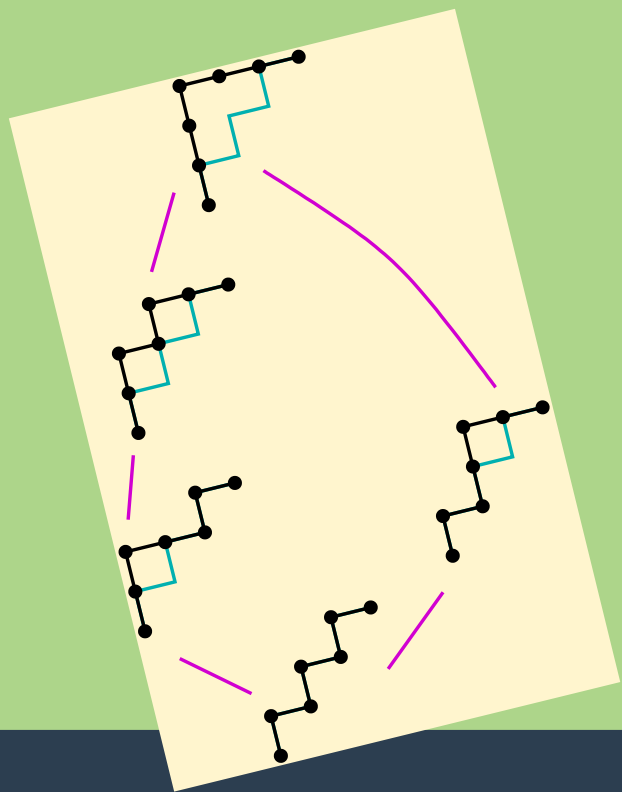


# The ascent lattice on Dyck paths

with Jean-Luc Baril, Sergey Kirgizov  
(LIB, Dijon) and Mehdi Naima (LIP6, Paris)



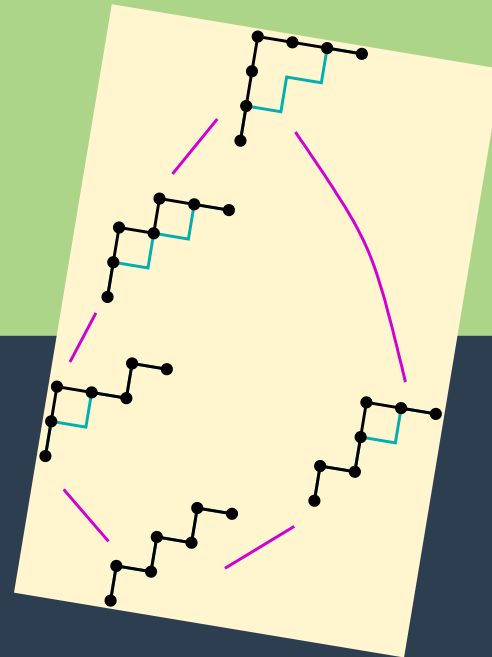
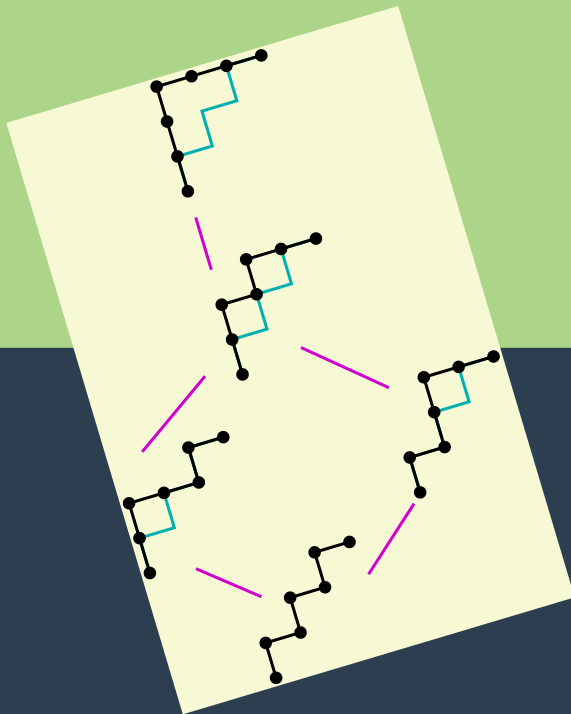
arXiv:2409.15982

Mireille Bousquet-Mélou  
CNRS, LaBRI, Université de Bordeaux

# Outline

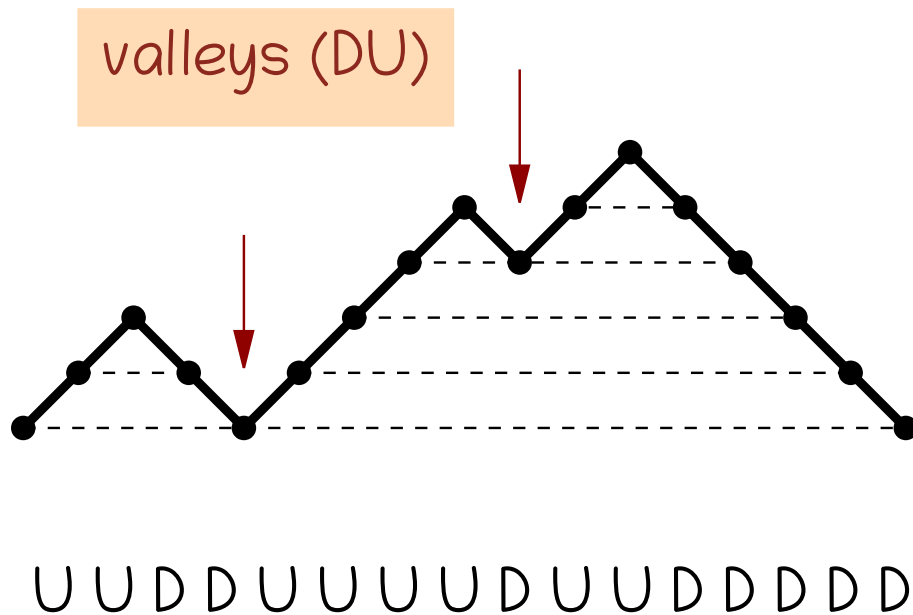
- Orders on Dyck paths --- The ascent order
- Lattice properties (the Nadeau-Tewari lattice)
- Counting intervals
- m-Dyck paths, and mirrored m-Dyck paths
- Connection with the sylvester congruence [Hivert, Novelli, Thibon 05]

# I. Orders on Dyck paths



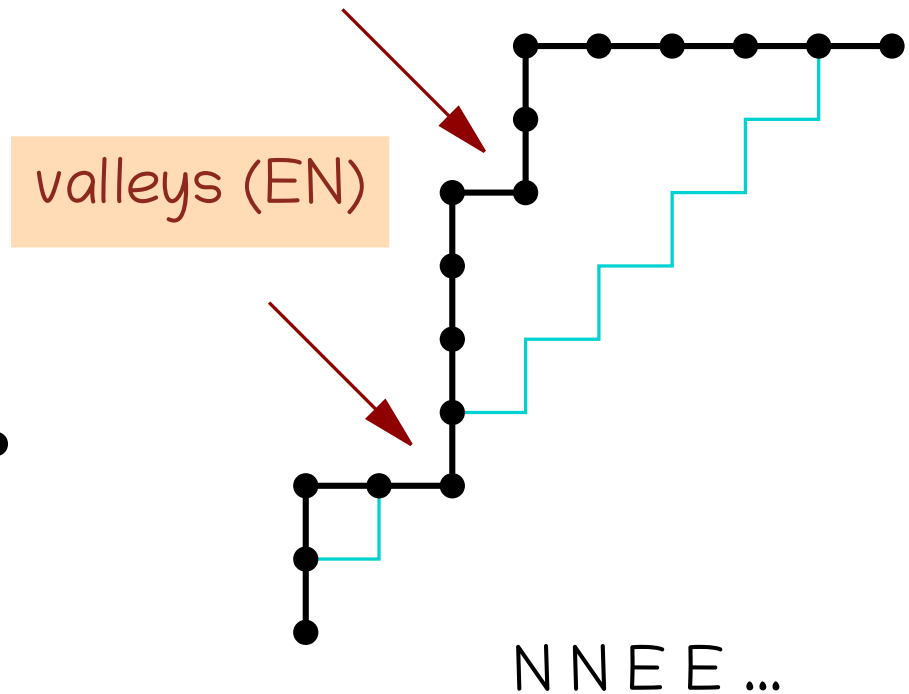
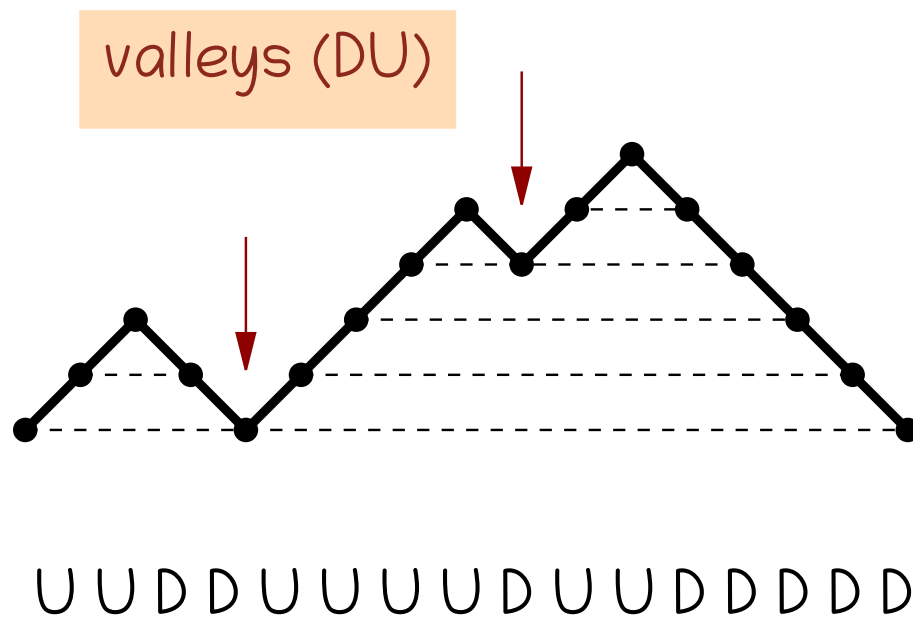
# Dyck paths

- A Dyck path of **size**  $n=8$  (size=number of up steps)



# Dyck paths

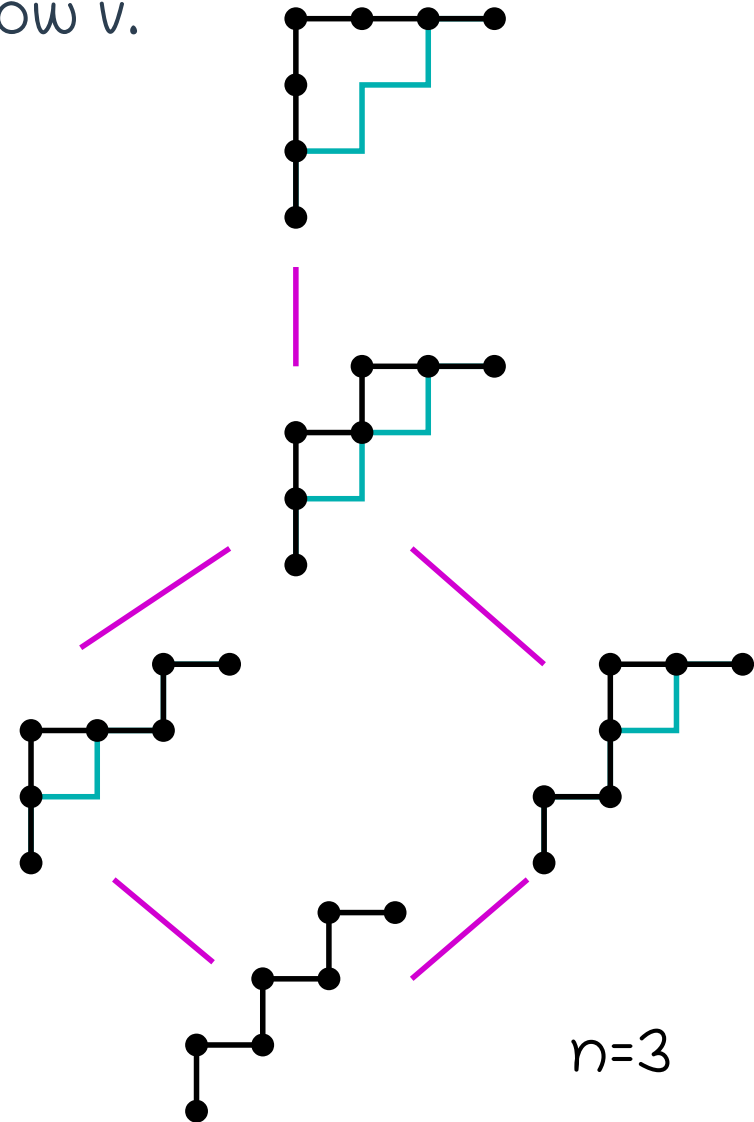
- A Dyck path of **size**  $n=8$  (size=number of up steps)



# Stanley's lattice $S_n$

A poset on Dyck paths of size  $n$

**Def.** The path  $u$  is smaller than  $v$  if  $u$  lies below  $v$ .

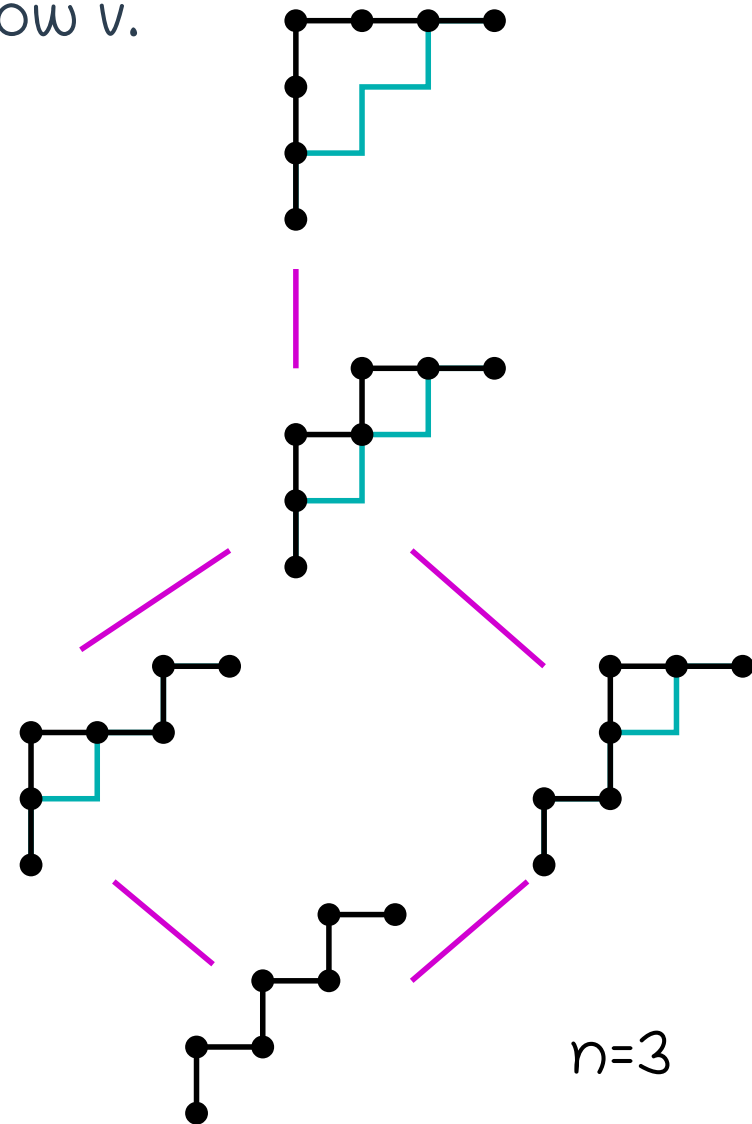


# Stanley's lattice $S_n$

## A poset on Dyck paths of size $n$

**Def.** The path  $u$  is smaller than  $v$  if  $u$  lies below  $v$ .

- Cover relations are given by  $EN \rightarrow NE$  (that is, valley  $\rightarrow$  peak)

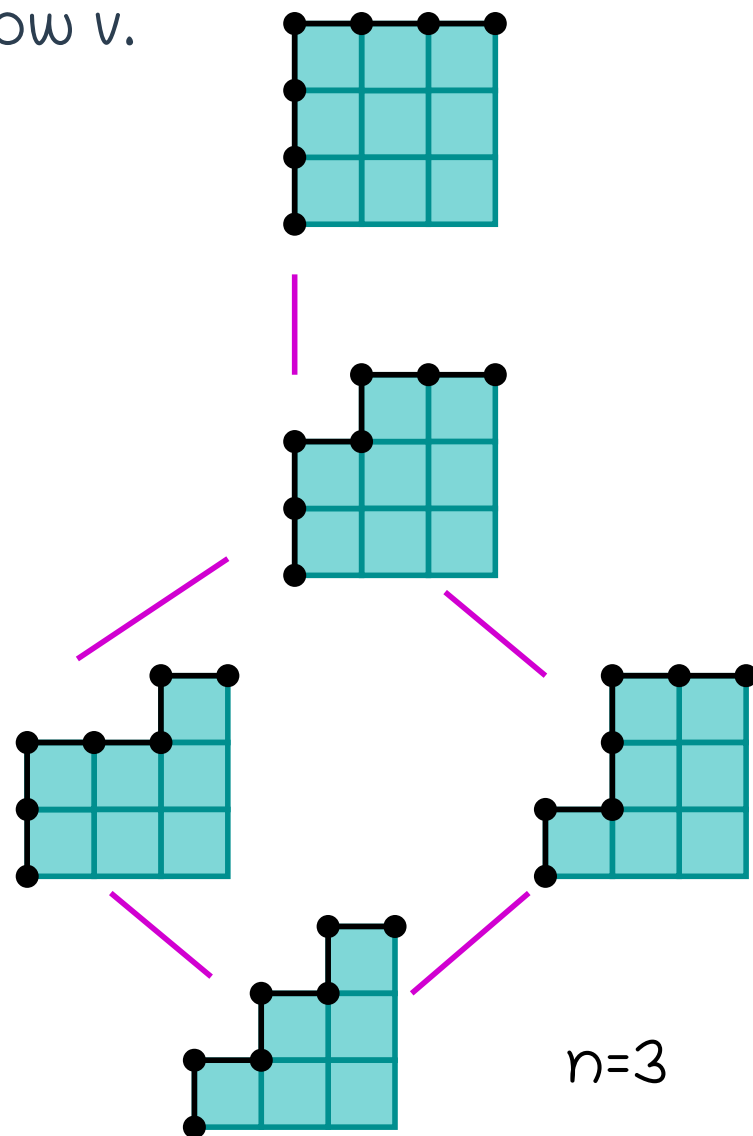


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 $EN \rightarrow NE$  (that is, valley  $\rightarrow$  peak)
- A sub-lattice of the Young lattice



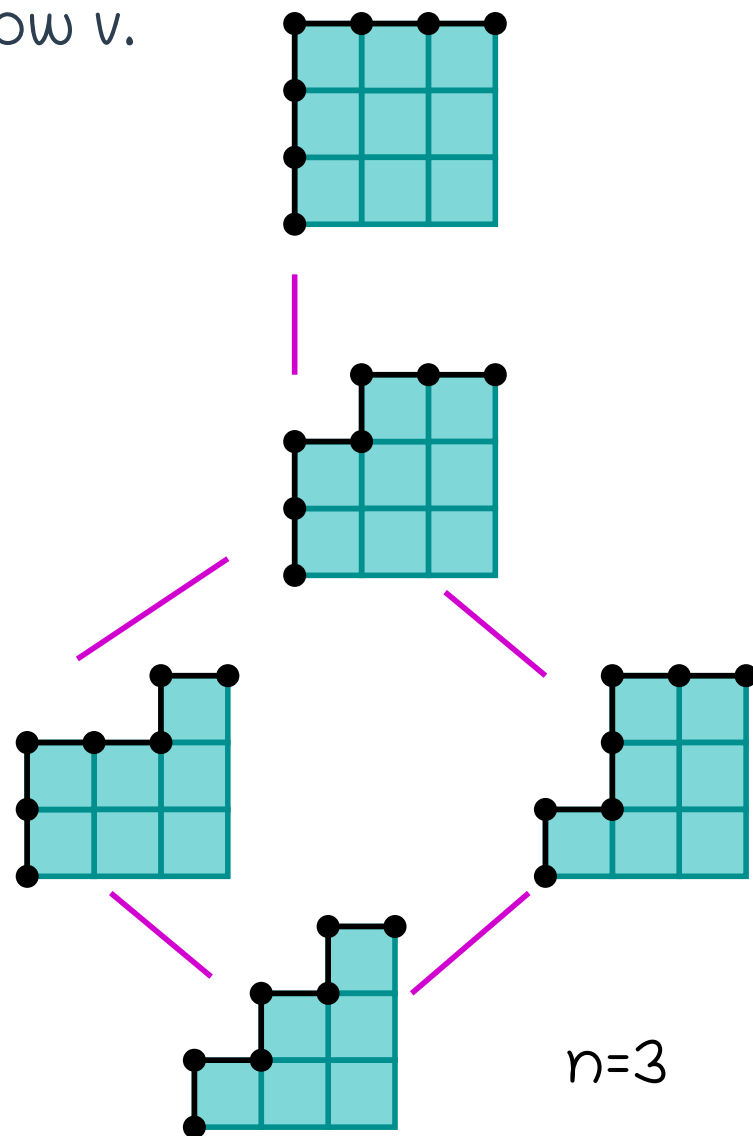


# Stanley's lattice $S_n$

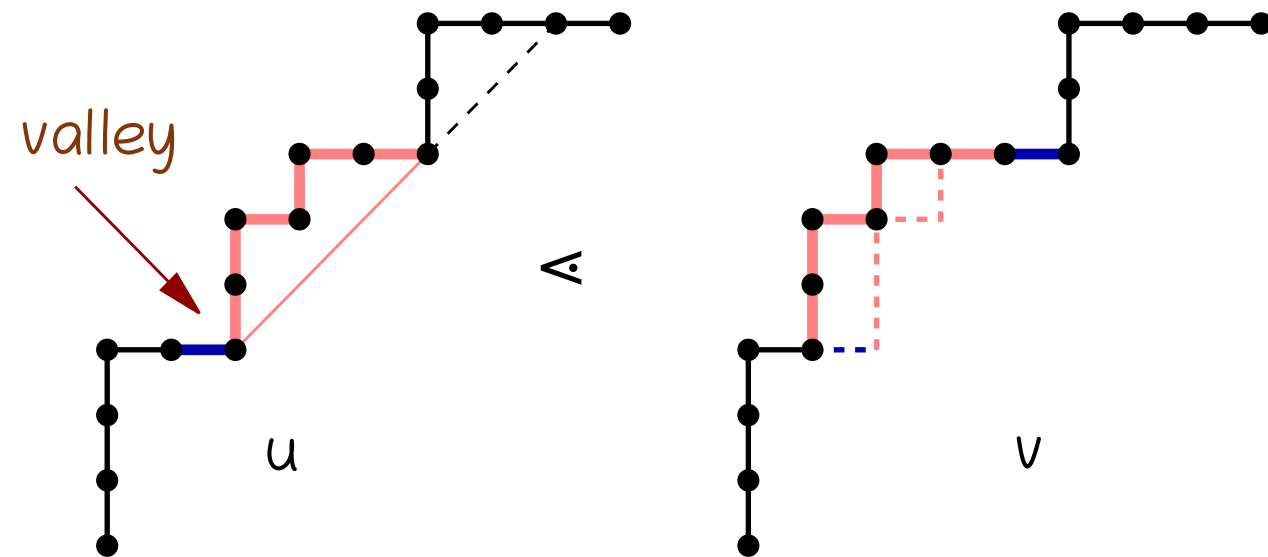
A poset on Dyck paths of size  $n$

**Def.** The path  $u$  is smaller than  $v$  if  $u$  lies below  $v$ .

- Cover relations are given by  
 $EN \rightarrow NE$  (that is, valley  $\rightarrow$  peak)
- A sub-lattice of the Young lattice
- Graded (by area)



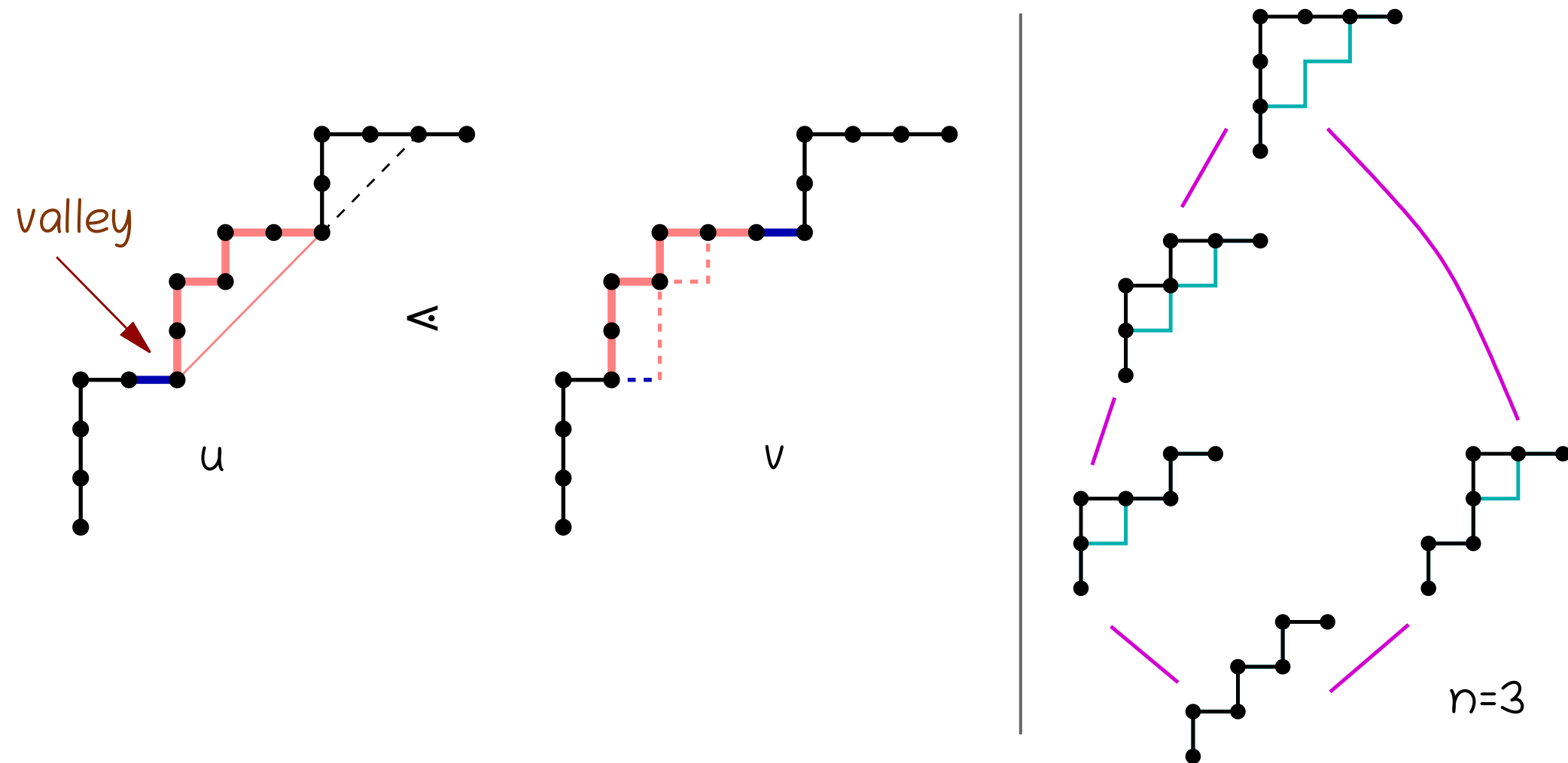
- A poset on Dyck paths with  $n$  up steps
- Cover relations: choose a valley in the path  $u$ .  
Swap the East step and the **shortest Dyck path** that follows.



# The Tamari lattice

[Tamari 5]

- A poset on Dyck paths with  $n$  up steps
- Cover relations: choose a valley in the path  $u$ .  
Swap the East step and the **shortest Dyck path** that follows.



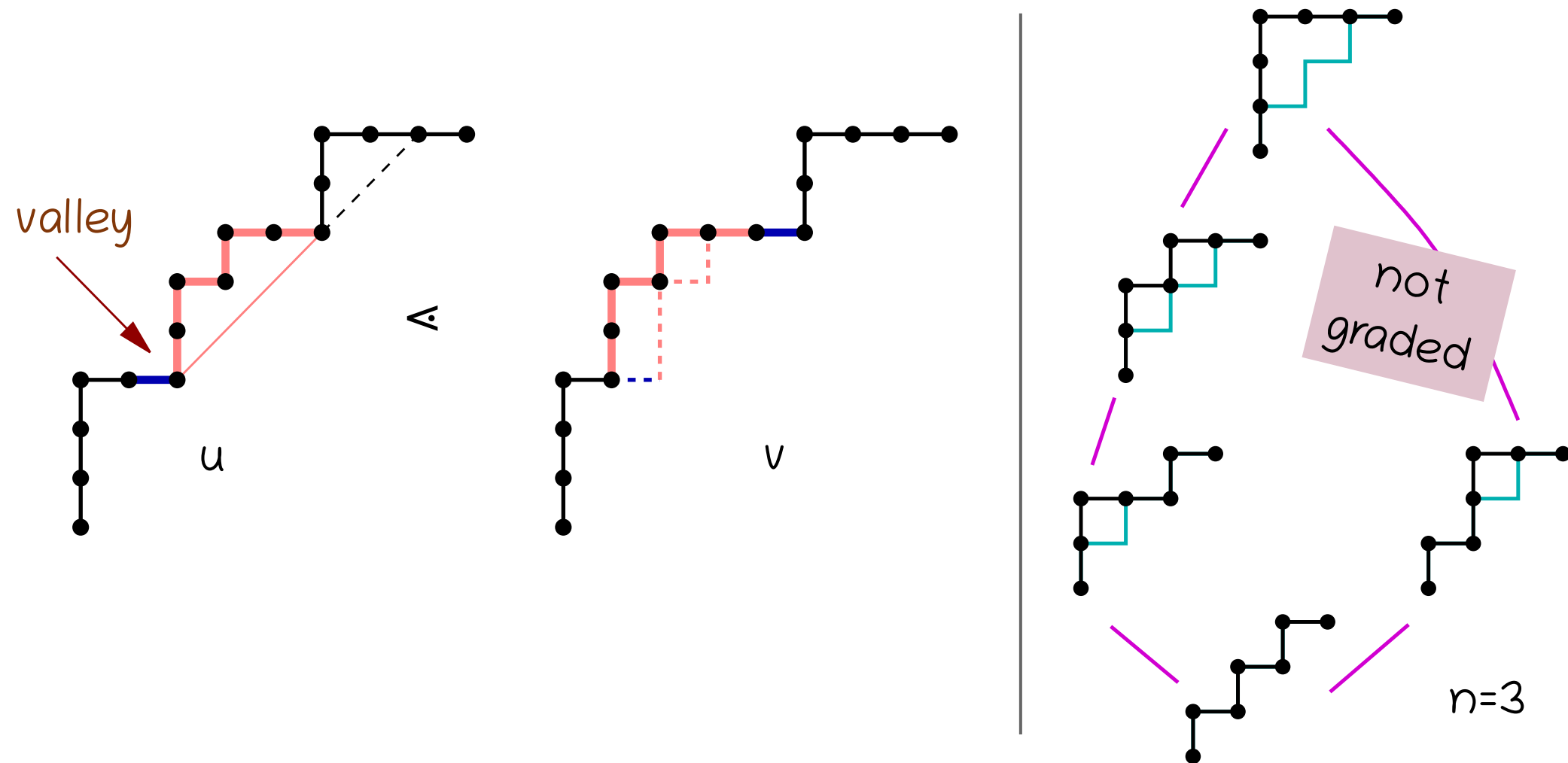
# The Tamari lattice

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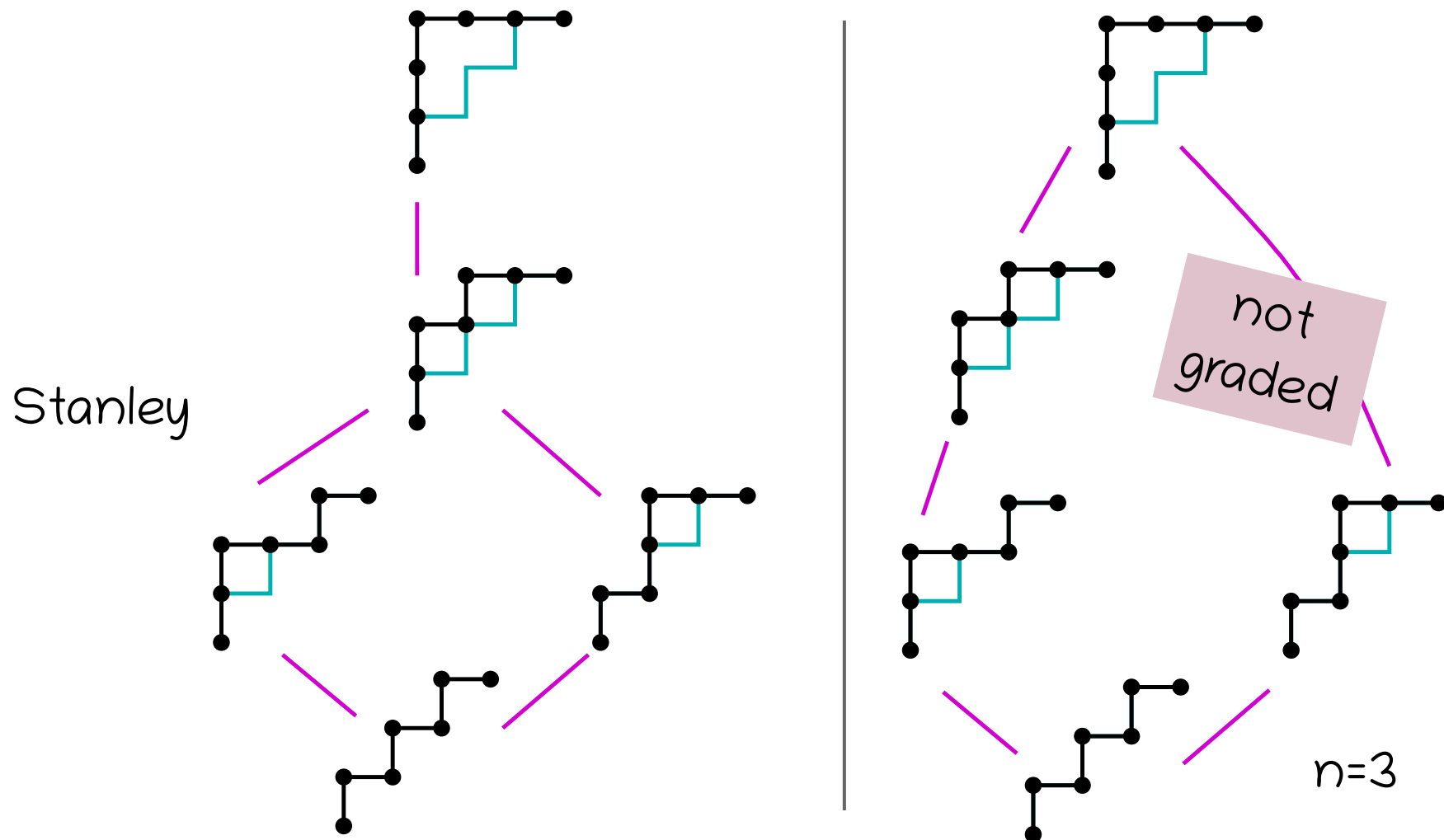


# The Tamari lattice

[Tamari 5]

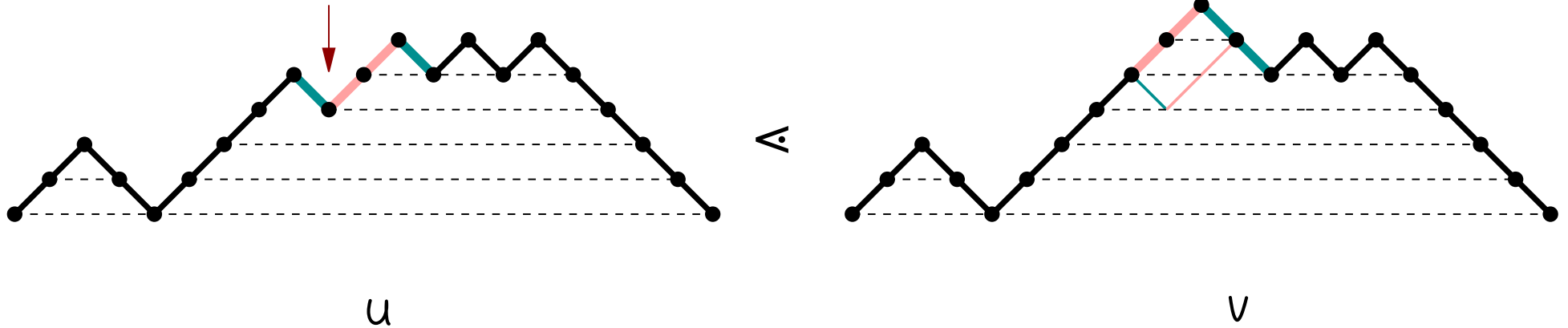
- A poset on Dyck paths with  $n$  up steps
- Cover relations: choose a valley in the path  $u$ .

Swap the East step and the **shortest Dyck path** that follows.



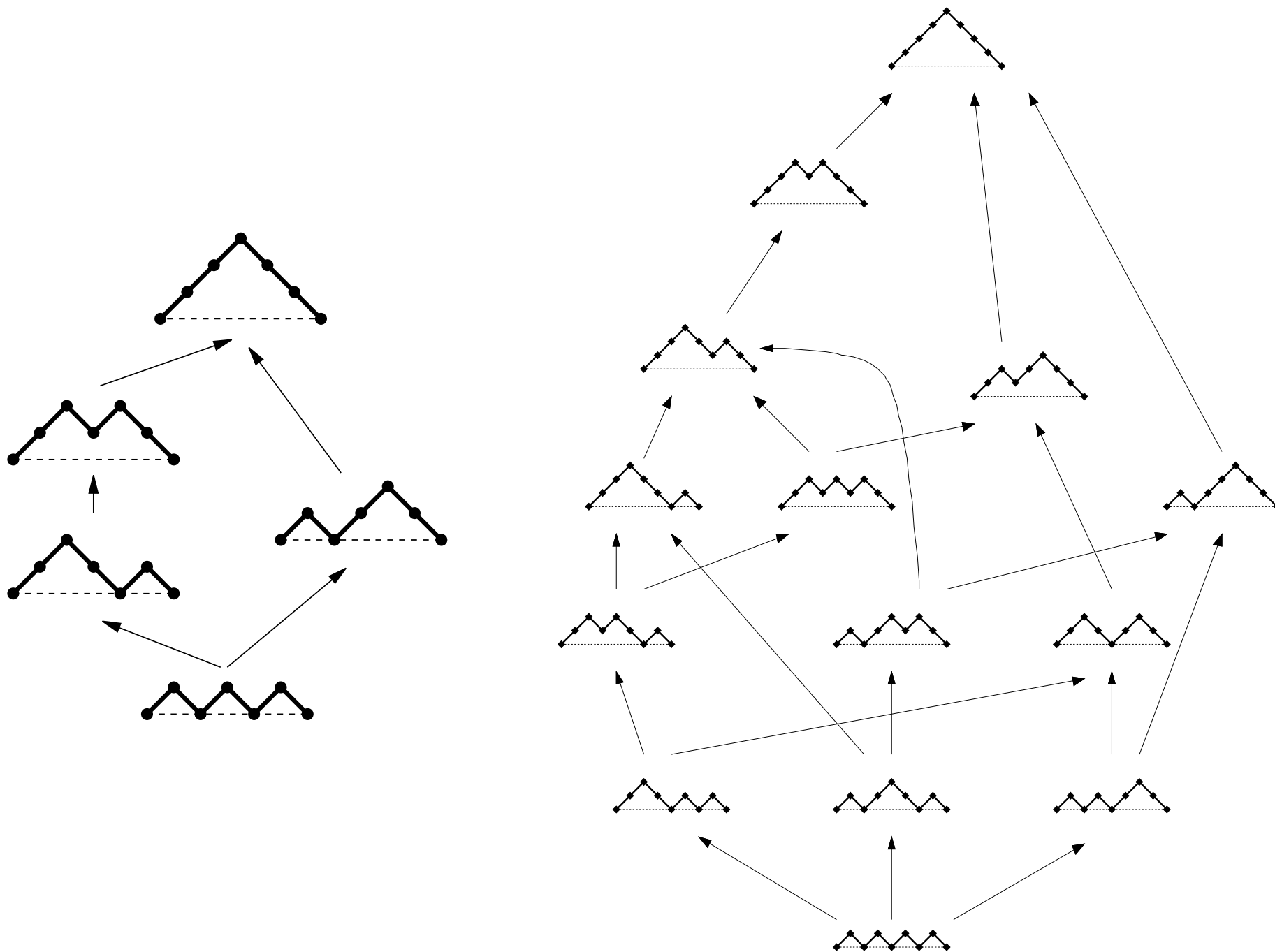
# The ascent poset (or: greedy Stanley?)

- A poset on Dyck paths with  $n$  up steps
- Cover relations: choose a valley in the path  $u$ .  
Swap the down step and the **ascent** that follows.  
(the path moves up)

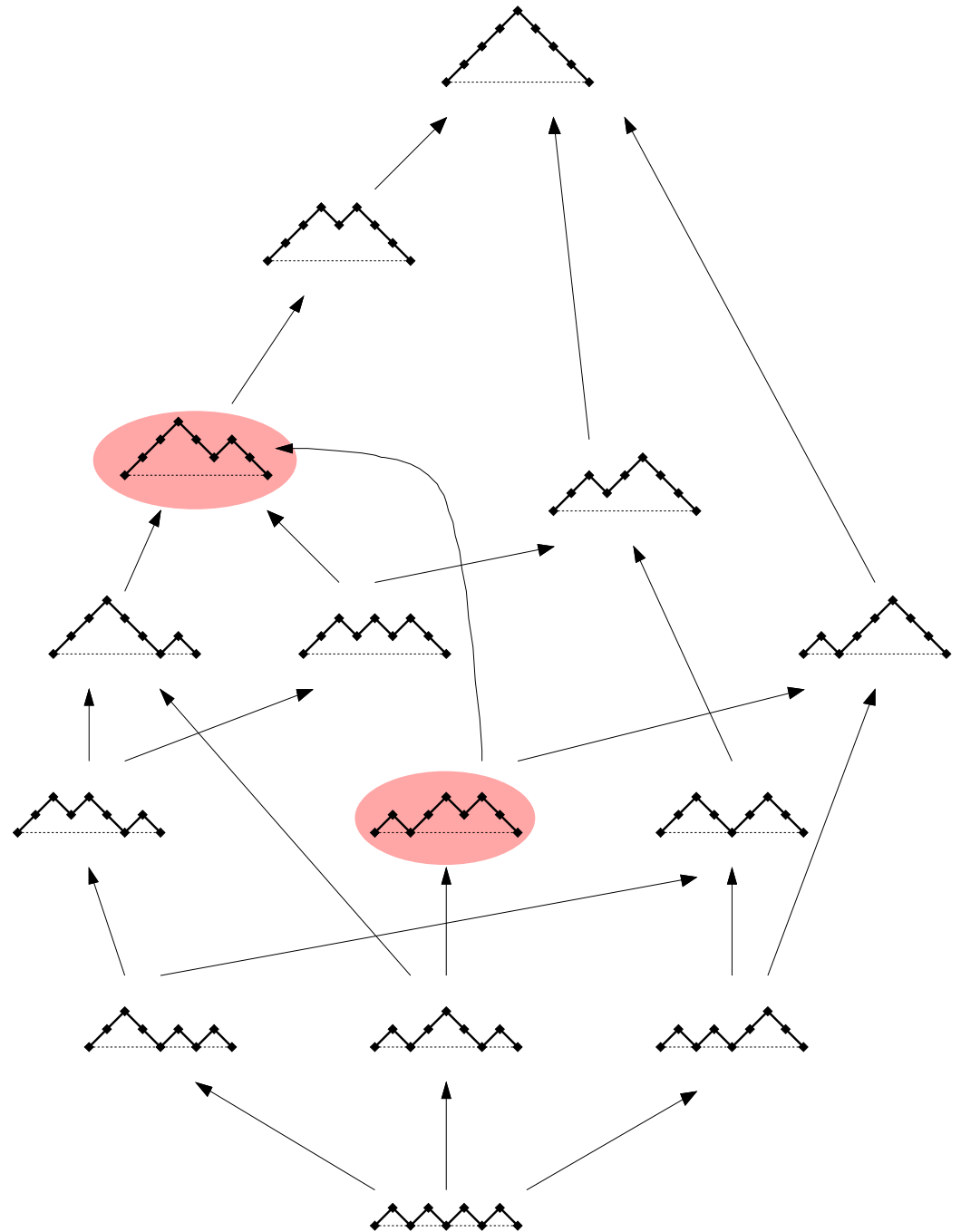
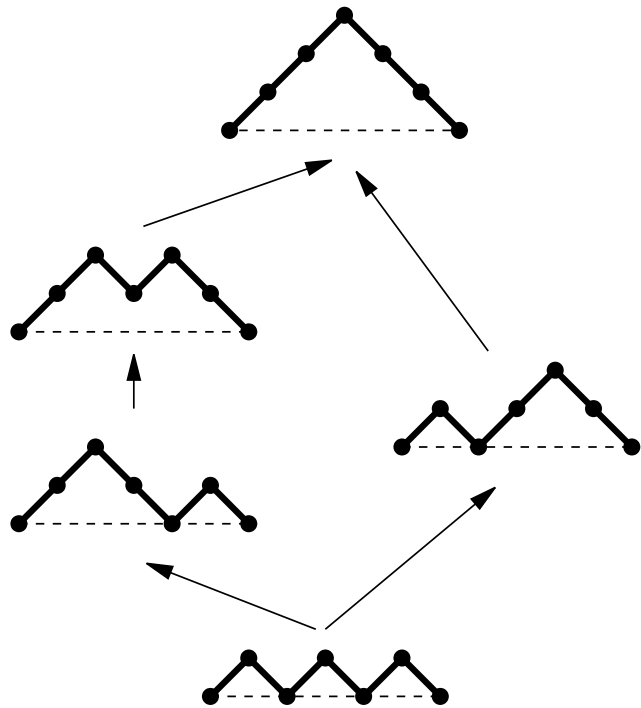


[Chenevière, Nadeau...]

# Ascent posets: $n = 3, 4$

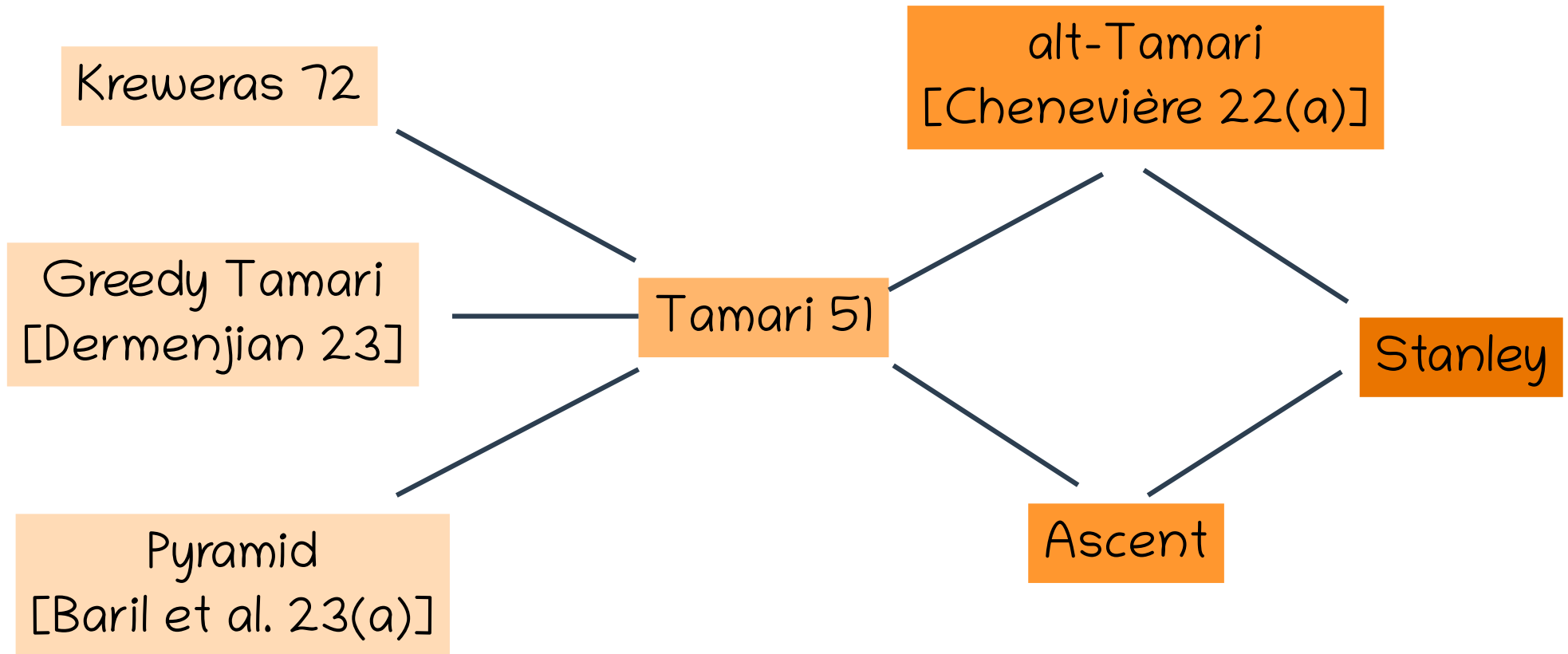


# Ascent posets: $n = 3, 4$





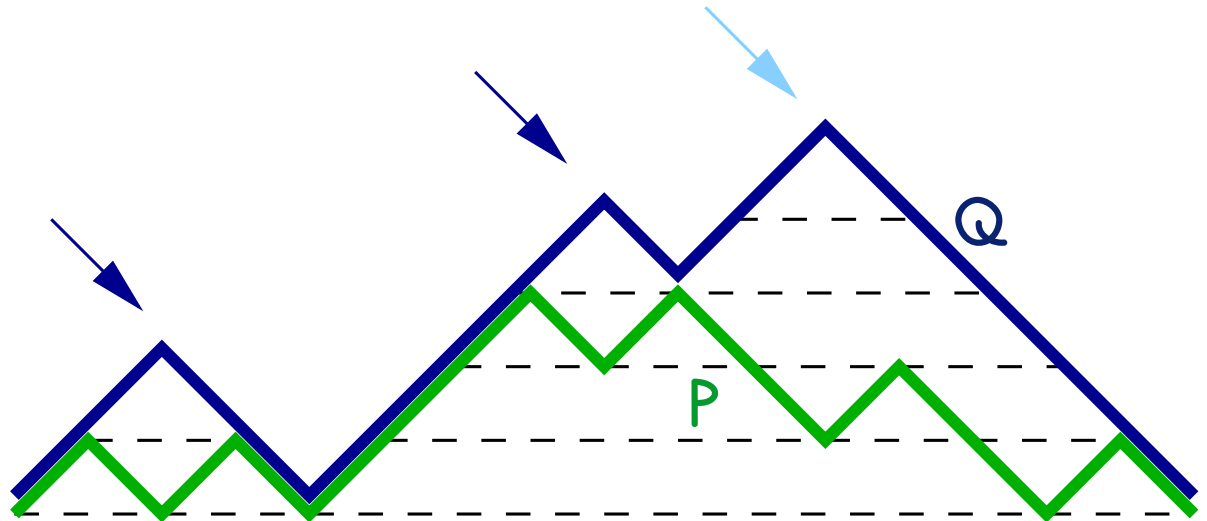
# Poset inclusion



# A characterization of the ascent order

**Proposition.** In the ascent poset,  $P \preceq Q$  iff

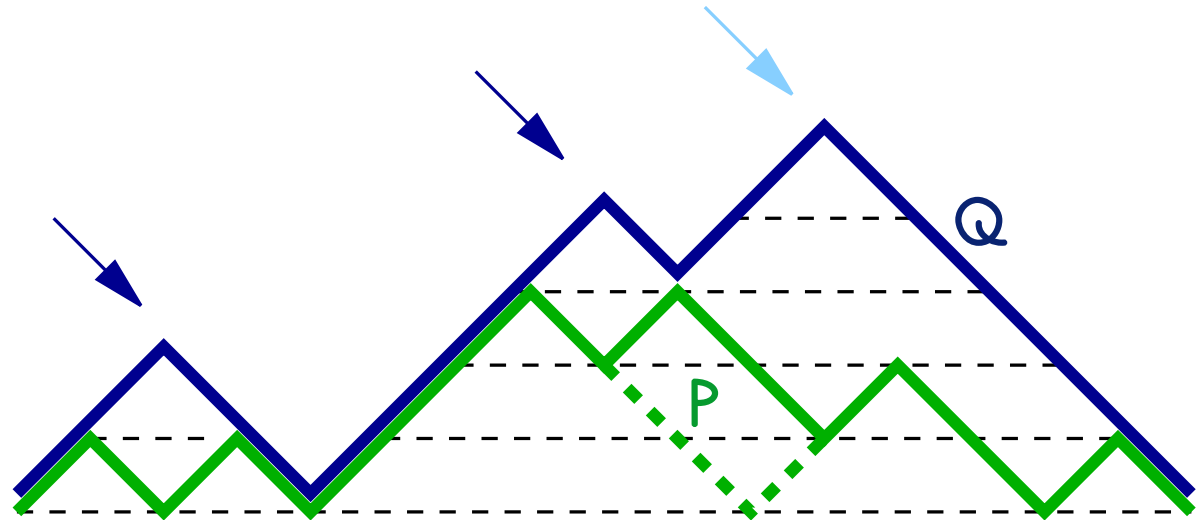
- ◆  $P$  lies below  $Q$
- ◆ every descent of  $Q$  **is** (i.e. lies on the same diagonal as) a descent of  $P$ .



# A characterization of the ascent order

**Proposition.** In the ascent poset,  $P \preceq Q$  iff

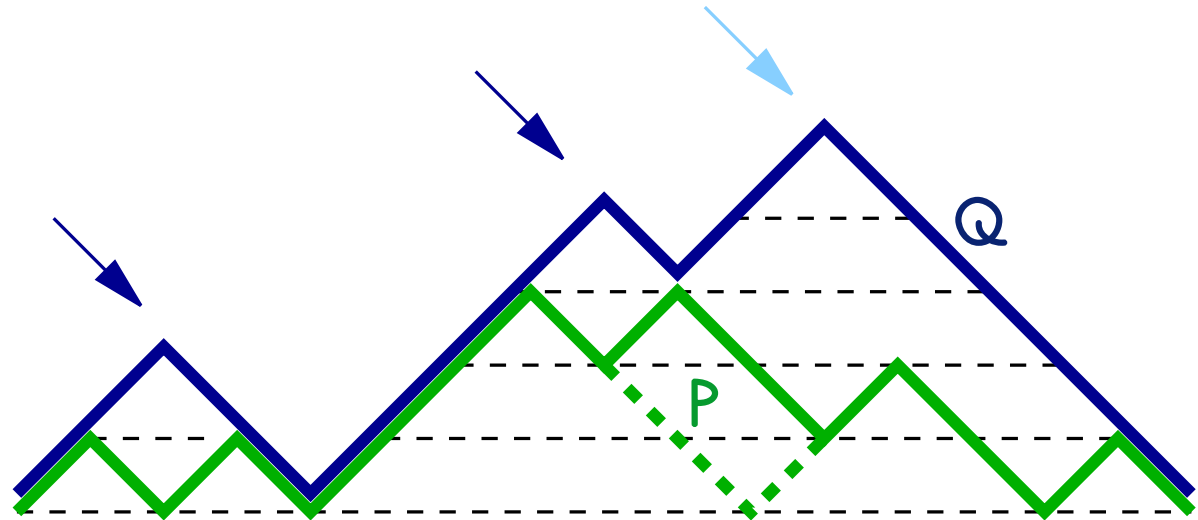
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# A characterization of the ascent order

**Proposition.** In the ascent poset,  $P \preceq Q$  iff

- ♦  $P$  lies below  $Q$
- ♦ every descent of  $Q$  **is** (i.e. lies on the same diagonal as) a descent of  $P$ .



## Applications:

- ♦ lattice structure
- ♦ recursive construction of intervals

## **II. Lattice structure**

(Each pair of elements has a sup and an inf)

# The Nadeau-Tewari poset

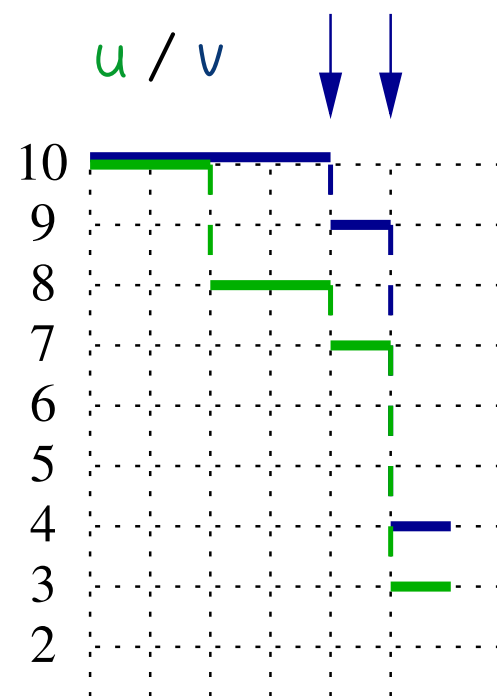
[2024]

**Def.** Let  $u=(u_1, \dots, u_n)$  and  $v=(v_1, \dots, v_n)$  be two nonincreasing sequences of integers. Then  $u \leq v$  for the NT order if

- $u$  lies below  $v$  ( $u_i \leq v_i$ )
- every descent of  $v$  is a descent of  $u$ .

$$v = (10, 10, 10, 10, 9, 4)$$

$$u = (10, 10, 8, 8, 7, 3)$$

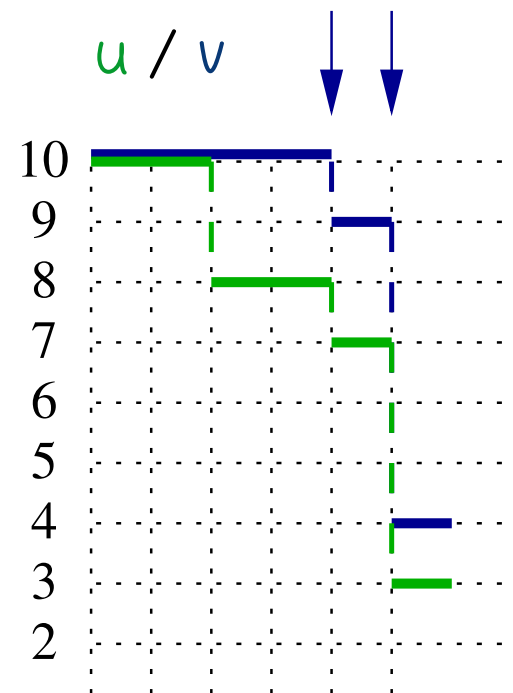
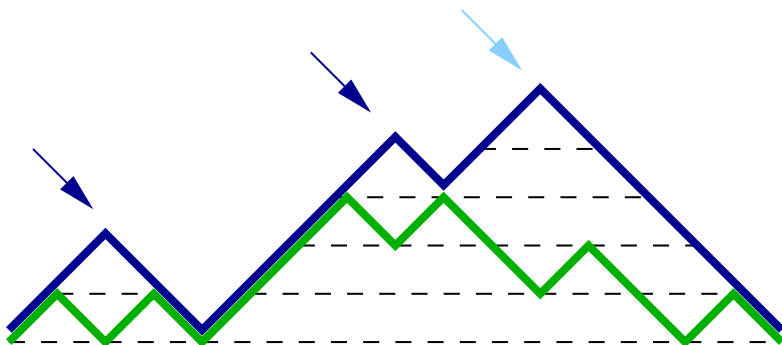


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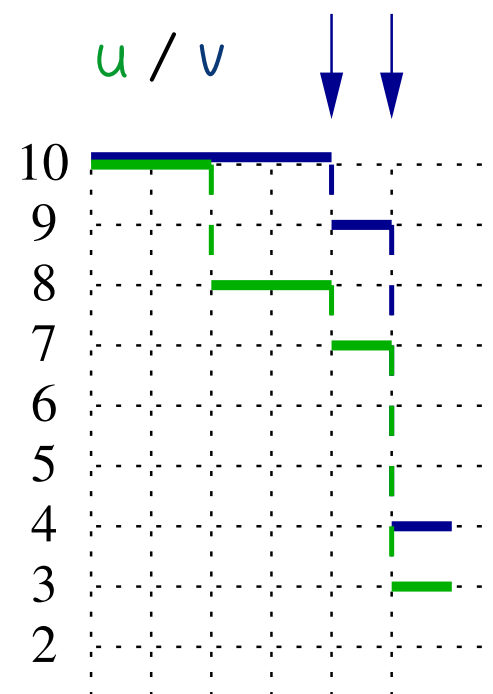
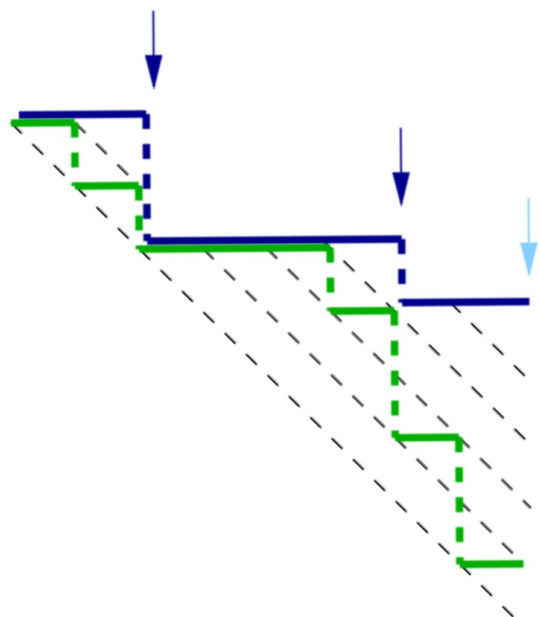


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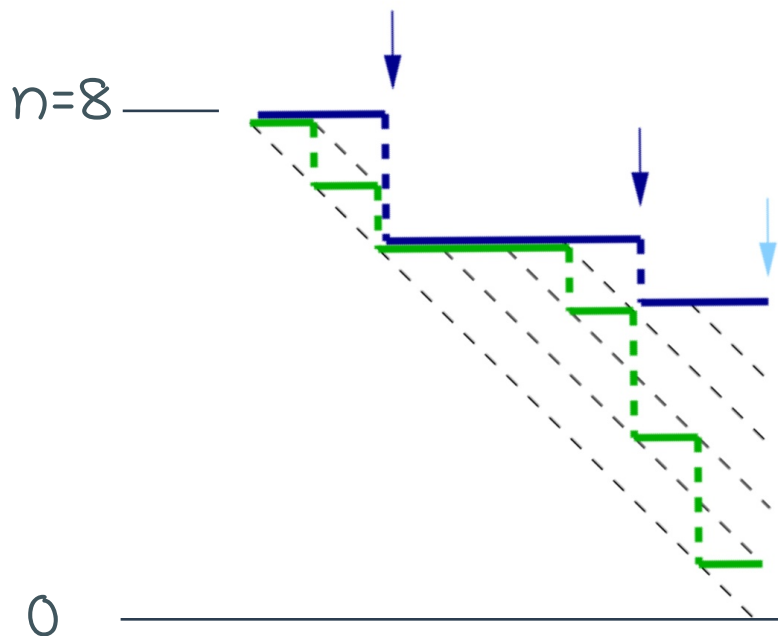


# The Nadeau-Tewari poset

[2024]

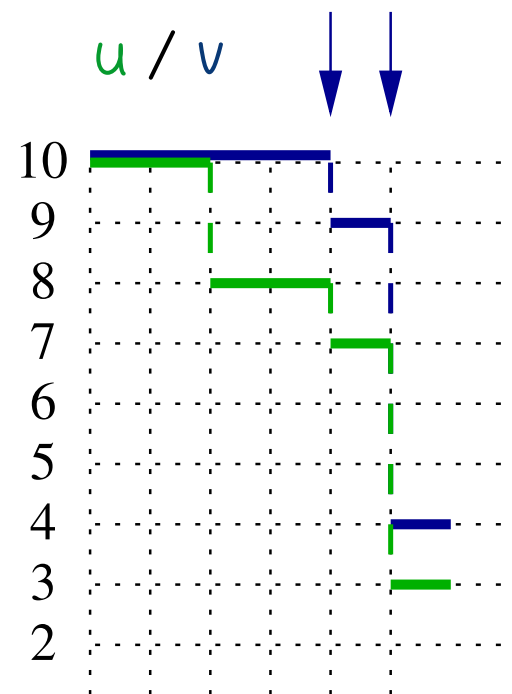
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- $u$  lies below  $v$  ( $u_i \leq v_i$ )
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$$v = (8, 8, 6, 6, 6, 6, 5, 5)$$

$$u = (8, 7, 6, 6, 6, 5, 3, 1)$$



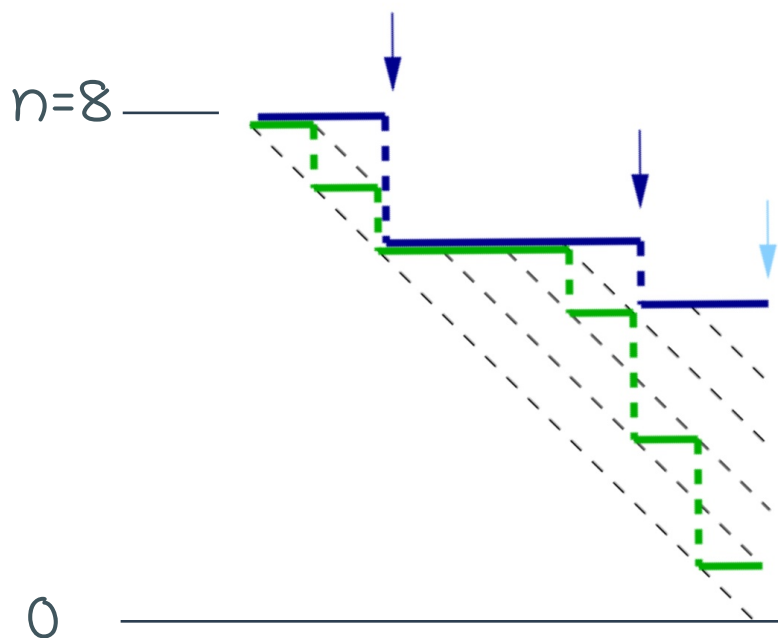
# The Nadeau-Tewari poset

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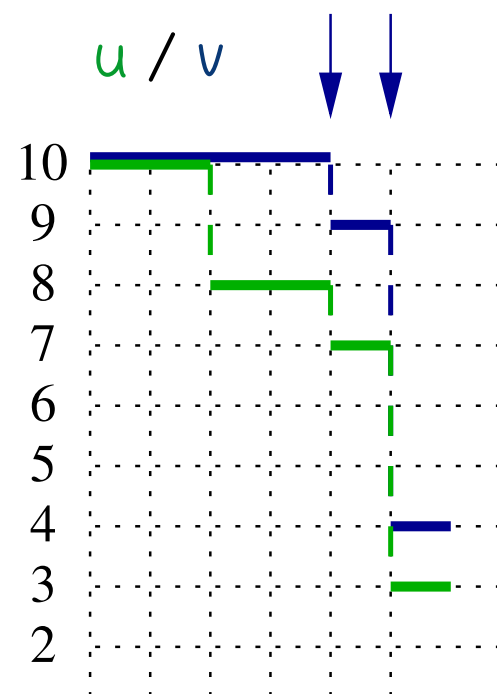
- $u$  lies below  $v$  ( $u_i \leq v_i$ )
- every descent of  $v$  is a descent of  $u$ .

**Observation:** Dyck paths of size  $n$  can be encoded by nonincreasing sequences of length  $n$ , and then the ascent order coincides with the NT order.



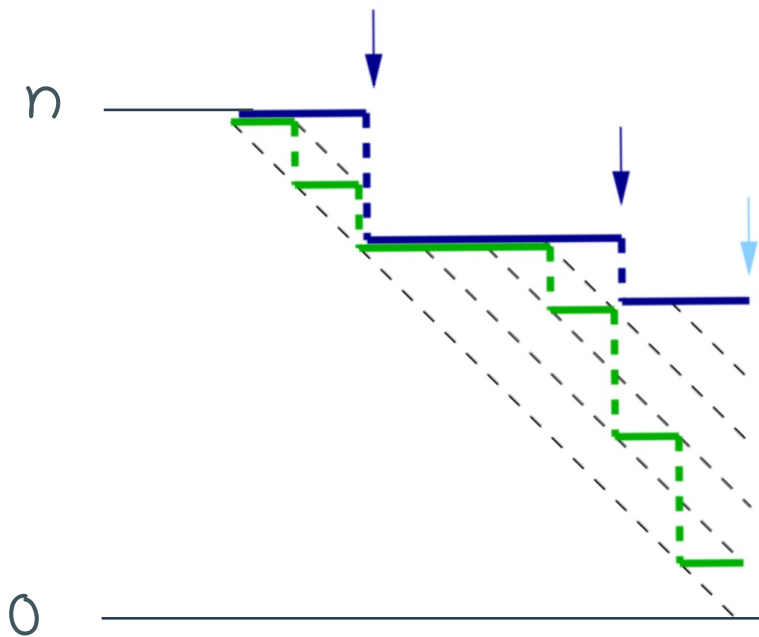
$$v = (8, 8, 6, 6, 6, 6, 5, 5)$$

$$u = (8, 7, 6, 6, 6, 5, 3, 1)$$



# The Nadeau-Tewari poset and the ascent order

**Observation:** The ascent order is the order induced by the NT order on sequences  $u=(u_1, \dots, u_n)$  such that  $n-i+1 \leq u_i \leq n$  for all  $i$ .



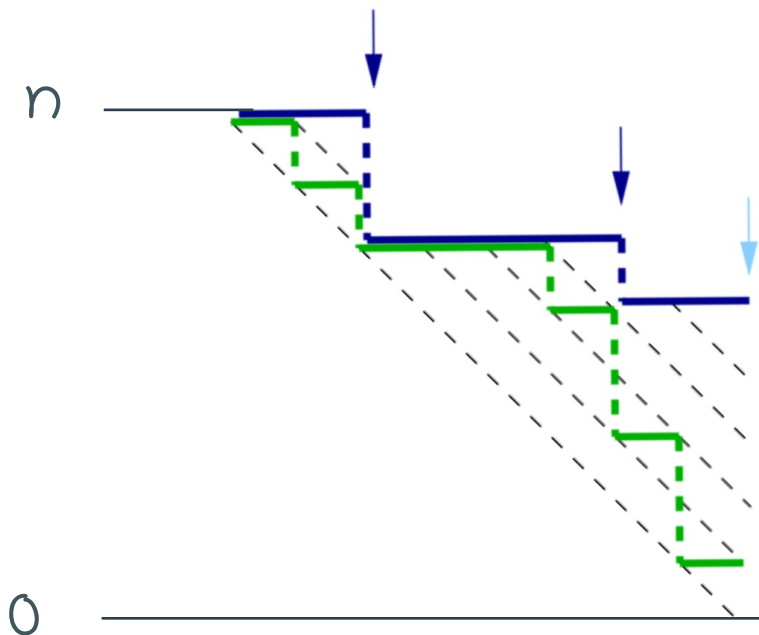
$$v = (8, 8, 6, 6, 6, 6, 5, 5)$$

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# The Nadeau-Tewari poset and the ascent order

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These sequences form an **interval** with  $u_{\min} = (n, n-1, \dots, 1)$  and  $u_{\max} = (n, n, \dots, n)$ .



$$v = (8, 8, 6, 6, 6, 6, 5, 5)$$

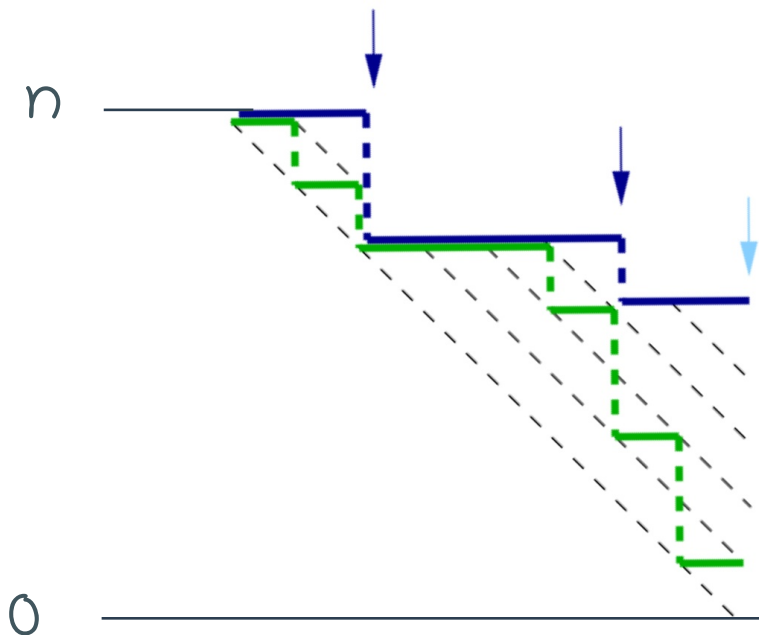
$$u = (8, 7, 6, 6, 6, 5, 3, 1)$$

# The Nadeau-Tewari poset and the ascent order

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**Proposition [NT 24]:** the NT poset is a **lattice**.



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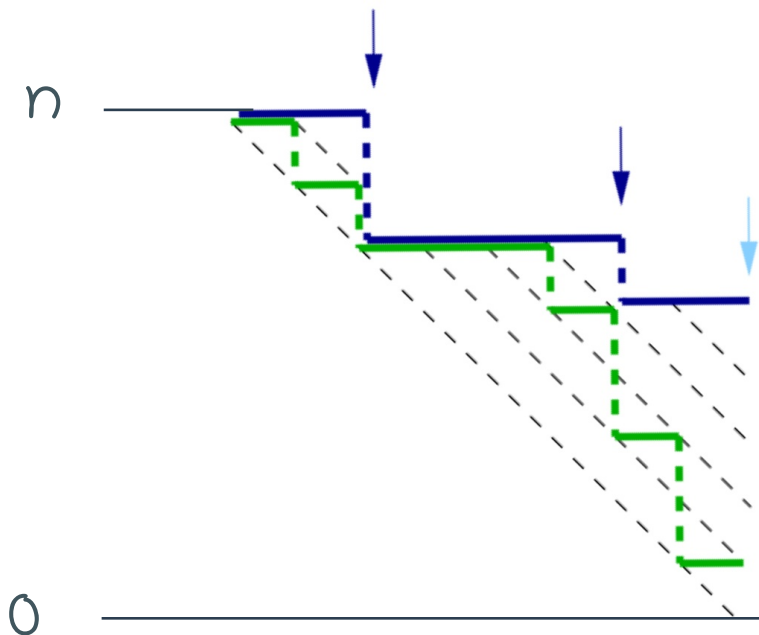
# The Nadeau-Tewari poset and the ascent order

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**Proposition [NT 24]:** the NT poset is a **lattice**.

**Corollary.** The ascent order on Dyck paths of size  $n$  defines a **lattice**.

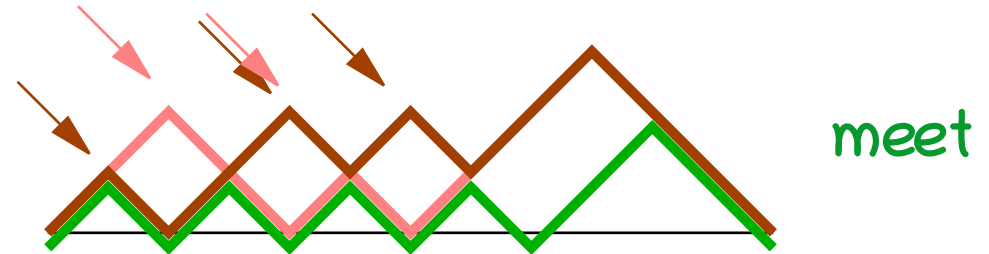
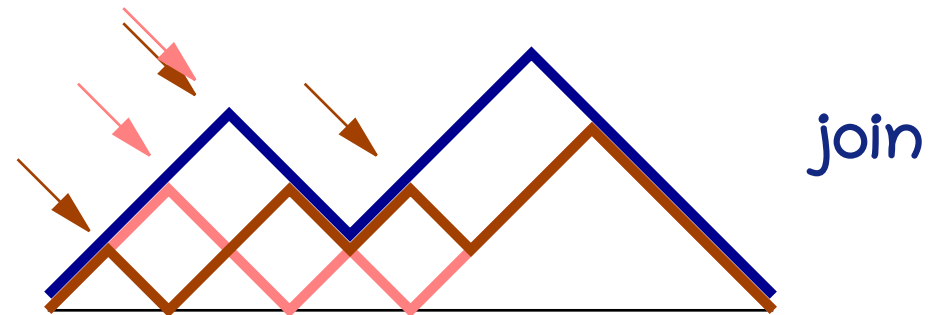
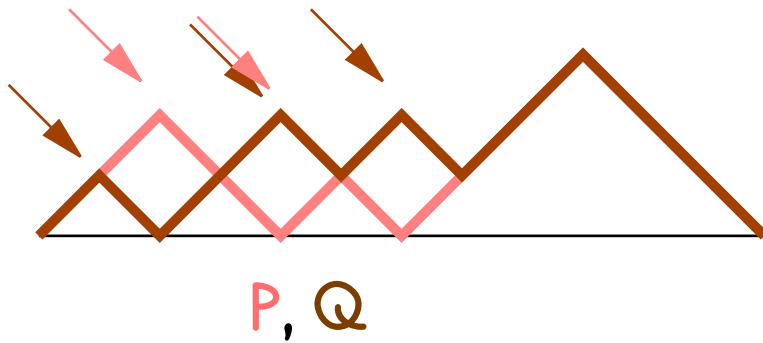


$$v = (8, 8, 6, 6, 6, 6, 5, 5)$$

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# The Nadeau-Tewari poset and the ascent order

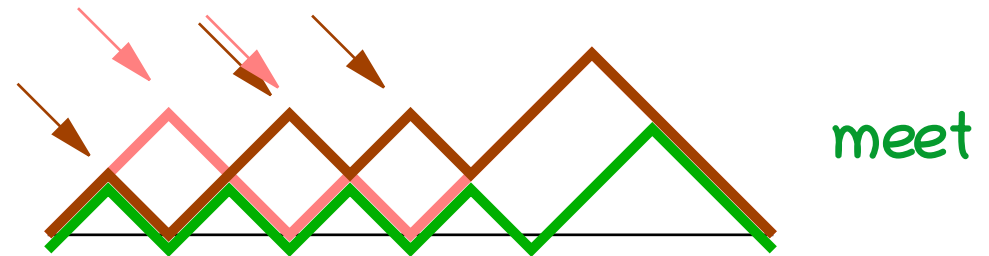
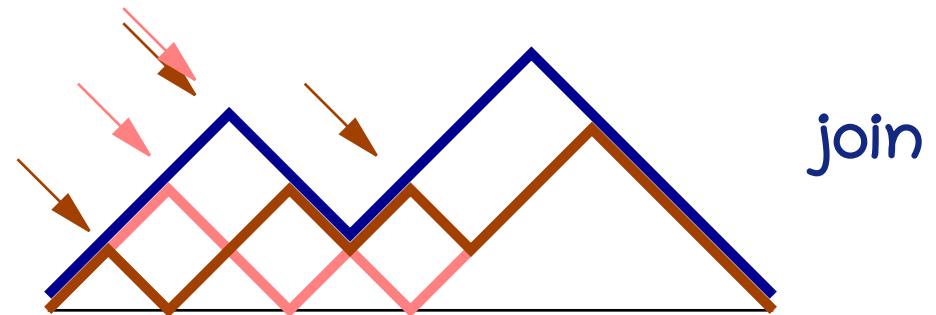
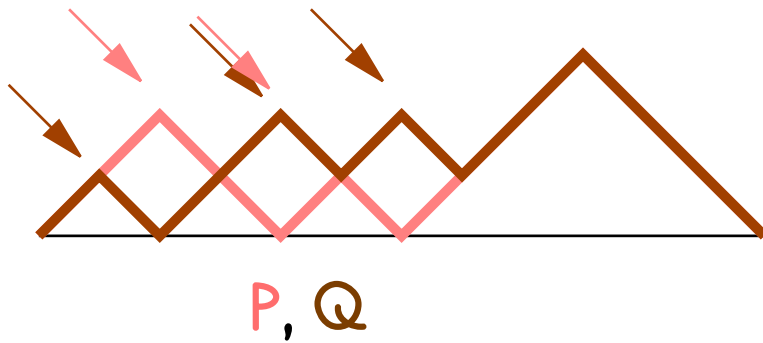
**Corollary.** The ascent order on Dyck paths of size  $n$  defines a **lattice**.



# The Nadeau-Tewari poset and the ascent order

**Corollary.** The ascent order on Dyck paths of size  $n$  defines a **lattice**.

**Join(P,Q):** the lowest Dyck path above P and Q whose descents are included in  $\text{des}(P)$  and in  $\text{des}(Q)$ .



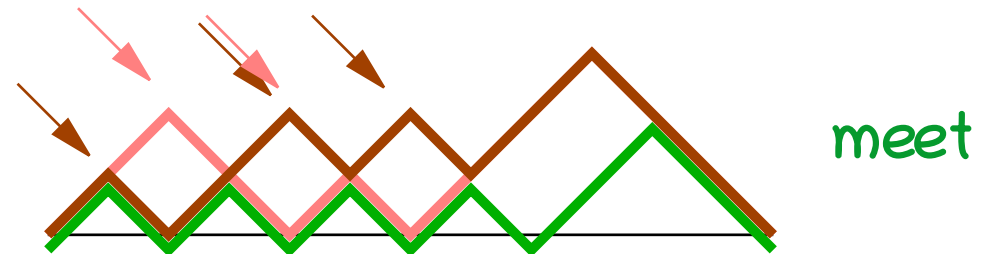
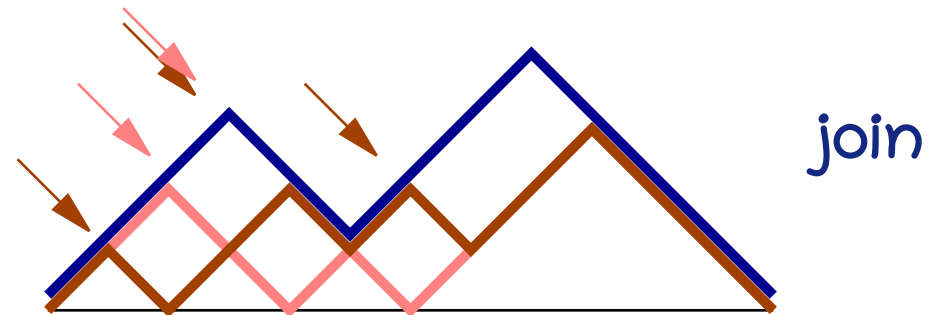
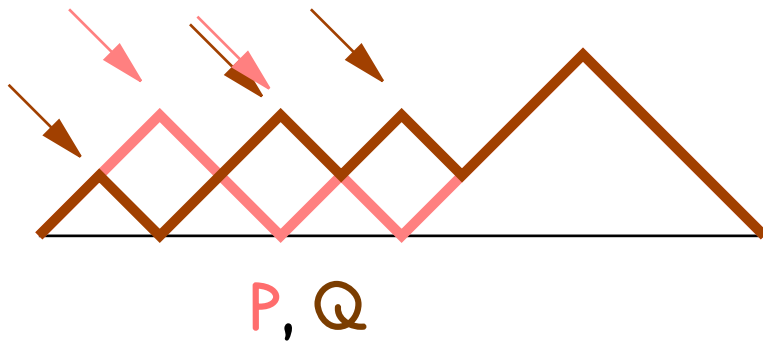


# The Nadeau-Tewari poset and the ascent order

**Corollary.** The ascent order on Dyck paths of size  $n$  defines a **lattice**.

**Join(P,Q):** the lowest Dyck path above P and Q whose descents are included in  $\text{des}(P)$  and in  $\text{des}(Q)$ .

**Meet(P,Q):** the highest Dyck path below P and Q whose descents contain those of P and Q.



### **III. The number of intervals**

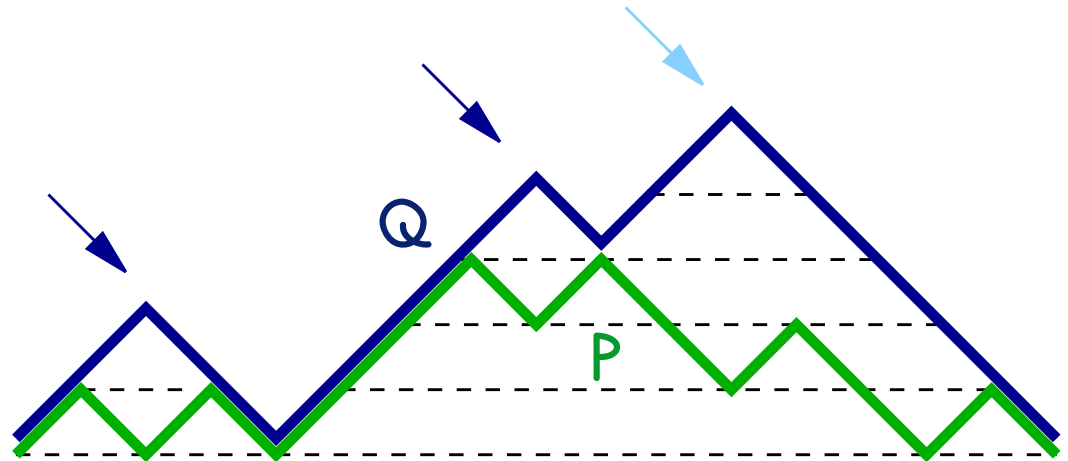
Interval  $[P, Q] \sim (P, Q)$  with  $P \leq Q$

# Recursive construction of ascent intervals

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**Proposition.** In the ascent poset,  $P \preceq Q$  iff

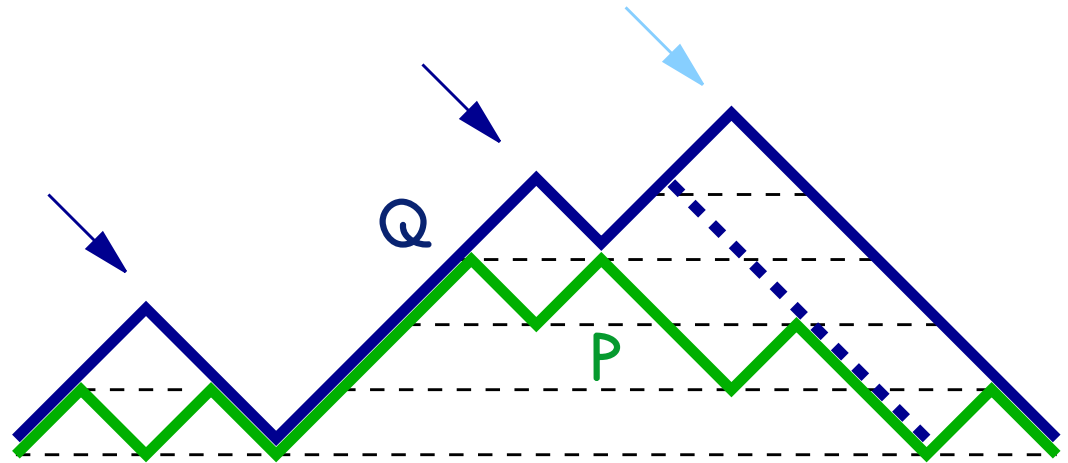
- ◆  $P$  lies below  $Q$
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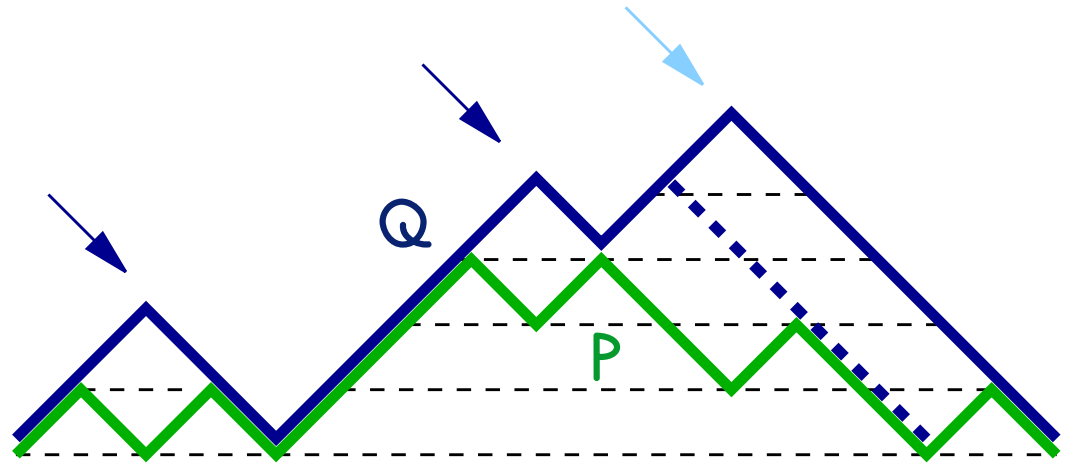
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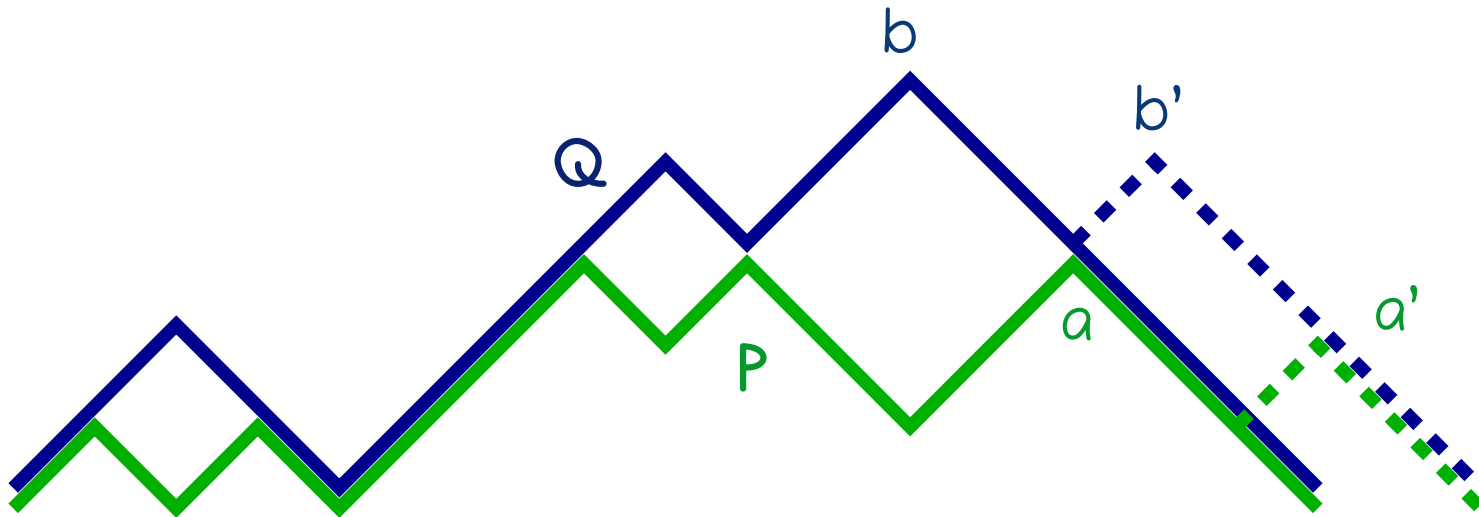
- ◆  $P$  lies below  $Q$
- ◆ every descent of  $Q$  **is** (i.e. lies on the same diagonal as) a descent of  $P$ .



**Corollary:** if  $[P, Q]$  is an interval, deleting the last peak of  $P$  and the last peak of  $Q$  gives a new interval.

# Recursive construction of ascent intervals

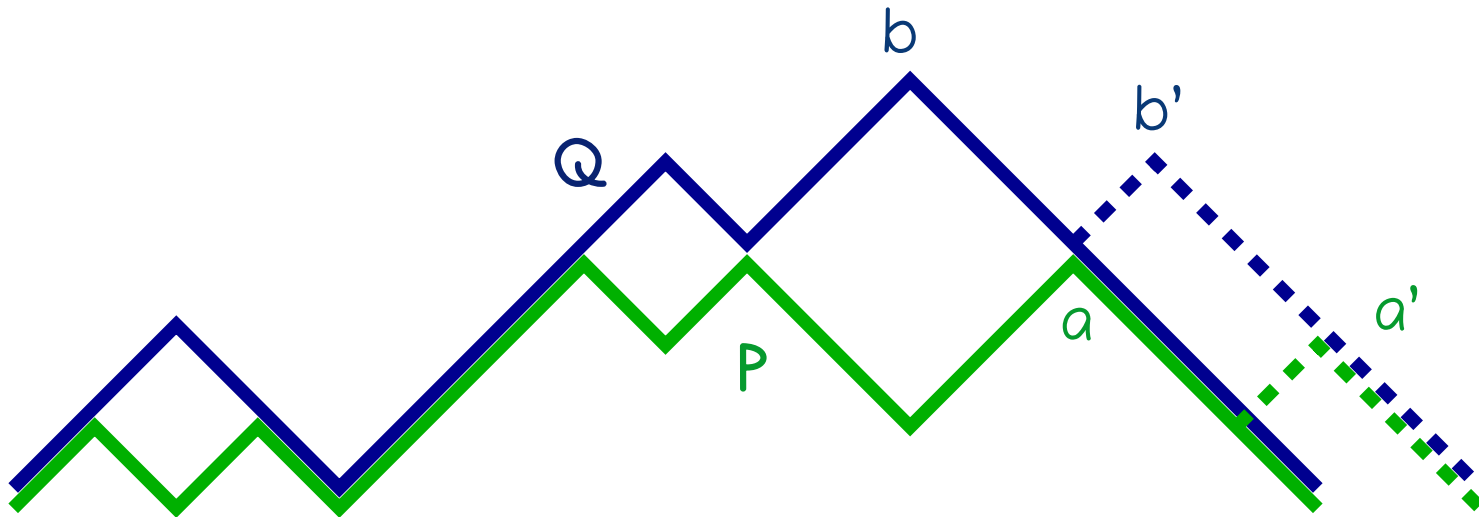
Conversely, starting from an interval  $[P, Q]$  with final peaks at heights  $a \leq b$ , adding peaks in  $P$  and  $Q$  at heights  $a'$  and  $b'$  gives an interval iff...



# Recursive construction of ascent intervals

Conversely, starting from an interval  $[P, Q]$  with final peaks at heights  $a \leq b$ , adding peaks in  $P$  and  $Q$  at heights  $a'$  and  $b'$  gives an interval iff...

- ◆  $1 \leq a' \leq a+1, \quad 1 \leq b' \leq b+1$

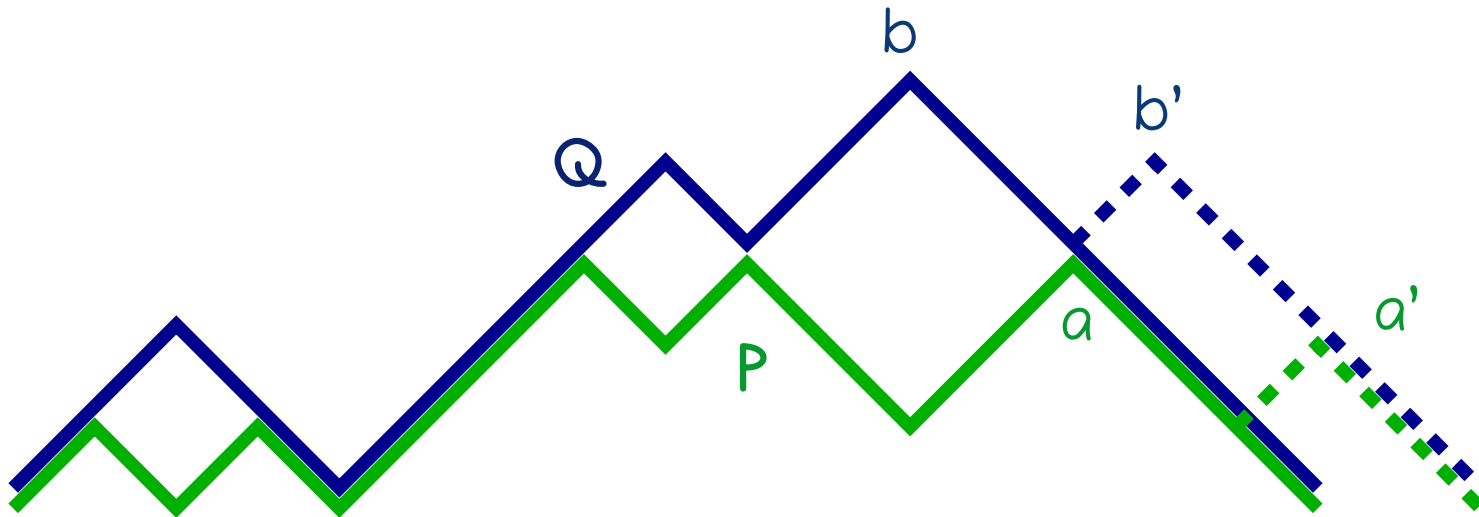




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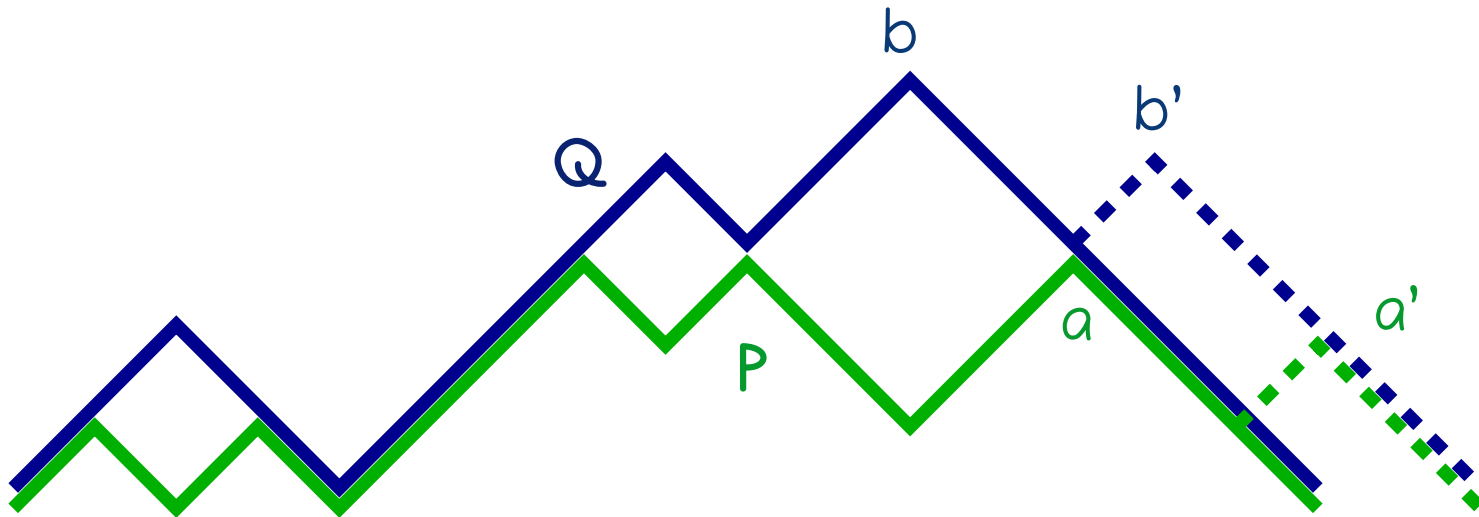
- ◆  $1 \leq a' \leq a+1, \quad 1 \leq b' \leq b+1$
- ◆  $a' \leq b'$



# Recursive construction of ascent intervals

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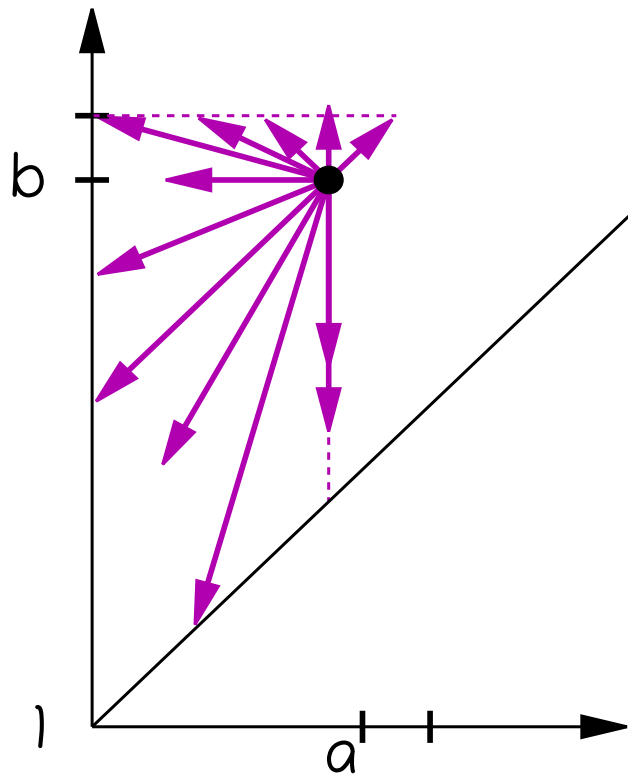
- ◆  $1 \leq a' \leq a+1, \quad 1 \leq b' \leq b+1$
- ◆  $a' \leq b'$
- ◆ if  $a' = a+1$  then  $b' = b+1$ .



# Recursive construction of ascent intervals

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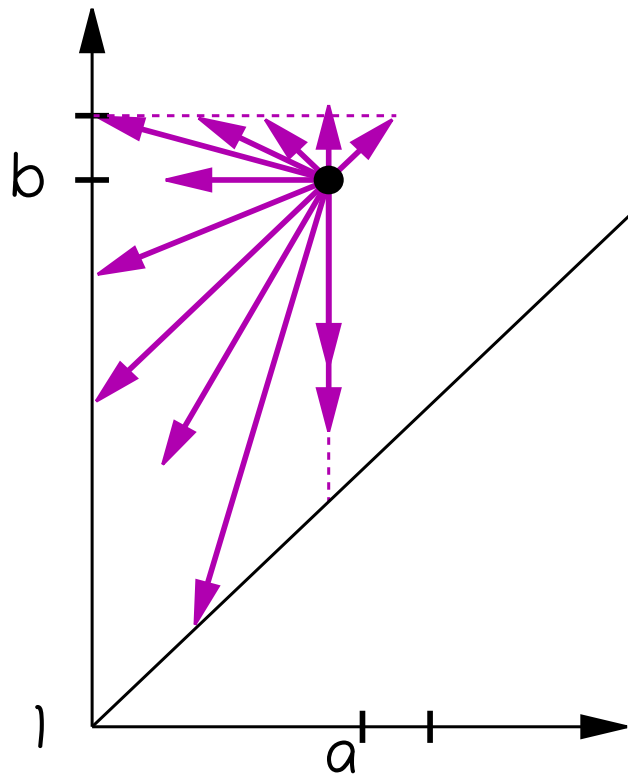
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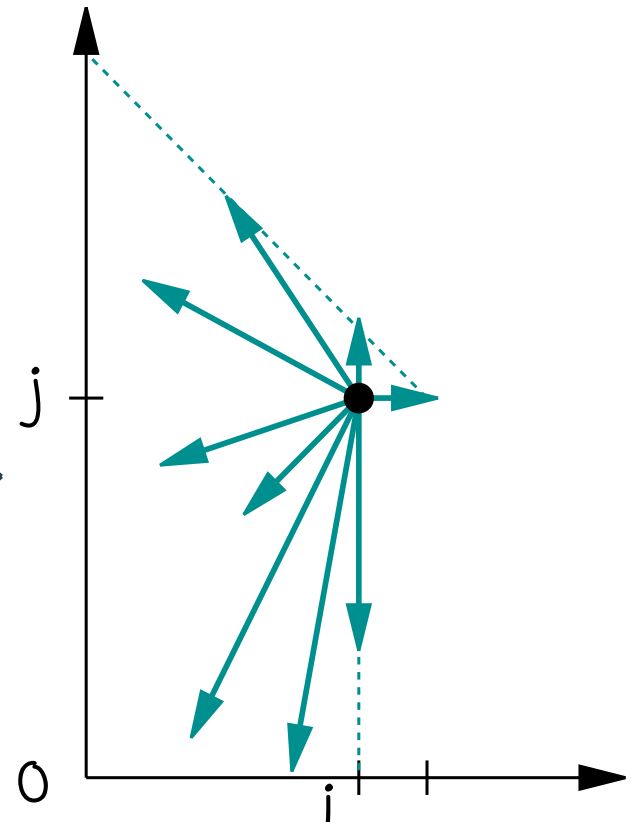
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$$\begin{array}{c} i=a-1 \\ \longrightarrow \\ j=b-a \end{array}$$

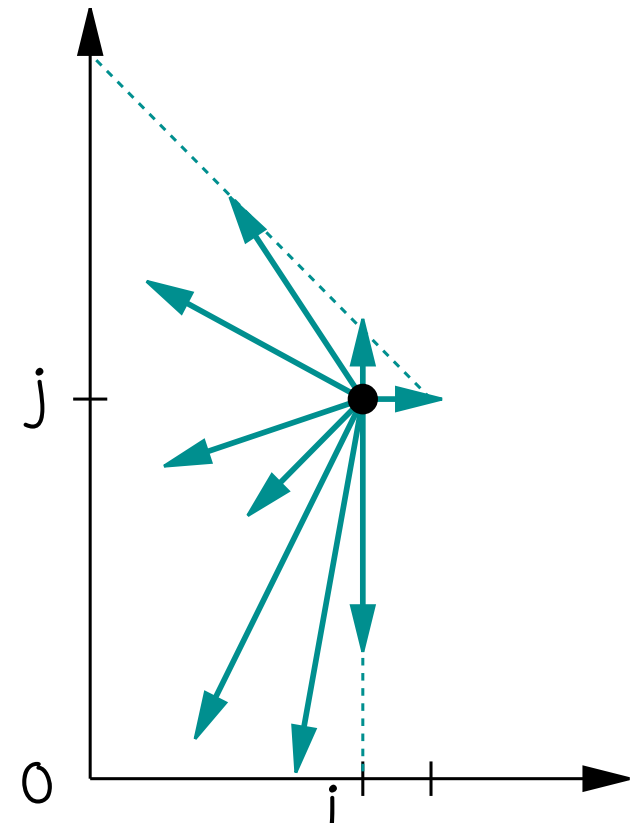


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quadrant walks of length  $n-1$  starting  
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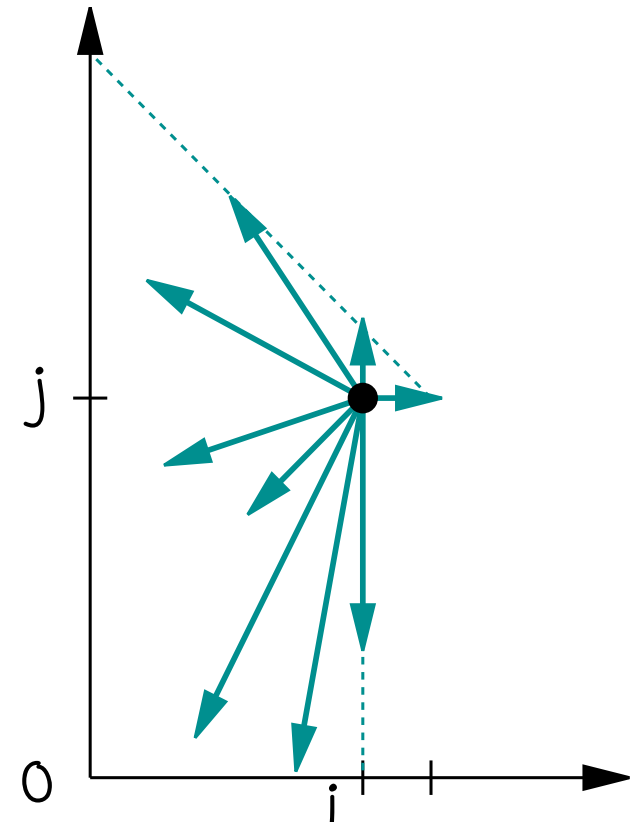
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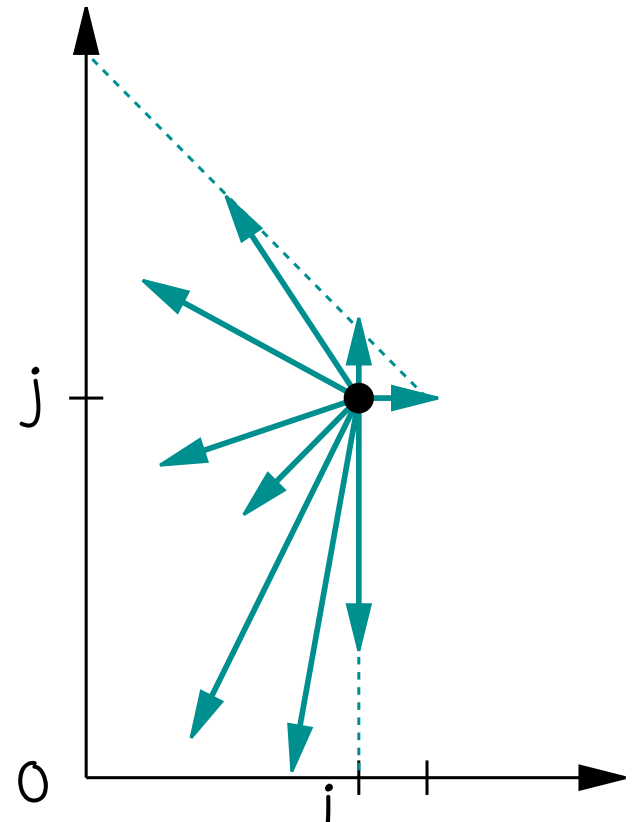
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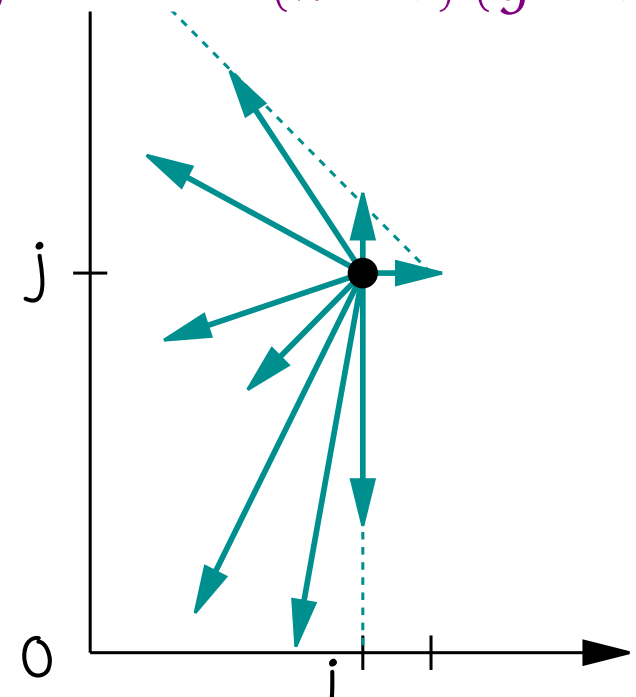
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**Thm.** Ascent intervals have **an algebraic GF**, namely

$$G = Z(1 - 2Z + 2Z^3), \quad \text{where} \quad Z = t(1 + Z)(1 + 2Z)^2.$$

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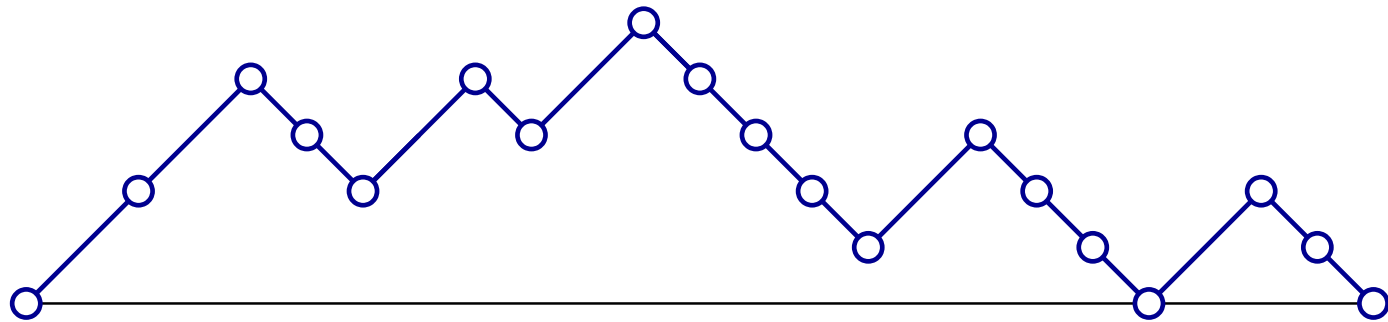
$$g(n) \sim \kappa \mu^n n^{-7/2}, \quad \text{with} \quad \mu = \frac{11 + 5\sqrt{5}}{2}.$$

## **IV. m-Dyck paths, and mirrored m-Dyck paths**

# m-Dyck paths and mirrored m-Dyck paths

In an **m-Dyck path**, the length of all **ascents** is a multiple of  $m$ .

$m=2$

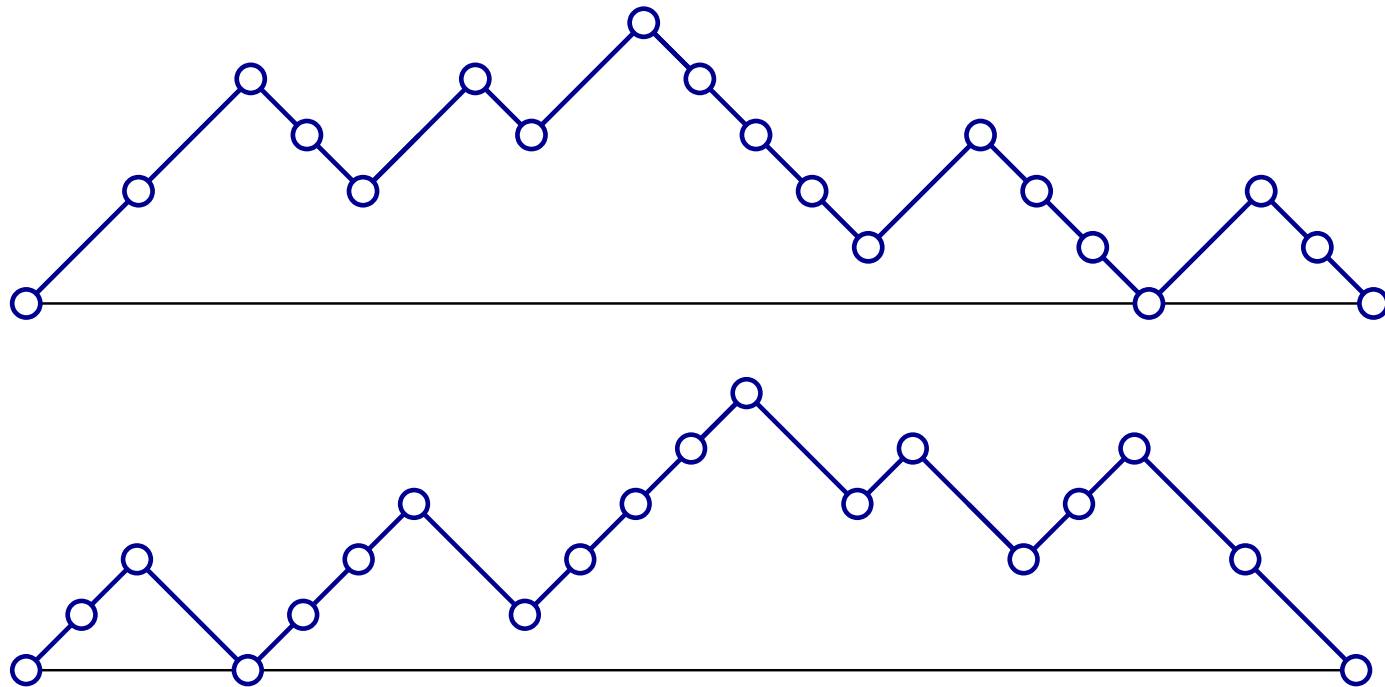


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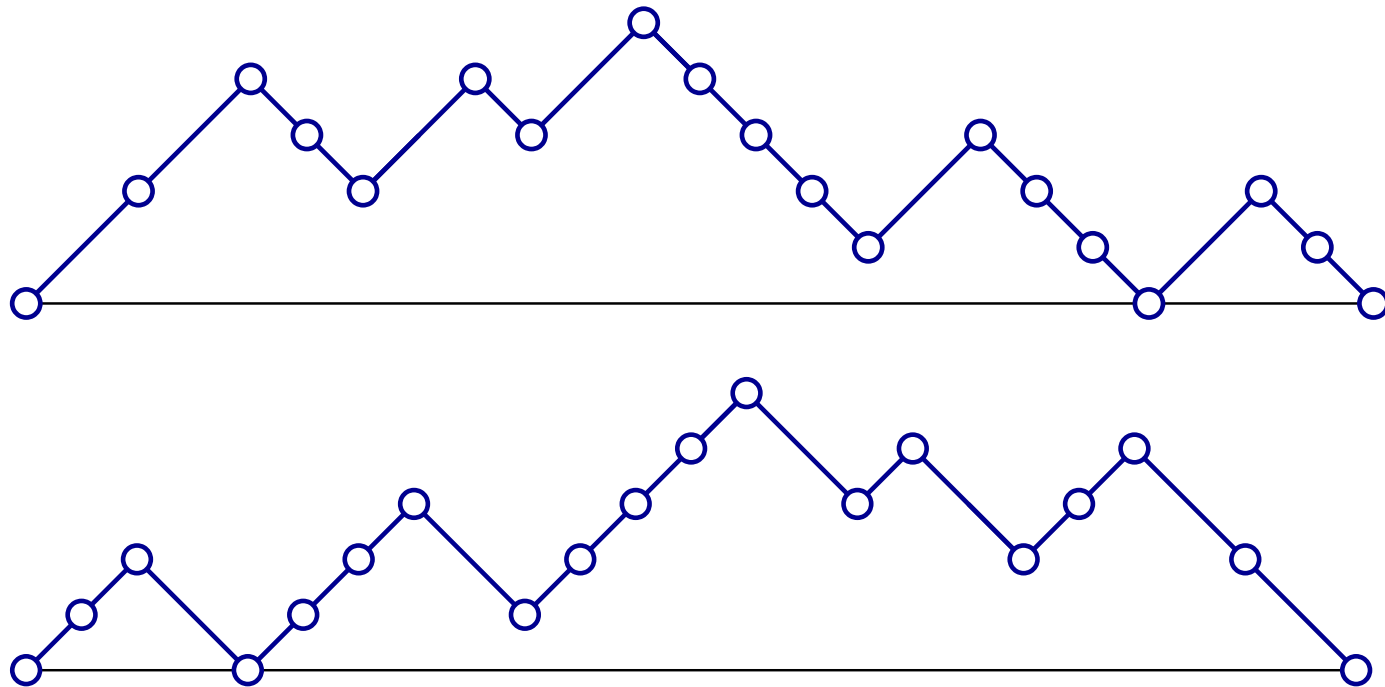


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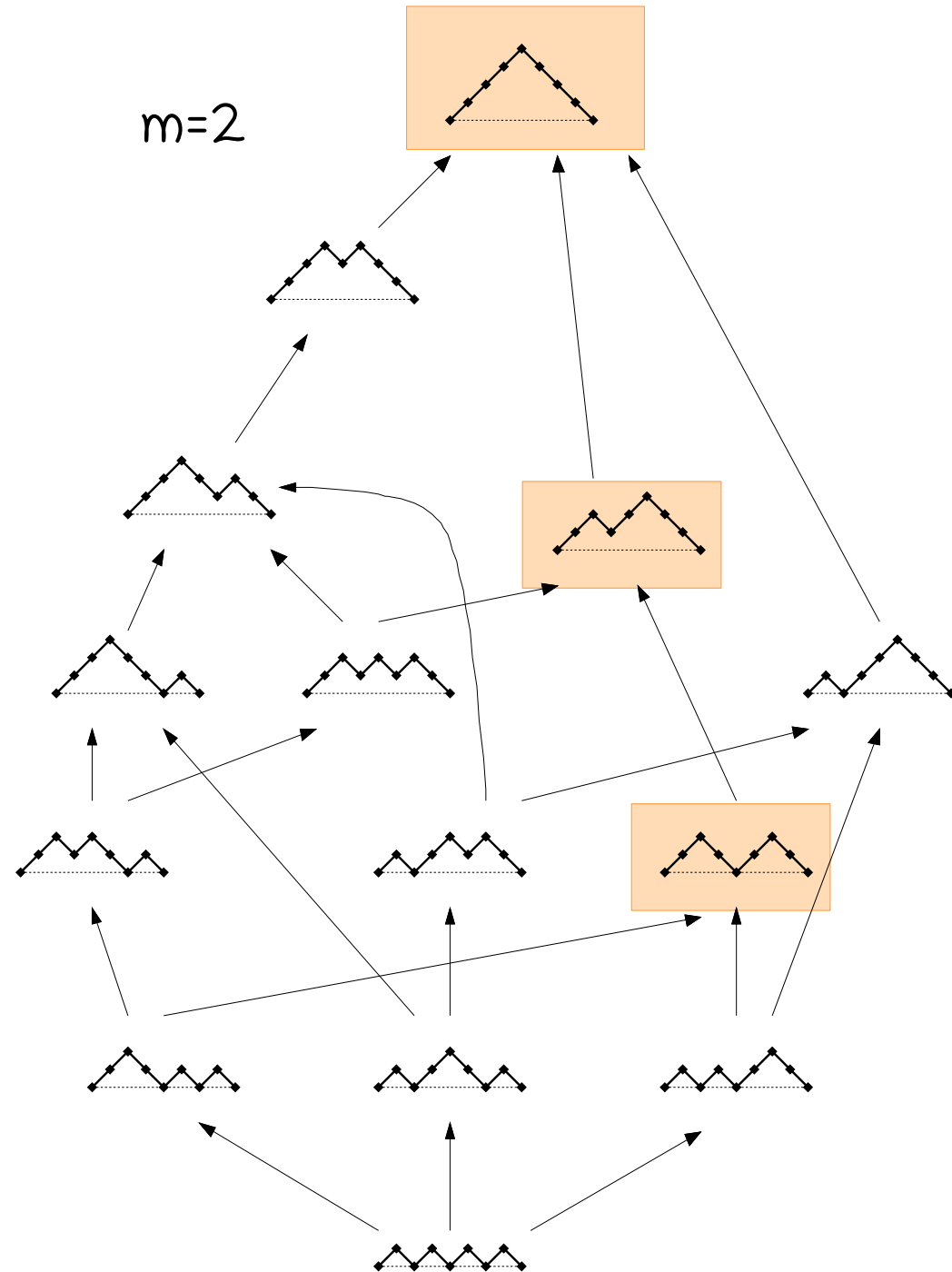
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→ Study the order induced by the ascent order on m-Dyck paths and mirrored m-Dyck paths.

# m-Dyck paths

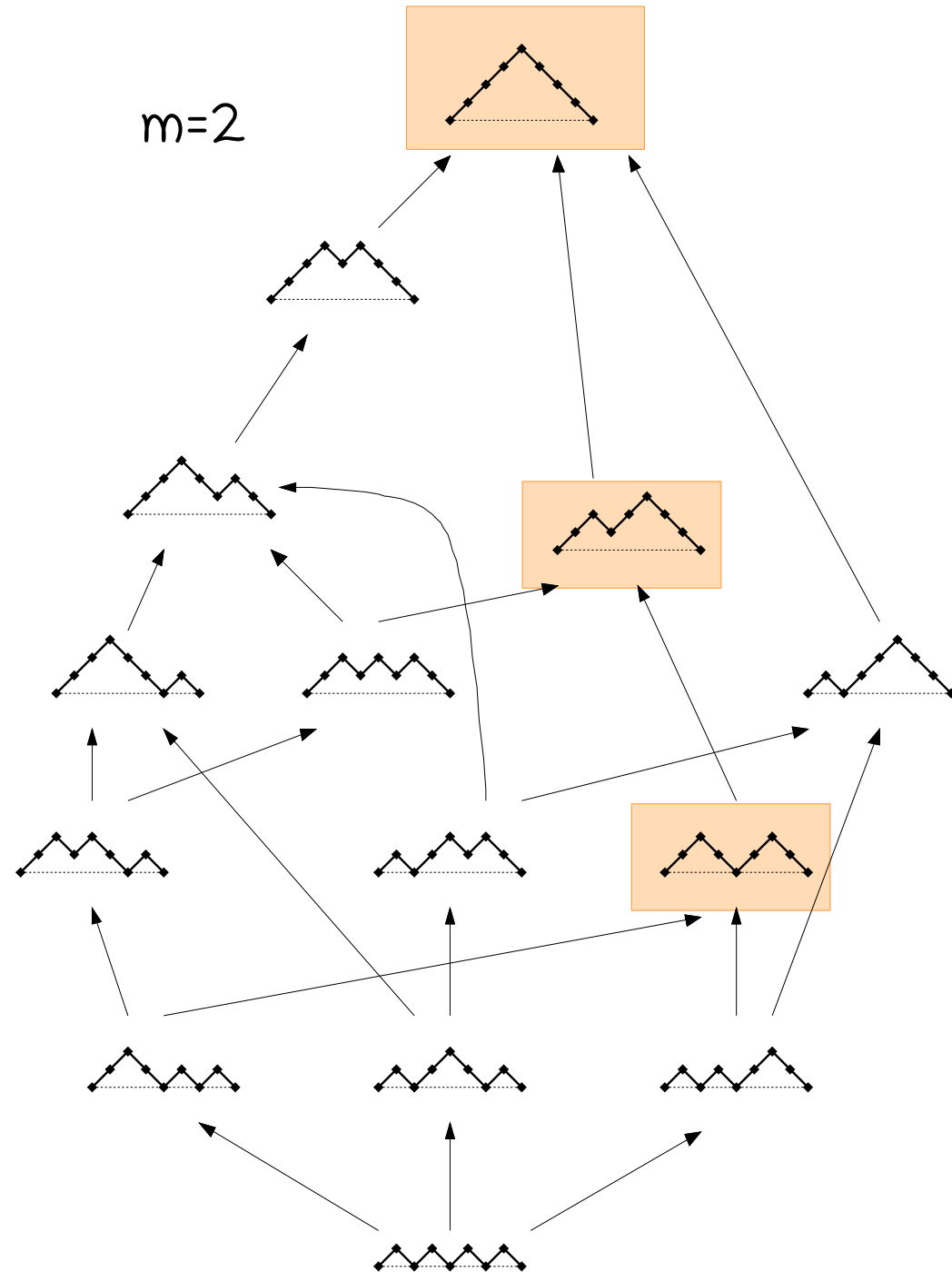
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# m-Dyck paths

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m-Dyck paths form an **interval** in the ascent lattice  $A_{mn}$ , with

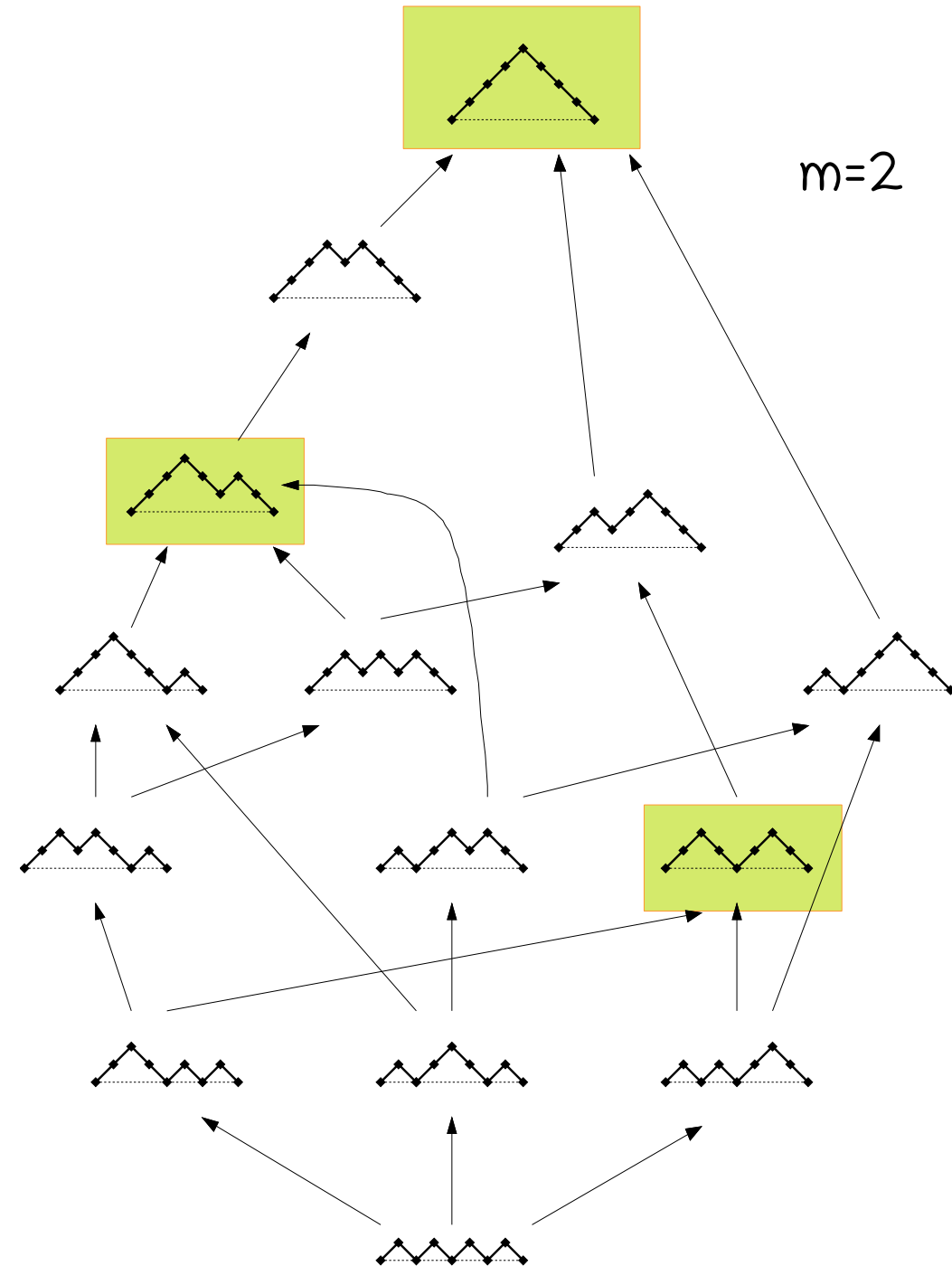
$$\min = (U^m D^m)^n$$

$$\max = U^{mn} D^{mn}.$$

In particular, it is a **lattice**.

# Mirrored m-Dyck paths

Mirrored m-Dyck paths only form a **join semi-lattice**.



# What about intervals?

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## Intervals in m-Dyck paths:

- Stanley lattice: **D-finite GF** (i.e., linear DE with pol. coeff's)

$$\frac{2(m+2)((m+1)n)!((m+1)(n+1))!}{n!(n+1)!(mn+2)!(m(n+2)+2)!}$$

- Tamari lattice: **algebraic GF** [mbm, Fusy, Préville-Ratelle 11]

$$\frac{m+1}{n(mn+1)} \binom{(m+1)^2n+m}{n-1}$$

Conj: Bergeron,  
Préville-Ratelle

- Greedy Tamari lattice: **algebraic GF** [mbm, Chapoton 24]

$$\frac{(m+2)(m+1)^{n-1}}{(mn+1)(mn+2)} \binom{(m+1)n}{n}.$$

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## V. Connection with the sylvester congruence

[Hivert, Novelli, Thibon 05]



# In the OEIS...

**Observation:** for  $m=1, 2, \dots, 5$ , the sequence  $\bar{g}_m(n)$  that counts intervals of mirrored  $m$ -Dyck paths appears in the OEIS.

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- **$m=1$ :** number of sylvester classes of 1-multiparking functions

Search: **seq:1,3,13,69,417,2759 id:243688**

Displaying 1-1 of 1 result found.

Sort: relevance | [references](#) | [number](#) | [modified](#) | [created](#)      Format: long | [short](#) | [data](#)

**A243688**      Number of Sylvester classes of 1-multiparking functions of length  $n$ .

**1, 3, 13, 69, 417, 2759**

([list](#); [graph](#); [refs](#); [listen](#); [history](#); [text](#); [internal format](#))

OFFSET      1,2

COMMENTS      See Novelli-Thibon (2014) for precise definition.

LINKS      [Table of  \$n, a\(n\)\$  for  \$n=1..6\$ .](#)

J.-C. Novelli, J.-Y. Thibon, [Hopf Algebras of  \$m\$ -permutations,  \$\(m+1\)\$ -ary trees, and  \$m\$ -parking functions](#), arXiv preprint arXiv:1403.5962, 2014. See Fig. 26.

KEYWORD      nonn,more

AUTHOR      [N. J. A. Sloane](#), Jun 14 2014

STATUS      approved

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Search: **seq:1,5,40,407,4797 id:243671**

Displaying 1-1 of 1 result found.

Sort: relevance | [references](#) | [number](#) | [modified](#) | [created](#)      Format: long | [short](#) | [data](#)

**A243671**      Number of Sylvester classes of 2-parking functions of length  $n$ .

**1, 5, 40, 407, 4797**

([list](#); [graph](#); [refs](#); [listen](#); [history](#); [text](#); [internal format](#))

OFFSET      1,2

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LINKS      [Table of  \$n\$ ,  \$a\(n\)\$  for  \$n=1..5\$ .](#)

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- and so on.

Search: **seq:1,5,40,407,4797 id:243671**

Displaying 1-1 of 1 result found.

Sort: relevance | [references](#) | [number](#) | [modified](#) | [created](#)      Format: long | [short](#) | [data](#)

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Link with the  
NT-lattice

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J.-C. Novelli, J.-Y. Thibon, [Hopf Algebras of  \$m\$ -permutations,  \$\(m+1\)\$ -ary trees, and  \$m\$ -parking functions](#), preprint arXiv:1403.5962, 2014. See Fig. 21.

KEYWORD      nonn,more

AUTHOR       [N. J. A. Sloane](#), Jun 14 2014

STATUS       approved

# The sylvester congruence

- Defined on words on the alphabet  $\mathbb{Z}$
- Generated by commutation relations:

$$ac \cdots b \equiv ca \cdots b, \quad a \leq b < c.$$

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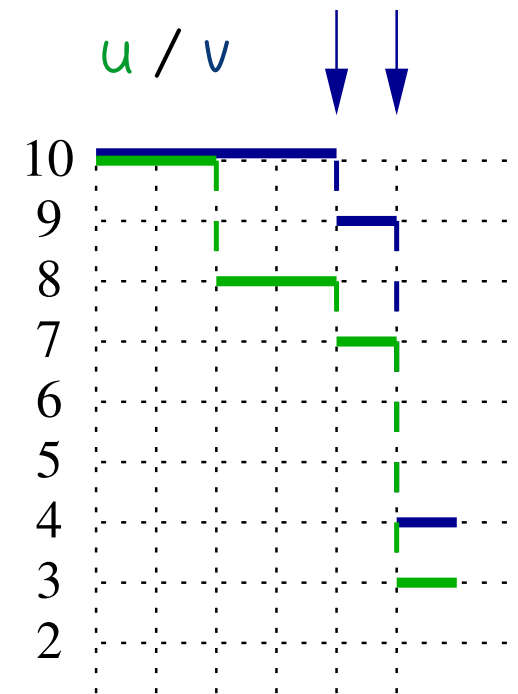
$$4\ \mathbf{2\ 5\ 3\ 3} \rightarrow 4\ \mathbf{5\ 2\ 3\ 3}, \text{ sylvester word}$$

**Def.** Let  $u=(u_1, \dots, u_n)$  and  $v=(v_1, \dots, v_n)$  be two nonincreasing sequences of integers. Then  $u \leq v$  for the NT order if

- $u$  lies below  $v$  ( $u_i \leq v_i$ )
- every descent of  $v$  is a descent of  $u$ .

$$v = (10, 10, 10, 10, 9, 4)$$

$$u = (10, 10, 8, 8, 7, 3)$$



# From sylvester words to Nadeau-Tewari intervals

**Example.** Fix  $n=6$  and a sylvester word  $w$  on the alphabet  $\{1, 2, \dots, n\}$ , containing the letter 1, say  $w = 3\ 2\ 2\ 2\ 2\ 5\ 1\ 1\ 1\ 5$ .

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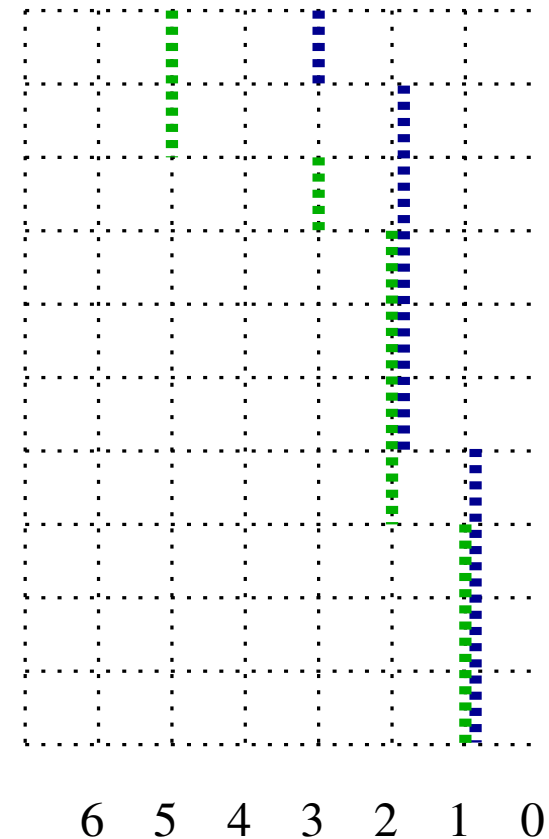
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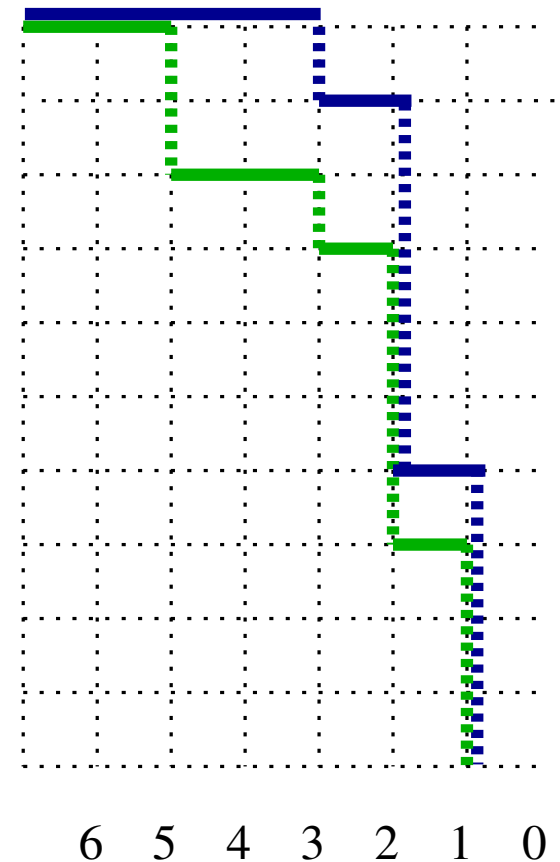
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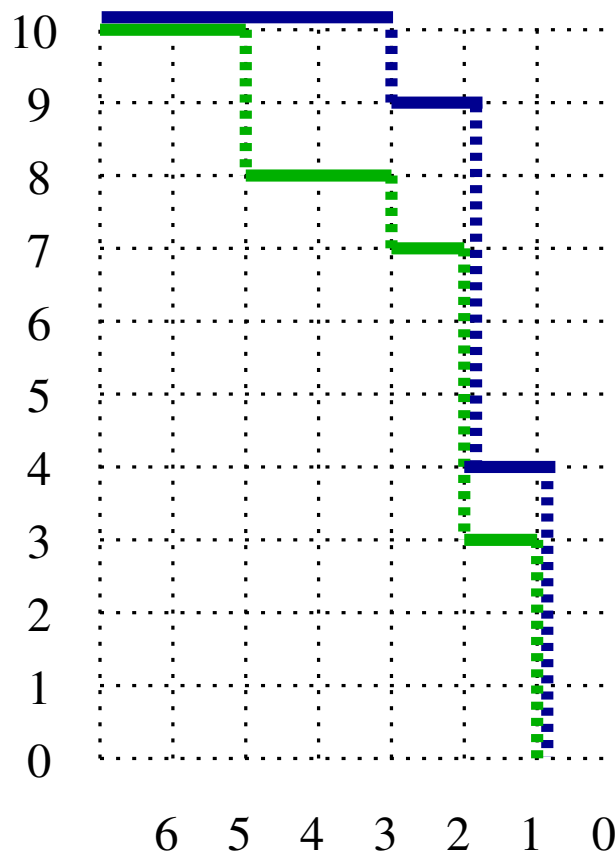
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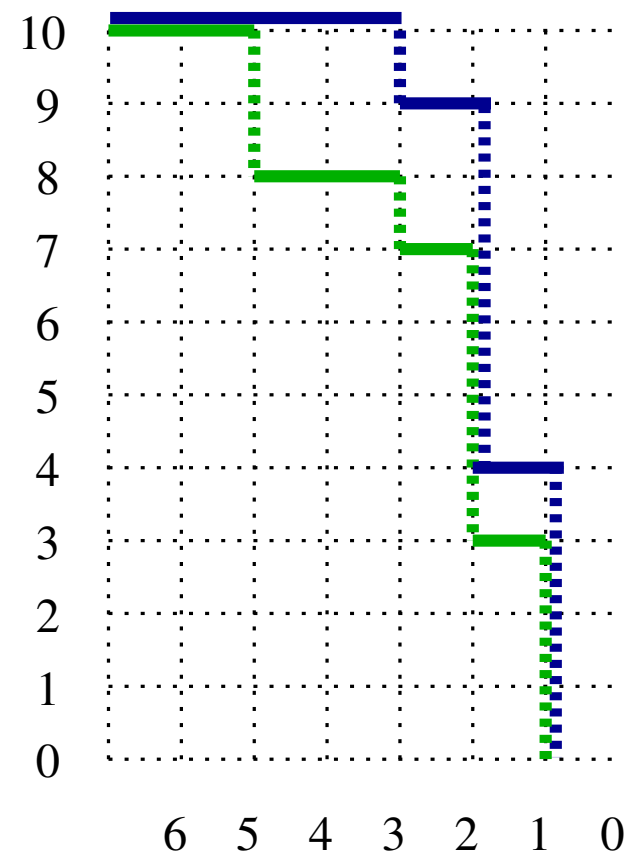
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**Proposition.** For any  $n$ , this construction gives a bijection between:

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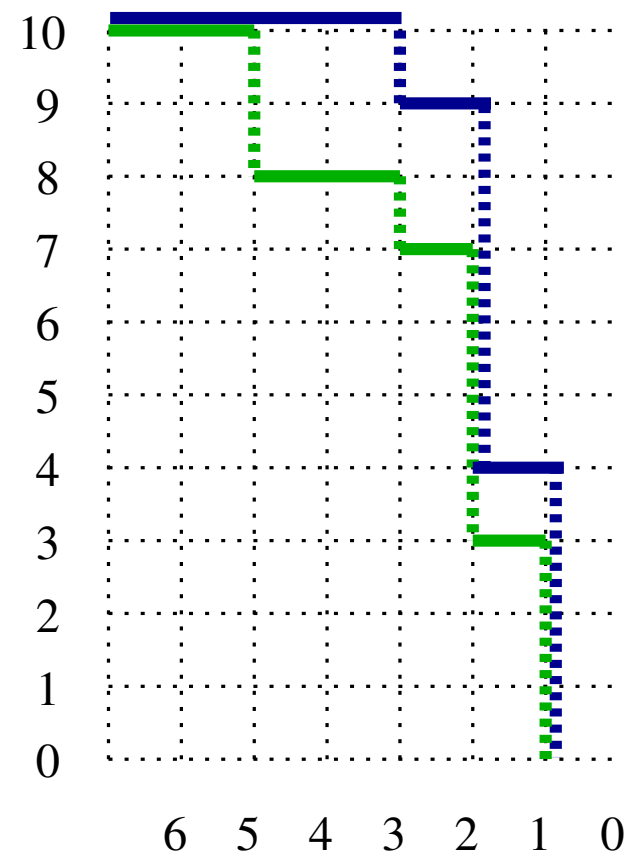
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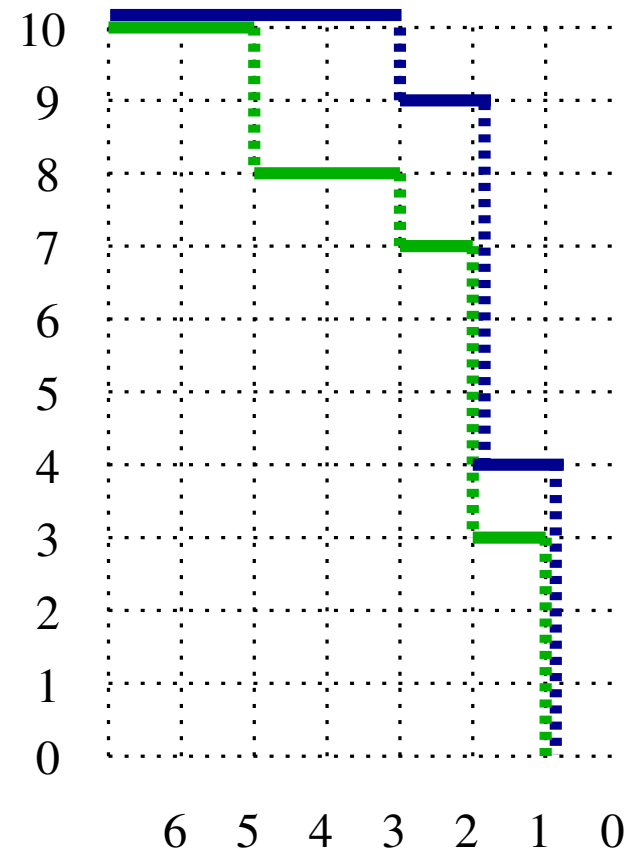
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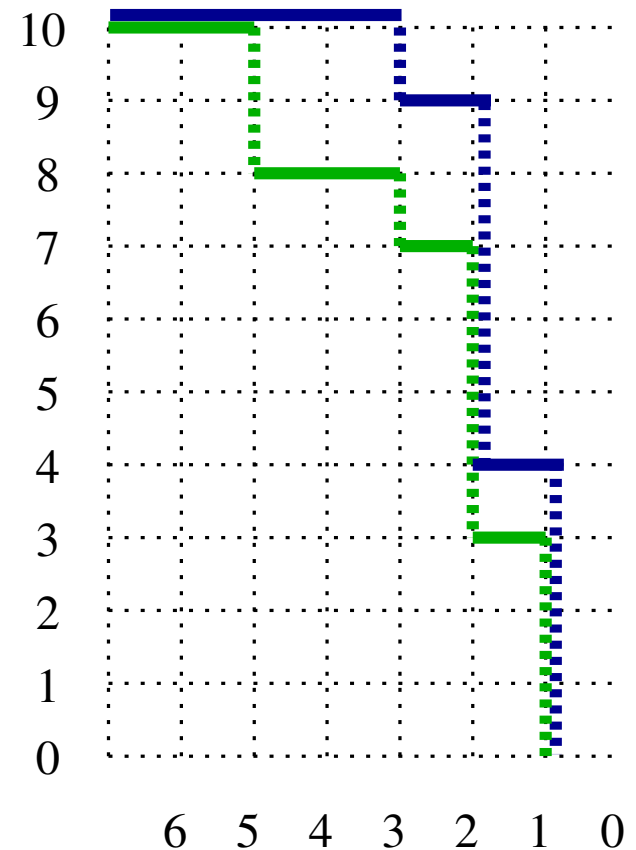
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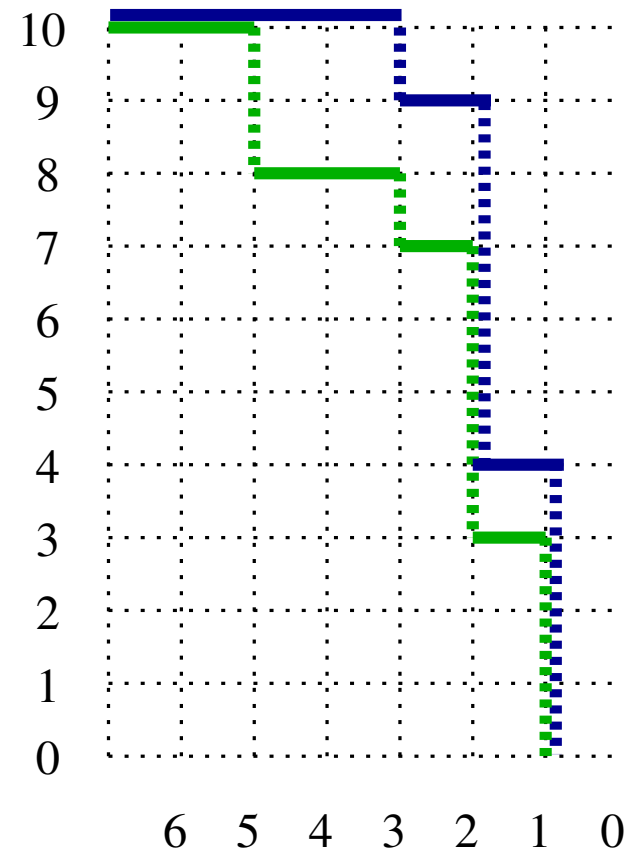
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**Specializations:** bijections between

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## **VI. Final remarks**

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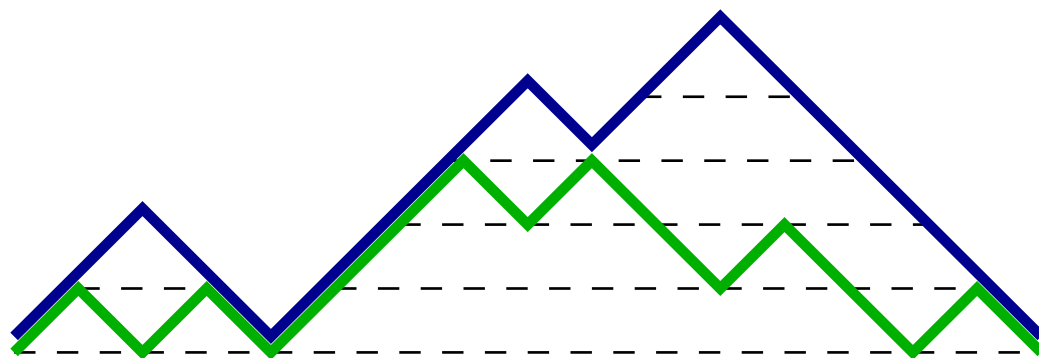
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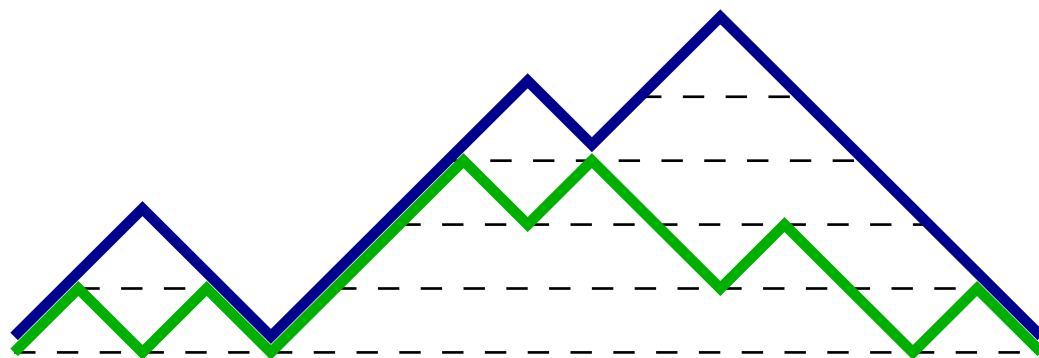
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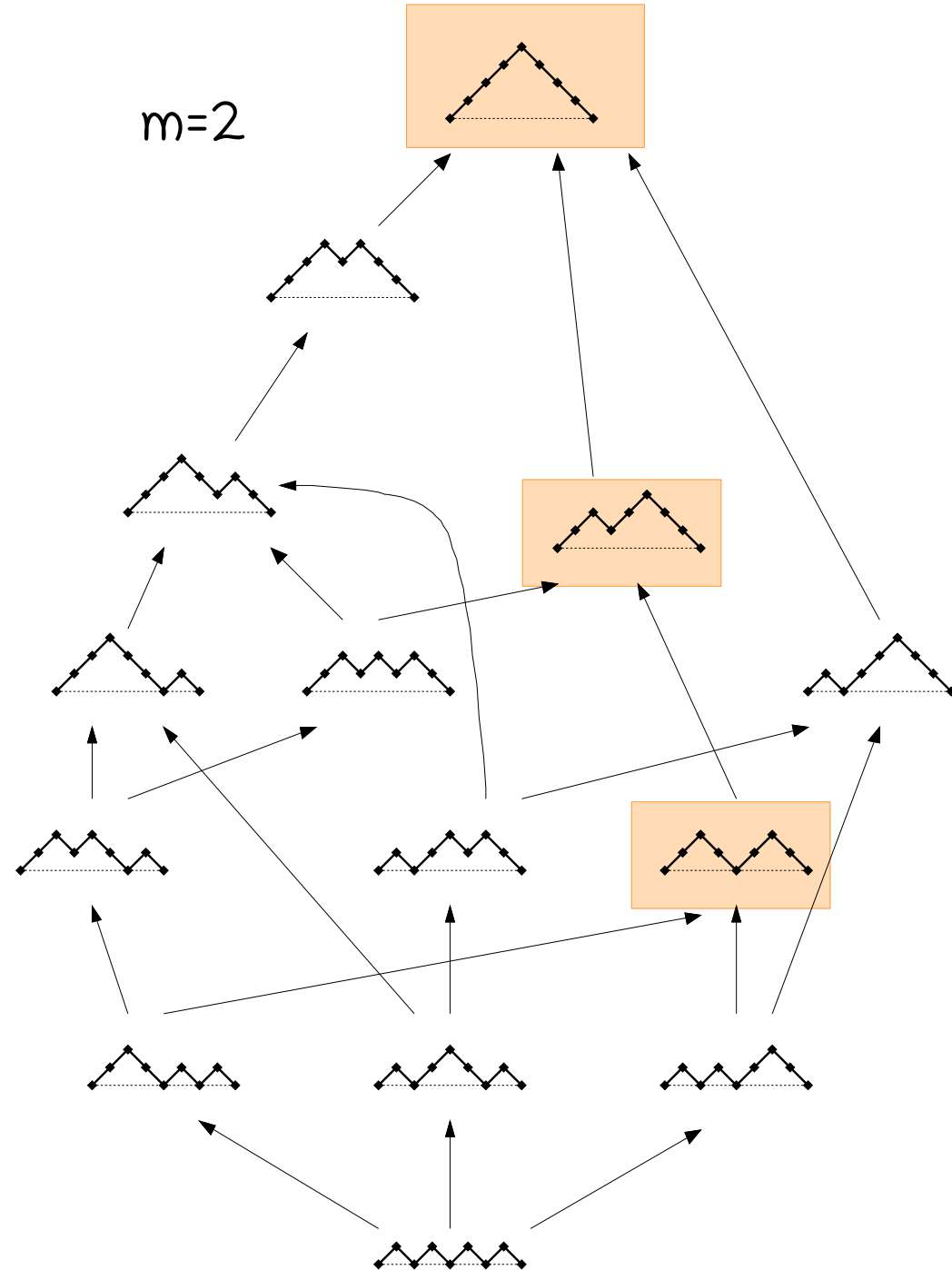
Thanks for your  
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# m-Dyck paths and mirrored m-Dyck paths

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