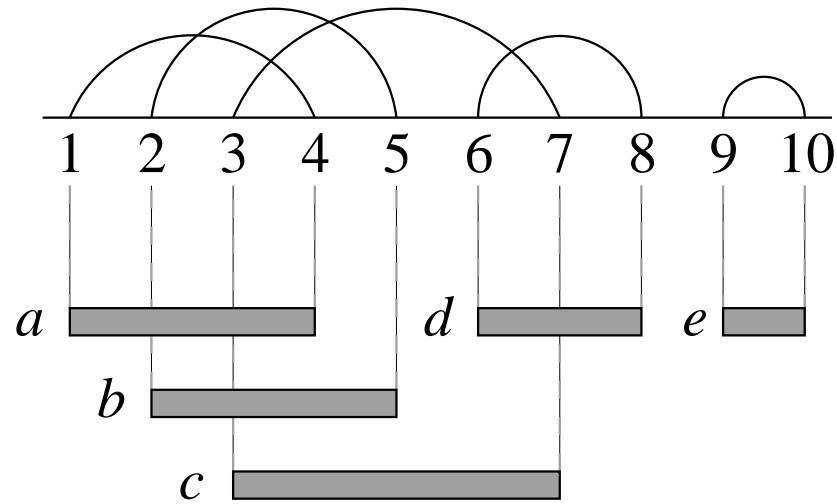
Counter-example: In



$D(c) = \{a\}$ is not comparable to $D(d) = \{b\}$.

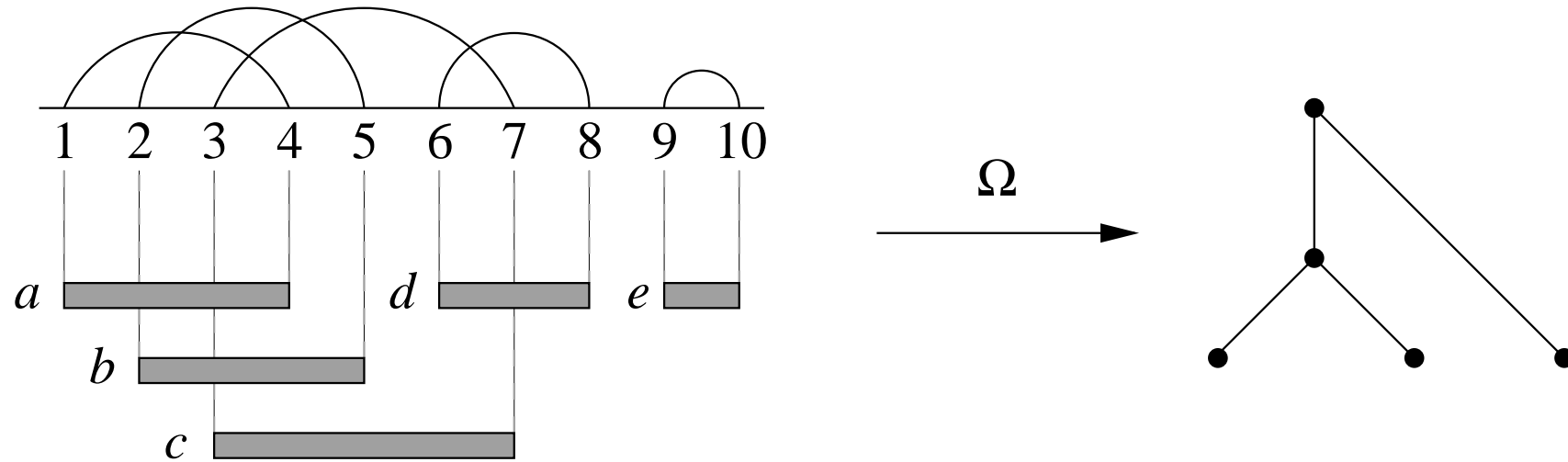
First bijection:

From involutions with no neighbour nesting to interval orders



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From involutions with no neighbour nesting to interval orders



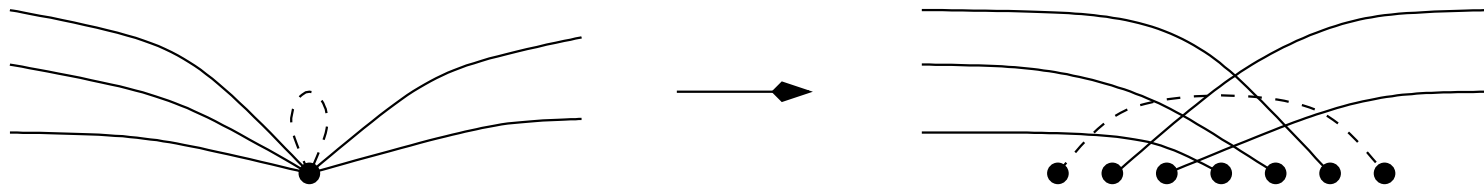
Proposition: Ω sends bijectively involutions on $2n$ points with no neighbour nesting onto interval orders on n unlabelled elements.

Remark. Reflecting σ across a vertical line \iff Taking the dual poset.

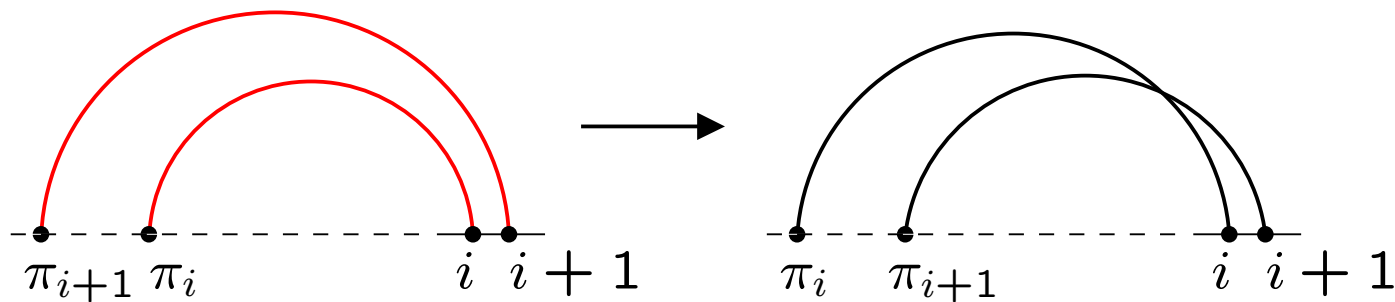
First bijection:

From involutions with no neighbour nesting to interval orders

- Ω is surjective. Let P be an interval order on n elements.
 - Take a collection of n intervals whose order is P
 - One can assume that the endpoints of these intervals are $2n$ distinct points



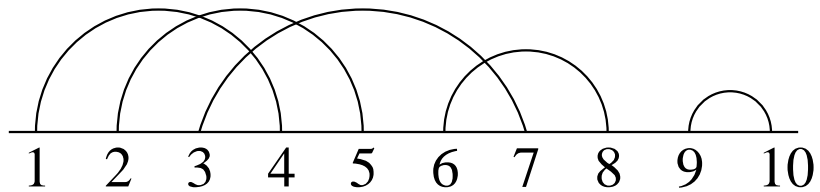
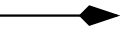
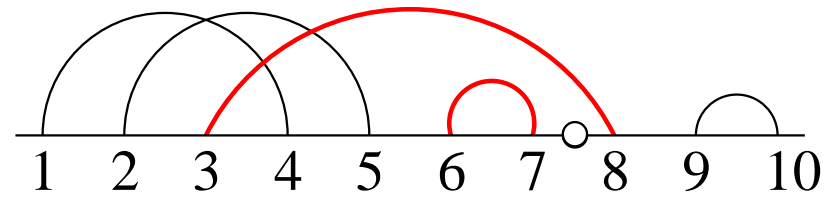
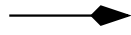
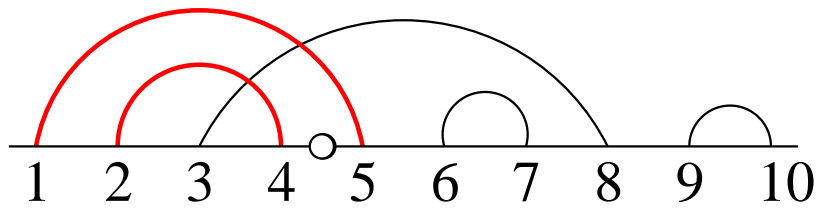
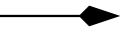
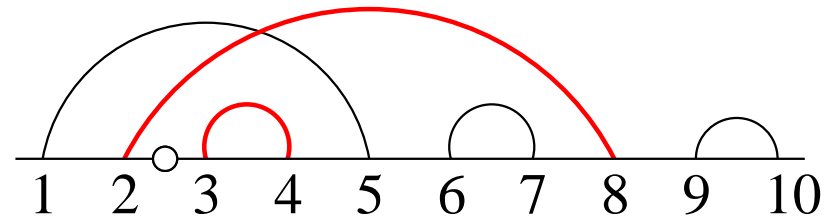
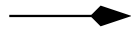
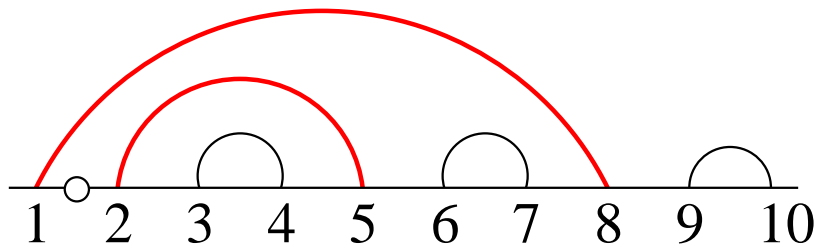
- One can assume that these points are $1, 2, \dots, 2n \Rightarrow$ a fixed point free involution
- Delete neighbour nestings without changing the order



(and symmetrically)

Surjectivity: an example

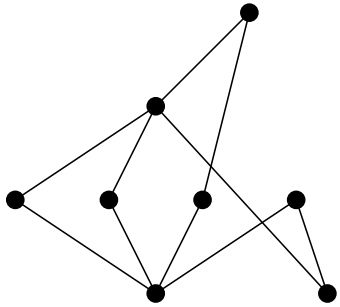
Collection of intervals: $[1, 8]$, $[2, 5]$, $[3, 4]$, $[6, 7]$, $[9, 10]$



No neighbour nesting

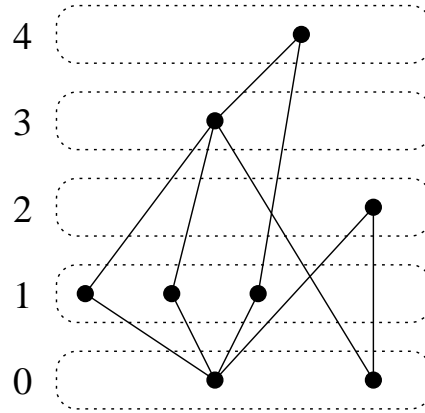
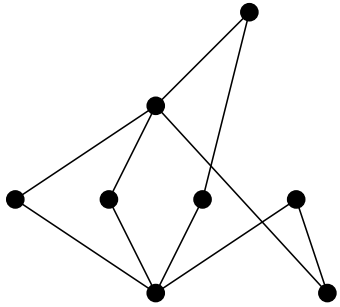
From involutions with no neighbour nesting to interval orders

- Ω is injective. Let $\sigma \in \mathcal{I}_{2n}$, and let $P = \Omega(\sigma)$. How can we reconstruct σ from P ?



From involutions with no neighbour nesting to interval orders

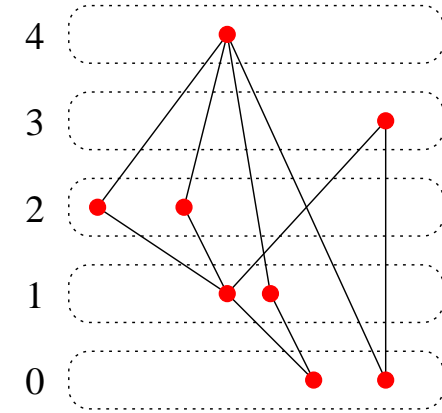
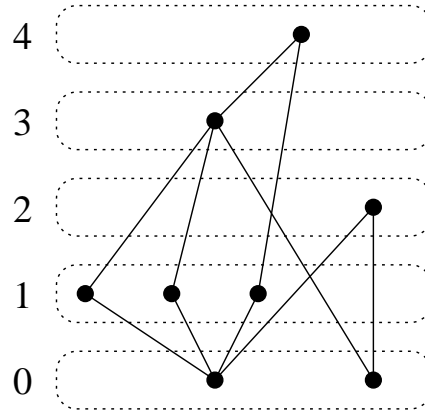
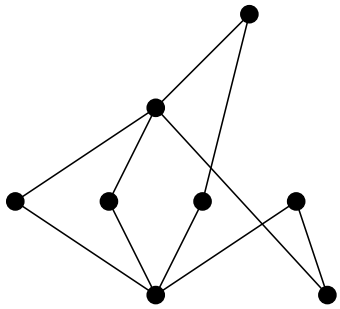
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- Display P in **canonical form**: if $\{D(x), x \in P\} = \{D_0, D_1, \dots, D_k\}$ with $\emptyset = D_0 \subset D_1 \subset \dots \subset D_k$, put x at level i if $D(x) = D_i$.

From involutions with no neighbour nesting to interval orders

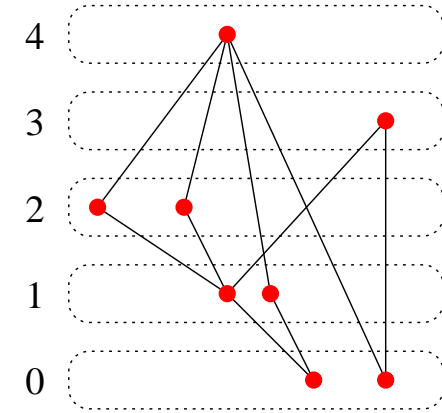
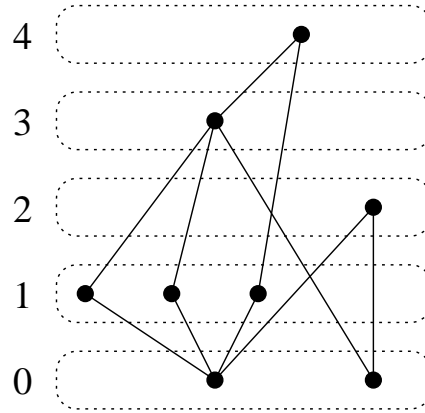
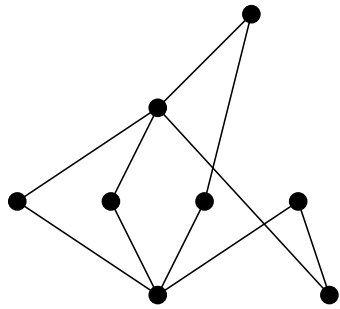
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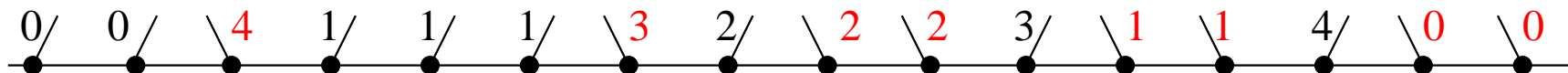
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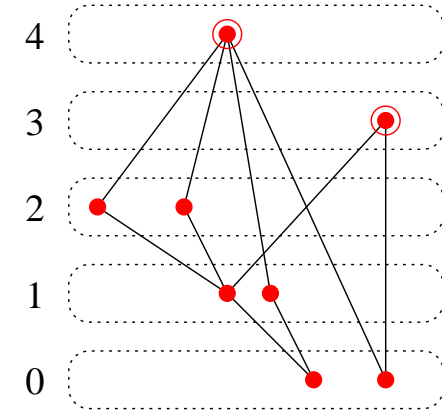
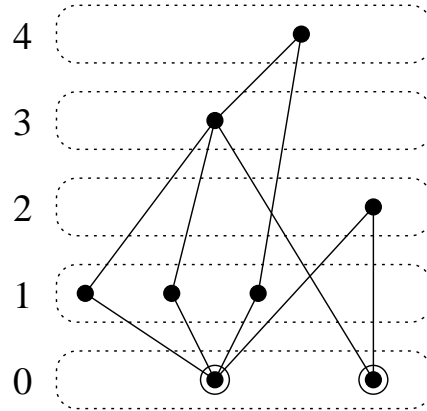
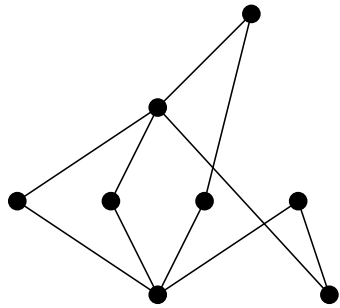


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- Display the dual poset P^* in canonical form
- Form runs of opening/**closing** arches by reading the levels in P (resp. P^*)

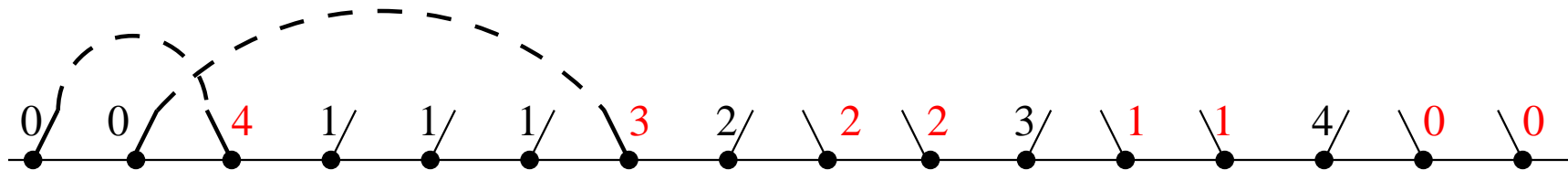


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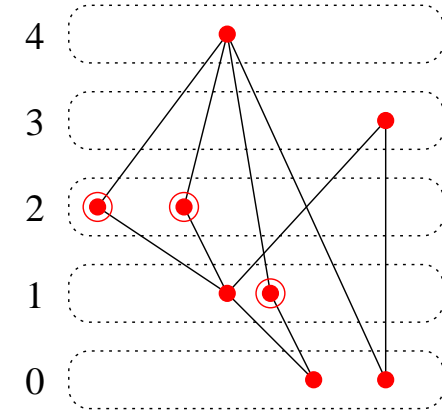
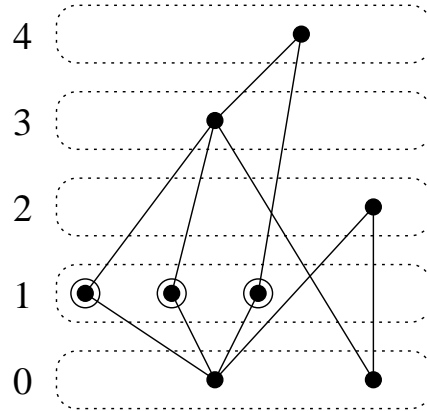
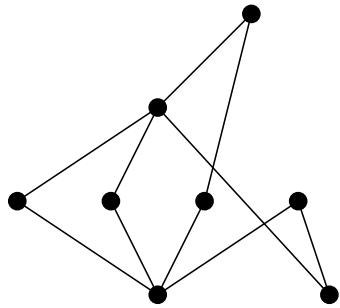
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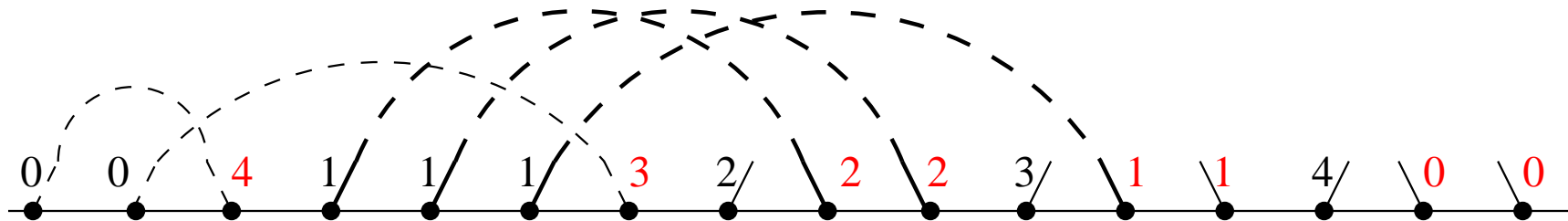
- Match half-arches by identifying points in P and P^*

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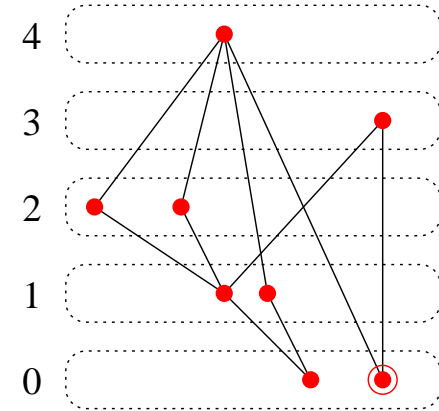
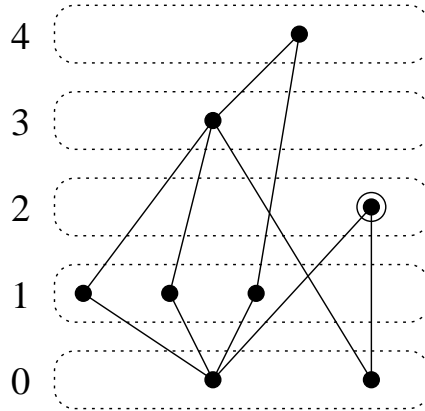
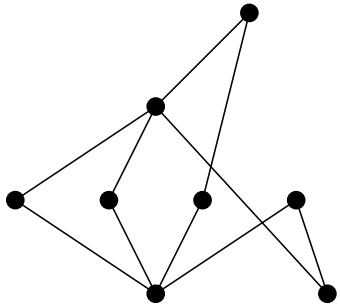
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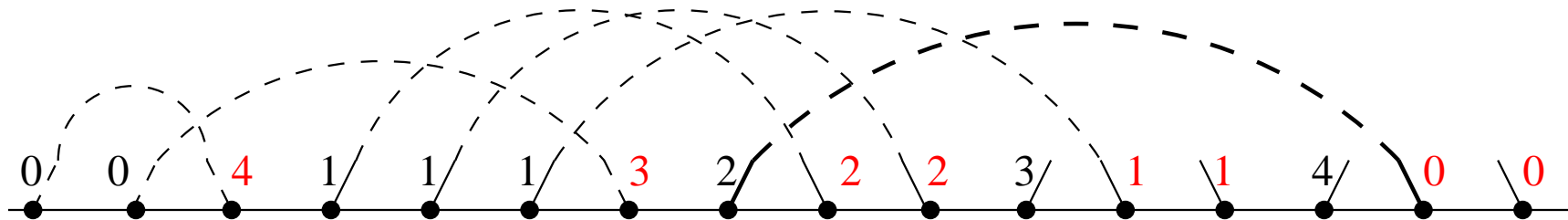
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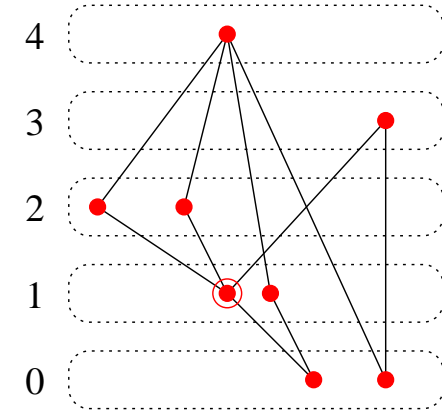
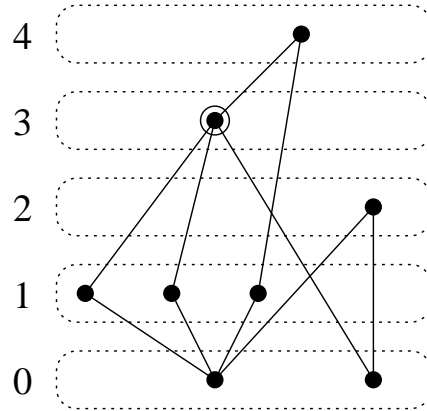
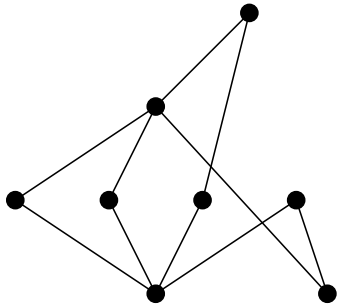
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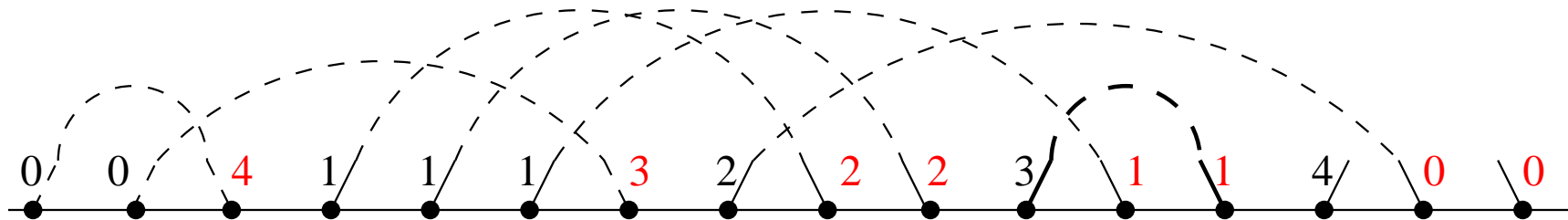
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From involutions with no neighbour nesting to interval orders

- Ω is injective. Let $\sigma \in \mathcal{I}_{2n}$, and let $P = \Omega(\sigma)$. How can we reconstruct σ from P ?



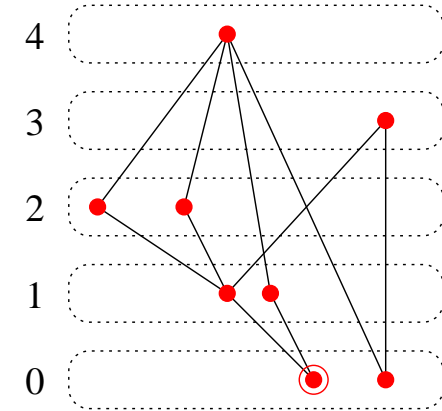
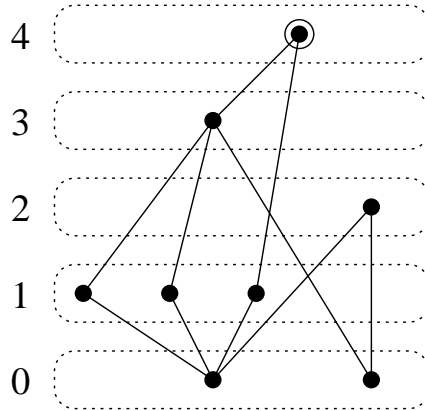
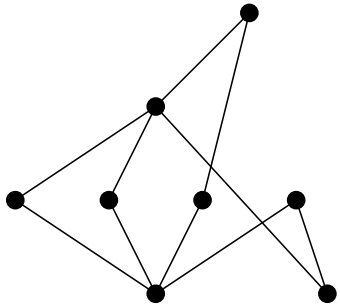
- Display P in **canonical form**: if $\{D(x), x \in P\} = \{D_0, D_1, \dots, D_k\}$ with $\emptyset = D_0 \subset D_1 \subset \dots \subset D_k$, put x at level i if $D(x) = D_i$.
- Display the dual poset P^* in canonical form
- Form runs of opening/**closing** arches by reading the levels in P (resp. P^*)



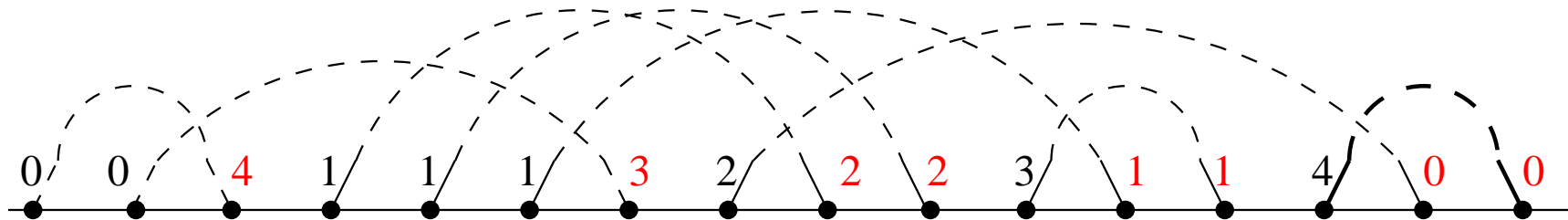
- Match half-arches by identifying points in P and P^*

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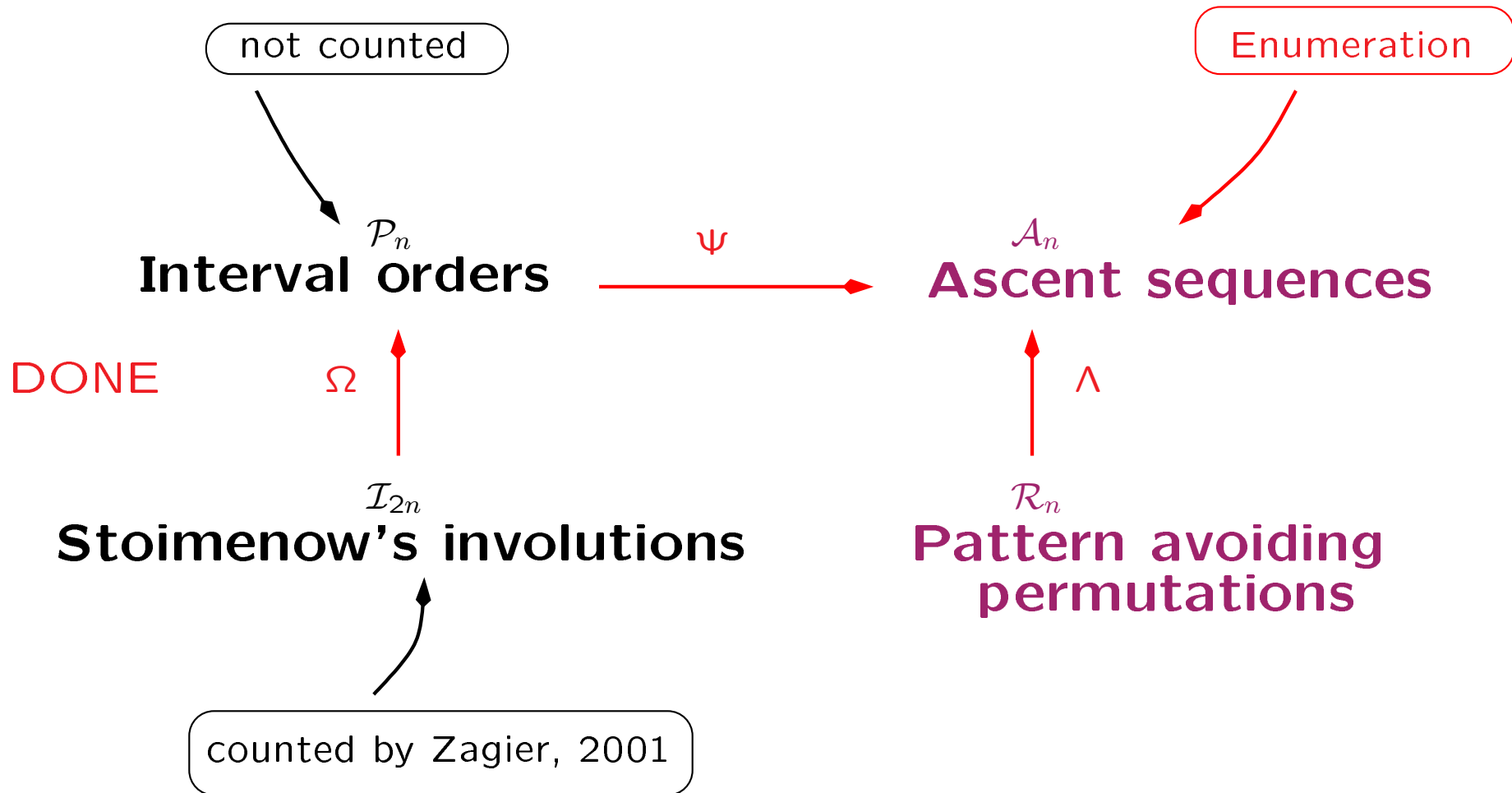


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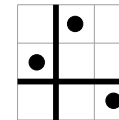


- Match half-arches by identifying points in P and P^*

Where we stand



III. From permutations avoiding
to ascent sequences



Decomposition of permutations of \mathcal{R}_n

- Consider $\pi = 61832547 \in \mathcal{R}_8$ and remove the largest entry:

$$\tau = 6132547 \in \mathcal{R}_7$$

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- Conversely, 8 could be inserted in the following **active sites** of τ :

$$\tau = \star 6 1 \star 3 2 \star 5 4 \star 7 \star$$

- **Active sites:**
 - to the left of τ ,
 - just after the entry i , provided $i - 1$ does not appear to the right of i .

Decomposition of permutations of \mathcal{R}_n

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- Labelling of the active sites:

$$0 6 1_1 3 2_2 5 4_3 7_4$$

The 8 appears in the active site with label 1

$\Rightarrow \pi \in \mathcal{R}_8$ may be recovered from the pair $(\tau, 1)$ where $\tau \in \mathcal{R}_7$.

Decomposition of permutations of \mathcal{R}_n

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$\Rightarrow \pi \in \mathcal{R}_8$ may be recovered from the pair $(\tau, 1)$ where $\tau \in \mathcal{R}_7$.

- Repeating this until ending up with the empty permutation, we have the sequence

$$\Lambda(\pi) = (0, 1, 1, 2, 2, 0, 3, 1)$$

from which π can be reconstructed.

Ascent sequences

Definition. A sequence (x_1, \dots, x_n) of non-negative integers is an **ascent sequence** if

- $x_1 = 0$, and
- $x_i \in [0, 1 + \text{asc}(x_1, \dots, x_{i-1})]$ for all $1 < i \leq n$.

Ascent number: $\text{asc}(0, 0, 1, 0, 1, 2, 0) = 3$

Examples:

All ascent sequences of length 3:

$(0, 0, 0)$ $(0, 0, 1)$ $(0, 1, 0)$ $(0, 1, 1)$ $(0, 1, 2)$

All ascent sequences of length 4:

$(0, 0, 0, 0)$ $(0, 0, 0, 1)$ $(0, 0, 1, 0)$ $(0, 0, 1, 1)$ $(0, 0, 1, 2)$ $(0, 1, 0, 0)$ $(0, 1, 0, 1)$
 $(0, 1, 0, 2)$ $(0, 1, 1, 0)$ $(0, 1, 1, 1)$ $(0, 1, 1, 2)$ $(0, 1, 2, 0)$ $(0, 1, 2, 1)$ $(0, 1, 2, 2)$
 $(0, 1, 2, 3)$

Second bijection:

from permutations avoiding  to ascent sequences

- Definition of Λ : if $\pi \in \mathcal{R}_n$ is obtained by inserting n in the active site labelled i of $\tau \in \mathcal{R}_{n-1}$ then

$$\Lambda(\pi) = (x_1, x_2, \dots, x_{n-1}, i) \quad \text{where} \quad \Lambda(\tau) = (x_1, x_2, \dots, x_{n-1}).$$

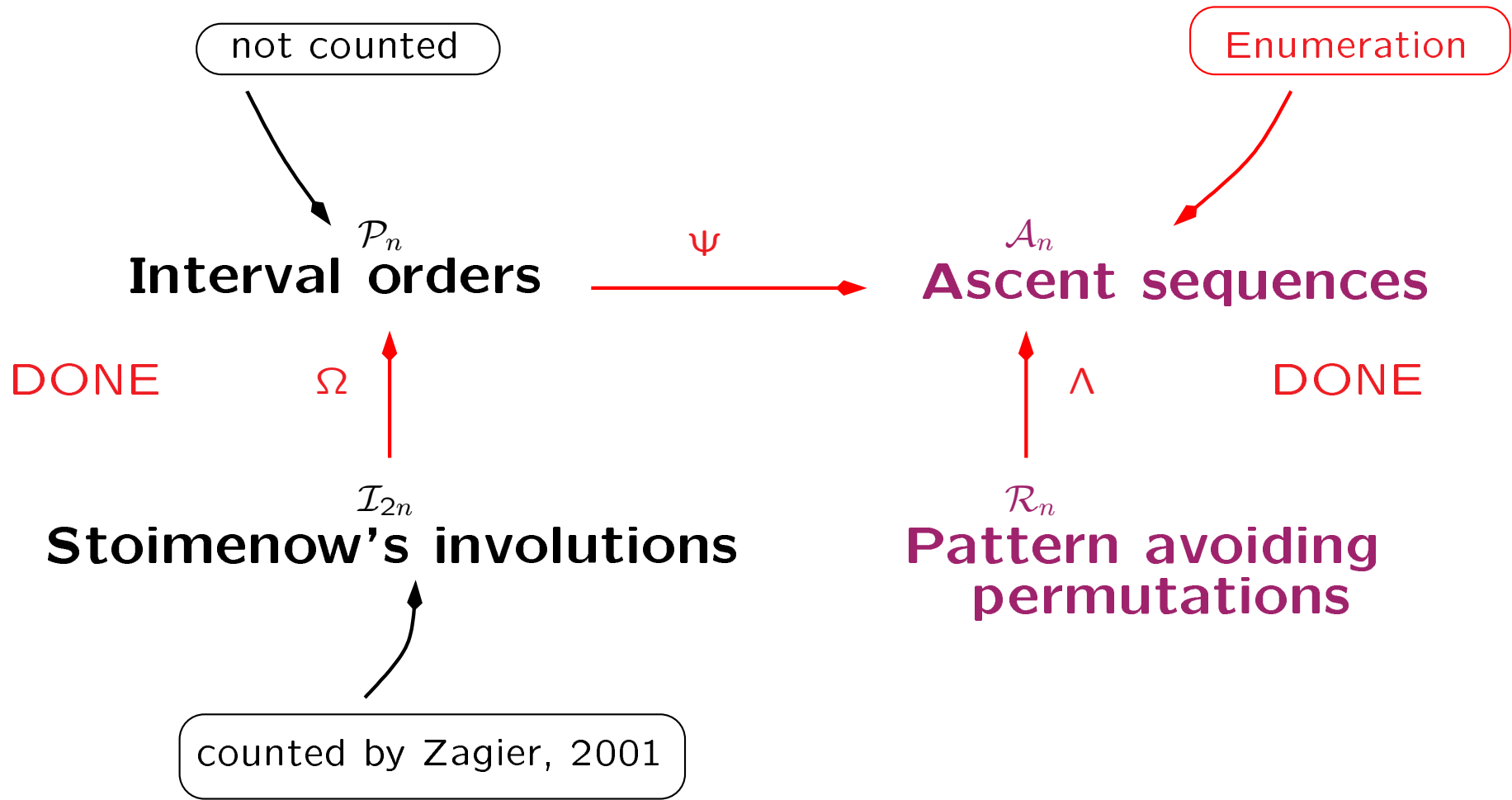
Theorem: The map Λ is a bijection between permutations of \mathcal{R}_n and ascent sequences of length n .

- Proof: by induction on n , using

$$s(\pi) = 2 + \text{asc}(x) \quad \text{and} \quad b(\pi) = x_n$$

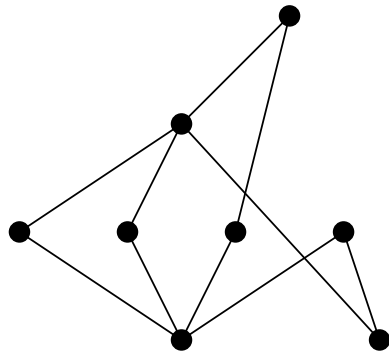
where $s(\pi)$ is the number of active sites, and $b(\pi)$ the label of the active site located just before the largest entry of π .

Where we stand

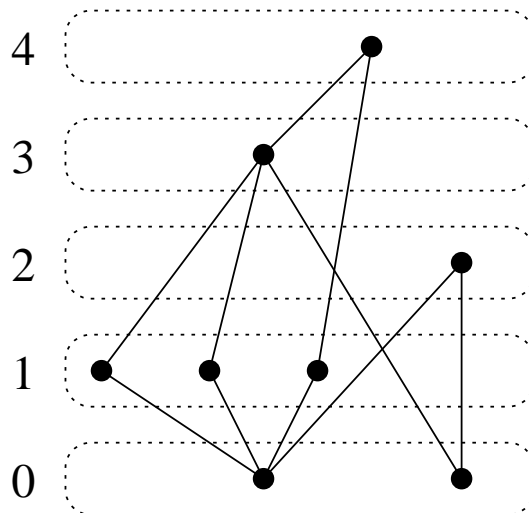


IV. From interval orders to ascent sequences

Third bijection: from interval orders to ascent sequences



Display P in **canonical form**: if $\{D(x), x \in P\} = \{D_0, D_1, \dots, D_k\}$ with $\emptyset = D_0 \subset D_1 \subset \dots \subset D_k$, put x at level i if $D(x) = D_i$. We denote $\ell(P) = k$.



Decomposition of $(2 + 2)$ -free posets

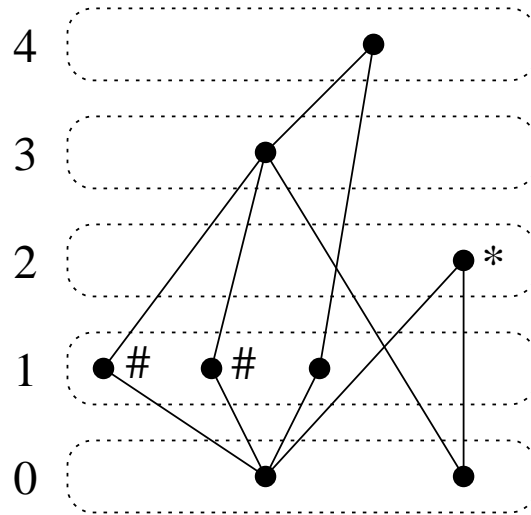
Let $\ell^*(P)$ be the minimum level containing a maximal element. Let $L_i(P)$ be the set of elements of P at level i . All the maximal elements located at level i are order-equivalent in P . We will remove one of them. Let Q be the poset that results from applying:

- (i) If $|L_i(P)| > 1$, or if $|L_i(P)| = 1$ and $i = \ell(P)$, then remove one of the maximal elements at level i .
- (ii) If $|L_i(P)| = 1$ and $i < \ell(P)$ then set $\mathcal{N} = D_{i+1}(P) \setminus D_i(P)$. Make each element in \mathcal{N} a maximal element of the poset by deleting from the order all relations $x < y$ where $x \in \mathcal{N}$. Finally, remove the unique element lying at level i .

- Definition of the map Ψ between posets and ascent sequences:

$$\Psi(P) = (x_1, \dots, x_{n-1}, i) \quad \text{where} \quad \Psi(Q) = (x_1, \dots, x_{n-1})$$

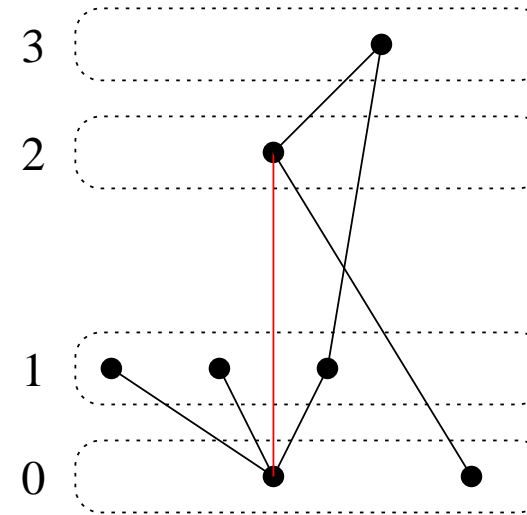
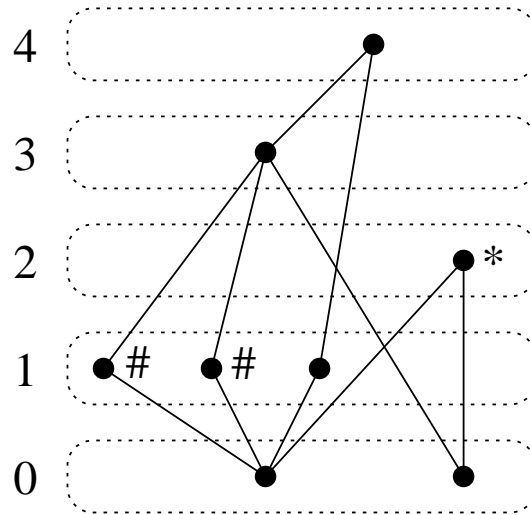
Decomposition of $(2 + 2)$ -free posets



$$\ell(P) = 4, \quad i = 2 < \ell(P) \quad \Rightarrow x_8 = 2$$

- (ii) If $|L_i(P)| = 1$ and $i < \ell(P)$ then set $\mathcal{N} = D_{i+1}(P) \setminus D_i(P)$. Make each element in \mathcal{N} a maximal element of the poset by deleting from the order all relations $x < y$ where $x \in \mathcal{N}$. Finally, remove the unique element lying at level i .

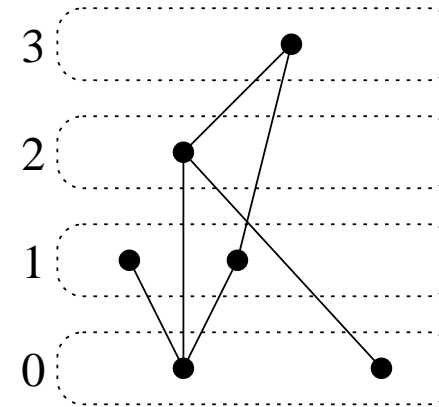
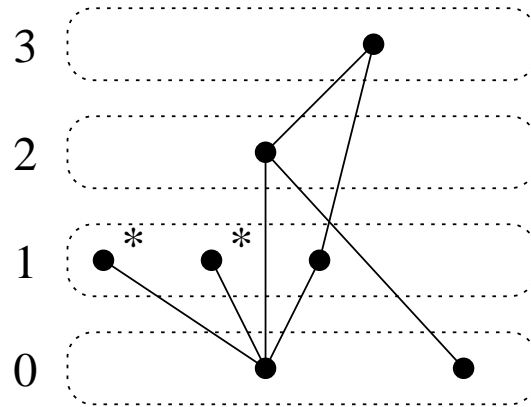
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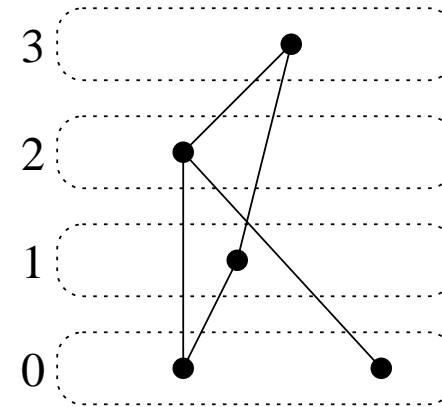
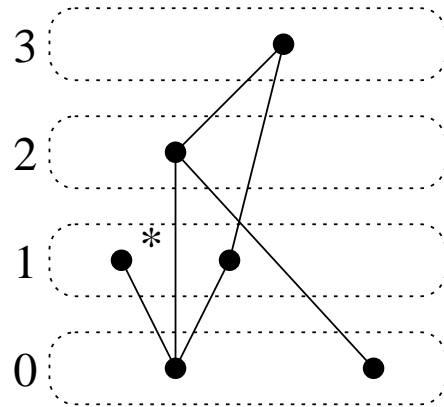
Decomposition of $(2 + 2)$ -free posets



$$\ell(P) = 3, \quad i = 1 < \ell(P) \quad \Rightarrow \quad x_7 = 1$$

- (i) If $|L_i(P)| > 1$, or if $|L_i(P)| = 1$ and $i = \ell(P)$, then remove one of the maximal elements at level i .

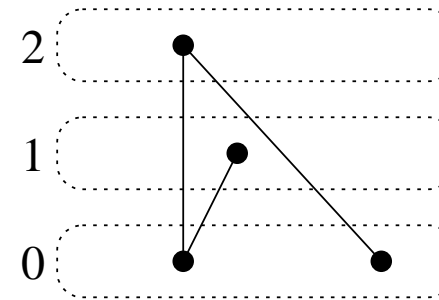
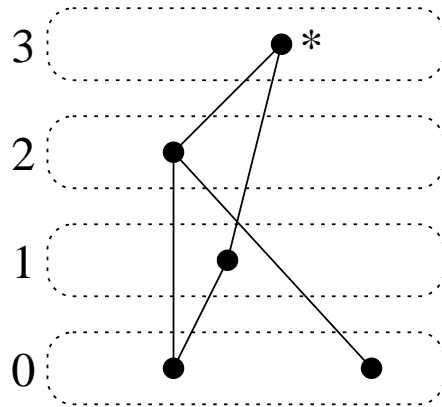
Decomposition of $(2 + 2)$ -free posets



$$\ell(P) = 3, \quad i = 1 < \ell(P) \quad \Rightarrow x_6 = 1$$

- (i) If $|L_i(P)| > 1$, or if $|L_i(P)| = 1$ and $i = \ell(P)$, then remove one of the maximal elements at level i .

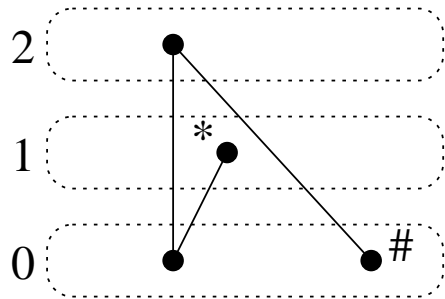
Decomposition of $(2 + 2)$ -free posets



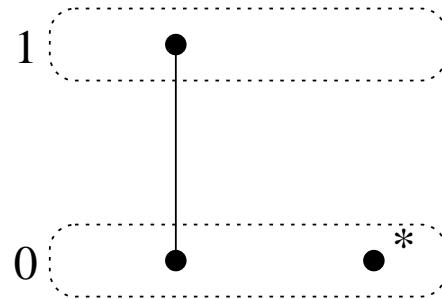
$$\ell(P) = 3, \quad i = 3 = \ell(P) \quad \Rightarrow \quad x_5 = 3$$

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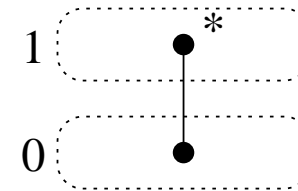
Third bijection: from interval orders to ascent sequences



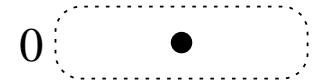
$$x_4 = 1$$



$$x_3 = 0$$



$$x_2 = 1$$



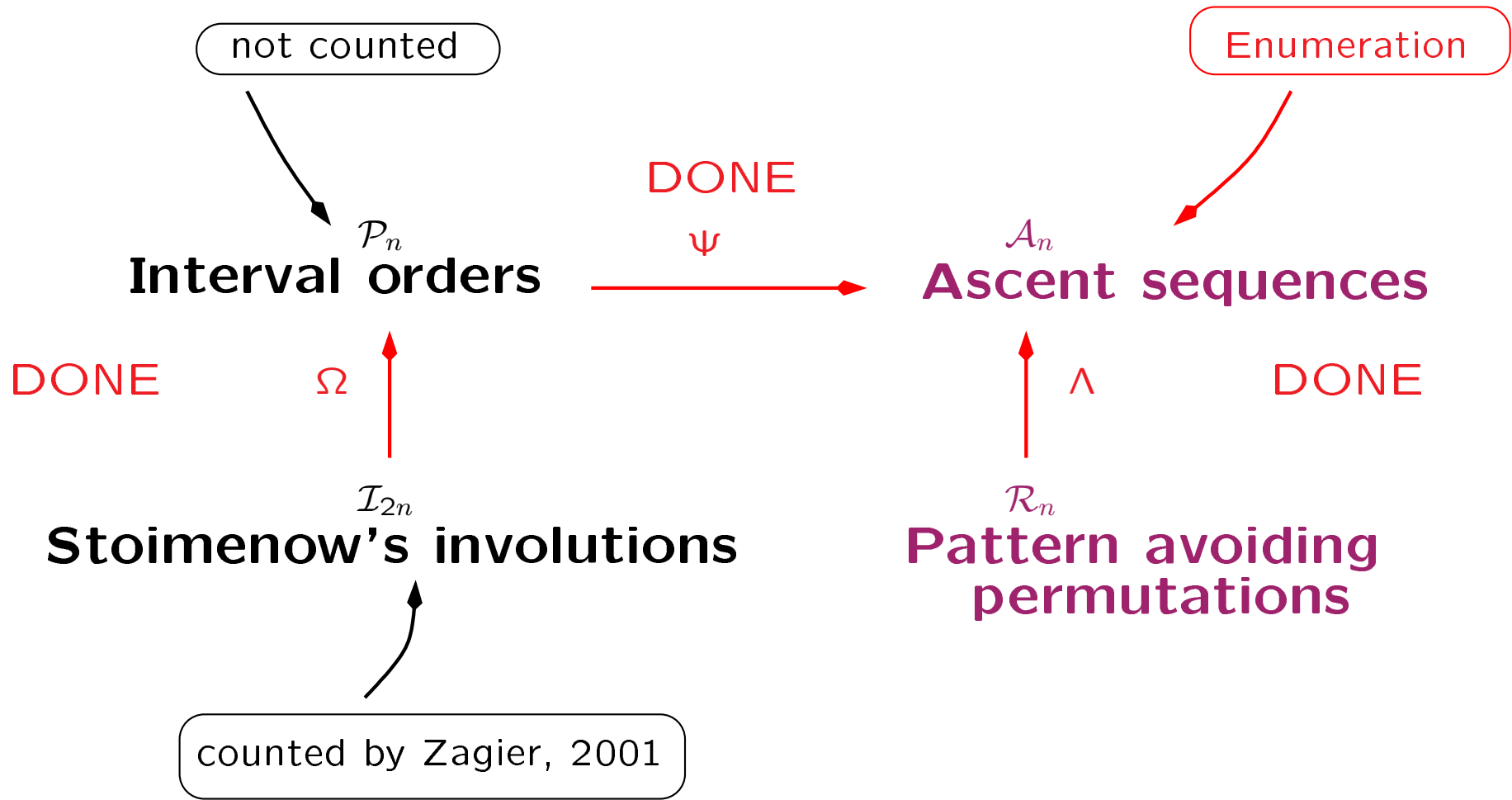
$$x_1 = 0$$

$$\Psi(P) = (0, 1, 0, 1, 3, 1, 1, 2)$$

Theorem: The map Ψ is a bijection between $(2 + 2)$ -free posets on n elements and ascent sequences of length n .

Remark: other decompositions given by [El-Zahar 89], [Khamis 04], [Haxel, McDonald, Thomasson 87]

Where we stand



Some properties of our bijections

Given an ascent sequence x , let P and π be the poset and permutation corresponding to x , and let σ be the involution corresponding to P . Then

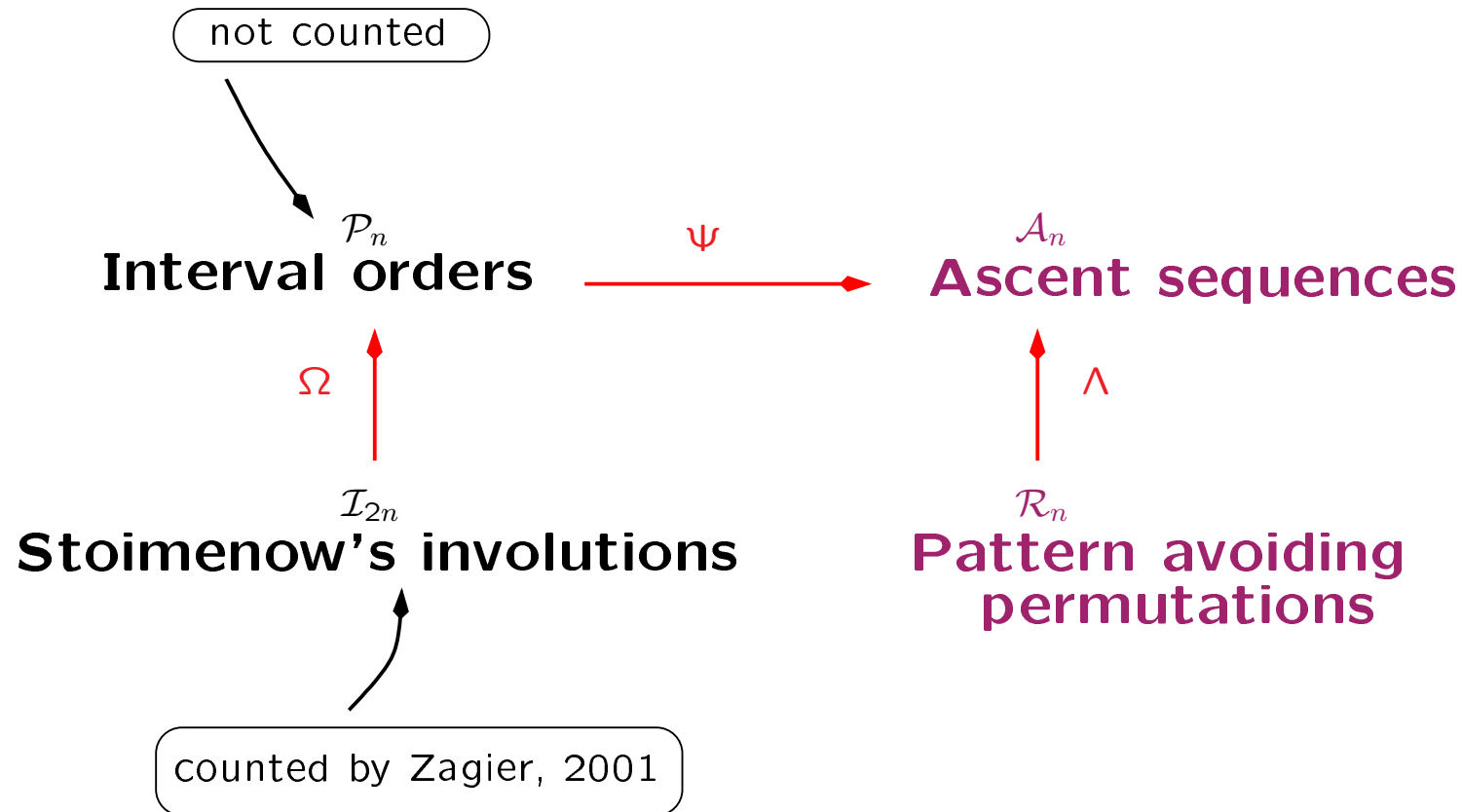
$$\begin{aligned} \text{lir}(\sigma) &= \min(P) = \text{zeros}(x) = \text{ldr}(\pi) \\ \ell^*(P) &= \text{last}(x) = b(\pi) \\ \text{or}(\sigma) - 1 &= \ell(P) = \text{asc}(x) = \text{asc}(\pi^{-1}) = s(\pi) - 2 \end{aligned}$$

with

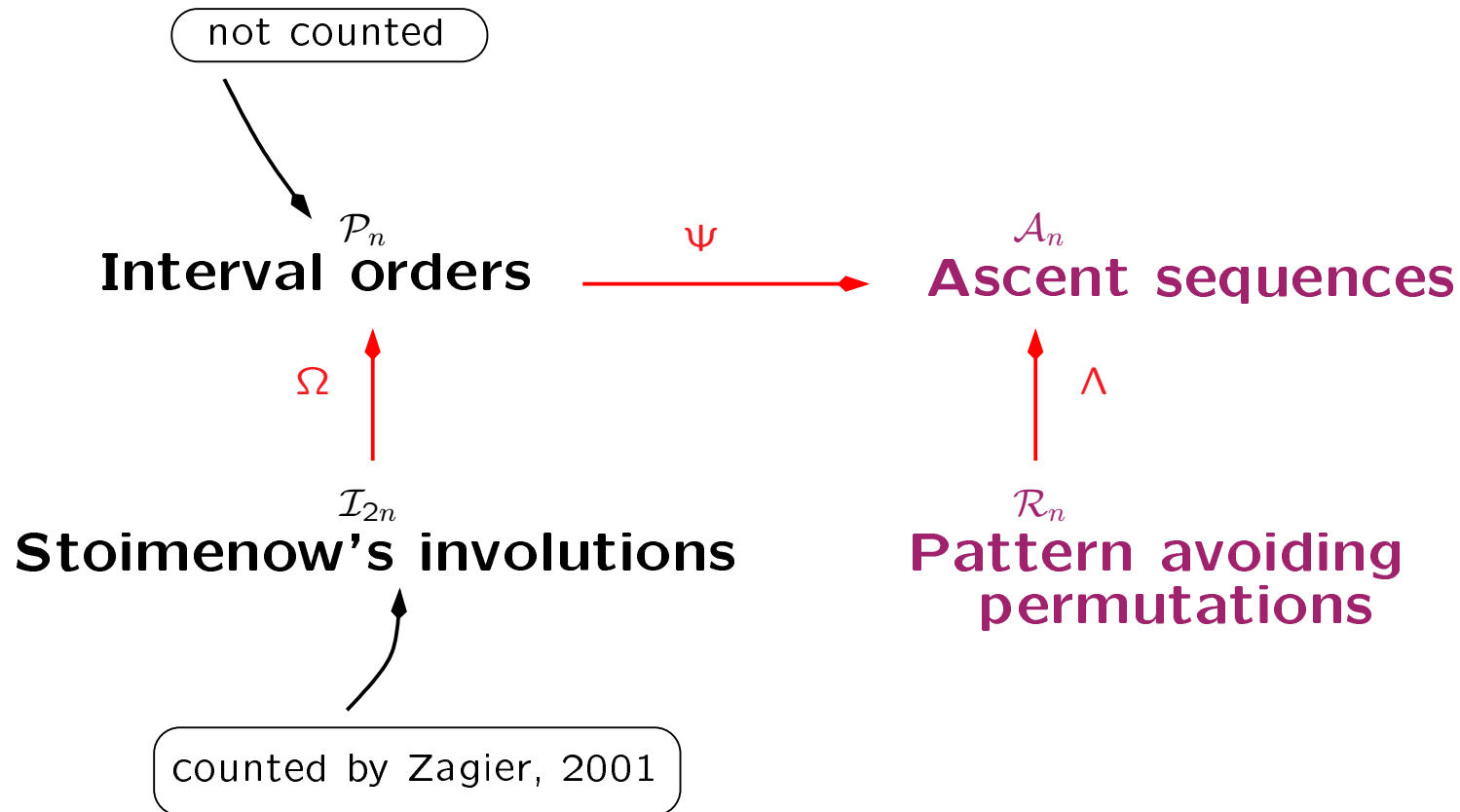
- $\text{lir}(\pi)$: leftmost increasing run (the largest integer i such that $\pi_1 < \pi_2 < \dots < \pi_i$)
- $\text{ldr}(\pi)$: leftmost decreasing run
- $\text{or}(\sigma)$: number of opening runs of σ
- $s(\pi)$: number of active sites
- $b(\pi)$: label of the active site located just before the largest entry of π .

V. Enumeration

Which is the most manageable class of objects?



Which is the most manageable class of objects?



Ascent sequences

Recursive description of ascent sequences

The only ascent sequence of length 1 is (0).

For $n \geq 1$, the sequence (x_1, \dots, x_n) is an ascent sequence iff

- (x_1, \dots, x_{n-1}) is an ascent sequence, and
- $x_n \in [0, 1 + \text{asc}(x_1, \dots, x_{n-1})]$.

\Rightarrow Obvious recursive description, keeping track of

- the number of ascents,
- the value of the last entry.

A functional equation

Let $F(t; u, v) \equiv F(u, v)$ be the generating function of ascent sequences, counted by length (t), number of ascents (u) and last entry (v). This is a formal power series in t with coefficients in $\mathbb{Q}[u, v]$. The first few terms of $F(t; u, v)$ are

$$F(t; u, v) = 1 + t + (1 + uv)t^2 + (1 + 2uv + u + u^2v^2)t^3 + O(t^4).$$

Proposition: The generating function $F(t; u, v)$ satisfies

$$(v - 1 - tv(1 - u)) F(u, v) = (v - 1)(1 - tuv) - tF(u, 1) + tuv^2F(uv, 1)$$

Remark: one unknown series only on the right-hand side

First expression of the solution

$$(v - 1 - tv(1 - u)) F(u, v) = (v - 1)(1 - tuv) - tF(u, 1) + tuv^2 F(uv, 1).$$

The **kernel** vanishes when $v = V(u)$, with $V(u) = 1/(1 - t + tu)$. Replacing v by $V(u)$ in the equation results in:

$$F(u, 1) = \frac{(1 - u)(1 - t)}{(1 - t + tu)^2} + \frac{u}{(1 - t + tu)^2} F\left(\frac{u}{1 - t + tu}, 1\right).$$

Iterating this equation gives:

Proposition: The series $F(t; u, 1)$ counting ascent sequences by their length and ascent number, seen as a series in u , has rational coefficients in t , and satisfies

$$F(t; u, 1) = \sum_{k \geq 1} \frac{(1 - u) u^{k-1} (1 - t)^k}{(u - (u - 1)(1 - t)^k) \prod_{i=1}^k (u - (u - 1)(1 - t)^i)}.$$

A formal series in u with rational coefficients in t !

Massaging the solution

$$F(t; u, 1) = \sum_{k \geq 1} \frac{(1-u) u^{k-1} (1-t)^k}{(u - (u-1)(1-t)^k) \prod_{i=1}^k (u - (u-1)(1-t)^i)}.$$

Lemma. For $m \geq 1$, let

$$S(t; u) = \sum_{k \geq 1} \frac{(u-1)^m u^{k-1} (1-t)^{mk}}{\prod_{i=1}^k (u - (u-1)(1-t)^i)}.$$

Then $S(t; u)$ is actually a **polynomial** in u and t :

$$S(t; u) = - \sum_{j=0}^{m-1} (u-1)^j u^{m-1-j} (1-t)^j \prod_{i=j+1}^{m-1} (1 - (1-t)^i).$$

Second expression of the solution

- **Theorem.** Let $n \geq 0$, and consider the following polynomial in t and u :

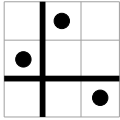
$$F_n(t; u) = \sum_{\ell=0}^n (u-1)^{n-\ell} u^\ell \sum_{m=\ell}^n (-1)^{n-m} \binom{n}{m} (1-t)^{m-\ell} \prod_{i=m-\ell+1}^m (1 - (1-t)^i).$$

Then $F_n(t; u)$ is a **multiple of t^n** .

The generating function of ascent sequences, counted by the length and the ascent number, is

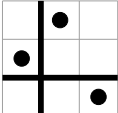
$$F(t; u, 1) = \sum_{n \geq 0} F_n(t; u).$$

- When $u = 1$, $F_n(t; 1) = \prod_{i=1}^n (1 - (1-t)^i)$, so that the generating function of

ascent sequences, $(2+2)$ -free posets, permutations avoiding , involutions with no neighbour nesting is:

$$F(t; 1, 1) = \sum_{n \geq 0} \prod_{i=1}^n (1 - (1-t)^i).$$

What a beautiful series!

The generating function of ascent sequences, $(2 + 2)$ -free posets, permutations avoiding , involutions with no neighbour nesting is:

$$P(t) = \sum_{n \geq 0} \prod_{i=1}^n (1 - (1-t)^i).$$

Obtained by Zagier for involutions with no neighbour nesting, using a decomposition of involutions found in [Stoimenow 98].

- Asymptotic behaviour:

$$\frac{p_n}{n!} \sim \kappa \left(\frac{6}{\pi^2}\right)^n \sqrt{n}, \quad \text{where} \quad \kappa = \frac{12\sqrt{3}}{\pi^{5/2}} e^{\pi^2/12}.$$

- A remarkable formula:

$$P(1 - e^{-24x}) = e^x \sum_{n \geq 0} \frac{T_n}{n!} x^n,$$

where

$$\sum_{n \geq 0} \frac{T_n}{(2n+1)!} x^{2n+1} = \frac{\sin 2x}{2 \cos 3x}.$$

[Zagier 01]