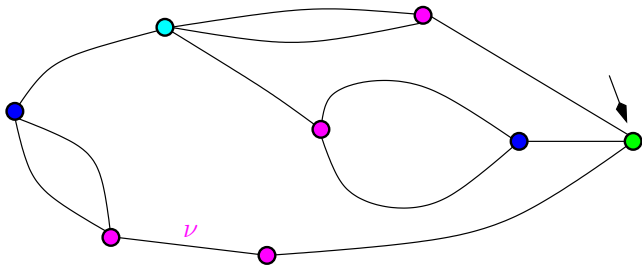


The Potts model on planar maps

Olivier Bernardi, Brandeis University, Boston
Mireille Bousquet-Mélou, CNRS, LaBRI, Bordeaux



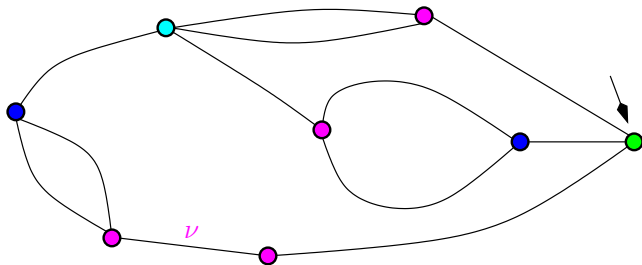
Outline

I. Planar maps and the Potts model

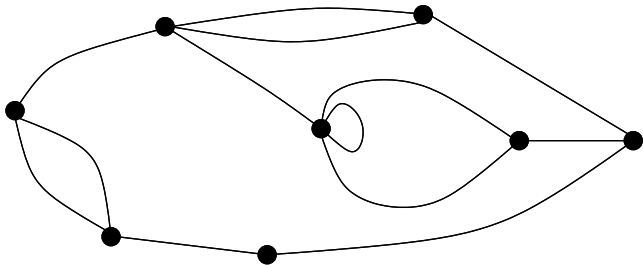
II. Main result

III. Where does it come from?

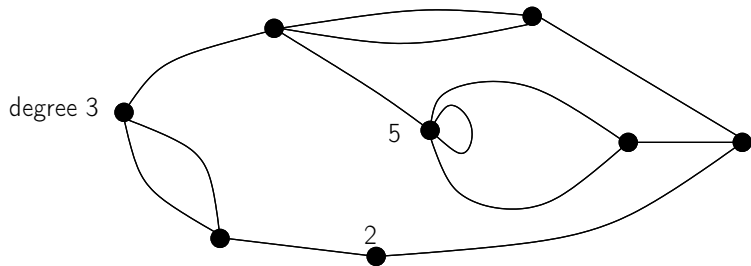
IV. Some special cases



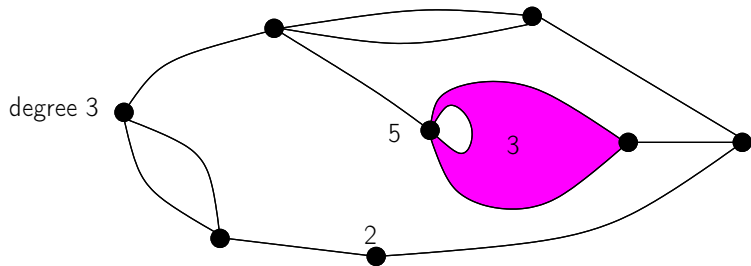
Planar maps



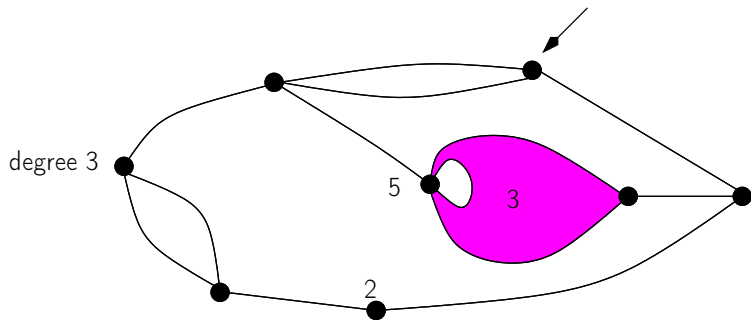
Planar maps



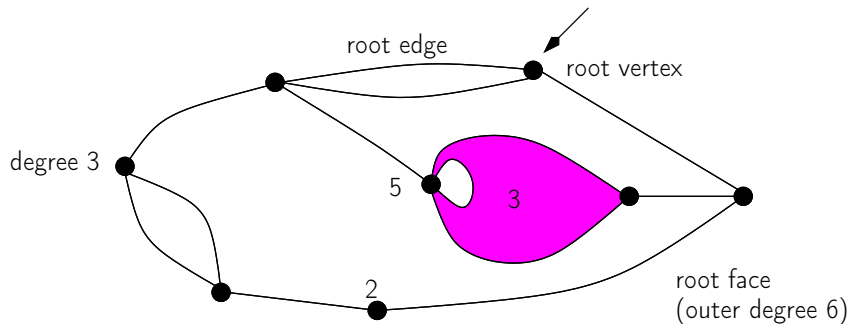
Planar maps



Planar maps: rooted version

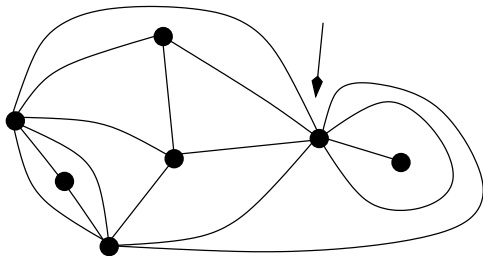


Planar maps: rooted version

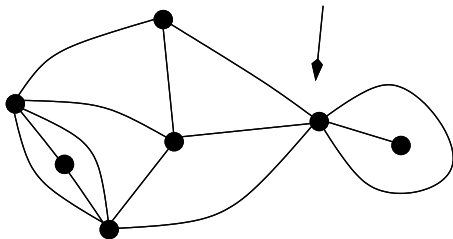


There is a finite number of maps with n edges

With degree constraints: rooted triangulations



With degree constraints: rooted triangulations



A near-triangulation with outer-degree 5

The Potts model on planar maps

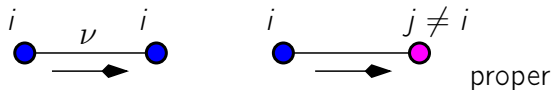
- The partition function of the q -state Potts model on a planar map M :

$$Z_M(q, \nu) = \sum_{c: V(M) \rightarrow \{1, 2, \dots, q\}} \nu^{m(c)}$$

where $m(c)$ is the number of monochromatic edges in the colouring c . In fact, $Z_M(q, \nu)$ is a polynomial in q (and ν), divisible by q .

Example: When M has one edge and two vertices,

$$Z_M(q, \nu) = q\nu + q(q-1)$$



The Potts model on planar maps

- Generating function:

$$\begin{aligned}M_1(q, \nu, w, t) &= \frac{1}{q} \sum_M Z_M(q, \nu) w^{\nu(M)} t^{e(M)} = \frac{1}{q} \sum_{M, c} w^{\nu(M)} t^{e(M)} \nu^{m(c)} \\ &= w + (\nu w + \nu w^2 + (q-1)w^2)t + O(t^2)\end{aligned}$$

“The Potts generating function of planar maps”

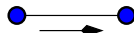
⇒ Enumeration of q -coloured planar maps, counted by vertices, edges, and monochromatic edges.



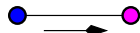
w



νw



νw^2



$(q-1)w^2$

A possible answer [Tutte 68]

- Consider the **refined** Potts generating function:

$$M(x, y) \equiv M(q, \nu, w, t; x, y) = \frac{1}{q} \sum_M Z_M(q, \nu) w^{\nu(M)} t^{e(M)} x^{dv(M)} y^{df(M)},$$

where $dv(M)$ (resp. $df(M)$) is the degree of the root-vertex (resp. root-face).

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where $dv(M)$ (resp. $df(M)$) is the degree of the root-vertex (resp. root-face).

- By deletion/contraction of the root-edge:

$$\begin{aligned} M(x, y) &= w + xyt((\nu - 1)(y - 1) + qy) M(x, y)M(1, y) \\ &\quad + xyt/w(x\nu - 1)M(x, y)M(x, 1) \\ &\quad + xywt(\nu - 1) \frac{xM(x, y) - M(1, y)}{x - 1} + xyt \frac{yM(x, y) - M(x, 1)}{y - 1}. \end{aligned}$$

A **discrete partial differential equation** with **two** catalytic variables

Are we happy?

The refined Potts generating function satisfies:

$$\begin{aligned}M(x, y) = & w + xyt ((\nu - 1)(y - 1) + qy) M(x, y)M(1, y) \\ & + xyt/w(x\nu - 1)M(x, y)M(x, 1) \\ & + xywt(\nu - 1) \frac{xM(x, y) - M(1, y)}{x - 1} + xyt \frac{yM(x, y) - M(x, 1)}{y - 1}.\end{aligned}$$

- compute coefficients efficiently (polynomial time)
- What is $M(1, 1) \equiv M(q, \nu, w, t; 1, 1)$?
- asymptotics? phase transitions?

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Experience makes us greedier: uncoloured maps
("pure gravity")

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- Let

$$G(t; y) \equiv G(y) = \sum_M t^{e(M)} y^{\text{df}(M)}$$

where $e(M)$ is the number of edges and $\text{df}(M)$ the degree of the outer face. Then by deletion of the root-edge [Tutte 68]:

$$G(y) = 1 + ty^2 G(y)^2 + ty \frac{yG(y) - G(1)}{y - 1}$$

A discrete differential equation with one catalytic variable, y .

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$$G(y) = 1 + ty^2 G(y)^2 + ty \frac{yG(y) - G(1)}{y - 1}$$

A **discrete differential equation** with **one** catalytic variable, y .

- The solution is an algebraic function!

$$G(1) = \sum_M t^{e(M)} = \frac{(1 - 12t)^{3/2} - 1 + 18t}{54t^2}$$

Equivalently,

$$27 t^2 G(1)^2 + (1 - 18 t) G(1) + 16 t - 1 = 0$$

Are we happy?

The refined Potts generating function satisfies:

$$\begin{aligned}M(x, y) = & w + xyt((\nu - 1)(y - 1) + qy)M(x, y)M(1, y) \\ & + xyt/w(x\nu - 1)M(x, y)M(x, 1) \\ & + xywt(\nu - 1)\frac{xM(x, y) - M(1, y)}{x - 1} + xyt\frac{yM(x, y) - M(x, 1)}{y - 1}.\end{aligned}$$

- compute coefficients efficiently (poly. time)
- What is $M(1, 1) \equiv M(q, \nu, w, t; 1, 1)$?
- asymptotics? phase transitions?
- Is $M(1, 1)$ algebraic?

An encouraging sign: the Ising model ($q = 2$)

Theorem

Let $\nu = (1 + \nu)/(1 - \nu)$. The Ising generating function of near-triangulations with outer degree 1 is

$$T_1(2, \nu, t) = \frac{(S - \nu)^2 (S - 2 + \nu) (-2\nu + \nu^2 - S\nu - S^2\nu + 3S^3)}{128t^4 (1 + \nu)^4 S^2},$$

where S is the unique series in t with constant term ν satisfying:

$$S = \nu + t^3 \frac{64(1 + \nu)^3 S^2}{(S - 2 + \nu)(2\nu - \nu^2 + 2S + S^2 - 4S^3)}.$$

In particular, it is algebraic.

\Rightarrow Asymptotics, exponents, transition at $\nu = 1 + 1/\sqrt{7}$

[Boulatov & Kazakov 87], [mbm & Schaeffer 02], [Bouttier, Di Francesco & Guitter 04]

Another special case: maps equipped with a spanning tree
(Potts in the limit $q \rightarrow 0, \nu \rightarrow 1$)

[Mullin 67]

Theorem

The GF of planar maps equipped with a spanning tree is

$$M_1(0, 1, 1, t) = \sum_{n \geq 0} \frac{1}{(n+1)(n+2)} \binom{2n}{n} \binom{2n+2}{n+1} t^n.$$

This series is **transcendental** (= non-algebraic), but **D-finite** (solution of a linear DE with polynomial coefficients).

⇒ Forget about algebraicity in general.

OK, but... is Potts D-finite?

A hierarchy of formal power series

- Rational series

$$A(t) = \frac{P(t)}{Q(t)}$$

- Algebraic series

$$P(t, A(t)) = 0$$

- Differentially finite series (D-finite)

$$\sum_{i=0}^d P_i(t) A^{(i)}(t) = 0$$

- D-algebraic series

$$P(t, A(t), A'(t), \dots, A^{(d)}(t)) = 0$$



Another special case: maps equipped with a connected subgraph

(Potts in the limit $q \rightarrow 0$)

Theorem [mbm & Courtiel 13(a)]

The generating function of triangulations equipped with a connected subgraph, counted by edges and by the size of the subgraph, is **not D-finite**.

But... it is **D-algebraic** (2nd order non-linear DE).

⇒ Forget about D-finiteness in general

OK, but... is Potts D-algebraic?

II. Main result

The Potts generating function is D-algebraic

Theorem

The Potts generating function of planar maps:

$$M_1(q, \nu, w, t) = \frac{1}{q} \sum_M Z_M(q, \nu) w^{\nu(M)} t^{e(M)},$$

is

- algebraic if $q = 2 + 2 \cos \frac{j\pi}{m}$, $q \neq 0, 4$ (includes $q = 2, 3$)
- D-algebraic (over $\mathbb{Q}(q, \nu, w, t)$) when q is an indeterminate

The same holds for triangulations.

[mbm-Bernardi 09(a)] Counting coloured planar maps: algebraicity results.

[mbm-Bernardi 15(a)] Counting coloured planar maps: differential equations

cf. [Eynard & Bonnet 99]: algebraicity w.r.t. the catalytic variable y (for near-triangulations)

An explicit differential system for Potts on planar maps

Let $D(t, u) = (q\nu + \beta^2)u^2 - q(\nu + 1)u + \beta t(q - 4)(wq + \beta) + q$,
with $\beta = \nu - 1$.

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- There exists a unique 11-tuple $(P_0(t), \dots, P_4(t), Q_0(t), \dots, Q_2(t), R_0(t), \dots, R_2(t))$ of series in t with coefficients in $\mathbb{Q}(q, \nu, w)$ such that

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$$P(0, u) = u^2(u - 1)^2 \quad \text{and} \quad Q(0, u) = u(u - 1).$$

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- The Potts GF of planar maps M_1 is an **explicit polynomial** in the P_i 's and Q_i 's.

Is this a differential system?

... or a partial differential equation?

$$\frac{1}{Q} \frac{\partial}{\partial t} \left(\frac{Q^2}{PD^2} \right) = \frac{1}{R} \frac{\partial}{\partial u} \left(\frac{R^2}{PD^2} \right),$$

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Both: The equation reads

$$2Q_t PD - QP_t D - 2QPD_t = 2R_u PD - RP_u D - 2RPD_u.$$

Extracting the coefficient of u^0, \dots, u^7 gives a system of 8 DEs (in t) between the 8 unknowns series. For u^7 for instance, one finds:

$$P'_3(t) - 2Q'_1(t) + 4(1 + \nu) - 4w(2\beta + q) = 0$$

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Elimination \Rightarrow existence of a DE of **order 5** for M_1 .

An explicit differential system for Potts on planar maps

Let $D(t, u) = (q\nu + \beta^2)u^2 - q(\nu + 1)u + \beta t(q - 4)(wq + \beta) + q$,
with $\beta = \nu - 1$.

- There exists a unique 11-tuple $(P_0(t), \dots, P_4(t), Q_0(t), \dots, Q_2(t), R_0(t), \dots, R_2(t))$ of series in t with coefficients in $\mathbb{Q}(q, \nu, w)$ such that

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$$\frac{1}{Q} \frac{\partial}{\partial t} \left(\frac{Q^2}{PD^2} \right) = \frac{1}{R} \frac{\partial}{\partial u} \left(\frac{R^2}{PD^2} \right),$$

with $P \equiv P(t, u) = P_0(t) + P_1(t)u + \dots + P_4(t)u^4$ and so on, and the initial conditions (at $t = 0$):

$$P(0, u) = u^2(u - 1)^2 \quad \text{and} \quad Q(0, u) = u(u - 1).$$

- The Potts GF of planar maps M_1 is an **explicit polynomial** in the P_i 's and Q_i 's \Rightarrow **DE of order 5**

An explicit differential system for Potts on triangulations

Let $D(t, u) = q\nu^2 u^2 + \beta(4\beta + q)u + q\beta\nu(q - 4)t + \beta^2$,
with $\beta = \nu - 1$.

- There exists a unique 9-tuple $(P_0(t), \dots, P_3(t), Q_0(t), \dots, Q_2(t), R_0(t), R_1(t))$ of series in t with coefficients in $\mathbb{Q}(q, \nu)$ such that

$$P_3(t) = 1, \quad Q_2(t) = 2\nu,$$

and

$$\frac{1}{Q} \frac{\partial}{\partial t} \left(\frac{Q^2}{PD^2} \right) = \frac{1}{R} \frac{\partial}{\partial u} \left(\frac{R^2}{PD^2} \right),$$

with $P \equiv P(t, u) = P_0(t) + P_1(t)u + \dots + P_3(t)u^3$ and so on, and the initial conditions (at $t = 0$):

$$P(0, u) = u^2(u + 1/4) \quad \text{and} \quad Q(0, u) = u(2\nu u + 1).$$

- The Potts GF T_1 of near-triangulations (outer degree 1) T_1 is an explicit polynomial in the P_i 's and Q_i 's \Rightarrow DE of order 4

III. Where does it come from?

$$\frac{1}{Q} \frac{\partial}{\partial t} \left(\frac{Q^2}{PD^2} \right) = \frac{1}{R} \frac{\partial}{\partial u} \left(\frac{R^2}{PD^2} \right)$$

A (vague) idea of the proof

In the footsteps of W. Tutte

- For the GF $T(x, y)$ of properly q -coloured triangulations:

$$T(x, y) = x(q - 1) + xyzT(x, y)T(1, y) \\ + xz \frac{T(x, y) - T(x, 0)}{y} - x^2yz \frac{T(x, y) - T(1, y)}{x - 1}$$

[Tutte 73] Chromatic sums for rooted planar triangulations: the cases $\lambda = 1$ and $\lambda = 2$

[Tutte 73] Chromatic sums for rooted planar triangulations, II: the case $\lambda = \tau + 1$

[Tutte 73] Chromatic sums for rooted planar triangulations, III: the case $\lambda = 3$

[Tutte 73] Chromatic sums for rooted planar triangulations, IV: the case $\lambda = \infty$

[Tutte 74] Chromatic sums for rooted planar triangulations, V: special equations

[Tutte 78] On a pair of functional equations of combinatorial interest

[Tutte 82] Chromatic solutions

[Tutte 82] Chromatic solutions II

[Tutte 84] Map-colourings and differential equations



[Tutte 95]: Chromatic sums revisited

Structure of the proof

1. Tutte's equation with two catalytic variables:

$$\begin{aligned} M(x, y) = & w + xyt((\nu - 1)(y - 1) + qy) M(x, y)M(1, y) \\ & + xyt/w(x\nu - 1)M(x, y)M(x, 1) \\ & + xywt(\nu - 1) \frac{xM(x, y) - M(1, y)}{x - 1} + xyt \frac{yM(x, y) - M(x, 1)}{y - 1} \end{aligned}$$

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$$\text{Pol}(M(x, y), M(x, 1), M(1, y), x, y) = 0$$

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2. For $q = 2 + 2 \cos(j\pi/m)$, there also exists an equation with only **one catalytic variable** defining $M(1, y)$: [B-mbm 09(a)]

$$\text{Pol}_{j,m}(M(1, y), M(1, 1), M_y(1, 1), M_{y,y}(1, 1), \dots, y) = 0$$

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3. Derive from it a **polynomial equation** for the Potts generating function $M_1 = M(1, 1)$. [mbm-Jehanne 06]

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$$\text{Pol}_{j,m}(M(1, y), M(1, 1), M_y(1, 1), M_{y,y}(1, 1), \dots, y) = 0$$

3. Derive from it a **differential system** for the Potts generating function $M_1 = M(1, 1)$.

Structure of the proof

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2. For $q = 2 + 2 \cos(j\pi/m)$, there also exists an equation with only **one catalytic variable** defining $M(1, y)$: [B-mbm 09(a)]

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Planar maps: algebraic solution

- The functional equation

$$G(y) = 1 + ty^2G(y)^2 + ty \frac{yG(y) - G(1)}{y - 1}$$

Planar maps: algebraic solution

- The functional equation, written with a square:

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- Algebraic consequence:** The discriminant of $\Delta(y)$ w.r.t. y is zero:

$$27t^2G_1^2 + (1 - 18t)G_1 + 16t - 1 = 0$$

Planar maps: differential solution

- The polynomial

$$\Delta(y) = (y - 1 - y^2t)^2 - 4ty^2(y - 1)^2 + 4t^2y^3(y - 1)G_1$$

has degree 4 in y , and admits a **double** root $Y(t)$:

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$$\frac{1}{Q} \frac{\partial}{\partial t} \left(\frac{Q^2}{P} \right) = \frac{1}{R} \frac{\partial}{\partial y} \left(\frac{R^2}{P} \right).$$

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- Writing $P(t; y) = P_0(t) + yP_1(t) + y^2P_2(t)$ and so on for Q and R , this gives a **system of differential equations in t** relating the series P_i , Q_i and R_i .
- Finally, the GF G_1 is a polynomial in the series P_i , Q_i , R_i .

IV. Special values of q and ν

Specializations: explicit DE for the Potts generating function

Specialization	general maps	triangulations
$q = 2, 3$	algebraic	algebraic
Proper colourings ($\nu = 0$)		$D = (4 - q)u + 1$ order 2
Four colours ($q = 4$)	$D = ((\nu + 1)u - 2)^2$ order 3	$D = (2\nu u + \beta)^2$ order 2
Connected subgraphs (spanning forests) ($q = 0$)		$D = \beta^2(1 + 4u)$ order 2
Self-dual model ($q = \beta^2, w = 1/\beta$)	$R_2 = 0$ order 3	

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Two combinatorial problems:

1. Properly 3-coloured planar maps [Bernardi-mbm 09(a)]

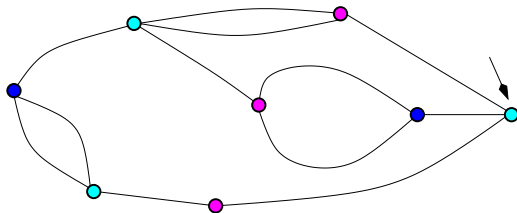
Let A be the quartic series in t defined by

$$A = t \frac{(1 + 2A)^3}{1 - 2A^3}.$$

Then the generating function of properly 3-coloured planar maps is

$$M_1(3, 0, 1, t) = \frac{(1 + 2A)(1 - 2A^2 - 4A^3 - 4A^4)}{(1 - 2A^3)^2}$$

The proof is at the moment h o r r i b l e ...



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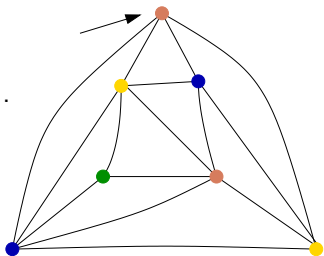
2. Properly q -coloured triangulations [Tutte 84]

The number $c(n)$ of properly q -coloured triangulations having n vertices satisfies:

$$(n+1)(n+2)c(n+2) = (q-4)(3n-1)(3n-2)c(n+1) + 2 \sum_{i=1}^n i(i+1)(3n-3i+1)c(i+1)c(n+2-i),$$

with the initial condition $c(2) = q - 1$.

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More questions are left...

A. The differential system

Simpler, and/or more combinatorial derivation? Done for

- the Ising model ($q = 2$)
[mbm & Schaeffer 02], [Bouttier, Di Francesco & Guitter 04]
- spanning forests/connected subgraphs ($q \rightarrow 0$)
[Bouttier et al. 07], [mbm-Courtiel 13(a)]

B. Asymptotics, phase transitions

- Asymptotic number of properly q -coloured triangulations when $q \in (28/11, 4) \cup [5, \infty)$ [Odlyzko-Richmond 83]
- Critical point of the Potts model on planar maps when $q \in (0, 4)$ (for the outer degree) [Borot et al. 12]
- Critical exponents for near-triangulations (for the outer degree) [Eynard & Bonnet 99]