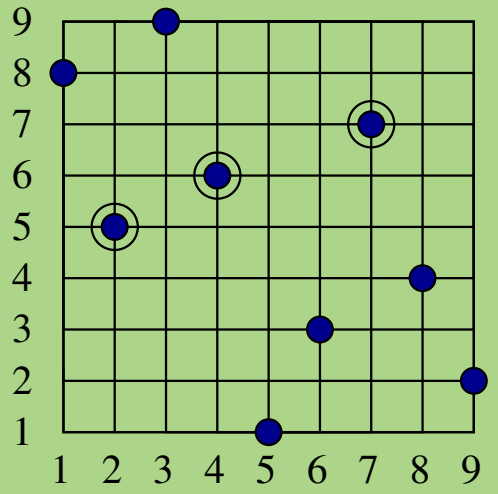
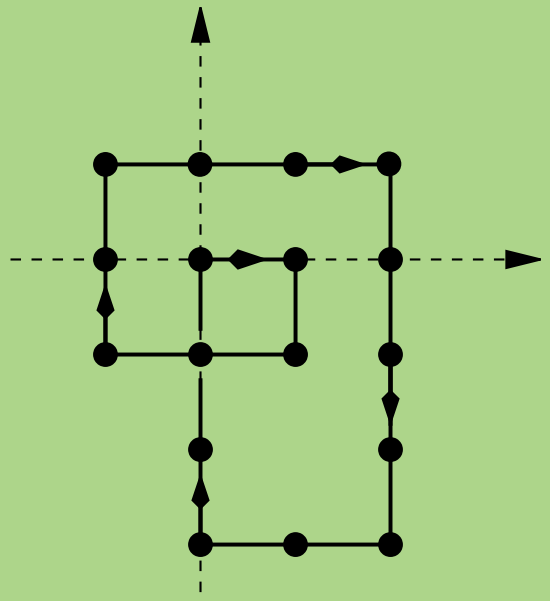
D-finite power series and diagonals in enumerative combinatorics





Mireille Bousquet-Mélou
CNRS, LaBRI, Université de Bordeaux, France

In this talk

I. Enumerative combinatorics and classes of generating functions

Main question:

How does one prove that a generating function is D-finite/algebraic/rational?

II. Proving D-finiteness, algebraicity, etc. : direct solutions

[interlude] "Positive" functional equations

III. Proving D-finiteness, algebraicity, etc. : undirect solutions

IV. Disproving D-finiteness, algebraicity, etc.

I. Enumerative combinatorics and generating functions

Enumerative combinatorics

- Let $\mathcal{A}$ be a set of discrete objects, equipped with an integer size:

$$|.| : \mathcal{A} \to \mathbb{N}$$

$$o \to |o|$$

and assume that the number of objects of size n is finite for any n. Denote this number by a(n).

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Def. The ordinary generating function (GF) of the objects of $\mathcal{A}$, counted by their size, is

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Multivariate enumeration: record additional (integer) statistics with additional variables $x_1, \ldots, x_d$.

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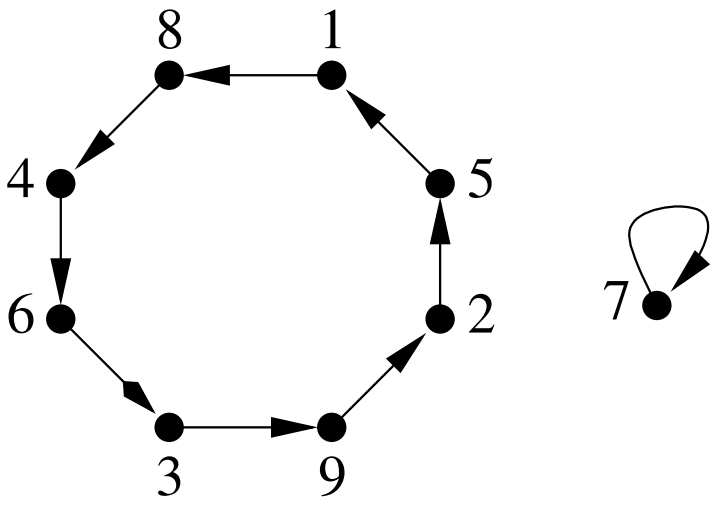

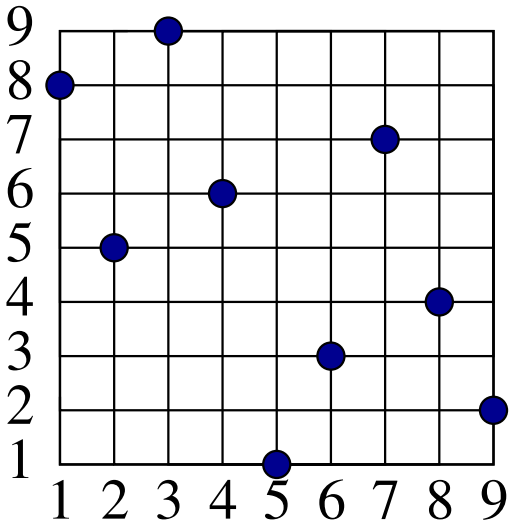

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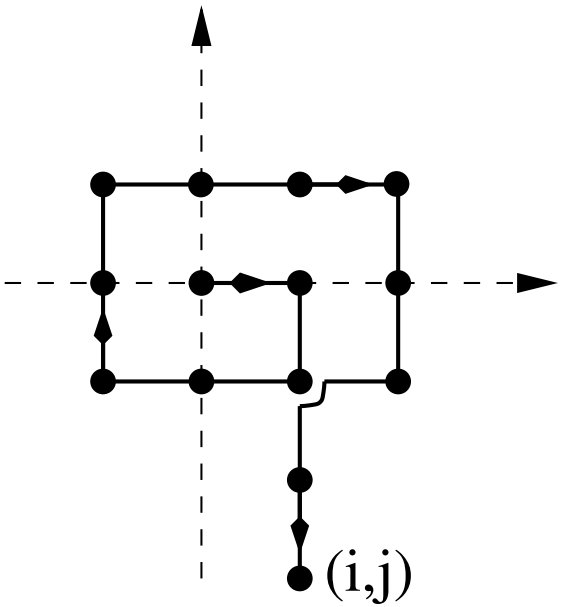

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$$A = \sum_{\text{walk}} x^i y^j t^{\text{length}}$$

$$= \sum_{n \geq 0} \left(x + \frac{1}{x} + y + \frac{1}{y} \right)^n t^n$$

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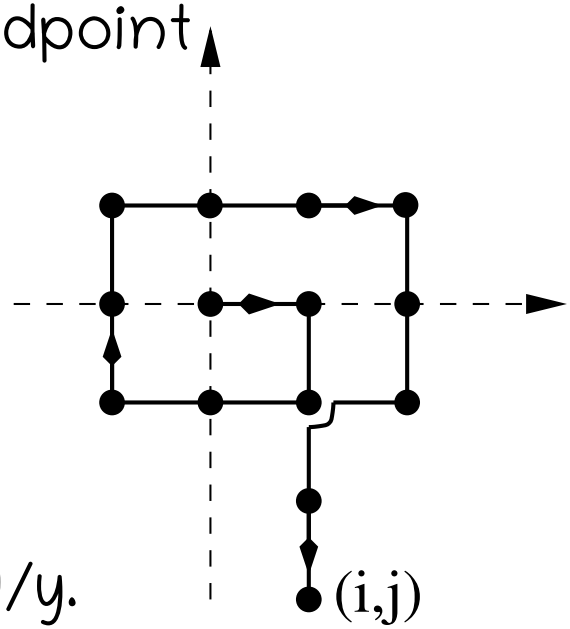
with $\bar{x}$=1/x and $\bar{y}$=1/y.



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D-finite

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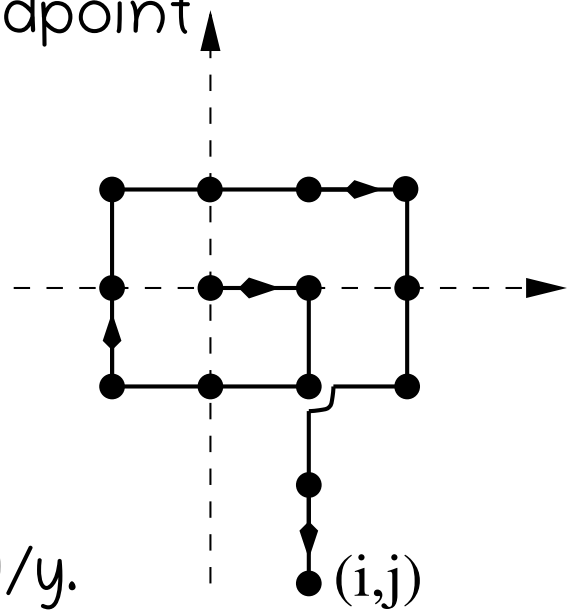

A hierarchy of formal power series

- Rational

$$A(t) = \frac{1-t}{1-t-t^2}$$

- Algebraic

$$1 - A(t) + tA(t)^2 = 0$$

- D-finite

$$t(1-16t)A''(t) + (1-32t)A'(t) - 4A(t) = 0$$

- D-algebraic

$$(2t + 5A(t) - 3tA'(t))A''(t) = 48t$$

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Inside the D-finite class (in one variable)

- **G-functions**: D-finite GFs with a **positive radius** of convergence

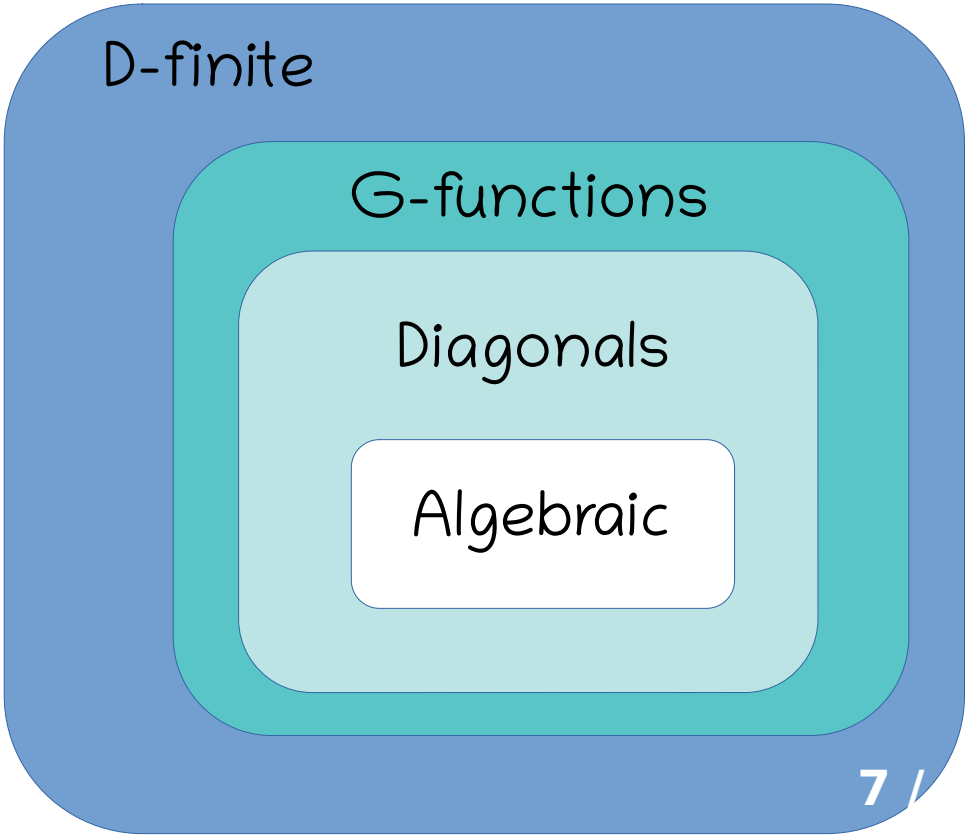
- **Diagonals (of rational series)**: for R a rational series in d variables,

$$R = \sum_{n_1,\dots,n_d \geq 0} a(n_1,\dots,n_d) x_1^{n_1} \cdots x_d^{n_d} \in \mathbb{Q}(x_1,\dots,x_d)$$

the **diagonal** of R is

$$\Delta R := \sum_{n \geq 0} a(n,n,\dots,n) t^n.$$

Recall: our GFs have integer coefficients.



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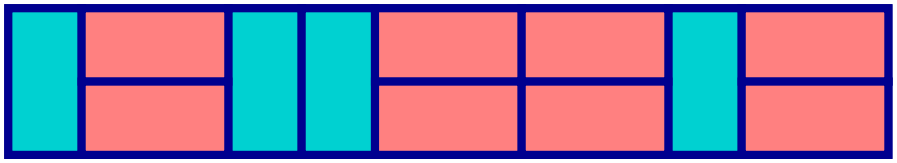
When a GF is a G-function, is it a diagonal?

II. Direct solutions

A basic rational example

Number of tilings of a 2 by n board with dominoes (here, n=12):



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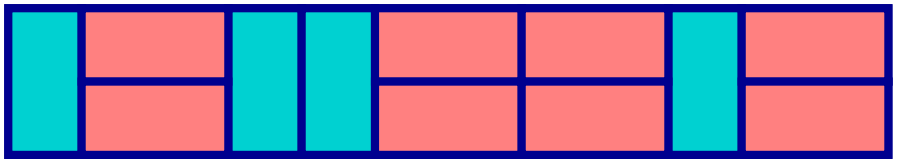

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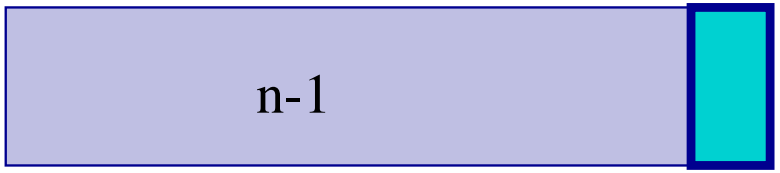

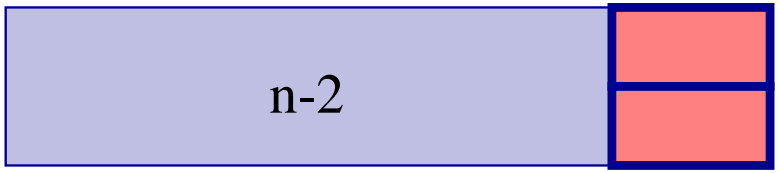

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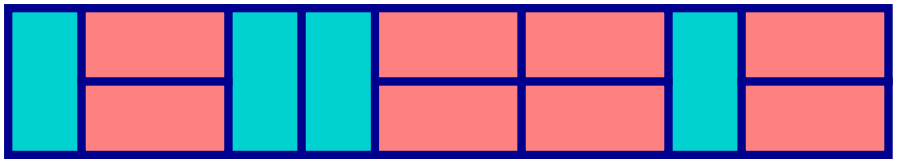

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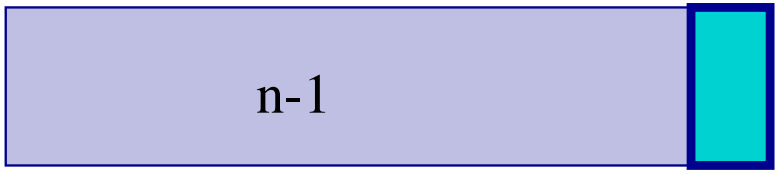

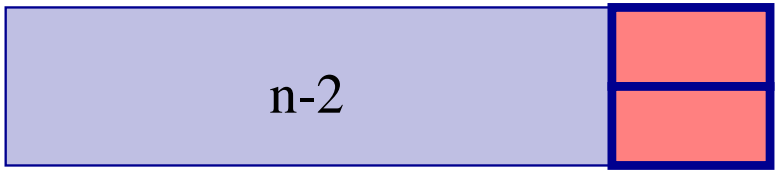

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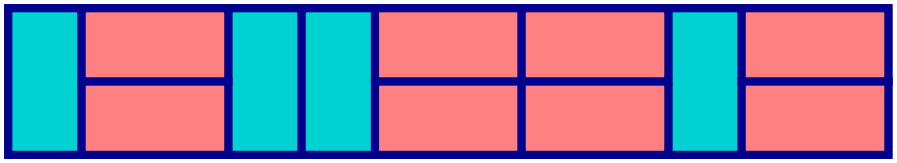

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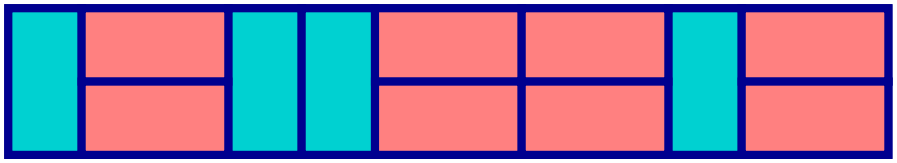

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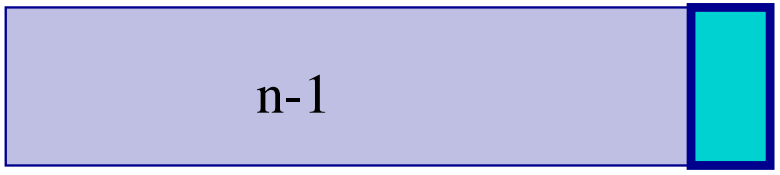

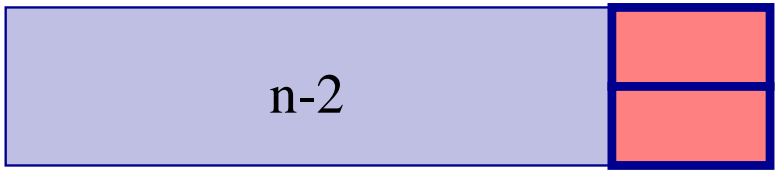

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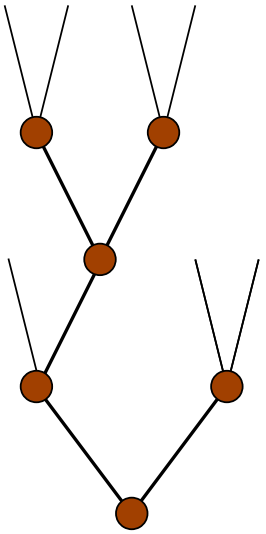
Binary trees with n internal nodes (here n=6)



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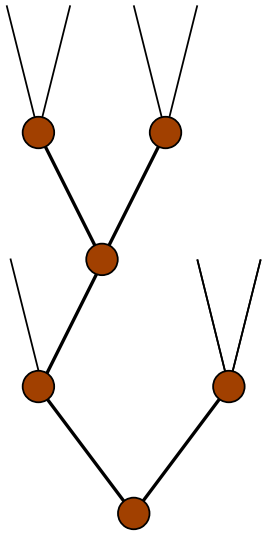

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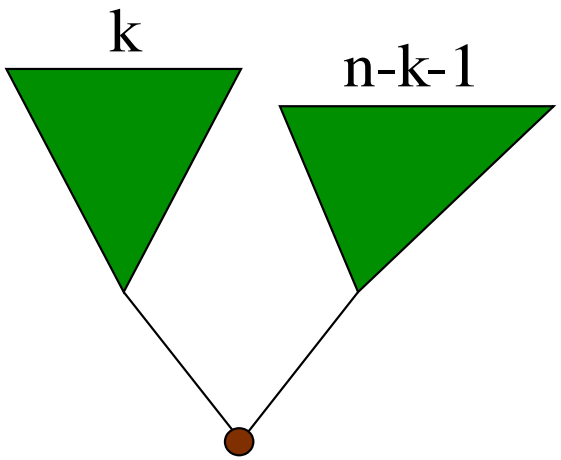

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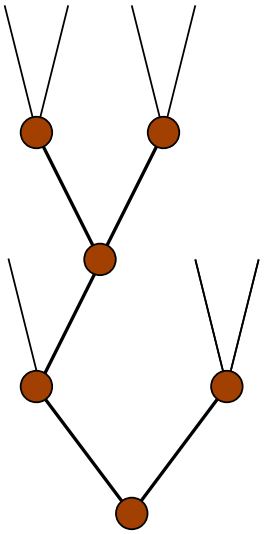

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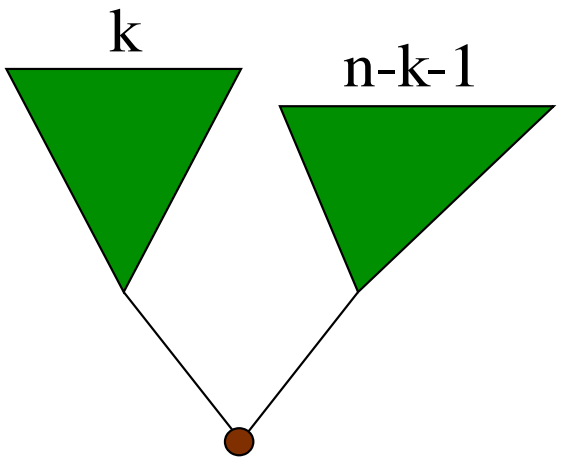

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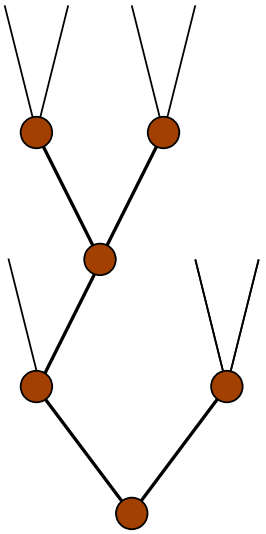

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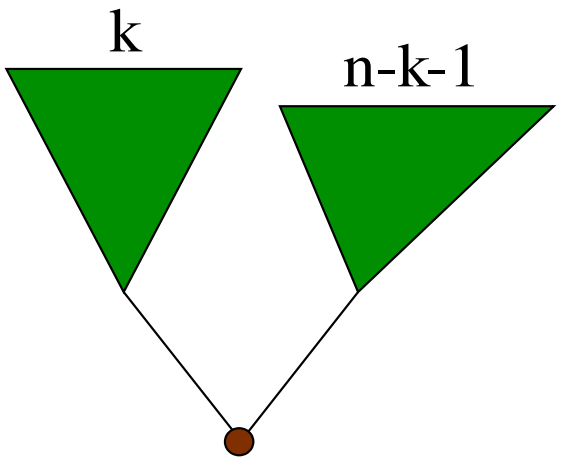

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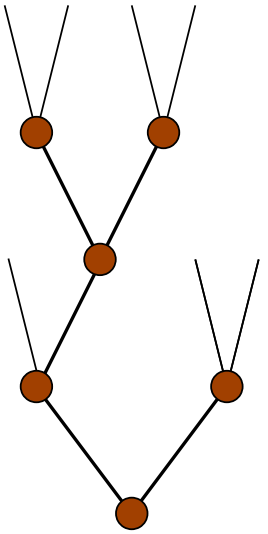

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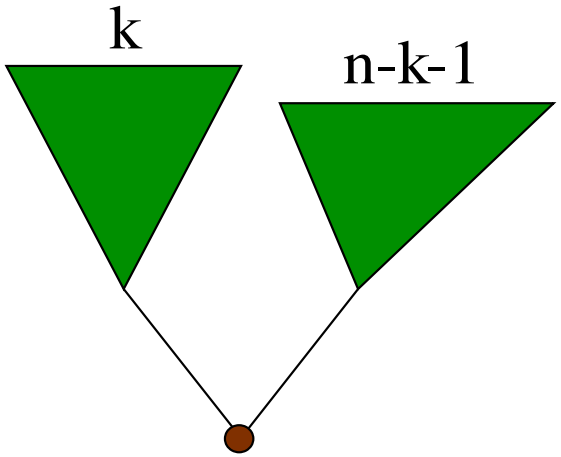

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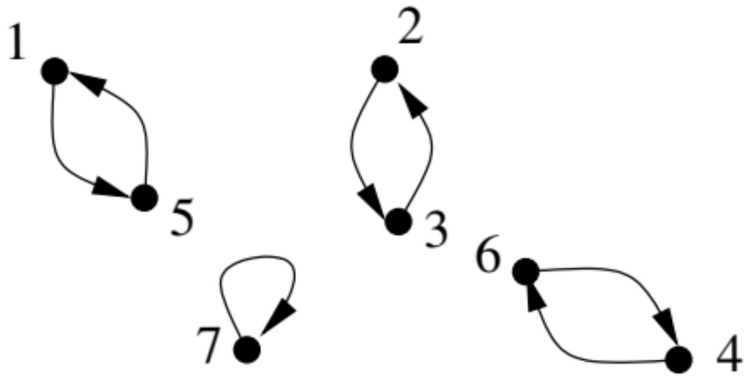
A basic D-finite example

Involutions on {1, 2, … , n} (here n=7)



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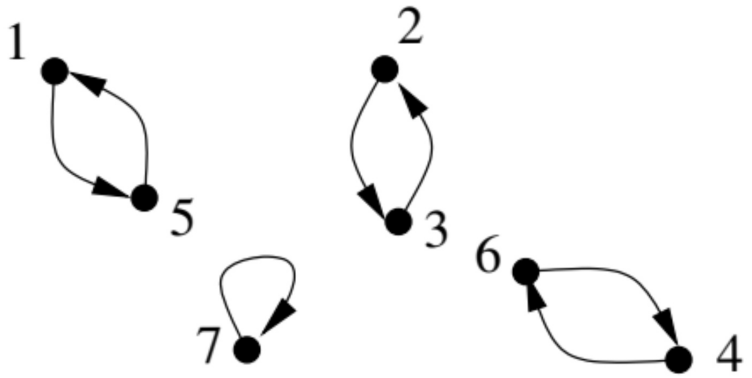

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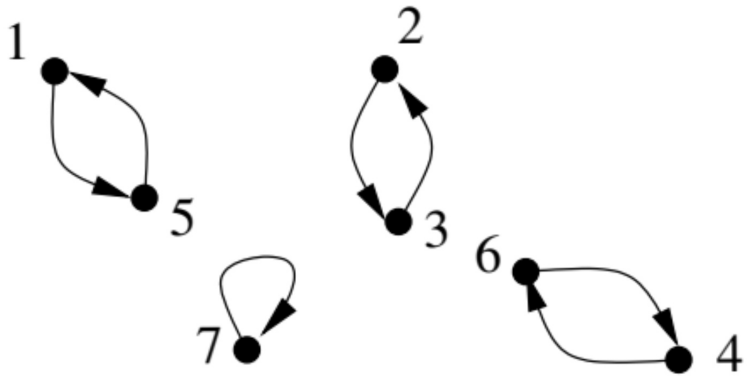

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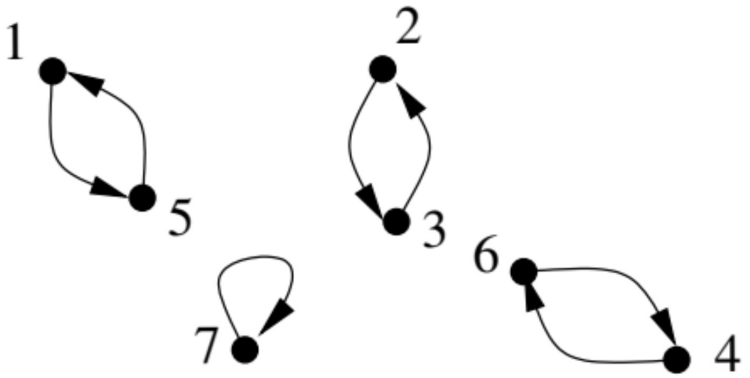

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Rem. This series is NOT a G-function, because $a(n)$ grows too

$$a(n) \sim \frac{1}{\sqrt{2}} \exp\left(-\frac{n}{2} + \sqrt{n} - \frac{1}{4}\right) n^{n/2}$$

Another basic D-finite example

All walks on the square lattice, counted by length (var. t) and final coordinates (vars. x,y):

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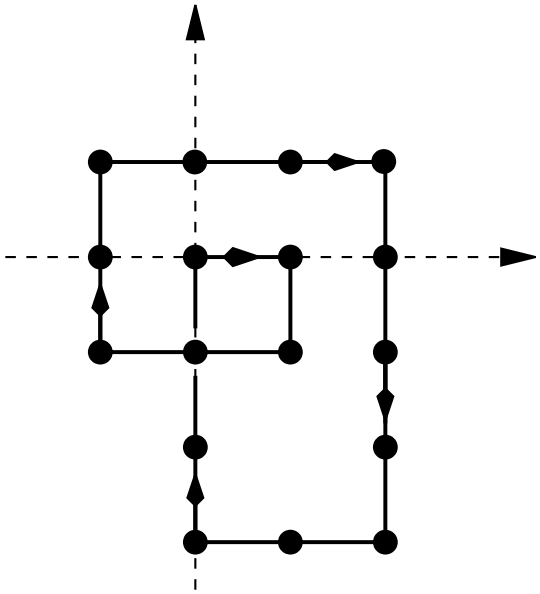
Closed walks (i.e., ending at (i,j)=(0,0)):



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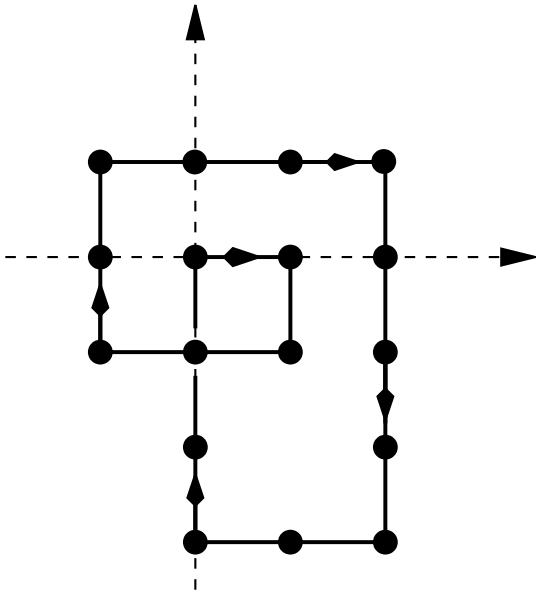

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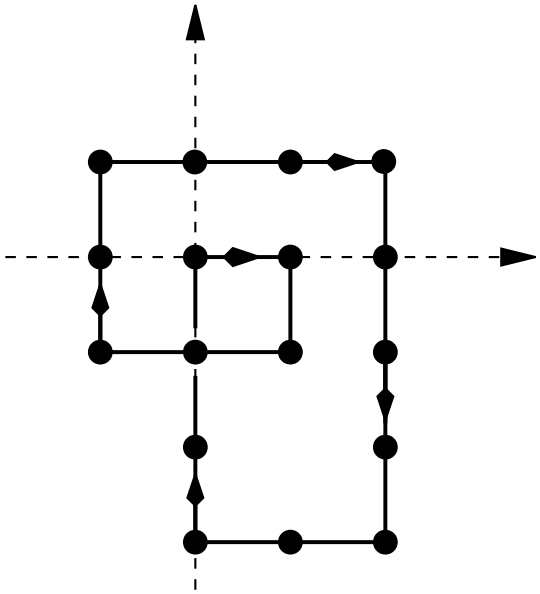

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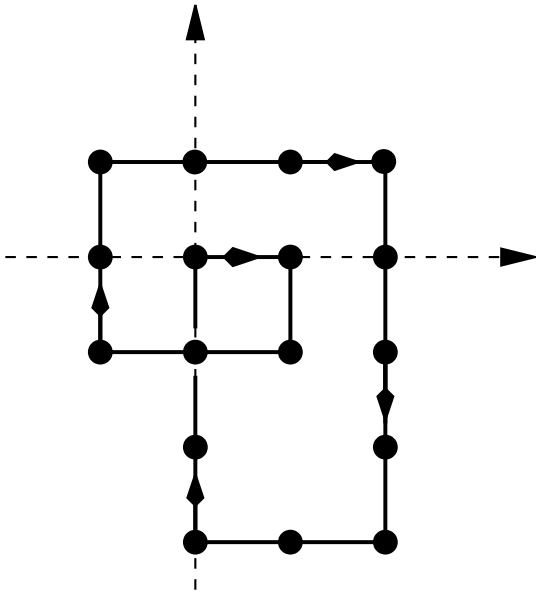
Intuition?



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Symmetric functions and D-finiteness

Closure property: the (partial) diagonals of a multivariate D-finite series are always D-finite.

Application: some classes of objects can be counted by extracting coefficients in series $A(x_1, x_2, \ldots)$ that are D-finite in the power sums

$$p_k = \sum_i x_i^k$$

Examples:

- d-regular graphs on vertices 1, 2, … , n, for d fixed
- square matrices of size n with non-negative coefficients summing to d on every row/column
- permutations of {1, 2, … , n} with no ascending sub-sequence of length d

[Gessel 90, Goulden & Jackson ~90 ; Chyzak, Mishna & Salvy 05]

Symmetric functions and D-finiteness: example

Let a(n) be the number of 3-regular graphs on vertices 1, 2, ... , n. Then

$$a(n) = [x_1^3 x_2^3 \cdots x_n^3] \prod_{1 \le i < j} (1 + x_i x_j) = [x_1^3 x_2^3 \cdots x_n^3] G$$

where

$$G = \prod_{1 \le i < j} (1 + x_i x_j) = \exp\left(\sum_{k \ge 1} (-1)^{k+1} \frac{p_k^2 - p_{2k}}{2k} \right)$$

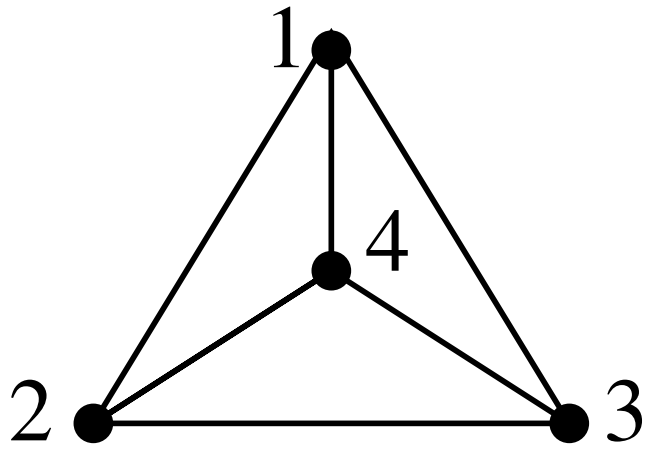
is D-finite is the p_k's, with

$$p_k = \sum_i x_i^k$$

The D-finiteness of

$$A = \sum_{n \ge 0} a(n) t^n$$

follows from closure properties.



[Interlude]

Some rational/algebraic/
D-finite series
...
are more combinatorial
than others

Reverse engineering

So far: a class of objects gives a **generating function**.

Here: some formal power series have a **natural combinatorial interpretation** as generating functions of certain objects.

Two "historical" families:

- $\mathbb{N}$-rational series
- $\mathbb{N}$-algebraic series

More recently:

- "natural" combinatorial families of **D-finite series**

$\mathbb{N}$-algebraic series

Example:

$$\begin{array}{lllllllll} A_1 & = & 1+2t & + & 4tA_1 & + & (12t+t^2)A_2 & + & t^2A_1A_2^2 \\ A_2 & = & & & 14tA_1 & + & 8tA_2 & + & 3tA_1^3 \end{array}$$

Definition: a series A_1 is **$\mathbb{N}$-algebraic** if it is the first component of a k-tuple $(A_1, \dots, A_k)$ satisfying a polynomial system of the form

$$\mathcal{A} = \begin{pmatrix} A_1 \\ \vdots \\ A_k \end{pmatrix} = \Phi(\mathcal{A}) = \begin{pmatrix} P_1(t, A_1, \dots, A_k) \\ \vdots \\ P_k(t, A_1, \dots, A_k) \end{pmatrix}$$

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where

- $P_j(t, x_1, \dots, x_k)$ is a polynomial **with coefficients in $\mathbb{N}$**, for each j, **of degree at most 1 in each x_i**
- the map Φ^m is a contraction for some m.

Combinatorial interpretations: $\mathbb{N}$-rational series

Let A_1 be an $\mathbb{N}$-rational series. Then $A_1 - a_1(0)$ counts **walks on a graph.**

Each series A_i will count walks from vertex 0 to vertex i on a directed finite graph.

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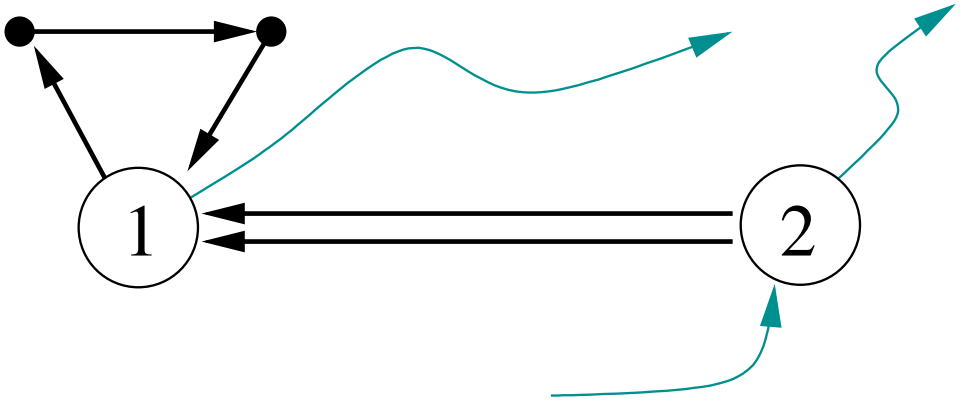

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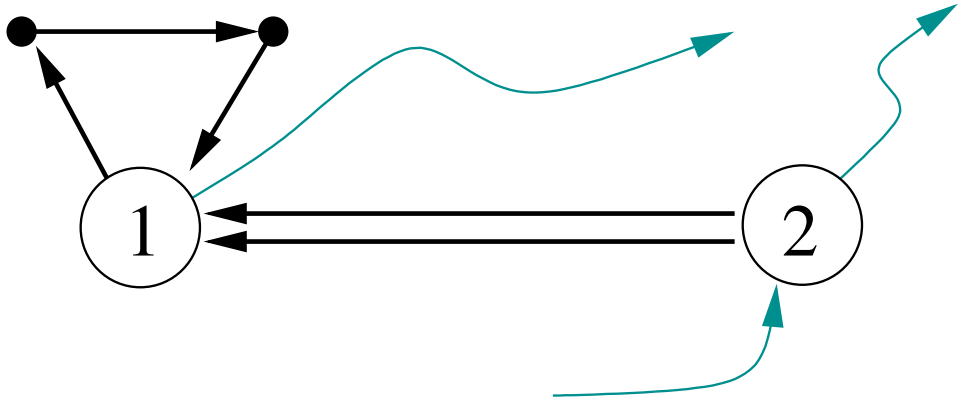

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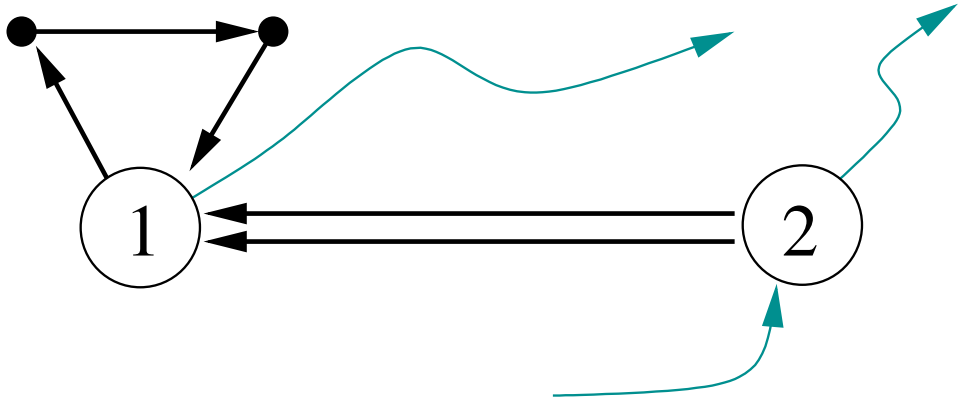

Walks on finite graphs have a **linear structure.**

In fact, all objects known to have a rational GF have an $\mathbb{N}$-rational GF.

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Let A_1 be an $\mathbb{N}$-algebraic series. Then $A_1 - a_1(0)$ counts by the number of nodes certain **coloured trees.**

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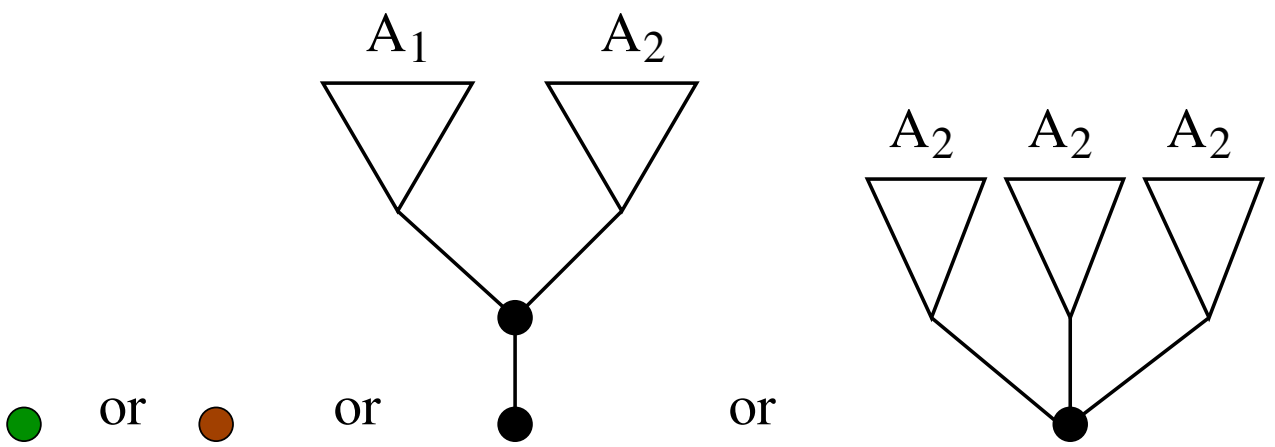

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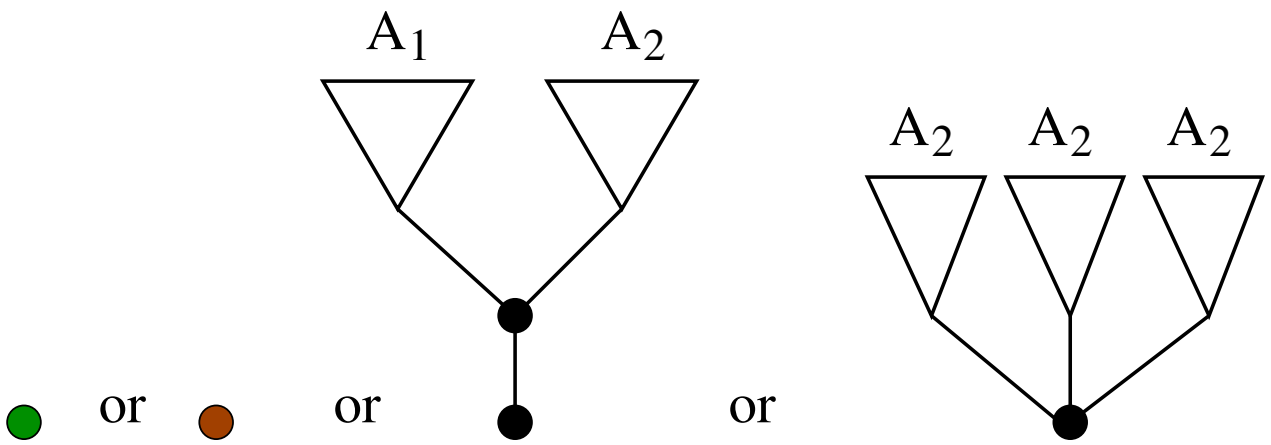

Trees of all kinds have a branching structure.

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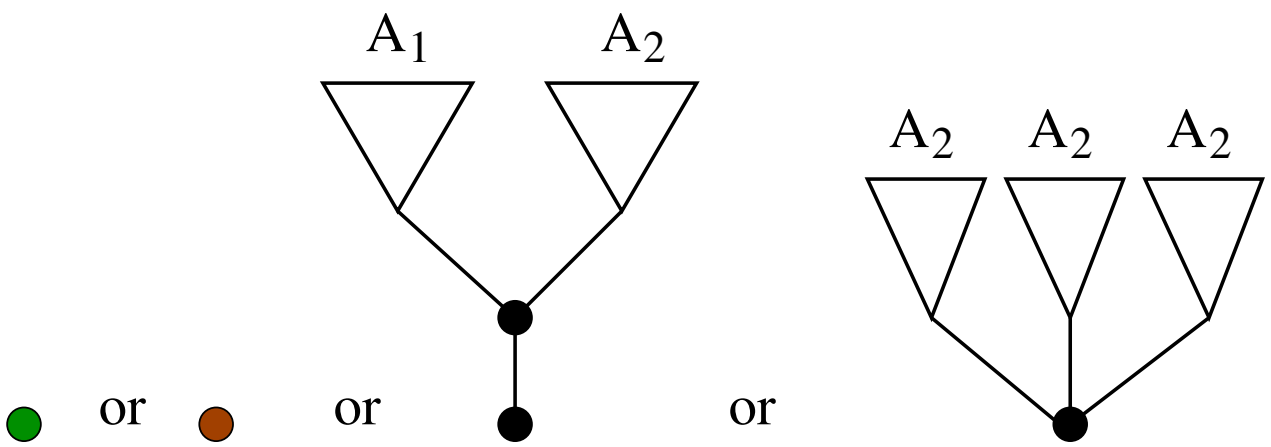

Trees of all kinds have a branching structure.

However, many objects have an algebraic, but NOT $\mathbb{N}$-algebraic, GF.

Some criteria

Thm. Let A be a rational series with coefficients in $\mathbb{N}$. It is **$\mathbb{N}$-rational** if and only if all its **poles of minimal modulus** are of the form $\rho e^{i\theta}$, where $e^{i\theta}$ is a root of unity. [Berstel 71, Soittola 76]

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No counterpart for $\mathbb{N}$-algebraic series...

A necessary asymptotic condition for $\mathbb{N}$-algebraicity: if

$$a(n) \sim \kappa \mu^n n^{\gamma-1}$$

then μ is algebraic and

$$\gamma \in \left\{-\frac{1}{2^k}, k \geq 1\right\} \cup \left\{\frac{m}{2^k}, m \geq 1, k \geq 0\right\}$$

NB. The algebraicity itself only implies that $\gamma \in \mathbb{Q} \setminus \{0, -1, -2, \ldots\}$.

[Banderier-Drmota 15]

$\mathbb{N}$-algebraic series vs. $\mathbb{N}$-rational series

Remark. $\mathbb{N}$-rationality and $\mathbb{N}$-algebraicity can be defined (via positive systems) for **multivariate series**.

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  Conjecture [Garrabrant-Pak 14(a)] Not true for $A=1+tA^2$.

Combinatorial D-finite series?

Several attempts:

- [Kotek-Makowsky 12]: an interpretation in terms of square lattice paths for some D-finite series
- [Garrabrant-Pak 14(a)]: a combinatorial interpretation for all **diagonals of ℕ-rational series** $R(x_1, \ldots, x_k)$ in terms of tilings of a strip with tiles of irrational sizes
- [Koechlin et al. 20+]: a combinatorial interpretation of the **same class** in terms of constrained walks on a graph
- [Koechlin et al. 20+]: a related combinatorial interpretation for **diagonals of ℕ-algebraic series**

III. Undirect solutions

A recursive description of the objects yields a recurrence relation for a(n), and a functional equation for the series A, that is not of rational/polynomial/differential type.

Additional variables

Instead of A=A(t), one considers a multivariate series $A(t;x_1, \dots, x_d)$, and proves that it satisfies some functional equation.

Examples:

- Integer partitions: a **q-equation** (or t-equation...)

$$A(t;x) = tx + \frac{1}{1-tx} A(t;xt)$$

- Set partitions (one variable only!): a **difference equation**

$$A(t) = 1 + \frac{t}{1-t} A\left(\frac{t}{1-t}\right)$$

- Walks on a half-line: a **discrete differential equation**

$$A(t;x) = 1 + txA(t;x) + t\frac{A(t;x) - A(t;0)}{x}$$

The variable x is "**catalytic**".

A basic example: walks on a half-line

Let a(n;k) be the number of n-step walks with steps +1 and -1, that start at 0, **remain non-negative**, and end at height k. Then

$$a(n;k) = a(n-1;k-1) + a(n-1;k+1)$$

with the boundary condition a(n; k)=0 for k<0. This translates into

$$A(t;x) = 1 + txA(t;x) + t\frac{A(t;x) - A(t;0)}{x},$$

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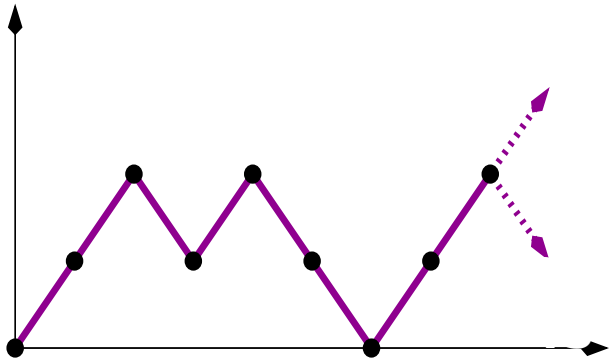

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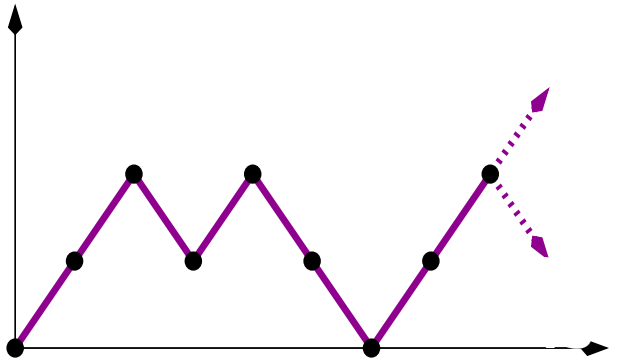
In short,

$$A(x) = 1 + txA(x) + t\frac{A(x) - A(0)}{x}.$$



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we can replace the series $A(t;0)$ by a series $B(t)$:

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A single equation may define several formal power series.

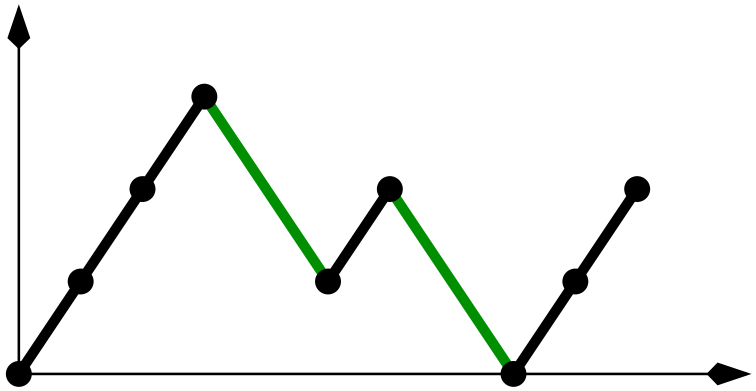
More examples in one catalytic variable

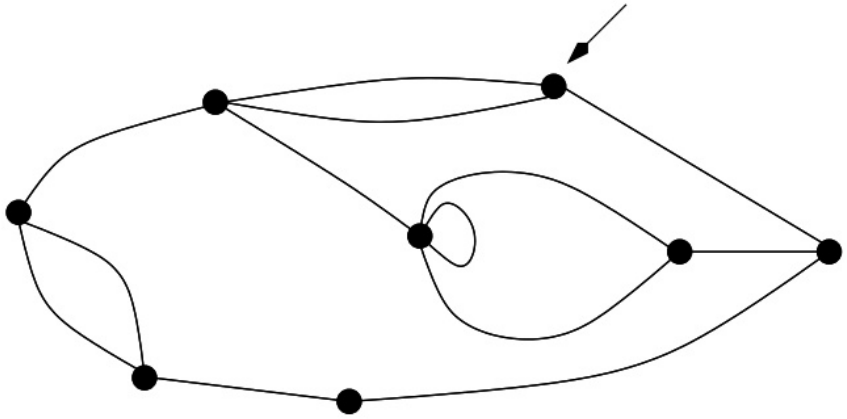
- **Walks with steps +1 and -2** on a half-line:

$$A(x) = 1 + txA(x) + t\frac{A(x) - A(0) - xA'(0)}{x^2}$$

- **Planar maps** [Tutte 68]: non-linear, one catalytic variable

$$A(x) = 1 + tx^2A(x)^2 + tx\frac{xA(x) - A(1)}{x - 1}$$





One catalytic variable: algebraicity

Theorem [MBM-Jehanne 06]

Let $P(A(x), B_1, B_2, \ldots, B_k, t, x)$ be a polynomial equation in one catalytic variable x. Under natural (combinatorial) assumptions, the series $A(x)=A(t;x)$ and the $B_i=B_i(t)$'s are **algebraic**.

Algebraic series

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The same holds for **systems** of such equations, defining series $A_1(x), \ldots, A_n(x)$ and $B_1, B_2, \ldots, B_k$.

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One catalytic variable: algebraicity

Theorem [MBM-Jehanne 06]

Let $P(A(x), B_1, B_2, \ldots, B_k, t, x)$ be a polynomial equation in one catalytic variable x. Under natural (combinatorial) assumptions, the series $A(x)=A(t;x)$ and the $B_i=B_i(t)$'s are **algebraic**.

[Popescu 85, Swan 98]

Extension [Notarantonio-Yurkevich 22(a)]

The same holds for **systems** of such equations, defining series $A_1(x), \ldots, A_n(x)$ and $B_1, B_2, \ldots, B_k$.

effective

Algebraic series

Example: planar maps

$$A(x) = 1 + tx^2A(x)^2 + tx\frac{xA(x) - A(1)}{x-1}$$

$$\Rightarrow 7A(1)^2t^2 + (1-18t)\,A(1) + 16t - 1 = 0$$

Popescu's theorem (in d+1 variables)

Here d=2, hence 3 variables t, x, y.

Thm. If a system of equations of the form

$$\mathrm{Pol}\,(A_1, \ldots, A_n, B_1, \ldots, B_m, C_1, \ldots, C_k, t, x, y) = 0$$

has a **unique solution** such that the following **diagonal conditions** hold

$$A_i \in \mathbb{C}[[t, x, y]], \quad B_i \in \mathbb{C}[[t, x]], \quad C_i \in \mathbb{C}[[t]],$$

then each of these series is **algebraic**.

Example:

$$A(t; x, y), \quad A(t; x, 0), \quad A(t; 0, 0) \qquad \cancel{A(t; 0, y)}$$

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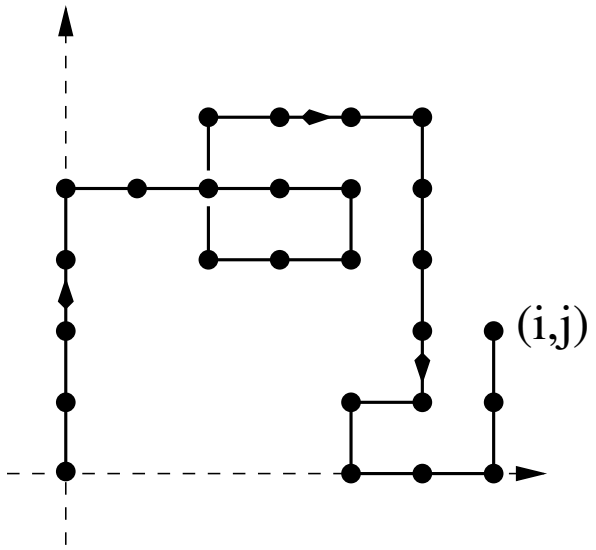
Alas... such equations do not occur much in combinatorics!

Two discrete derivatives: some examples

- Square lattice walks confined to a quadrant: linear equation

$$Q(x,y) = 1 + t(x+y)Q(x,y) + t\frac{Q(x,y) - Q(x,0)}{y} + t\frac{Q(x,y) - Q(0,y)}{x}$$



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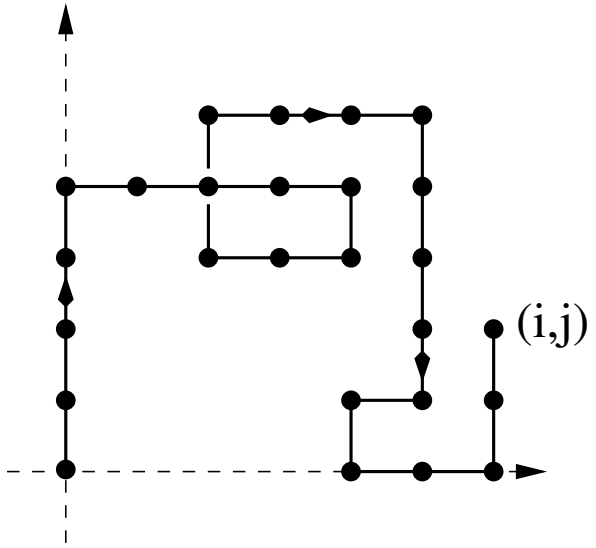

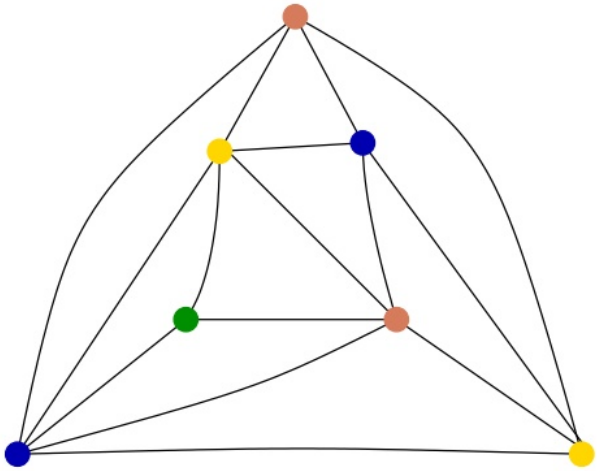

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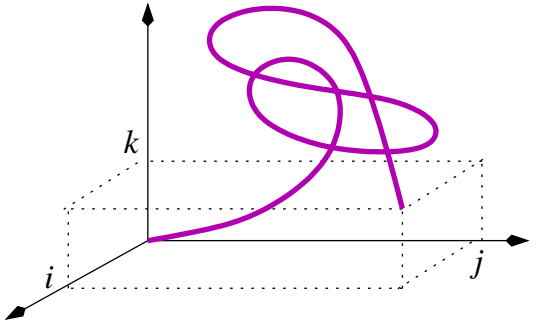
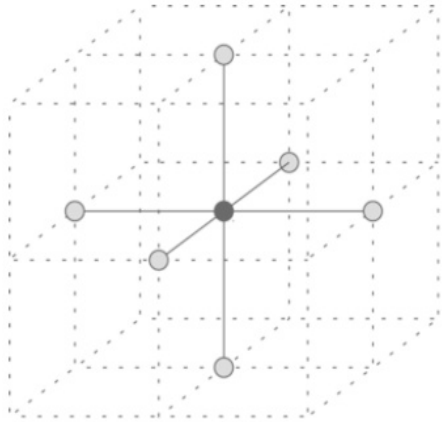
- Three-stack sortable permutations [Defant, Elvey Price, Guttmann 21]

$$P(x,y) = t(x+1)^2(y+1)^2 + ty(1+x)P(x,y) + t(1+x)\frac{P(x,y) - P(x,0)}{y}\left((1+y)^2 + y\frac{P(x,y) - P(0,y)}{x}\right)$$

Arbitrarily many discrete derivatives

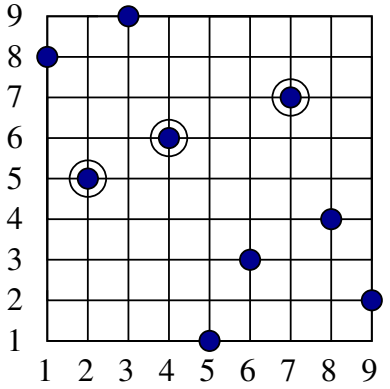
- Walks in $\mathbb{N}^d$ with unit steps $(0,\dots,0,\pm 1,0,\dots,0)$:





- Permutations with no ascending sub-sequence of length (d+2)

[Guibert 95], [mbm 11]



A natural question

Solutions of discrete differential equations (with derivatives with respect to **one variable** only) are **algebraic**.

$$A(t;x) = 1 + txA(t;x) + t\frac{A(t;x) - A(t;0)}{x}$$

Is it always true for solutions of discrete PDEs (with derivatives with respect to **several variables**)?

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Well, sometimes

The story of walks confined to the quadrant

About twenty years ago...

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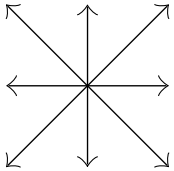
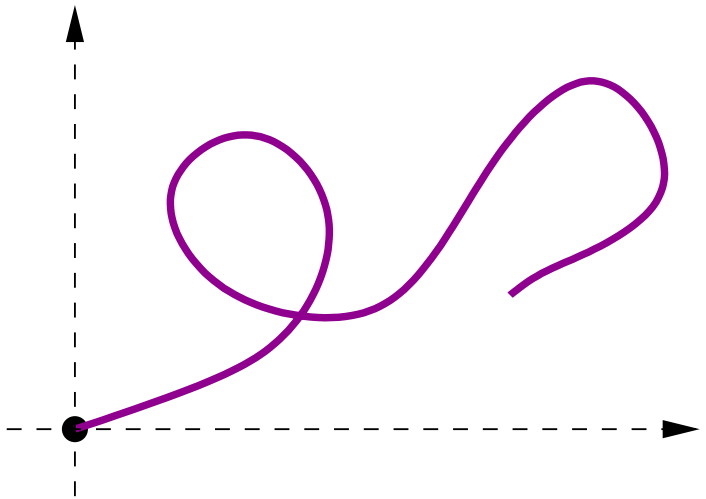
Set of steps ("model") in 



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⇒ 79 really interesting and distinct small step models [mbm-Mishna10]

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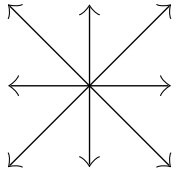
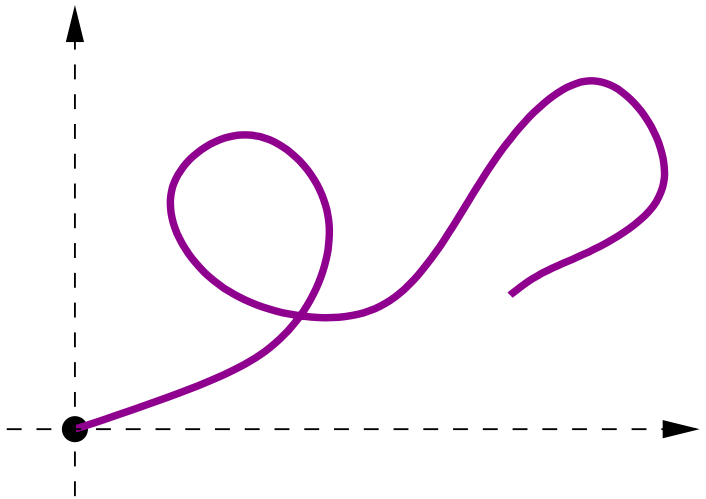
Set of steps ("model") in 



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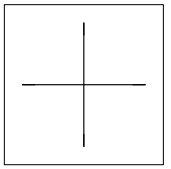

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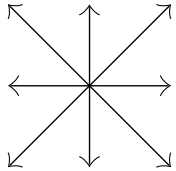
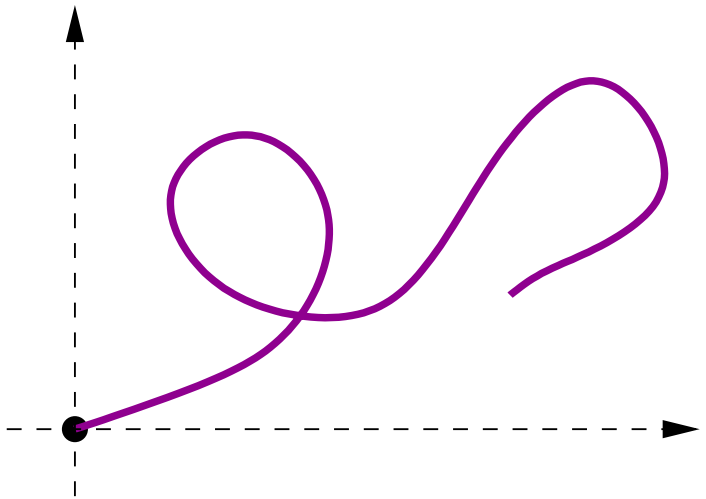
Set of steps ("model") in 



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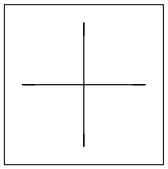

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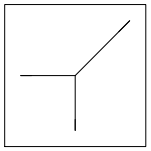
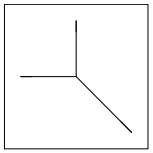
Four examples



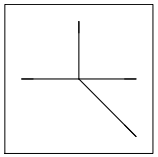
Algebraic [Kreweras 65, Gessel 86]

$$(1 - t(\bar{x} + \bar{y} + xy))xyQ(x,y) = xy - tyQ(0,y) - txQ(x,0)$$



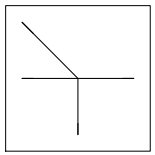
D-finite transcendental [Gessel 90]

$$(1 - t(y + \bar{x} + x\bar{y}))xyQ(x,y) = xy - tyQ(0,y) - tx^2Q(x,0)$$



D-algebraic, but not D-finite [Bernardi, mbm, Raschel 17(a)]

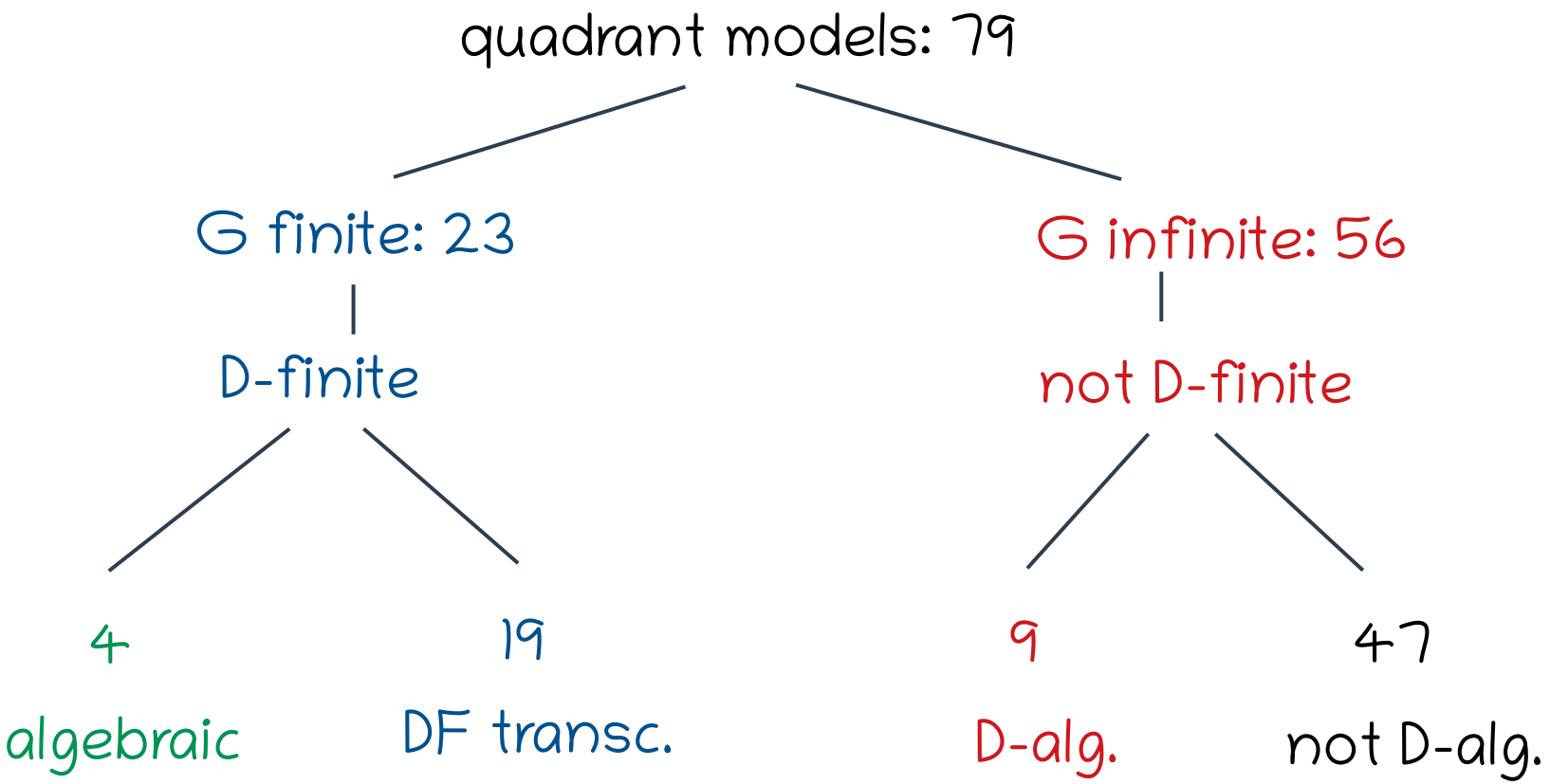
$$(1 - t(x + \bar{x} + y + x\bar{y}))xyQ(x,y) = xy - tyQ(0,y) - tx^2Q(x,0)$$



Not D-algebraic [Dreyfus, Hardouin, Roques, Singer 17]

$$(1 - t(x\bar{y} + \bar{x} + \bar{y} + y))xyQ(x,y) = xy - tyQ(0,y) - tx(1+x)Q(x,0)$$

Twenty years later: classification of quadrant walks



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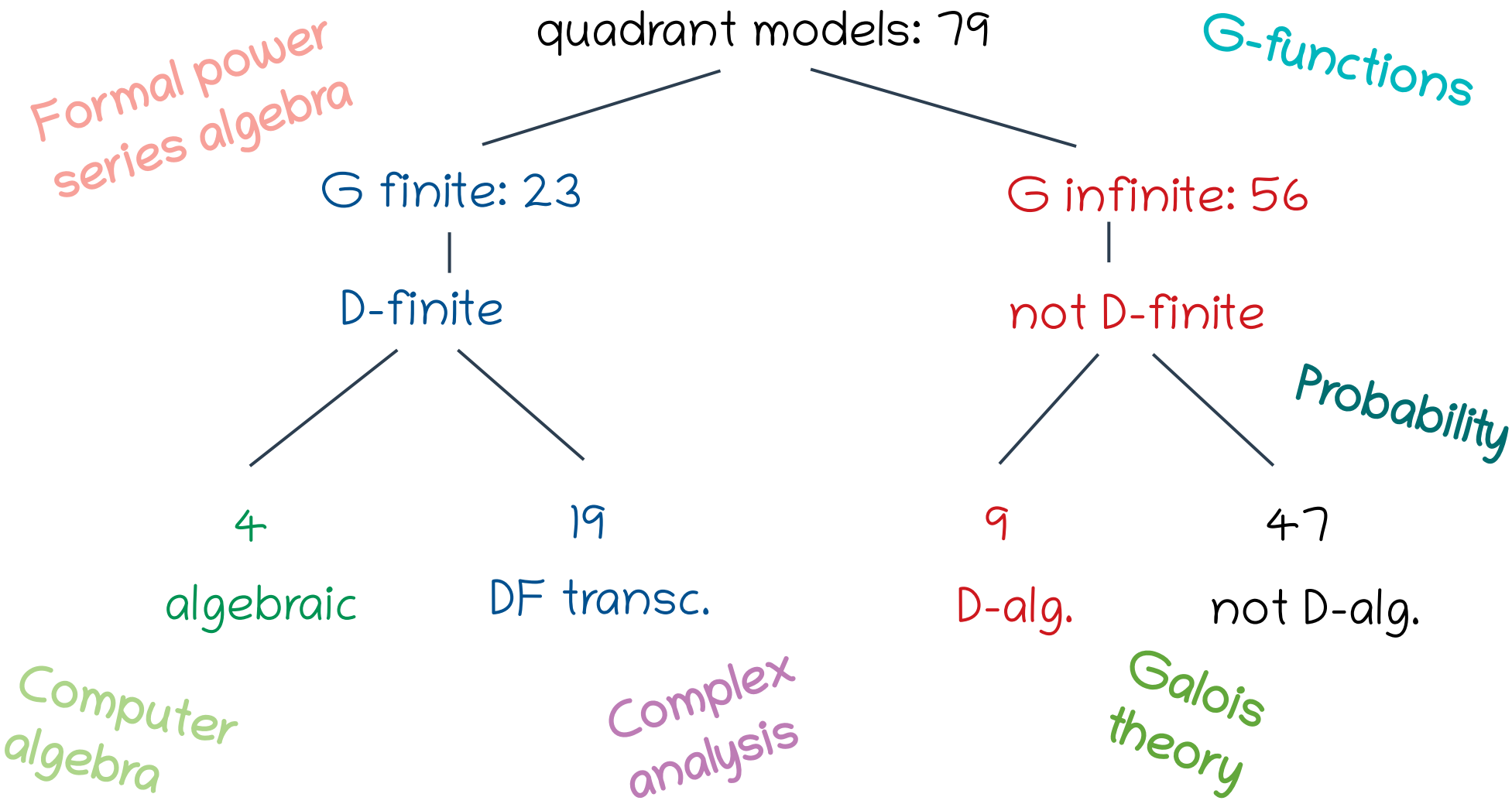

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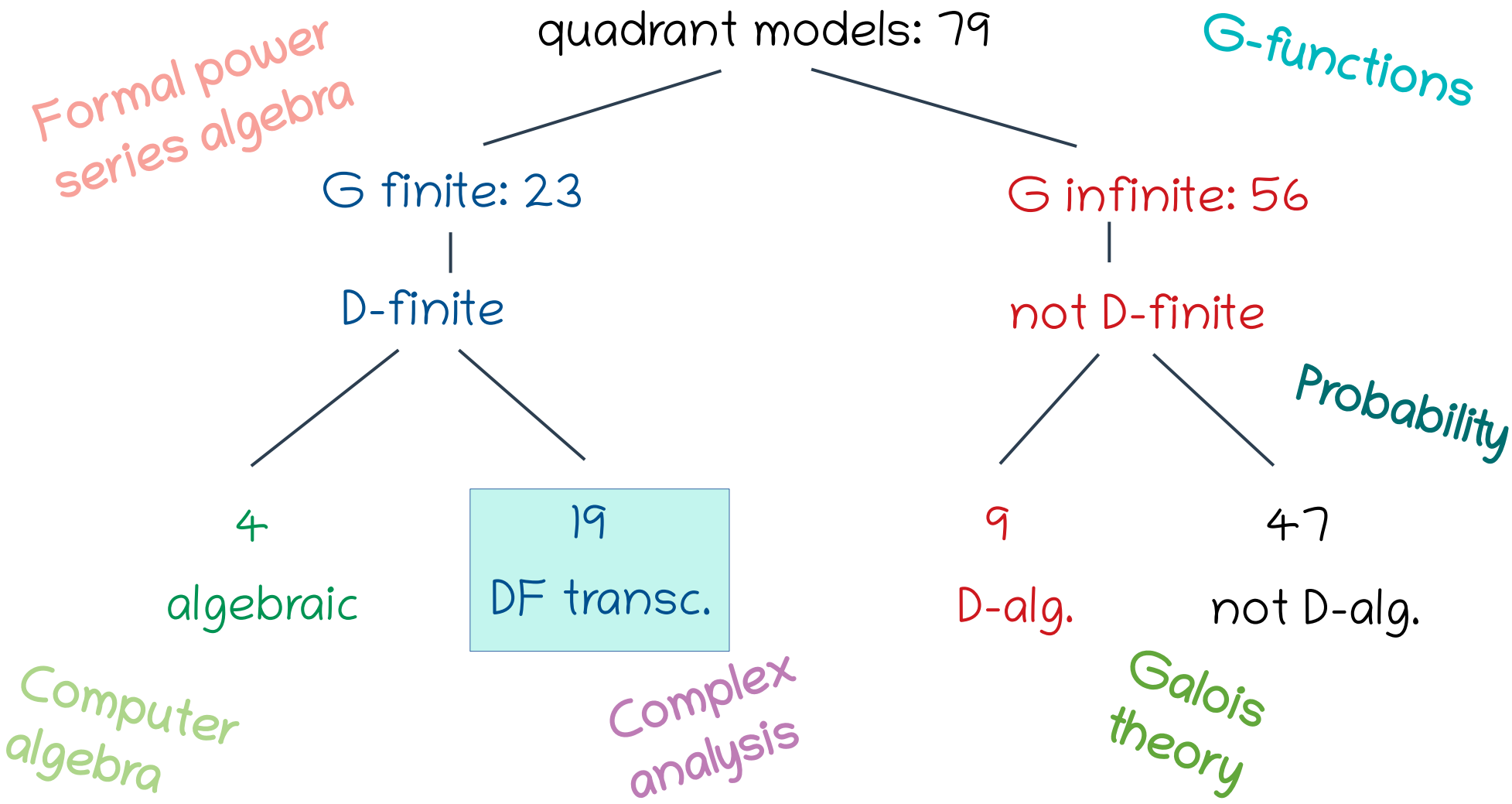

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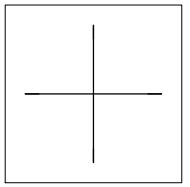
Linear discrete PDEs and diagonals

Walks on the square lattice confined to the first quadrant:

$$(1 - t(x + \bar{x} + y + \bar{y}))xyQ(x, y) = xy - txQ(x, 0) - tyQ(0, y).$$



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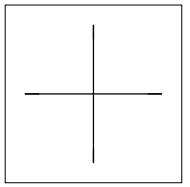

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By playing with the symmetries $x \to 1/x$ and $y \to 1/y$, one obtains

$$xyQ(x,y) - \bar{x}yQ(\bar{x},y) + \bar{x}\bar{y}Q(\bar{x},\bar{y}) - x\bar{y}Q(x,\bar{y}) = \frac{xy - \bar{x}y + \bar{x}\bar{y} - x\bar{y}}{1 - t(x + \bar{x} + y + \bar{y})}$$

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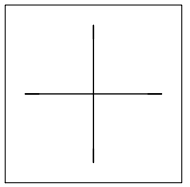

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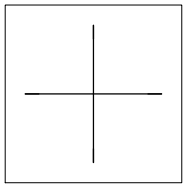

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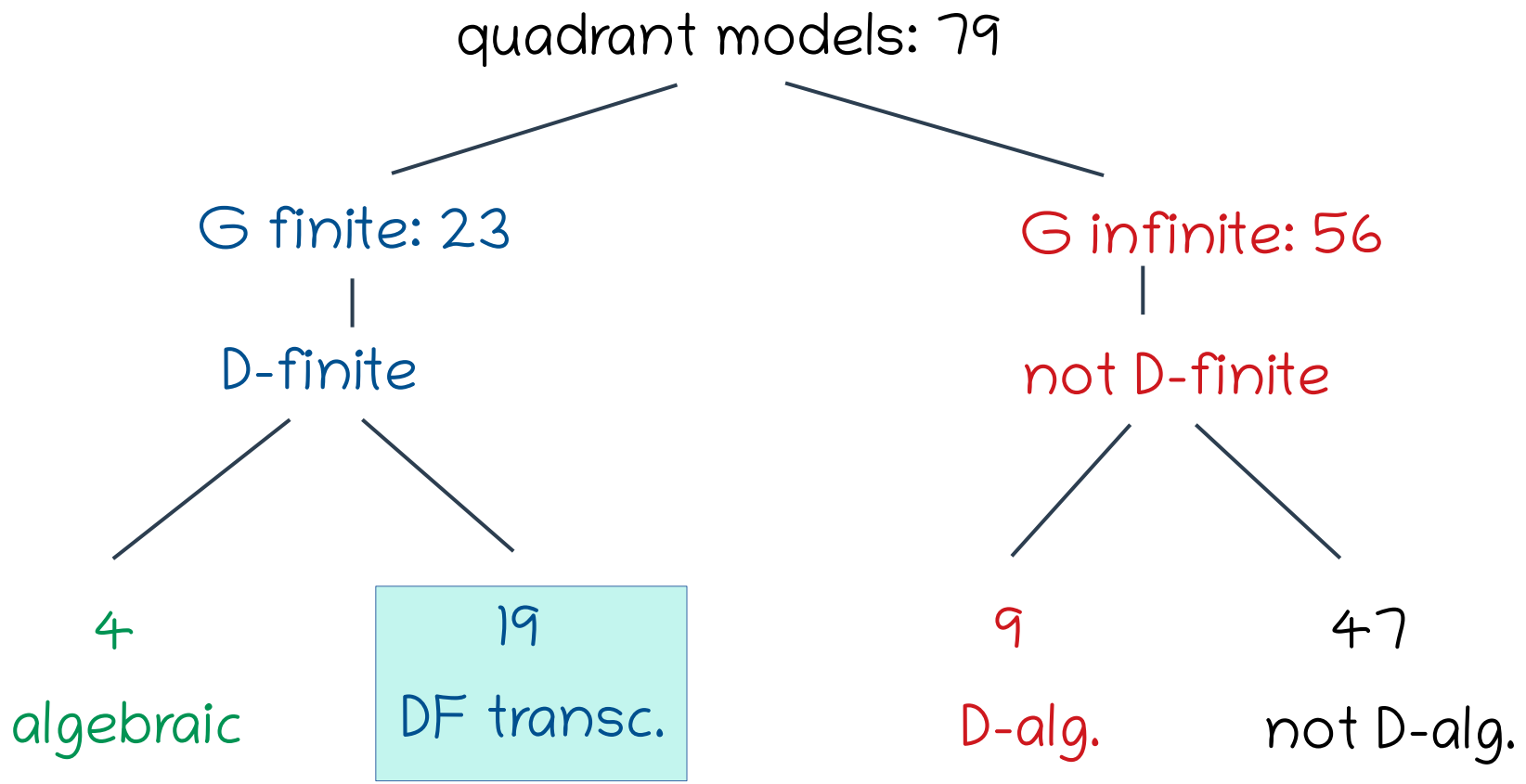
Equivalently,

$$Q(u, v) = [x^0y^0]\frac{1}{(1 - u\bar{x})(1 - v\bar{y})}\frac{1 - \bar{y}^2 + \bar{x}^2\bar{y}^2 - \bar{x}^2}{1 - t(x + \bar{x} + y + \bar{y})},$$

and Q(u,v) can also be written as a diagonal.

Twenty years later: classification of quadrant walks



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- **2D walks** confined to the non-negative quadrant, with steps in {(1,0), (-1,0), (0,-1), (-2,1)}: [Bostan, mbm, Melczer 21]

$$Q(x,y) = [x^{\geq 0} y^{\geq 0}] \frac{(x^2+1)\,(x+y)\,(y-x)\,(x^2y-2x-y)\,(x^3-x-2y)}{x^7y^3\,(1-t(x+\bar{x}+\bar{x}^2y+\bar{y}))}$$

- **3D walks** confined to the non-negative orthant, with steps in {(-1,-1,-1), (-1,-1,1), (-1,1,0),(1,0,0)}: [Bostan, mbm, Kauers, Melczer 16]

$$O(x,y,z) = [x^{\geq 0} y^{\geq 0} z^{\geq 0}] \frac{(x-\bar{x}y-\bar{x}\bar{y}z-\bar{x}\bar{y}\bar{z})(y-\bar{y}z-\bar{y}\bar{z})(z-\bar{z})}{x^2y^2z^2(1-t(\bar{x}\bar{y}\bar{z}+\bar{x}\bar{y}z+\bar{x}y+x))}$$

Linear discrete PDEs and diagonals

In fact, all linear equations* in two or more catalytic variables whose solution has been proved D-finite (transcendental) have been solved in terms of **diagonals of rational functions**.

- For m=2ℓ+1, the GF of **involutions with no ascending sub-sequence** of length m+1 is:

$$I_m(t) = [x_1 x_2^2 \cdots x_\ell^\ell] \frac{\det(x_j^i - \bar{x}_j^i)_{1\le i,j\le \ell}}{1 - t(1 + x_1 + \cdots + x_\ell + \bar{x}_1 + \cdots + \bar{x}_\ell)}$$

- The GF of **permutations with no ascending sub-sequence** of length m+1 is:

$$[x_1^0 \cdots x_m^0] \frac{\det(z_j^{i-j})_{1\le i,j\le m}}{1 - t(\bar{z}_1 + \cdots + \bar{z}_m)} \sum_{i=0}^{m} \prod_{j=1}^{i} \frac{x_j}{1 - x_j}$$

with $z_j = x_j - x_{j-1}$.

[mbm ו"ך

Linear discrete PDEs and diagonals

In fact, all linear equations* in two or more catalytic variables whose solution has been proved D-finite (transcendental) have been solved in terms of diagonals of rational functions.

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- Quadrant walks with steps in {(-1,-1), (-1,1), (-1,0), (-1,0), (1,0), (1,1)} are D-finite transcendental, but this has only been proved by a guess & check computer approach [Bostan, mbm, Kauers, Melczer 16].

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- Some quadrant walks are **conjectured** D-finite, with no proof at all. E.g. with steps in {(1,0), (0,1), (-2,-1), (-1,-2)} [Bostan, mbm, Melczer 21]

IV. Disproving D-finiteness, algebraicity, etc.

Not *really* a combinatorial question!

Some tools

Asymptotics. Assume that the (integer) sequence a(n) satisfies

$$a(n) \sim \kappa \mu^n n^\gamma, \qquad n \to \infty.$$

If the associated series A(t) is **rational**/**algebraic**/**D-finite** then

- μ is algebraic
- γ belongs to $\mathbb{N}$, $\mathbb{Q}\backslash\{-1, -2, \ldots\}$, $\mathbb{Q}$.

G-functions

[Katz 70, Chudnovsky^2 85, André 89... Bostan, Raschel, Salvy 14]

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Functional equations. Start from a difference/differential/etc. equation --- **Galois theory**

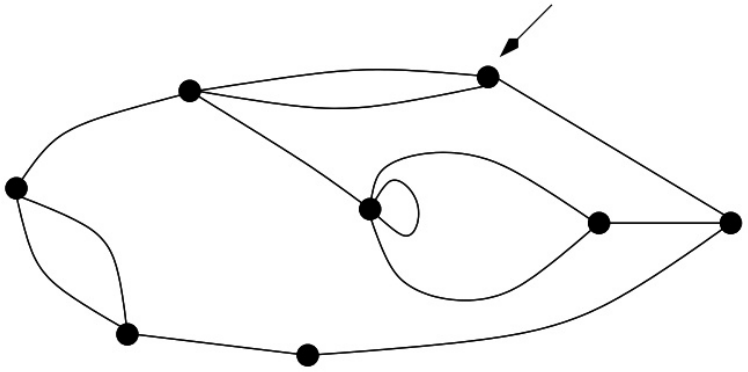
Quadrant walks [Dreyfus, Hardouin, Roques, Singer 17]

Final comments

- It sometimes takes **a long time** to find a "direct" solution once an "undirect" one has been found.

  Planar maps: algebraicity proved in the 60's using a catalytic variable [Tutte]; first direct proof in 81 [Cori-Vauquelin], complete understanding in the 90's [Schaeffer]

- Many algebraicity/D-finiteness results still wait for a **direct proof**



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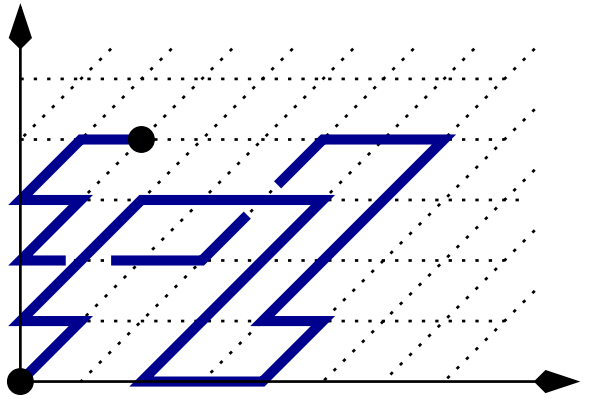

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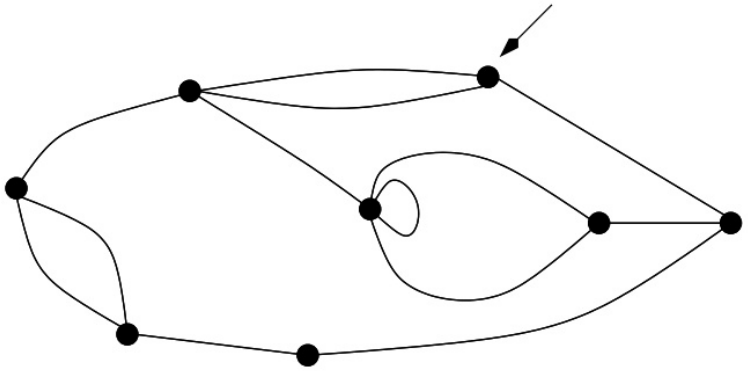

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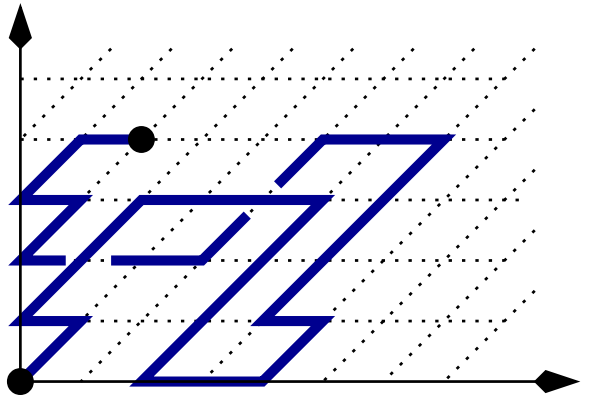

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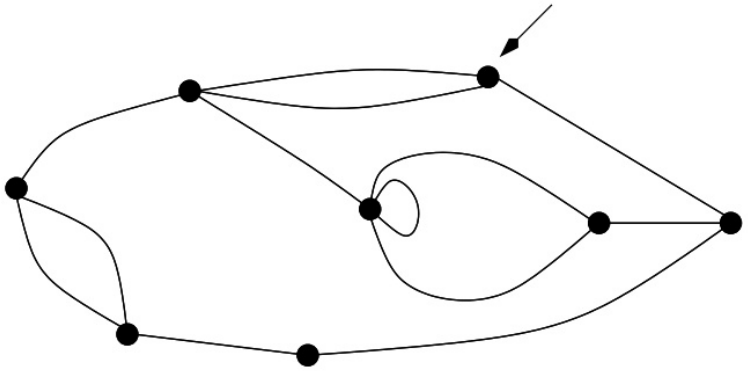

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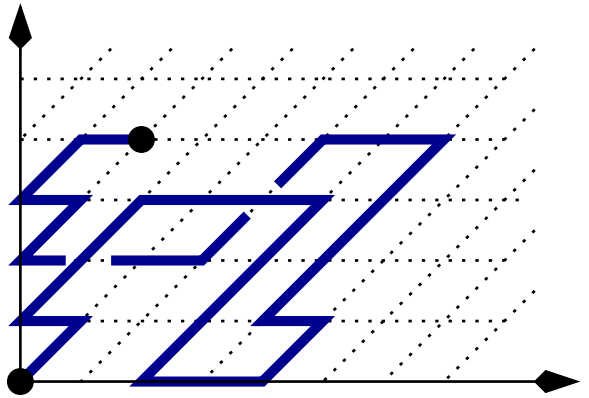

- Interesting questions on **$\mathbb{N}$-rational and $\mathbb{N}$-algebraic series**
- Combinatorialists need specialists of functional equations