

Prudent self-avoiding walks

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Enumerative combinatorics

Let \mathcal{A} be a set of discrete objects equipped with an integer size, and assume that for all n , the number of objects of \mathcal{A} of size n is finite. Let $a(n)$ be this number.

- **Exact enumeration:** determine the sequence $a(n)$

$$a(n) = \dots \quad \text{or} \quad A(t) := \sum_{n \geq 0} a(n)t^n = \dots$$

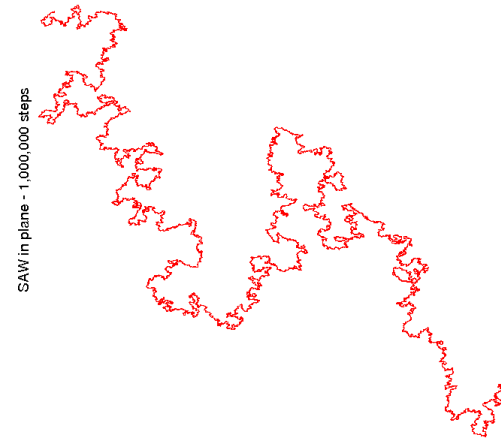
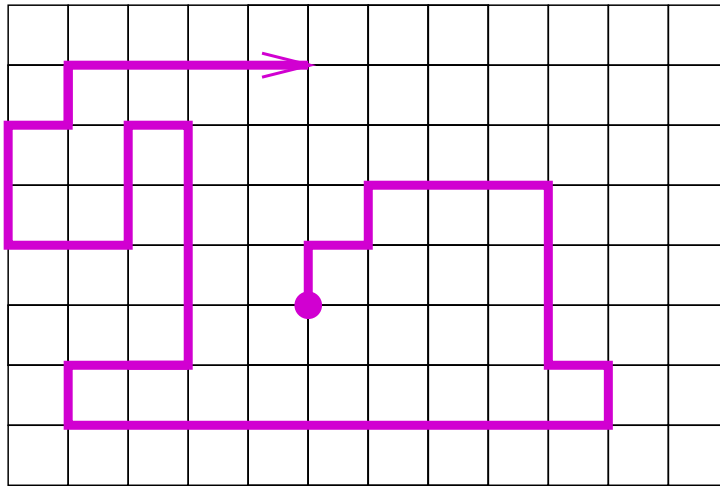
where $A(t)$ is the **generating function of the objects of \mathcal{A}** .

- **Asymptotic enumeration:** estimate the numbers $a(n)$, as $n \rightarrow \infty$

$$a(n) \sim \dots \quad \text{or} \quad a(n) = O(\dots) \quad \text{or} \quad a(n)^{1/n} \rightarrow \dots$$

- **Applications:** probability, algebra, computer science (analysis of algorithms), statistical physics... and curiosity!

Self-avoiding walks (SAW)



Conjectures ($d = 2$)

- The number of n -step SAW is equivalent to $(\kappa) \mu^n n^{11/32}$ for n large.
- The endpoint lies on average at distance $n^{3/4}$ from the starting point.
- The scaling limit of SAW is $SLE_{8/3}$ (proved under an assumption of conformal invariance [Lawler, Schramm, Werner 02]).

Theorem [Hara-Slade 92]

- In dimension $d \geq 5$, the number of n -step SAW is equivalent to μ^n .
- The endpoint lies on average at distance $n^{1/2}$ from the starting point.

This is too hard!

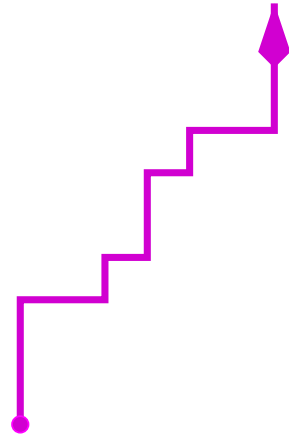
... for exact enumeration

⇒ **Study of toy models**, that should be as general as possible, but still tractable

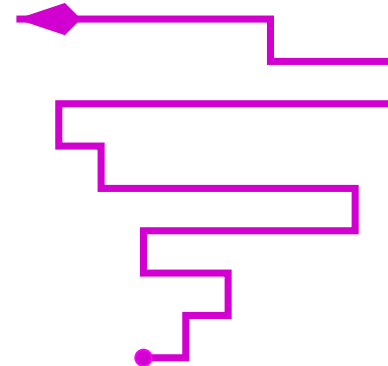
- develop new techniques in exact enumeration
- solve better and better approximations of real SAW

Two toy models

Directed walks (N,E)



Partially directed walks (N,W,E)



Memory-less

$$a(n) = 2^n$$

$$\sum_n a(n)t^n = \frac{1}{1-2t}$$

$$\mathbb{E}(X_n) = \mathbb{E}(Y_n) = n/2$$

Markovian with memory 1

$$a(n) \sim (1 + \sqrt{2})^n \sim (2.41\dots)^n$$

$$\sum_n a(n)t^n = \frac{1+t}{1-2t-t^2}$$

$$\mathbb{E}(|X_n|) \sim \sqrt{n}, \quad \mathbb{E}(Y_n) \sim n$$

A hierarchy of formal power series

- The formal power series $A(t)$ is **rational** if it can be written

$$A(t) = \frac{P(t)}{Q(t)}$$

where $P(t)$ and $Q(t)$ are polynomials in t .

- The formal power series $A(t)$ is **algebraic** (over $\mathbb{Q}(t)$) if it satisfies a (non-trivial) polynomial equation:

$$P(t, A(t)) = 0.$$

- The formal power series $A(t)$ is **D-finite** if it satisfies a (non-trivial) linear differential equation with polynomial coefficients:

$$P_0(t)A^{(k)}(t) + P_1(t)A^{(k-1)}(t) + \cdots + P_k(t)A(t) = 0.$$

$$\text{Rat} \subset \text{Alg} \subset \text{D-finite}$$

I. Prudent self-avoiding walks

Self-directed walks [Turban-Debierre 86], Exterior walks [Préa 97],
Outwardly directed SAW [Santra-Seitz-Klein 01]

Prudent walks [Duchi 05], [Dethridge, Guttmann, Jensen 07], [mbm 08]

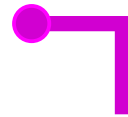
Prudent self-avoiding walks

Every new step can be repeated indefinitely without creating intersections.



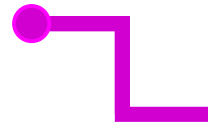
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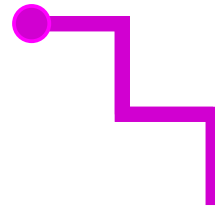
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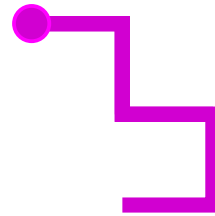
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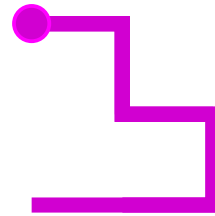
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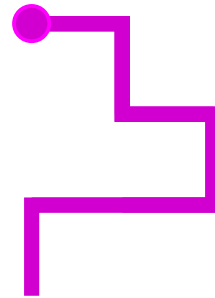
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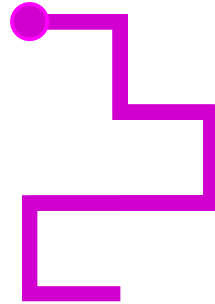
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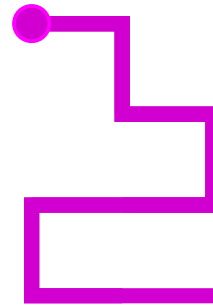
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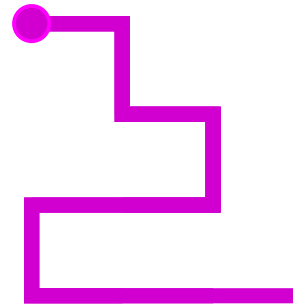
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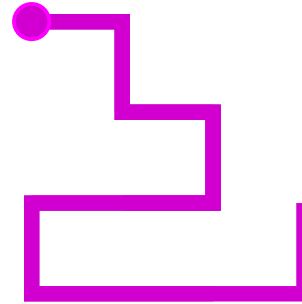
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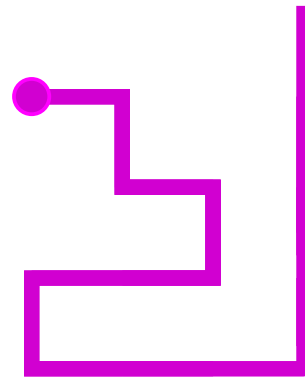
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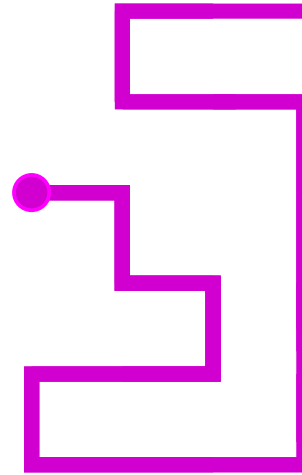
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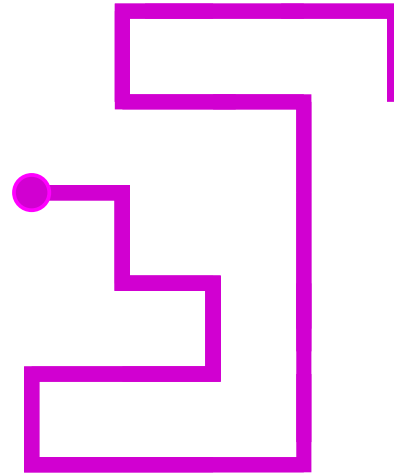
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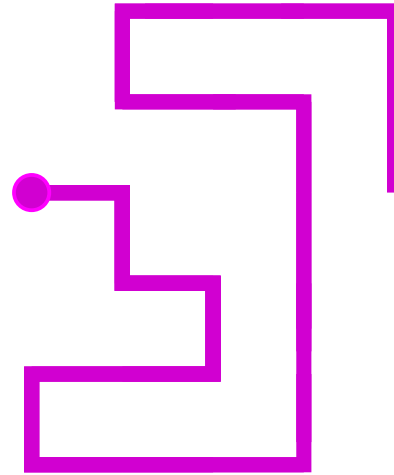
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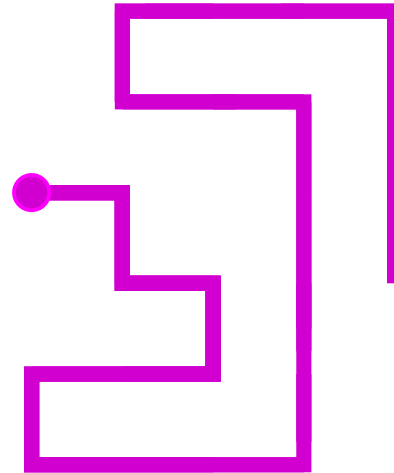
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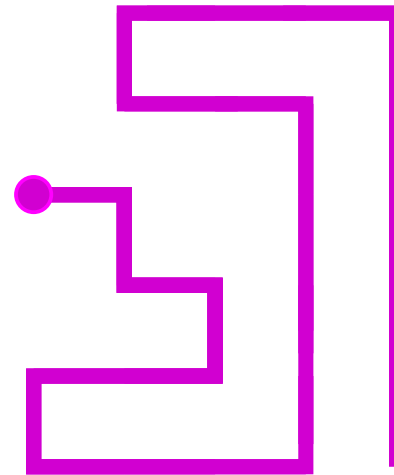
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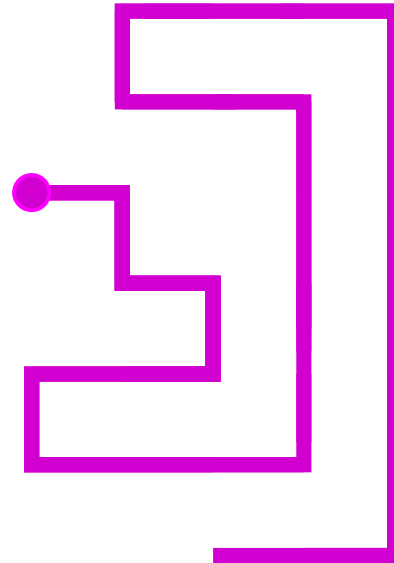
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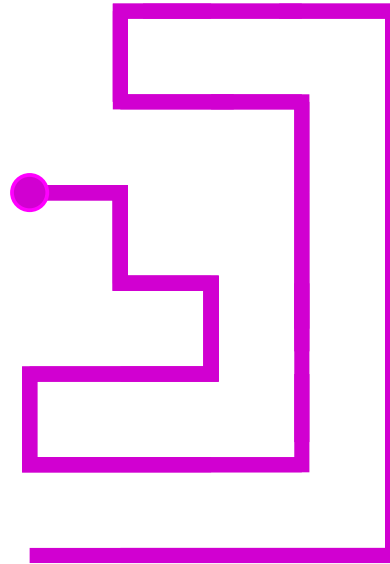
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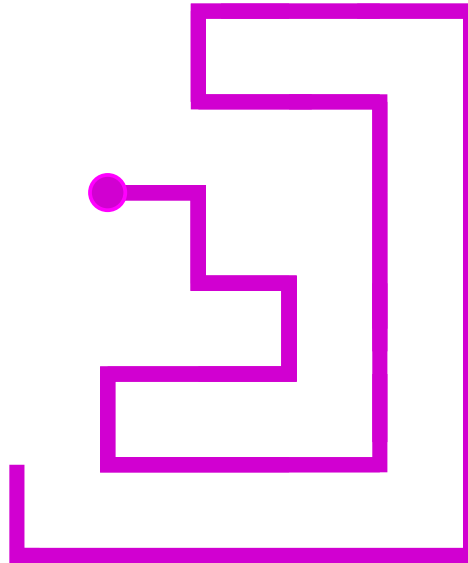
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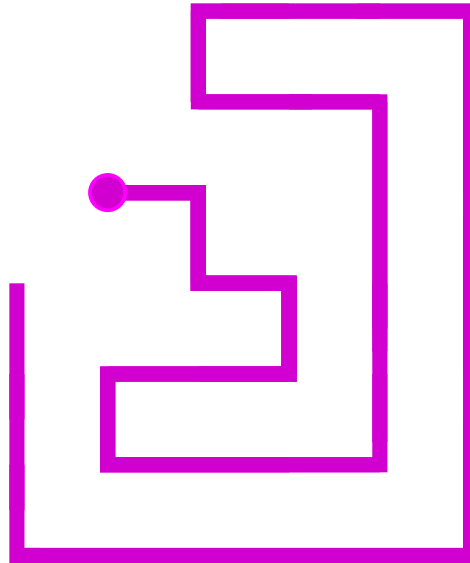
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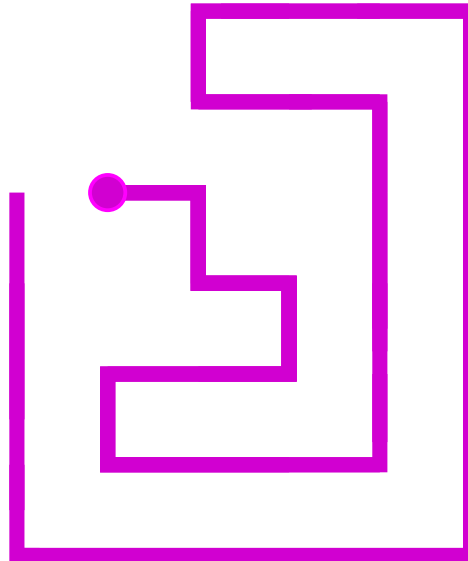
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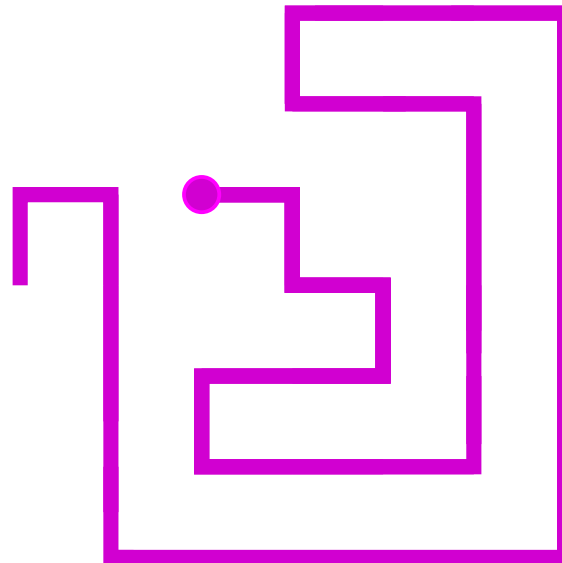
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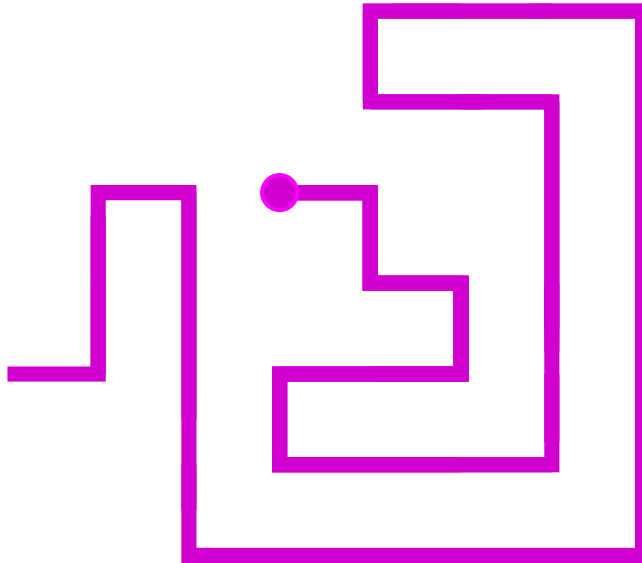
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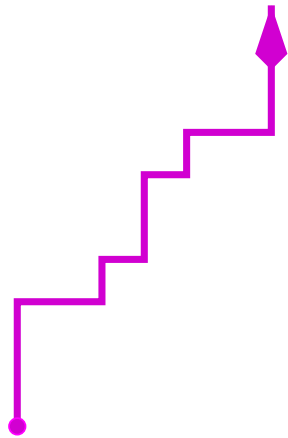


Prudent self-avoiding walks

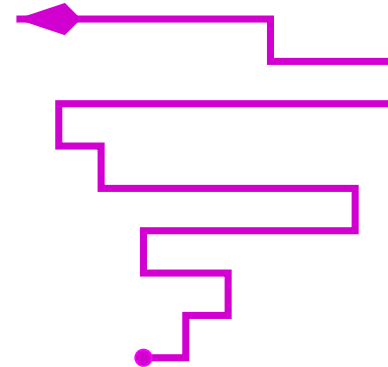
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Examples of prudent walks



Directed walks (N, E)



Partially directed walks (N, E, W)

Some properties of prudent walks



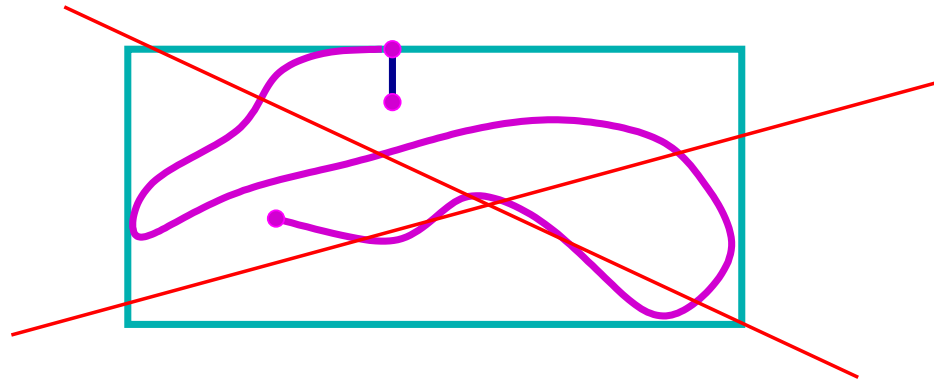
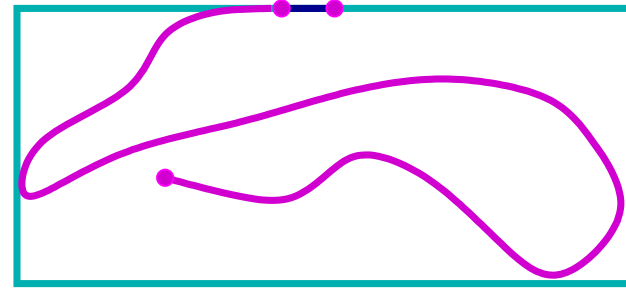
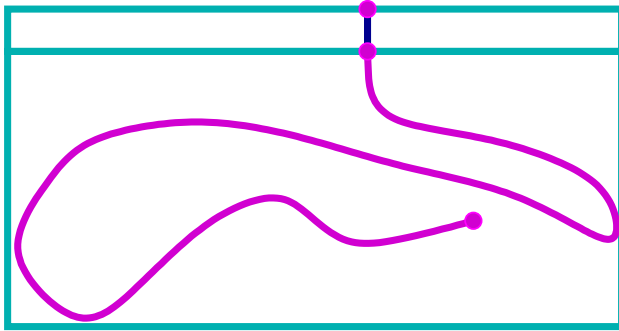
Some properties of prudent walks



The **box** of a prudent walk

Some properties of prudent walks

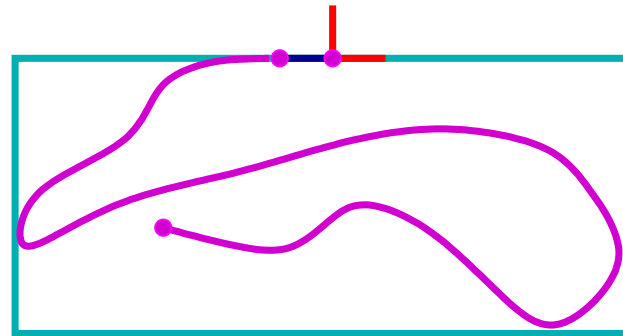
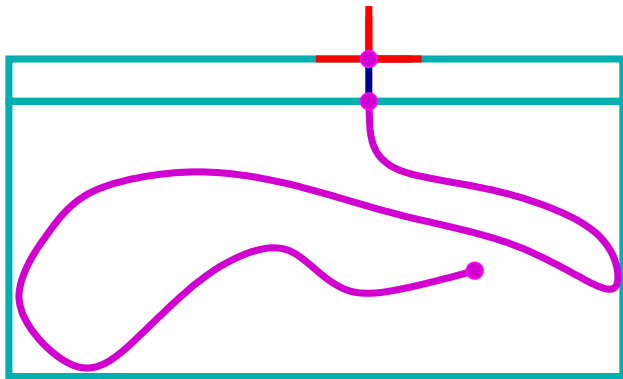
The endpoint of a prudent walk is always on the border of the box.



Each new step either **inflates** the box or walks (prudently) **along the border**.

Some properties of prudent walks

Each new step either **inflates** the box or walks (prudently) **along the border**.



- After an inflating step, **3** possibilities
- Otherwise, only **2**.

A comparison with partially directed walks (N, E and W steps)

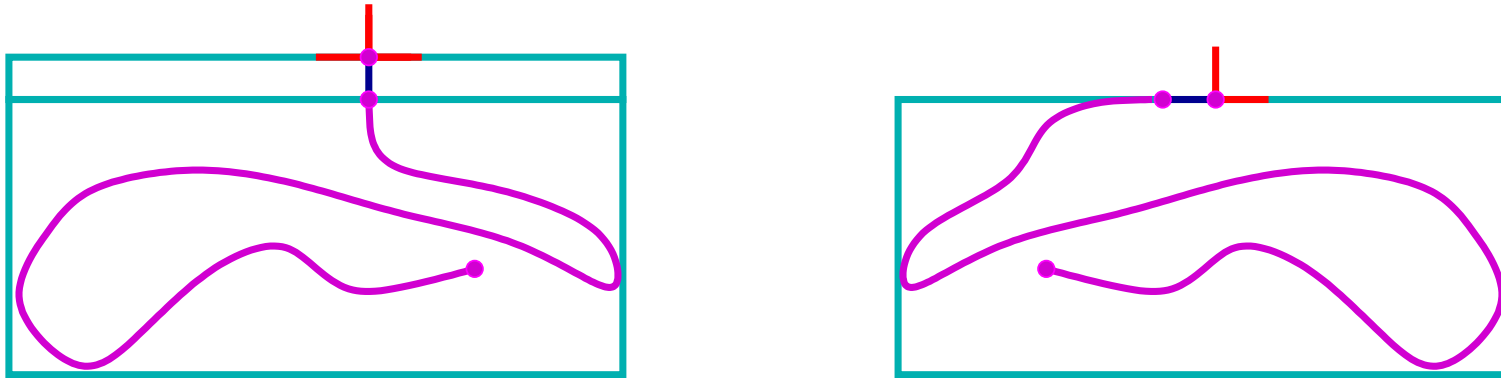


- After a North step, 3 possibilities (N, E, W)
- Otherwise, only 2 (N, or repeat)

⇒ Count partially directed walks by looking for the last North step

Some properties of prudent walks

Each new step either **inflates** the box or walks (prudently) **along the border**.



- After an inflating step, **3** possibilities

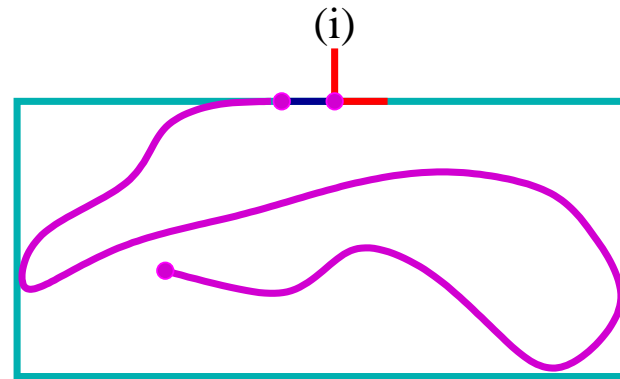
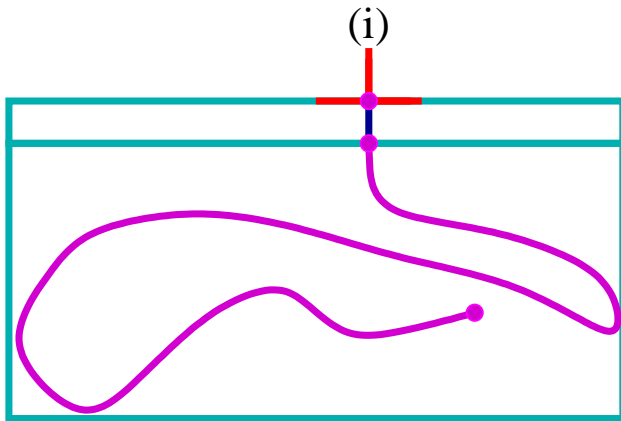
- Otherwise, only **2**.

⇒ Count prudent walks by looking for the last inflating step

When do we create an inflating step?

Some properties of prudent walks

Each new step either **inflates** the box or walks (prudently) **along the border**.

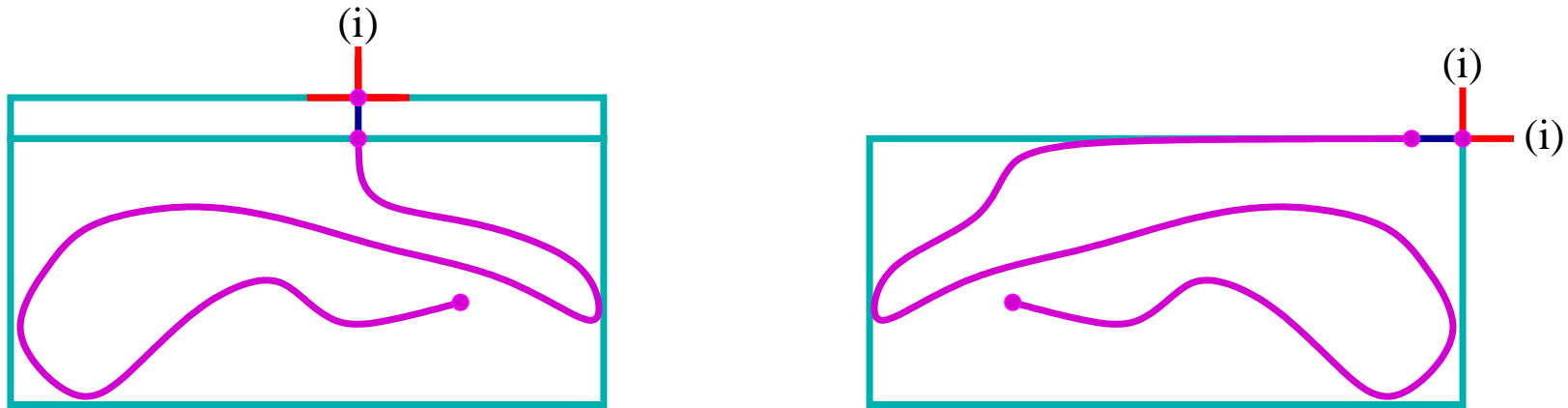


- After an inflating step, **3** possibilities
 - Otherwise, only **2**.
- ⇒ Count prudent walks by looking for the last inflating step

When do we create an inflating step?

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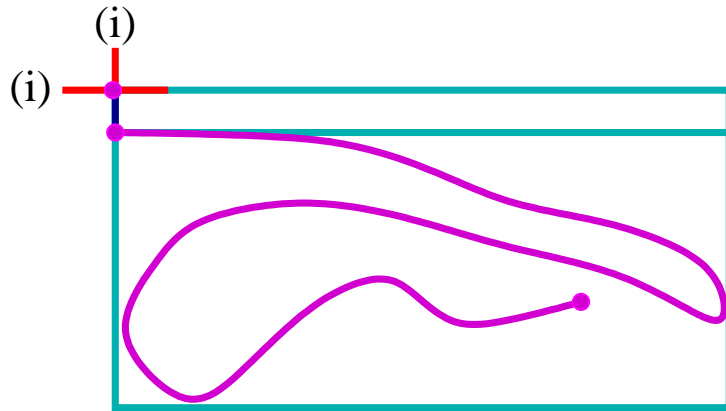
- After an inflating step, **3** possibilities
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⇒ Count prudent walks by looking for the last inflating step

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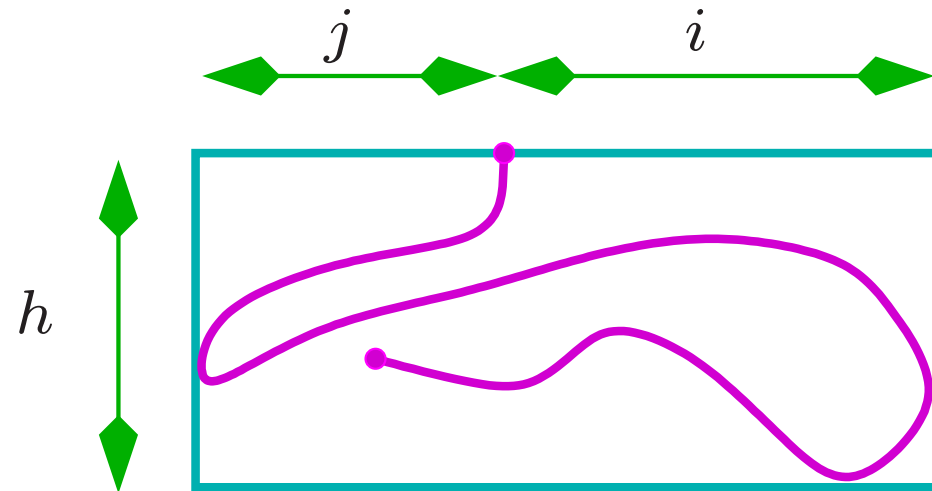
- After an inflating step, **3** possibilities

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⇒ Count prudent walks by looking for the last inflating step

When do we create an inflating step?

Some properties of prudent walks



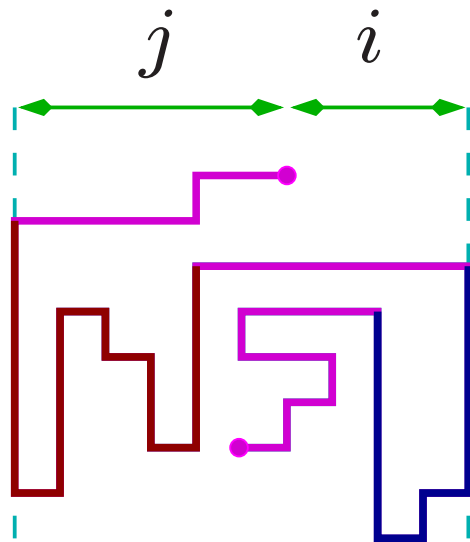
If one knows:

- the direction of the last step,
- whether it is inflating or not,
- the distances i , j and h ,

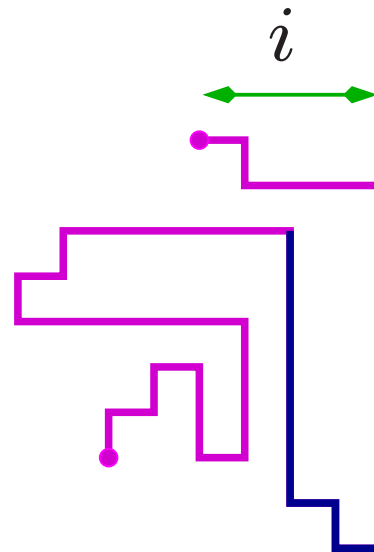
then one can decide which steps can be appended to the walk, and the new values of these parameters.

⇒ Count prudent walks by looking for the last inflating step, keeping track of the distances i, j, h

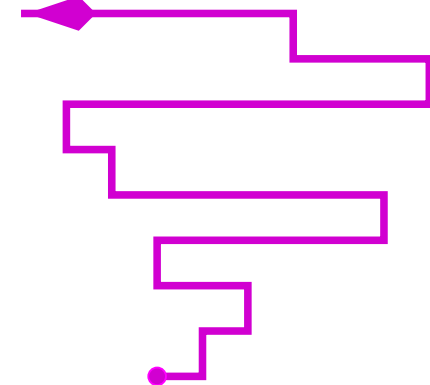
Simpler families of prudent walks [Préa 97]



3-sided



2-sided



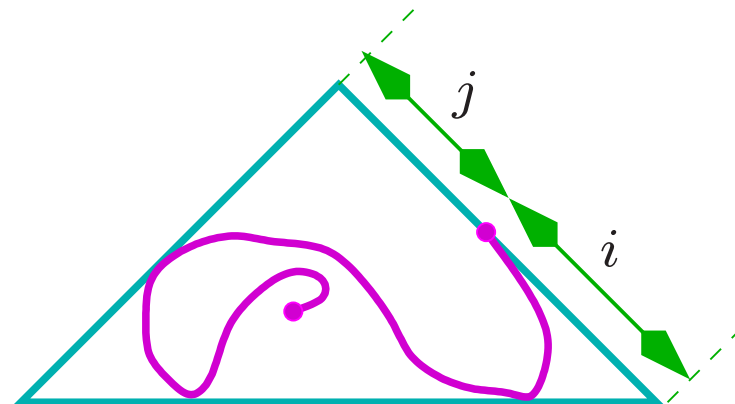
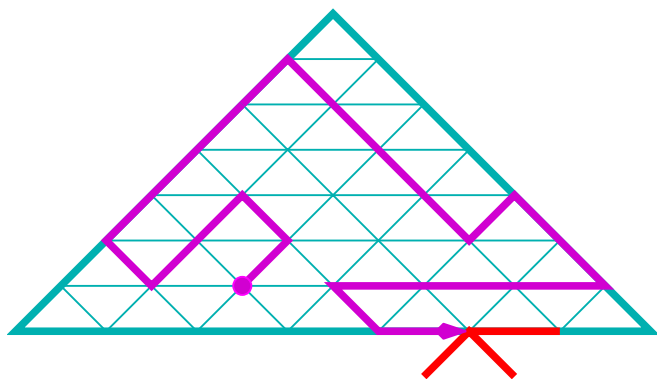
1-sided

- The endpoint of a **3-sided walk** lies always on the top, right or left side of the box
- The endpoint of a **2-sided walk** lies always on the top or right side of the box
- The endpoint of a **1-sided walk** lies always on the top side of the box (= partially directed!)

Non-isotropic models

An isotropic model with only two additional parameters: triangular prudent walks

Definition: Every new step either inflates the box or walks (prudently) along the border



II. Results

Summary of the results

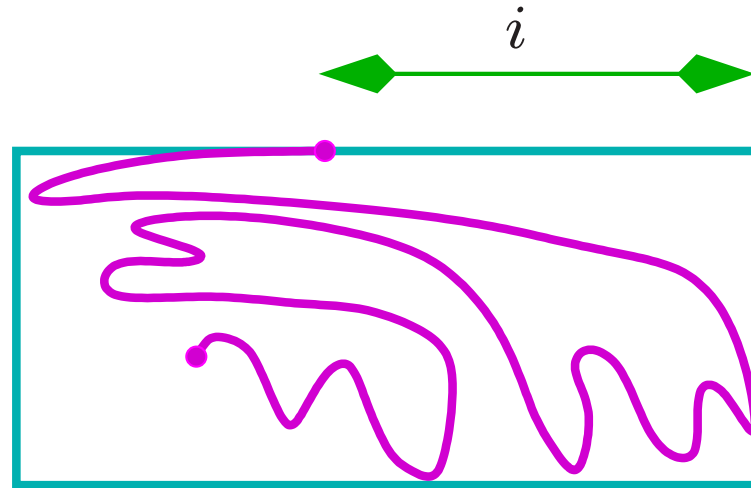
	Nature of the g.f.	Asympt. growth	End-to-end distance
1-sided (part. dir)	Rat.	$(2.41\dots)^n$	n
2-sided	Alg. [Duchi 05]	$(2.48\dots)^n$	n
3-sided	not D-finite	$(2.48\dots)^n$	n
4-sided (general)	not D-finite	$(2.48\dots)^n$	n
square lattice SAW	?	$(2.63\dots)^n n^{11/32}$	$n^{3/4}$
triangular prudent	not D-finite	$(3.56\dots)^n$	n
triangular SAW	?	$(4.15\dots)^n n^{11/32}$	$n^{3/4}$

Some tools

- Functional equations with “catalytic” variables
- The kernel method
- Singularity analysis and asymptotics

A functional equation for two-sided prudent walks

One keeps track of the distance between the endpoint and the NE corner of the box:



Generating function for walks ending on the top side of the box::

$$T(t; u) = \sum_{n; i} a(n; i) t^n u^i = \sum_i T_i(t) u^i \equiv T(u)$$

- For 2-sided walks: series $P(t; u) \equiv P(u)$

A functional equation for two-sided prudent walks

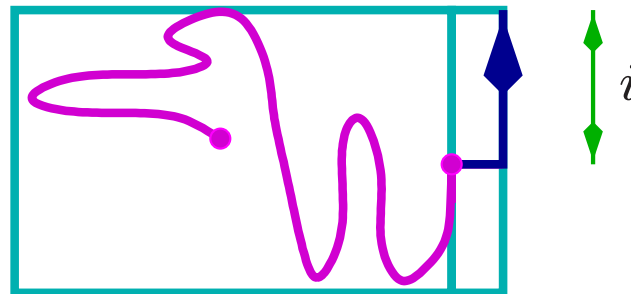
Walks ending on the top side of the box: where is the last step that moved the right or top side of the box?

- Either there was no such step:

$$\frac{1}{1 - tu}$$



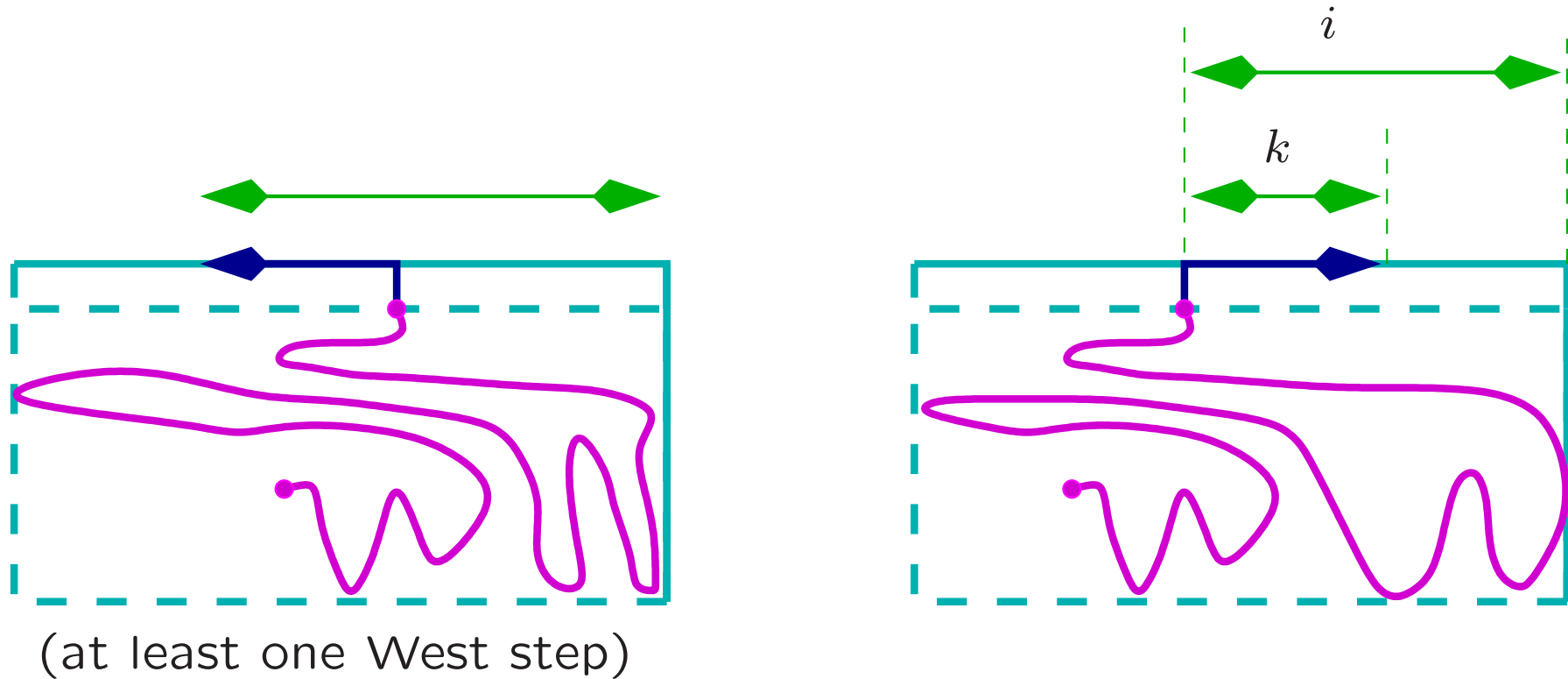
- ... or an East step moved the right side:



$$t \sum_i T_i(t) t^i = tT(t; t)$$

A functional equation for two-sided prudent walks

... or a North step moved the top side:



$$\frac{t^2 u}{1 - tu} T(t; u) + t \sum_{i \geq 0} T_i(t) \sum_{k=0}^i t^k u^{i-k} = \frac{t^2 u}{1 - tu} T(t; u) + \frac{t}{u - t} (uT(t; u) - tT(t; t)).$$

A functional equation for two-sided prudent walks

- The generating function $T(t; u)$ of 2-sided walks ending on the top side of their box satisfies

$$\left(1 - \frac{tu(1-t^2)}{(1-tu)(u-t)}\right) T(t; u) = \frac{1}{1-tu} + t \frac{u-2t}{u-t} T(t; t)$$

- The g.f. that counts all 2-sided walks is

$$P(t; u) = 2T(t; u) - T(t; 0)$$

(inclusion-exclusion)

The more general the class, the more additional variables

1. Two-sided walks : **one** catalytic variable

$$\left(1 - \frac{tu(1-t^2)}{(1-tu)(u-t)}\right) T(t; u) = \frac{1}{1-tu} + t \frac{u-2t}{u-t} T(t; t).$$

2.a. Three-sided walks : **two** catalytic variables

$$\left(1 - \frac{uvt(1-t^2)}{(u-tv)(v-tu)}\right) T(t; u, v) = 1 + \dots - \frac{t^2v}{u-tv} T(t; tv, v) - \frac{t^2u}{v-tu} T(t; u, tu)$$

2.b. Triangular prudent walks: **two** catalytic variables

$$\left(1 - \frac{tuv(1-t^2)(u+v)}{(u-tv)(v-tu)}\right) R(t; u, v) = 1 + tu(1+t) \frac{v-2tu}{v-tu} R(t; u, tu) + tv(1+t) \frac{u-2tv}{u-tv} R(t; tv, v).$$

3. General prudent walks : **three** catalytic variables

$$\left(1 - \frac{uvw(1-t^2)}{(u-tv)(v-tu)}\right) T(t; u, v, w) = 1 + \mathcal{G}(w, u) + \mathcal{G}(w, v) - tv \frac{\mathcal{G}(v, w)}{u-tv} - tu \frac{\mathcal{G}(u, w)}{v-tu}$$

with $\mathcal{G}(u, v) = tvT(t; u, tu, v)$.

Two-sided walks: the kernel method

The generating function of 2-sided walks ending on the top side of their box satisfies:

$$\left(1 - \frac{tu(1-t^2)}{(1-tu)(u-t)}\right) T(t; u) = \frac{1}{1-tu} + t \frac{u-2t}{u-t} T(t; t).$$

The g.f. that counts all 2-sided walks is

$$P(t; u) = 2T(t; u) - T(t; 0).$$

⇒ the kernel method

[Knuth 72], [mbm-Petkovšek 2000]

Two-sided walks: the kernel method

$$\left((1 - tu)(u - t) - tu(1 - t^2) \right) T(t; u) = u - t + t(u - 2t)(1 - tu)T(t; t).$$

- If $u = U(t)$ cancels $(1 - tu)(u - t) - tu(1 - t^2)$, then

$$U(t) - t + t(U(t) - 2t)(1 - tU(t))T(t; t) = 0,$$

that is,

$$T(t; t) = \frac{t - U(t)}{t(U(t) - 2t)(1 - tU(t))}$$

- We know such a series $U(t)$:

$$U(t) = \frac{1 - t + t^2 + t^3 - \sqrt{(1 - t^4)(1 - 2t - t^2)}}{2t}$$

Two-sided walks

- The length generating function of 2-sided walks is

$$P(t) = \frac{1}{1 - 2t - 2t^2 + 2t^3} \left(1 + t - t^3 + t(1 - t) \sqrt{\frac{1 - t^4}{1 - 2t - t^2}} \right)$$

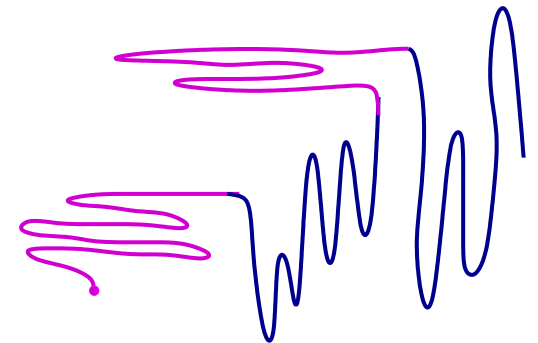
[Duchi 05]

- Dominant singularity: a simple pole for $1 - 2t - 2t^2 + 2t^3 = 0$, that is, $t_c = 0.40303\dots$. Asymptotically,

$$p(n) \sim \kappa (2.48\dots)^n$$

Compare with 2.41... for partially directed walks.

- Another approach: factorization of walks [Duchi 05]



Triangular prudent walks: two catalytic variables

- Functional equation for $R(t; u, v) \equiv R(u, v)$:

$$\begin{aligned} & \left((u - tv)(v - tu) - tuv(1 - t^2)(u + v) \right) R(u, v) = \\ & (u - tv)(v - tu) + tu(1 + t)(u - tv)(v - 2tu)R(u, tu) + tv(1 + t)(v - tu)(u - 2tv)R(v, tv). \end{aligned}$$

Let
$$K(u, v)R(u, v) = A(u, v) + B(u, v)\Phi(u) + B(v, u)\Phi(v)$$

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- Cancellation of the kernel: $K(u, V(u)) = 0$ for a series $V(u) \equiv V(t; u)$

$$\Phi(u) = -\frac{A(u, V(u))}{B(u, V(u))} - \frac{B(V(u), u)}{B(u, V(u))} \Phi(V(u))$$

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- If it is possible to iterate (...), denote $V^{(k)} = V(V(V(\dots(u))))$ (k iterations):

$$\Phi(u) = \sum_{k \geq 0} (-1)^{k-1} \frac{B(V^{(1)}, u)B(V^{(2)}, V^{(1)}) \dots B(V^{(k)}, V^{(k-1)})A(V^{(k)}, V^{(k+1)})}{B(u, V^{(1)})B(V^{(1)}, V^{(2)}) \dots B(V^{(k-1)}, V^{(k)})B(V^{(k)}, V^{(k+1)})}$$

Sum of algebraic series — iteration of algebraic functions

Triangular prudent walks

The length generating function of triangular prudent walks is

$$P(t; 1) = \frac{6t(1+t)}{1-3t-2t^2} \left(1 + t(1+2t)R(t; 1, t)\right)$$

with

$$R(t; 1, t) = (1+Y)(1+tY) \sum_{k \geq 0} \frac{t^{\binom{k+1}{2}} (Y(1-2t^2))^k}{(Y(1-2t^2); t)_{k+1}} \left(\frac{Yt^2}{1-2t^2}; t\right)_k$$

and

$$Y = \frac{1-2t-t^2 - \sqrt{(1-t)(1-3t-t^2-t^3)}}{2t^2}$$

Notation:

$$(a; q)_n = (1-a)(1-aq) \cdots (1-aq^{n-1}).$$

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- Dominant singularity: a simple pole for $1 - 3t - 2t^2 = 0$. Asymptotically,

$$p(n) \sim \kappa \left(\frac{3 + \sqrt{17}}{2}\right)^n$$

- The series $P(t; 1)$ is neither algebraic, nor even D-finite (infinitely many poles at $Yt^k(1-2t^2) = 0$)

Three-sided prudent walks

The generating function of three-sided prudent walks is:

$$P(t; 1) = \frac{1}{1 - 2t - t^2} \left(\frac{1 + 3t + tq(1 - 3t - 2t^2)}{1 - tq} + 2t^2 q T(t; 1, t) \right)$$

where

$$q \equiv q(t) = \frac{1 - t + t^2 + t^3 - \sqrt{(1 - t^4)(1 - 2t - t^2)}}{2t},$$

$$T(t; 1, t) = \sum_{k \geq 0} (-1)^k \frac{\prod_{i=0}^{k-1} \left(\frac{t}{1-tq} - U(q^{i+1}) \right)}{\prod_{i=0}^k \left(\frac{tq}{q-t} - U(q^i) \right)} \left(1 + \frac{U(q^k) - t}{t(1 - tU(q^k))} + \frac{U(q^{k+1}) - t}{t(1 - tU(q^{k+1}))} \right)$$

with

$$U(w) = \frac{1 - tw + t^2 + t^3 w - \sqrt{(1 - t^2)(1 + t - tw + t^2 w)(1 - t - tw - t^2 w)}}{2t}$$

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- Dominant singularity: (again) a simple pole for $1 - 2t - 2t^2 + 2t^3 = 0$. Asymptotically,

$$p(n) \sim \kappa (2.48\dots)^n$$

- The series $P(t; 1)$ is neither algebraic, nor even D-finite (infinitely many poles at $\frac{tq}{q-t} = U(q^i)$)

General prudent walks: three catalytic variables

$$\left(1 - \frac{uvwt(1-t^2)}{(u-tv)(v-tu)}\right) T(u, v, w) = 1 + \mathcal{G}(w, u) + \mathcal{G}(w, v) - tv \frac{\mathcal{G}(v, w)}{u-tv} - tu \frac{\mathcal{G}(u, w)}{v-tu}$$

with $\mathcal{G}(u, v) = tvT(u, tu, v)$.

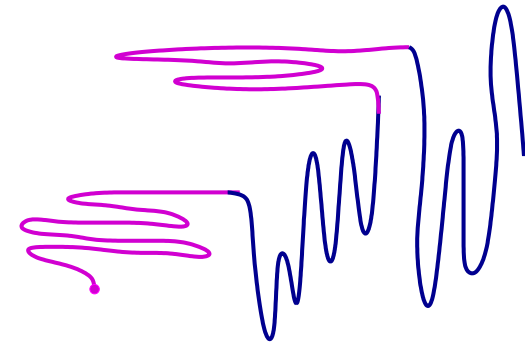
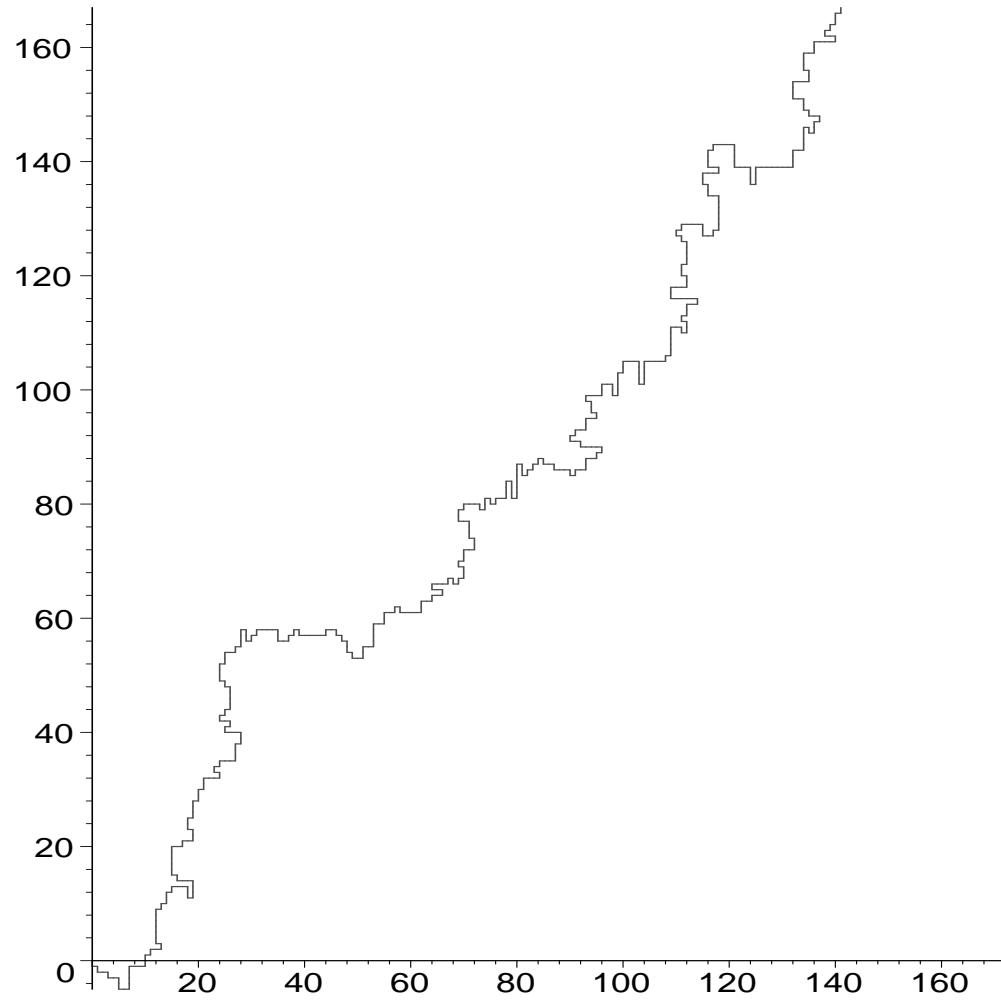
?

Random generation

(step by step construction)

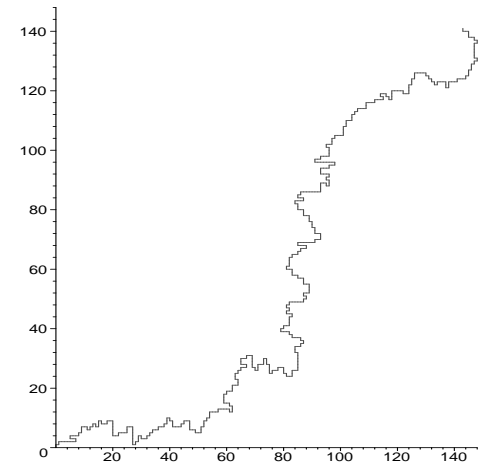
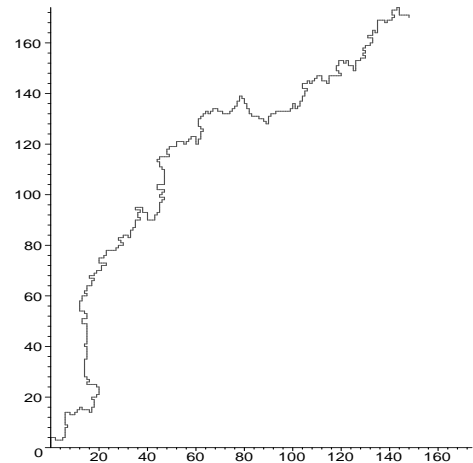
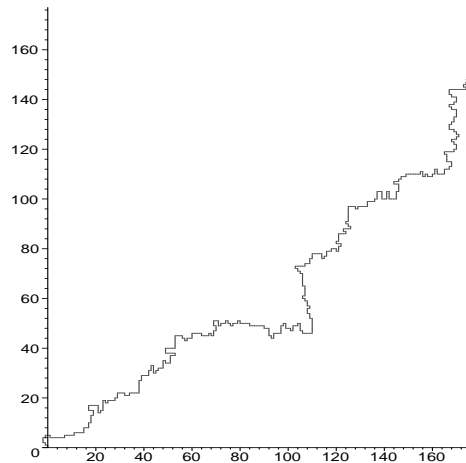
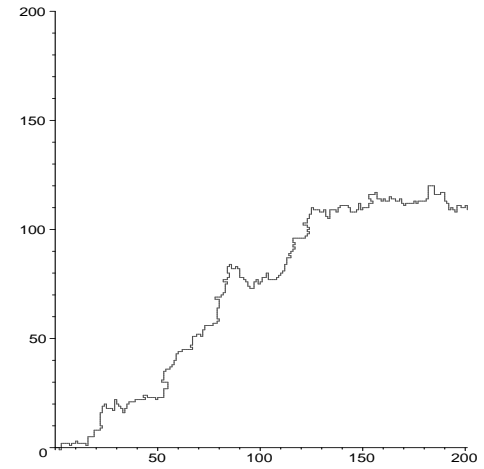
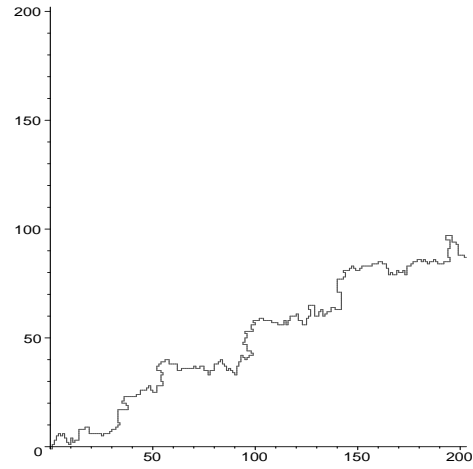
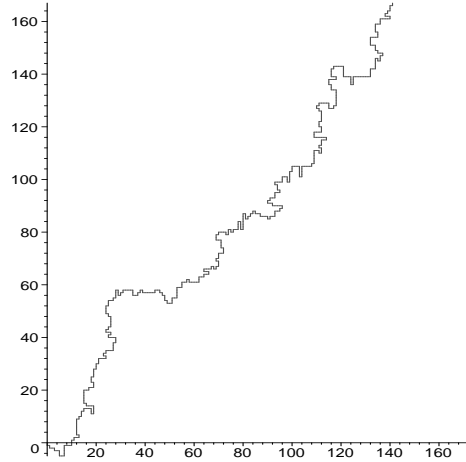
Two-sided prudent walks

500 steps



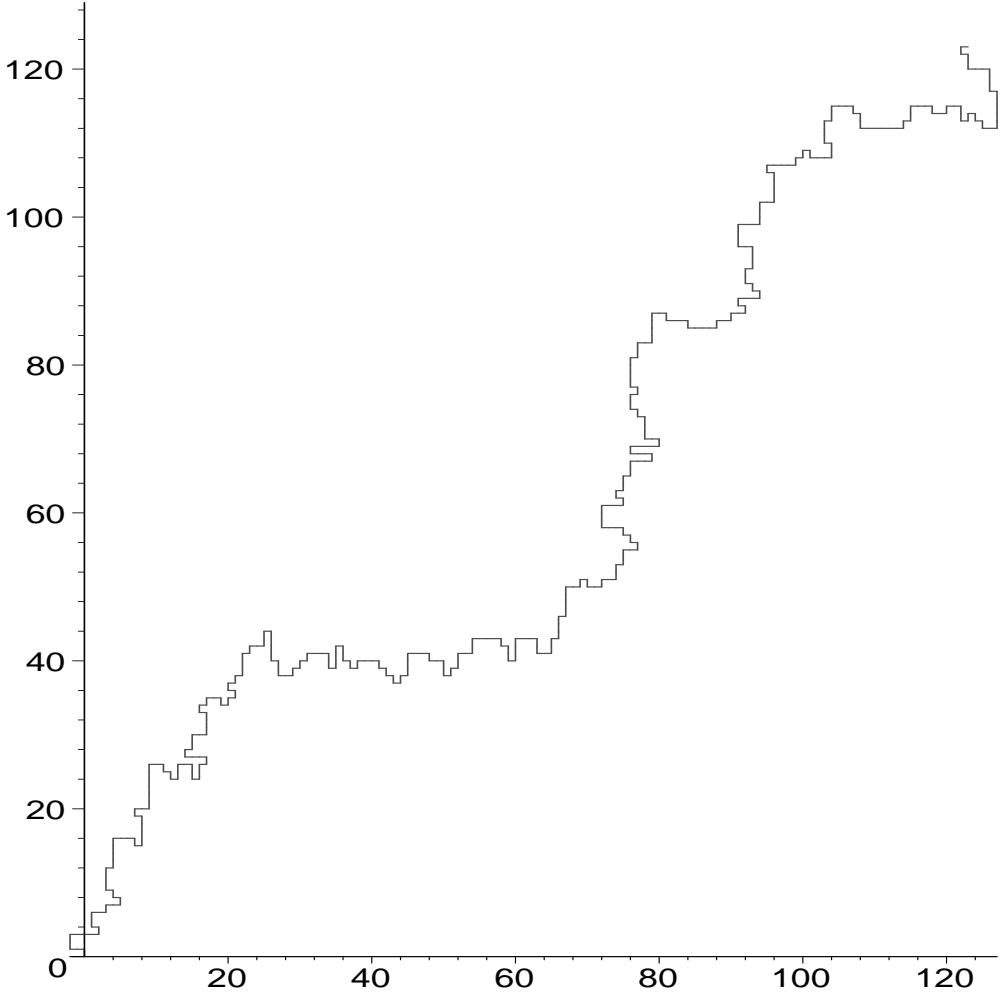
Two-sided prudent walks

500 steps



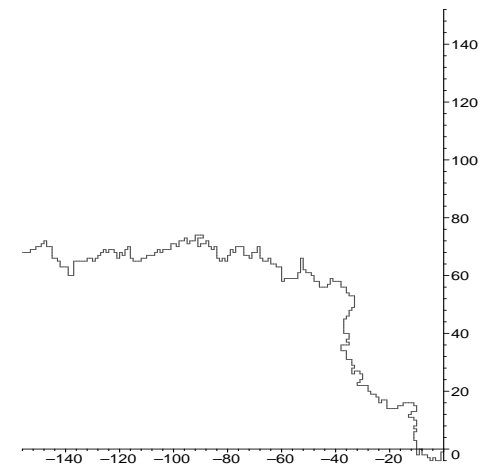
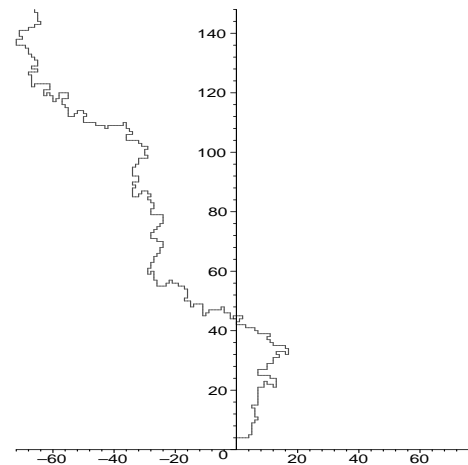
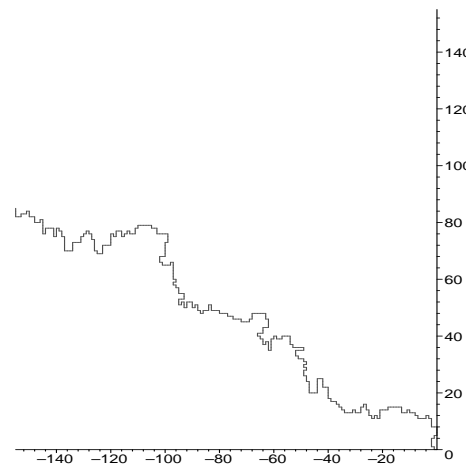
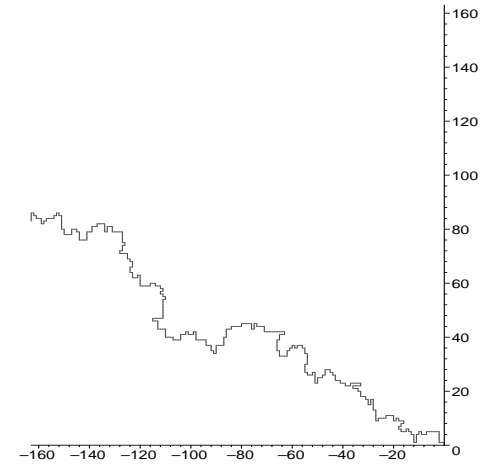
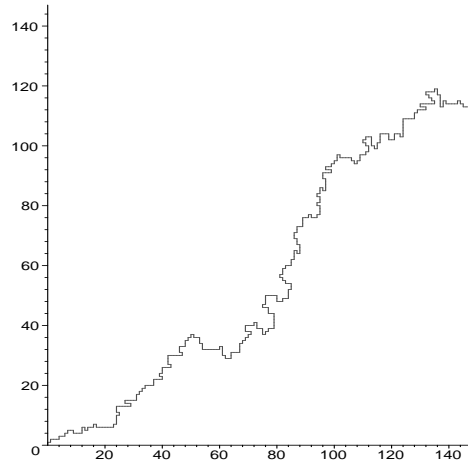
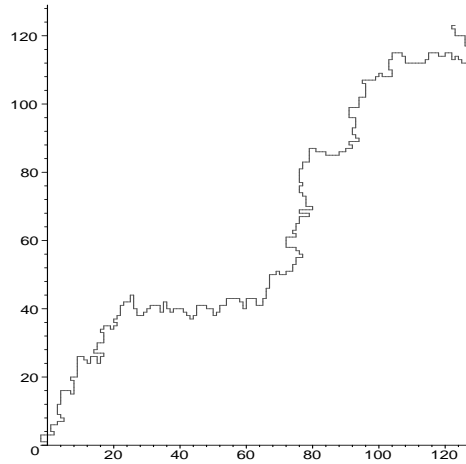
Three-sided prudent walks

400 steps



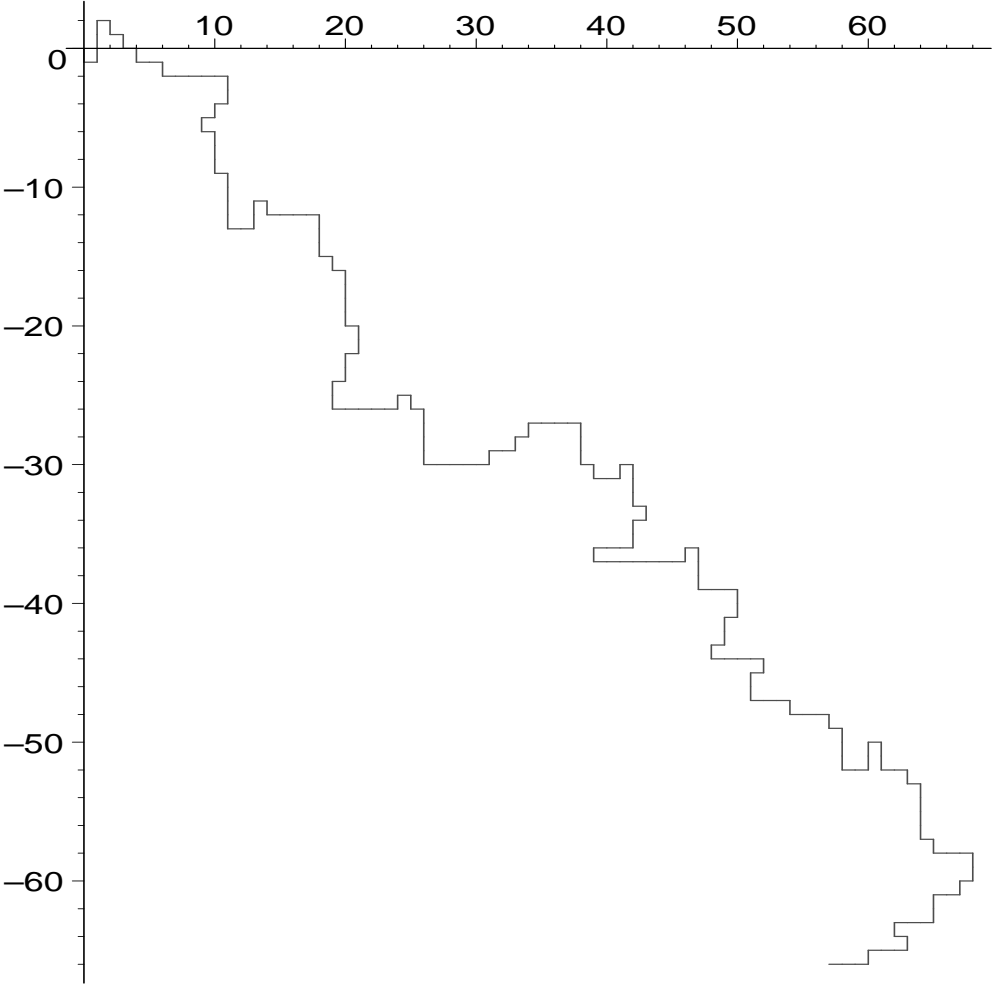
Three-sided prudent walks

400 steps



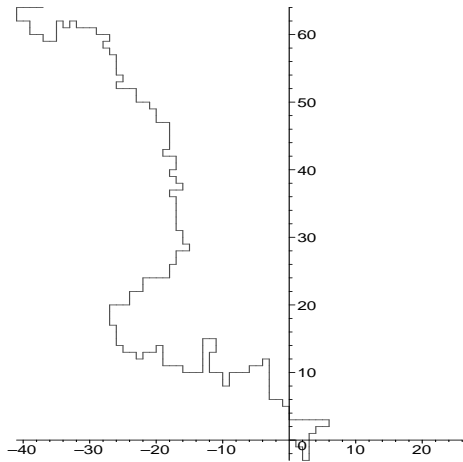
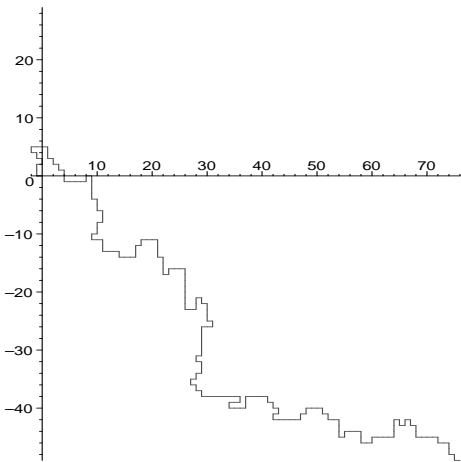
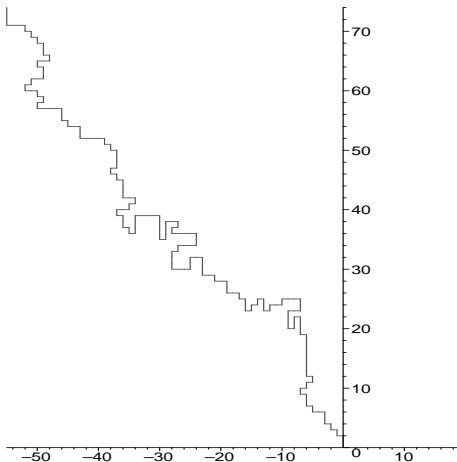
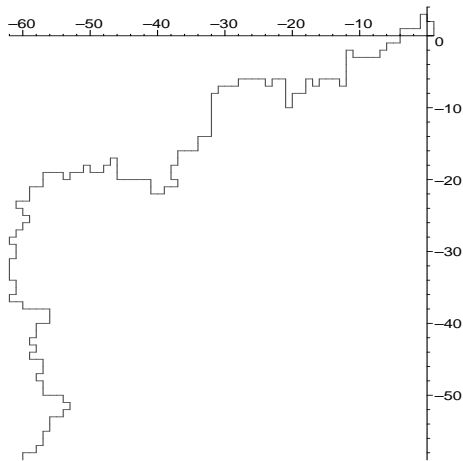
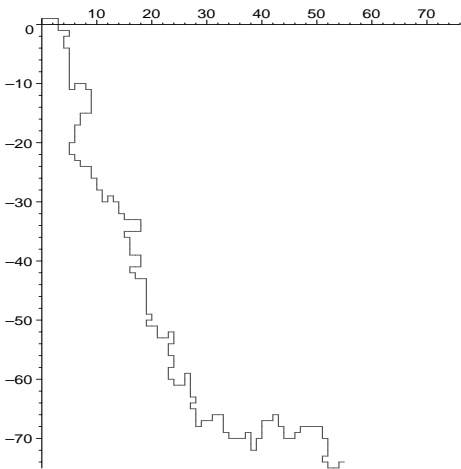
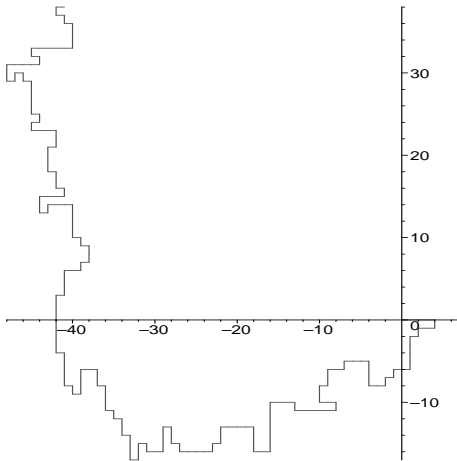
General prudent walks

195 steps (sic)



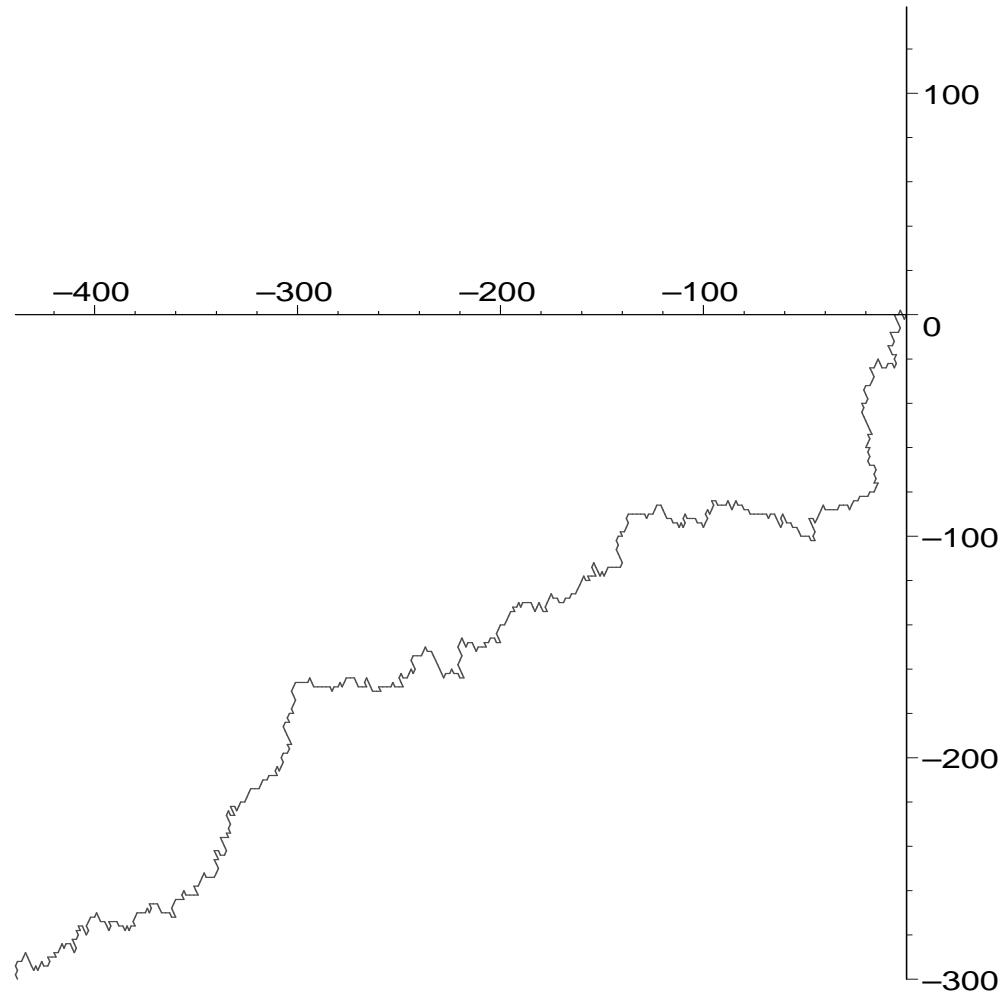
General prudent walks

195 steps



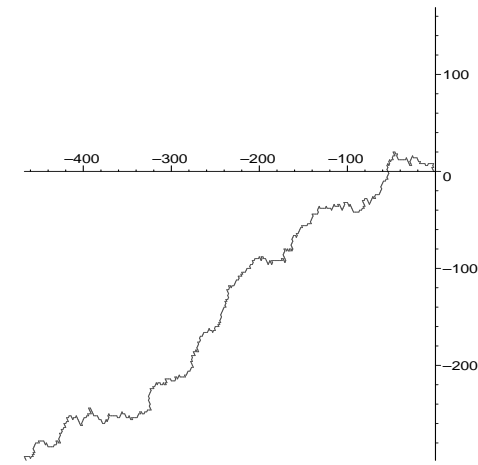
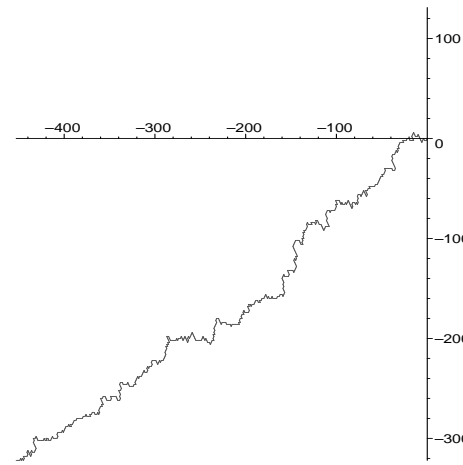
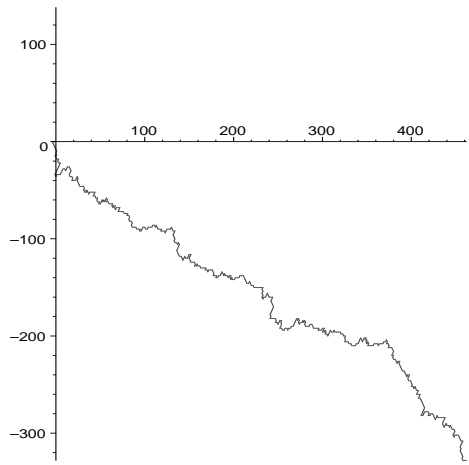
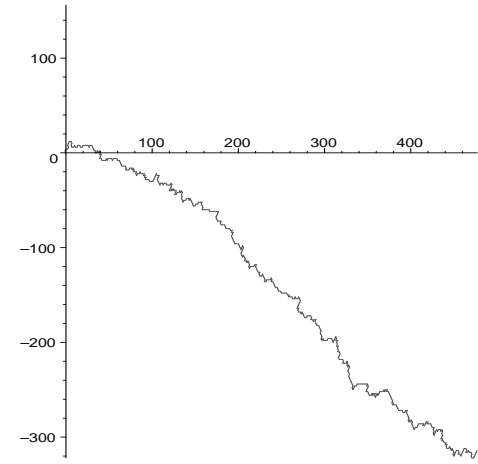
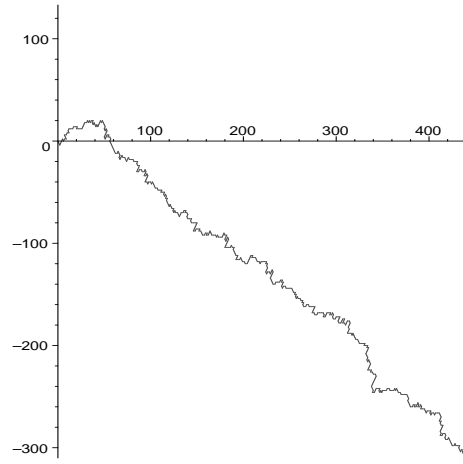
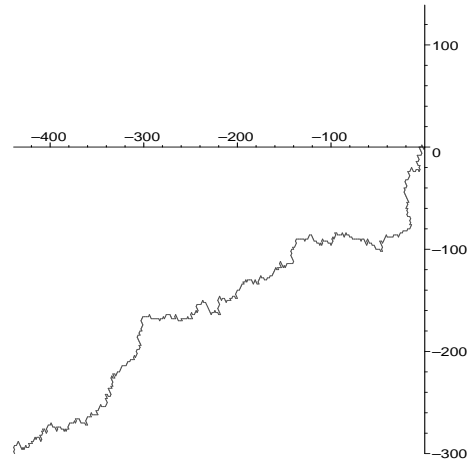
Triangular prudent walks

500 steps



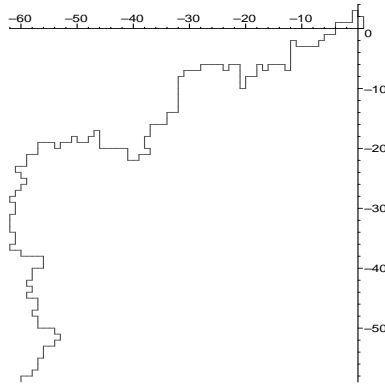
Triangular prudent walks

500 steps



Some questions

- General prudent walks on the square lattice: growth constant? Exact enumeration?
- More efficient procedures for random generation (maximal length 200 for general prudent walks...)



- The number of triangular prudent walks whose box has size k is

$$2^{k-1}(k+1)(k+2)!$$

Combinatorial explanation?

- Limit processes?