

Journées X. G. Viennot

28 et 29 juin 2012, LaBRI / U. Bordeaux, France

Ω (*creation, annihilation*)

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THE CLASSICAL
CREATION AND ANNIHILATION OPERATORS
IN COMBINATORICS

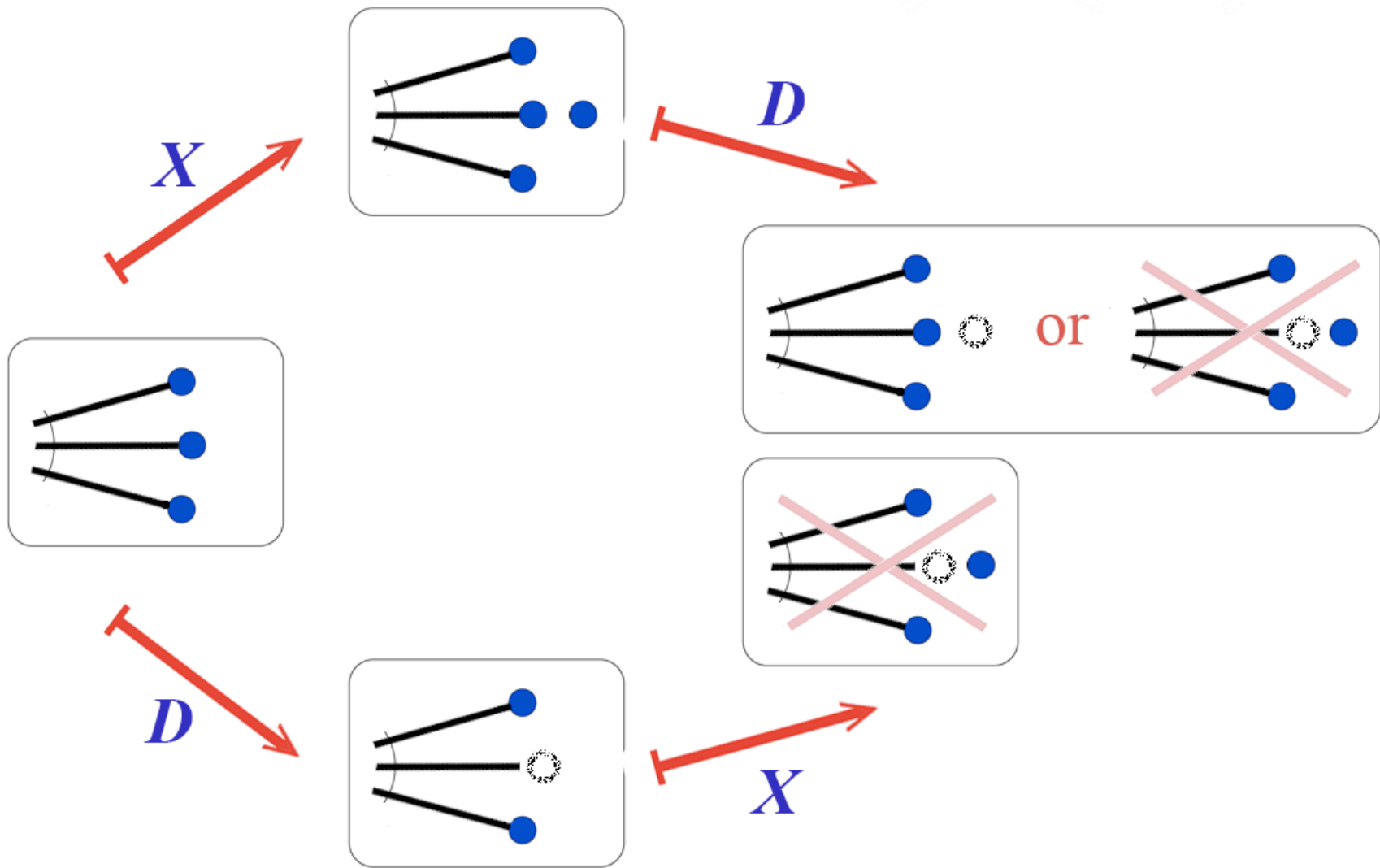
CREATION OPERATOR : X (= multiplication by X)

ANNIHILATION OPERATOR : D (= differentiation w/r to X)

These operators do not commute !

$$[D, X] \stackrel{\text{def}}{=} DX - XD = 1$$

$$DX - XD = 1$$



Creation-annihilation in the context of combinatorial classes



P. Blasiak and P. Flajolet, *Combinatorial models of creation-annihilation*, Journal électronique du Séminaire Lotharingien de Combinatoire, No. 65, Article B65c (2011).

+ many others (including Xavier)

THEORY OF COMBINATORIAL SPECIES

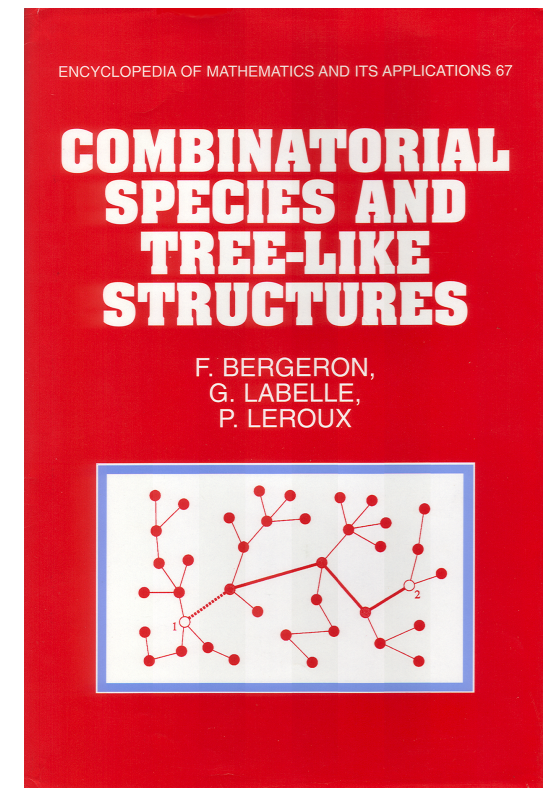
Introduced by André Joyal,



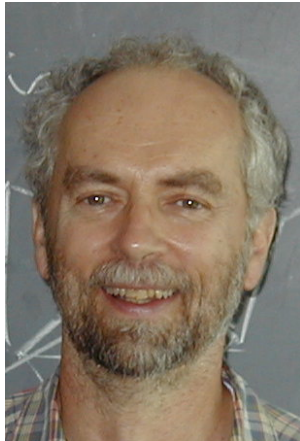
LaCIM, 1981

André Joyal, *Une théorie combinatoire des séries formelles*, *Advances in Mathematics* 42:1-82 (1981)

Followed by publications by many authors including,



Differential equations in the context of *linear* species



P. Leroux and G. Viennot, *Combinatorial resolution of systems of differential equations, I: Ordinary differential equations*, dans *Combinatoire énumérative*, UQAM 1985, Proceedings, Lecture Notes in Mathematics, vol. 1234, Springer-Verlag, 1986, 210-245.

P. Leroux et G.X. Viennot, *Résolution combinatoire des systèmes d'équations différentielles, II: Calcul intégral combinatoire*, *Ann. Sci. Math. Qué.* Vol. 12 (1988), 233-253.

P. Leroux et X.G. Viennot, *A combinatorial approach to non linear functional expansions : An introduction with an example*, *Theoretical Computer Science*, 79 (1991), 179-193.

P. Leroux et G.X. Viennot, *Combinatorial resolution of systems of differential equations, IV: Separation of variables*, *Discrete Math.* 72 (1988), 237-250.

Differential equations in the context of *ordinary species*



G. Labelle, *Éclosions combinatoires appliquées l'inversion multidimensionnelle des séries formelles*. J. Comb.Theory-A, 39, no. 1, (1985), 52-82.

G. Labelle, *On Combinatorial Differential Equations*, J. of Math. Analysis and Applications, 113, no.2, (1986), pp. 344-381.

G. Labelle, *Dérivées directionnelles et développements de Taylor combinatoires*, Discrete Mathematics 79, (1989/90), 279-297.

Y. Chiricota et G. Labelle, *Familles de solutions combinatoires de l'équation différentielle $Y' = 1 + Y^2$, $Y(0) = 0$ et des équations différentielles autonomes*, Discrete Mathematics 115 (1993) 77-93.

G. Labelle and C. Lamathe, *General Combinatorial Differential Operators*, Journal électronique du Séminaire Lotharingien de Combinatoire - numéro spécial à la mémoire de Pierre Leroux, No. 61A, Article B61Ag (2009).

Subject of the present talk :

Creation-annihilation in the context of *ordinary* species

Translation:

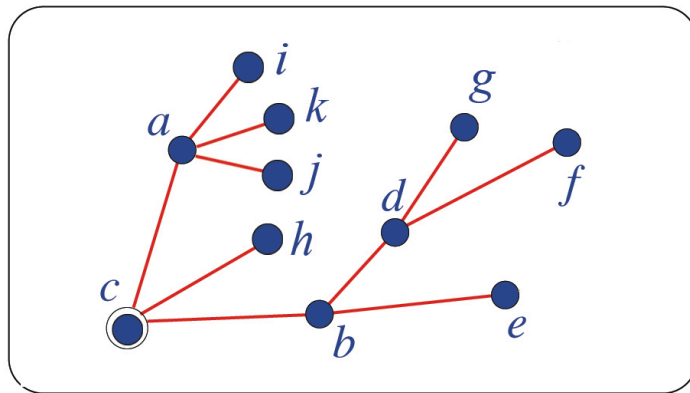
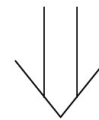
The combinatorial calculus of operators of the form

$$\Omega(X, D)$$

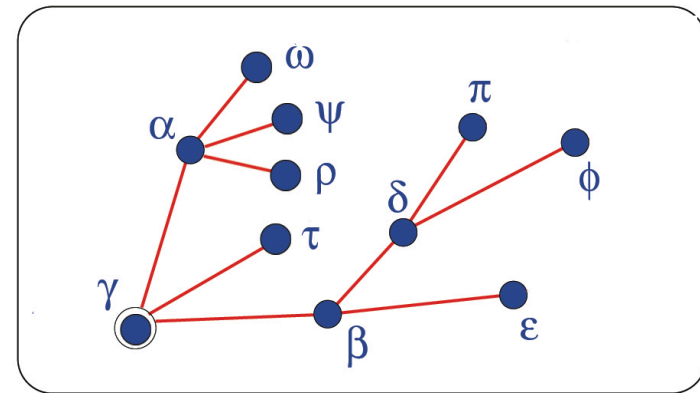
where $\Omega(X, T)$ is a (possibly weighted and/or complex) species on two sorts, X and T , of elements in their underlying structures.

THE CLASS OF ROOTED TREES WEIGHTED BY $t^{\# \text{ of leaves}}$

$$\sigma = \begin{pmatrix} a, b, c, d, e, f, g, h, i, j, k \\ \alpha, \beta, \gamma, \delta, \varepsilon, \phi, \pi, \tau, \omega, \rho, \psi \end{pmatrix} \text{ isomorphism}$$



relabelling
(transporting)
along σ



$$\text{weight} = t^7$$

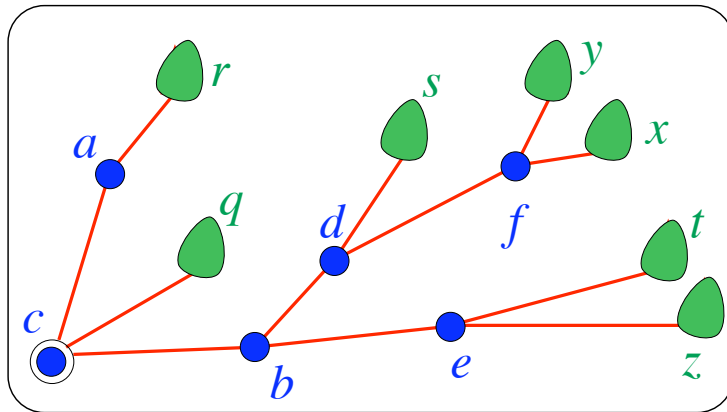
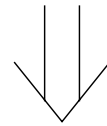
$$\text{weight} = t^7$$

Isomorphic structures

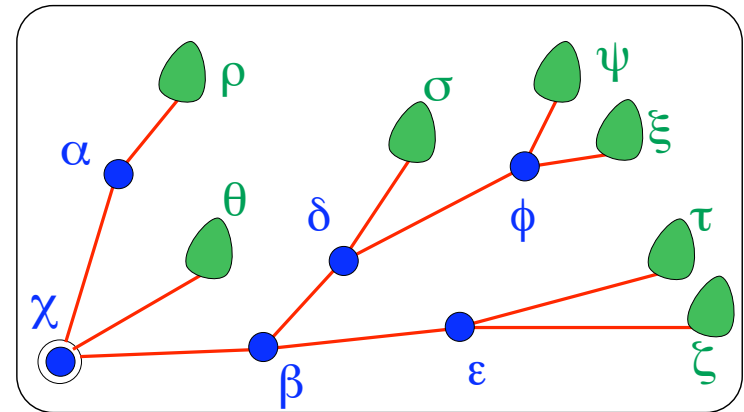
THE CLASS OF ROOTED TREES ON TWO SORTS OF ELEMENTS (inner nodes and leaves) WEIGHTED BY

$$t_0^{\# \text{ elements of degree } 0} \cdot t_1^{\# \text{ elements of degree } 1} \cdot t_2^{\# \text{ elements of degree } 2} \dots$$

$$\sigma = \left(\begin{array}{c|c} a, b, c, d, e, f & q, r, s, t, x, y, z \\ \alpha, \beta, \chi, \delta, \varepsilon, \phi & \theta, \rho, \sigma, \tau, \xi, \psi, \zeta \end{array} \right) \text{ isomorphism}$$



relabelling
(transporting)
along σ



$$\text{weight} = t_1^7 \cdot t_2 \cdot t_3^5$$

$$\text{weight} = t_1^7 \cdot t_2 \cdot t_3^5$$

WHAT IS A COMBINATORIAL SPECIES ?

Informal definition :

A species is a class of weighted combinatorial structures built on arbitrary finite sets made of various « sorts » of elements.

The class must be closed under arbitrary weight and sort preserving bijections between the underlying sets of elements of its structures.

We assume that the weight of a structure is a monomial in some « weight variables ».

WHAT IS A COMBINATORIAL SPECIES ?

Formal (and more precise) definition :

A species is a functor $F : (\mathbf{B} \times \mathbf{B} \times \dots)_{\text{weak}} \longrightarrow \mathbf{Sets}_{\text{summable weighted}}$

where \mathbf{B} is the category of finite sets and bijections and

$\mathbf{Sets}_{\text{summable weighted}}$ is a category of summable weighted sets.

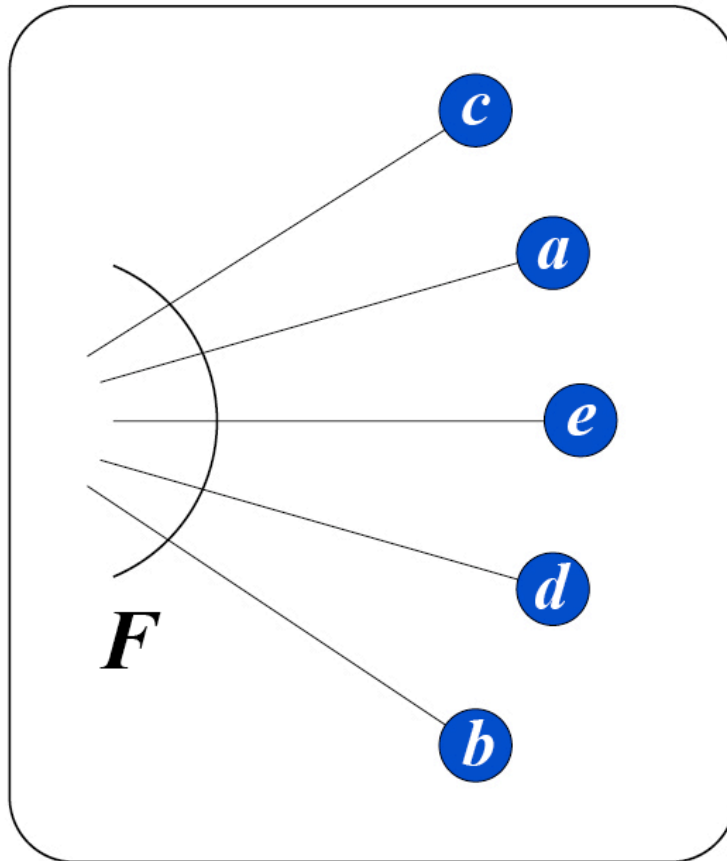
Weights are assumed to be monomials in some weight variables $t, u, t_0, t_1, t_2, \dots$

$$(\mathbf{B} \times \mathbf{B} \times \dots)_{\text{weak}} \ni (U_1, U_2, \dots) \xrightarrow{F} F[U_1, U_2, \dots] \in \mathbf{Sets}_{\text{summable weighted}}$$

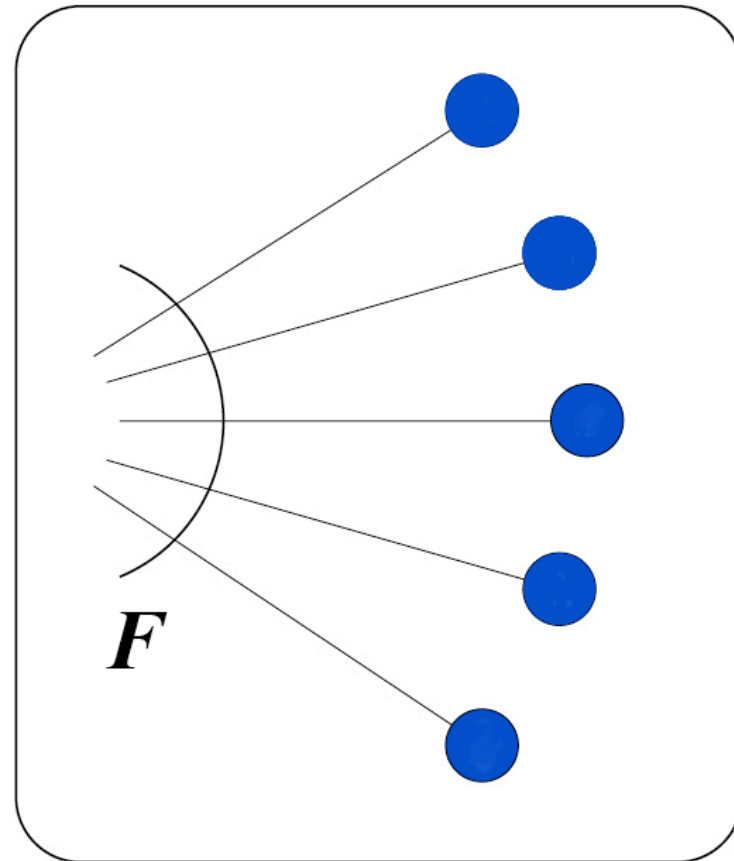
An element $s \in F[U_1, U_2, \dots]$ is called a F -structure on (U_1, U_2, \dots)

An element $u_i \in U_i$ is called a singleton of sort i

GRAPHICAL CONVENTION - 1

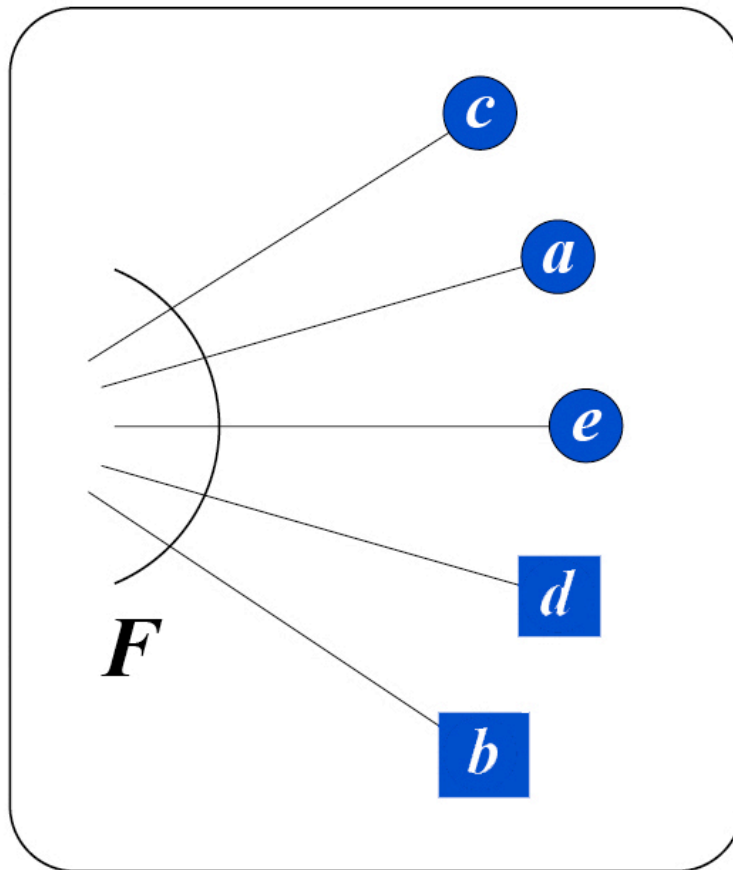


An F -structure on
 $\{a, b, c, d, e\}$

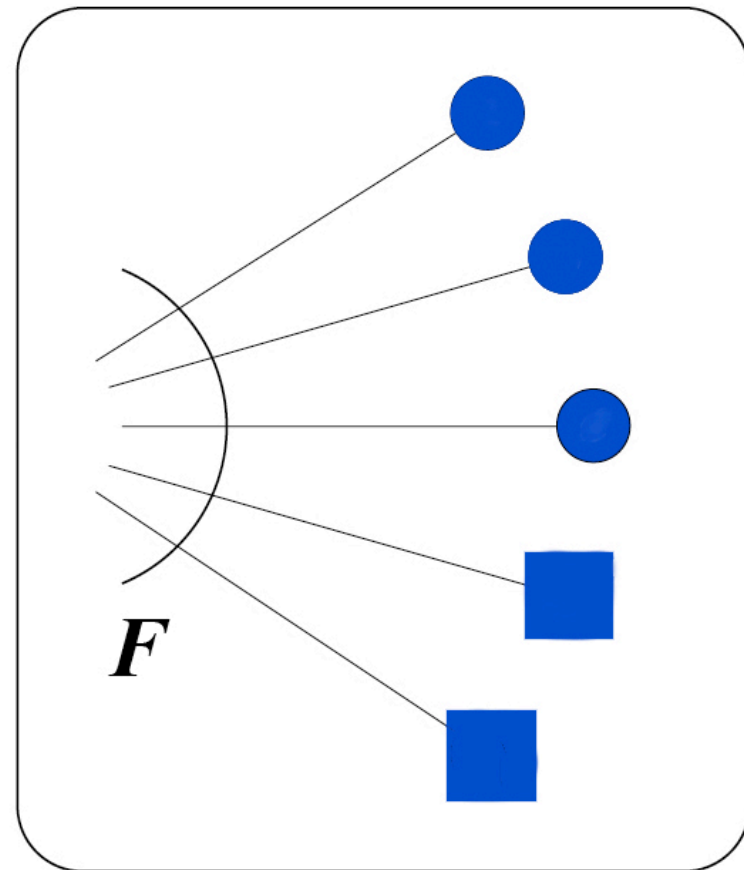


An F -structure on
a 5-element set

GRAPHICAL CONVENTION - 2

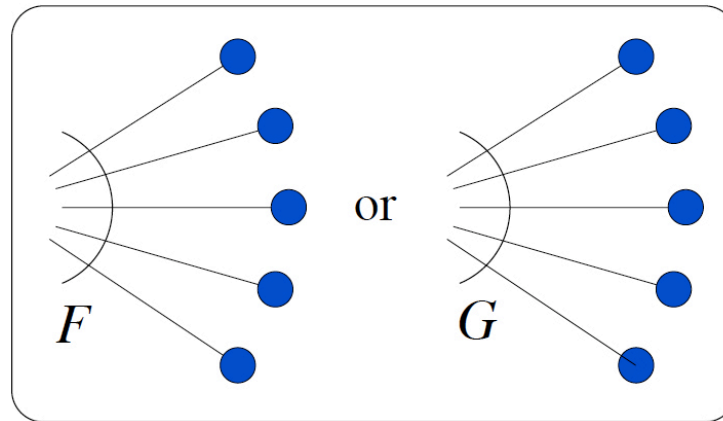


An F -structure on
 $\{a, c, e \mid d, b\}$

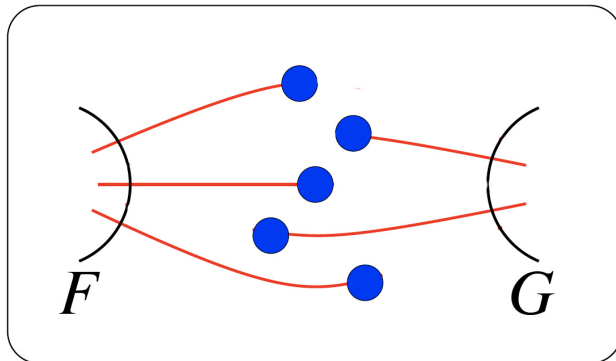


An F -structure on
a $(3, 2)$ -element set

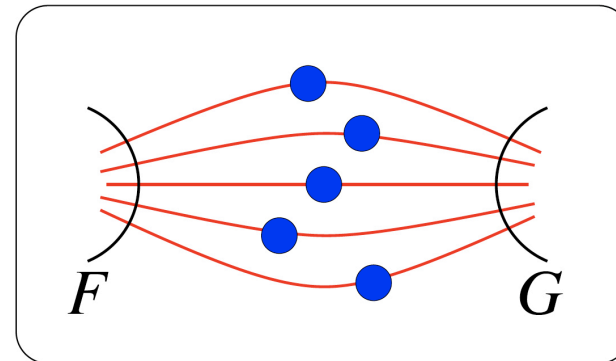
BASIC OPERATIONS ON SPECIES – 1



$(F+G)$ – structure

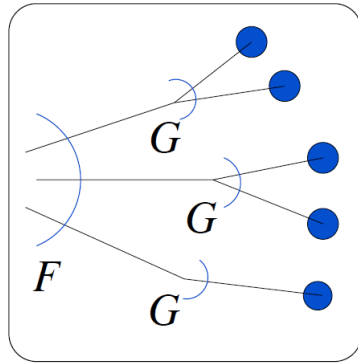


$F \cdot G$ – structure

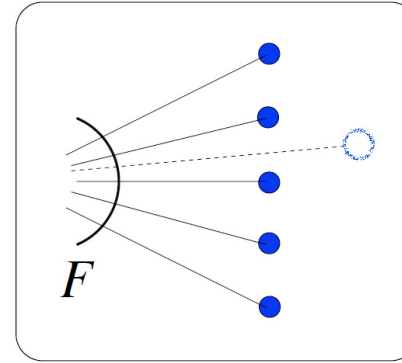


$F \times G$ – structure

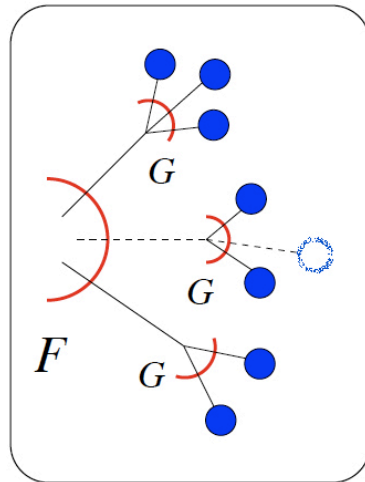
BASIC OPERATIONS ON SPECIES – 2



$F \circ G$ - structure

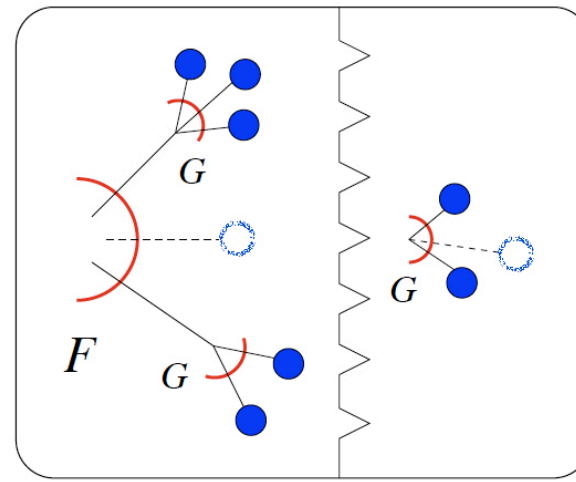


DF - structure

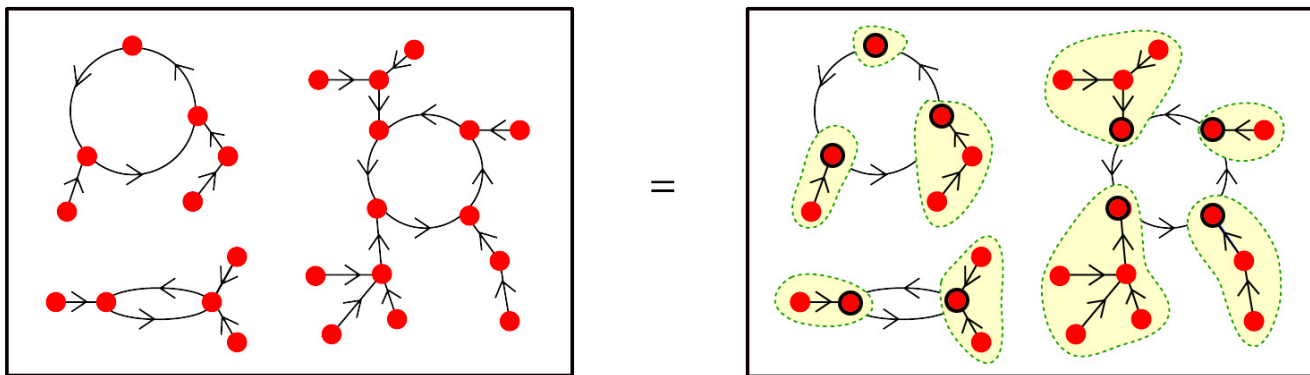


$D(F \circ G)$

=



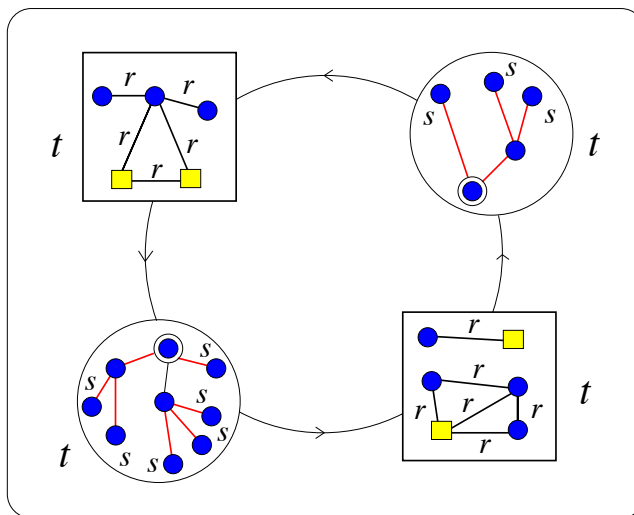
$((DF) \circ G) \cdot DG$



$\text{End}(X)$

$=$

$S(A(X))$



$C_{alt,t}(A_s(X), G_r(X+T))$

Cycle index series

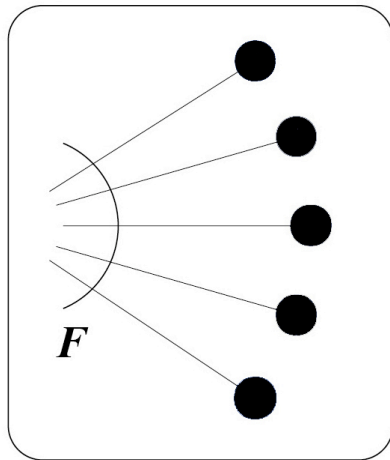
$$F(X) \mapsto Z_F(x_1, x_2, x_3, \dots) = \sum_{n \geq 0} \frac{1}{n!} \sum_{\sigma \in S_n} \text{weight Fix}F[\sigma] \frac{x_1^{\sigma_1} x_2^{\sigma_2} x_3^{\sigma_3} \dots}{1^{\sigma_1} \sigma_1! 2^{\sigma_2} \sigma_2! 3^{\sigma_3} \sigma_3! \dots}$$

$F \mapsto Z_F$ is compatible with the combinatorial operations

$Z_F(x, 0, 0, \dots) = F(x)$ weights (or counts) labelled structures

$Z_F(x, x^2, x^3, \dots) = \tilde{F}(x)$ weights (or counts) unlabelled structures

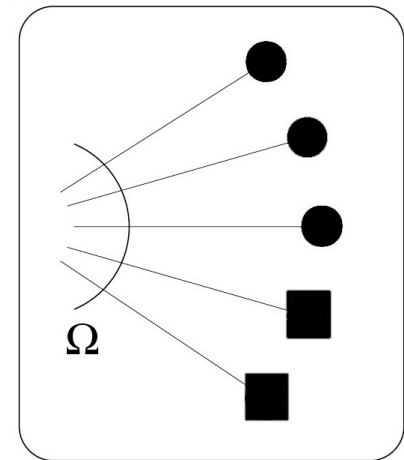
$$\Omega(X, T) \mapsto Z_\Omega(x_1, x_2, \dots; t_1, t_2, \dots) = \sum_{n, k \geq 0} \frac{1}{n! k!} \sum_{(\sigma, \tau) \in S_{n, k}} \text{weight Fix}F[\sigma, \tau] \frac{x_1^{\sigma_1} x_2^{\sigma_2} \dots t_1^{\tau_1} t_2^{\tau_2} \dots}{1^{\sigma_1} \sigma_1! 2^{\sigma_2} \sigma_2! \dots 1^{\tau_1} \tau_1! 2^{\tau_2} \tau_2! \dots}$$



$$F = F(X)$$

$X =$ the species of \bullet - singletons

$T =$ the species of \blacksquare - singletons



$$\Omega = \Omega(X, T)$$

$X^n =$ the species of n - lists of singletons

X^5 - structure : $\bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet$

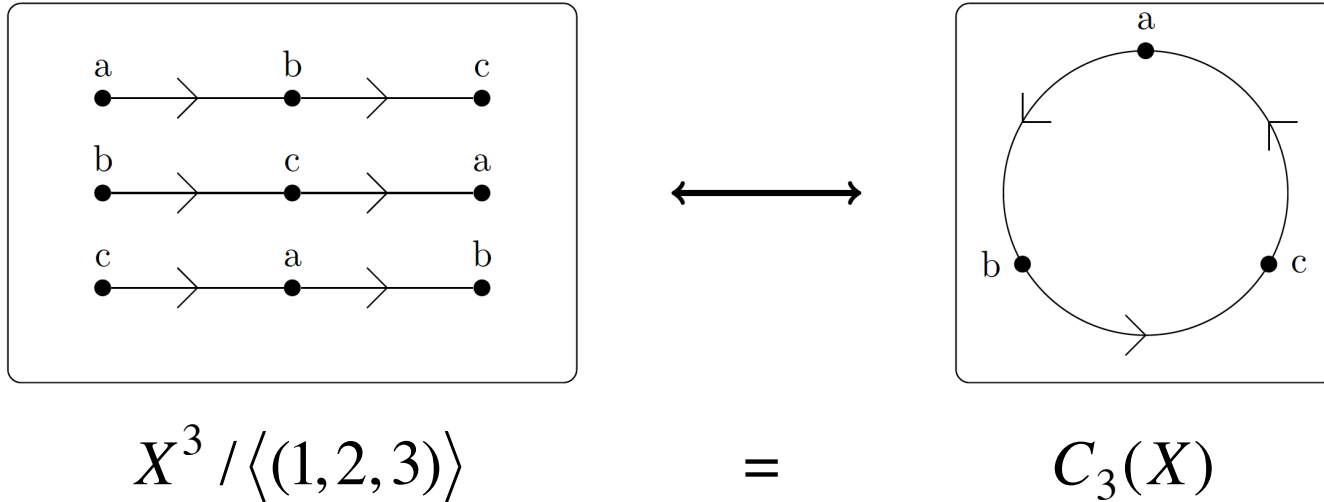
More generally, let $H \leq S_n$ then we can consider the species

$$X^n / H$$

of n - lists of singletons modulo H

EXAMPLE of X^n / H

Let $H = \langle (1,2,3) \rangle \leq S_3$

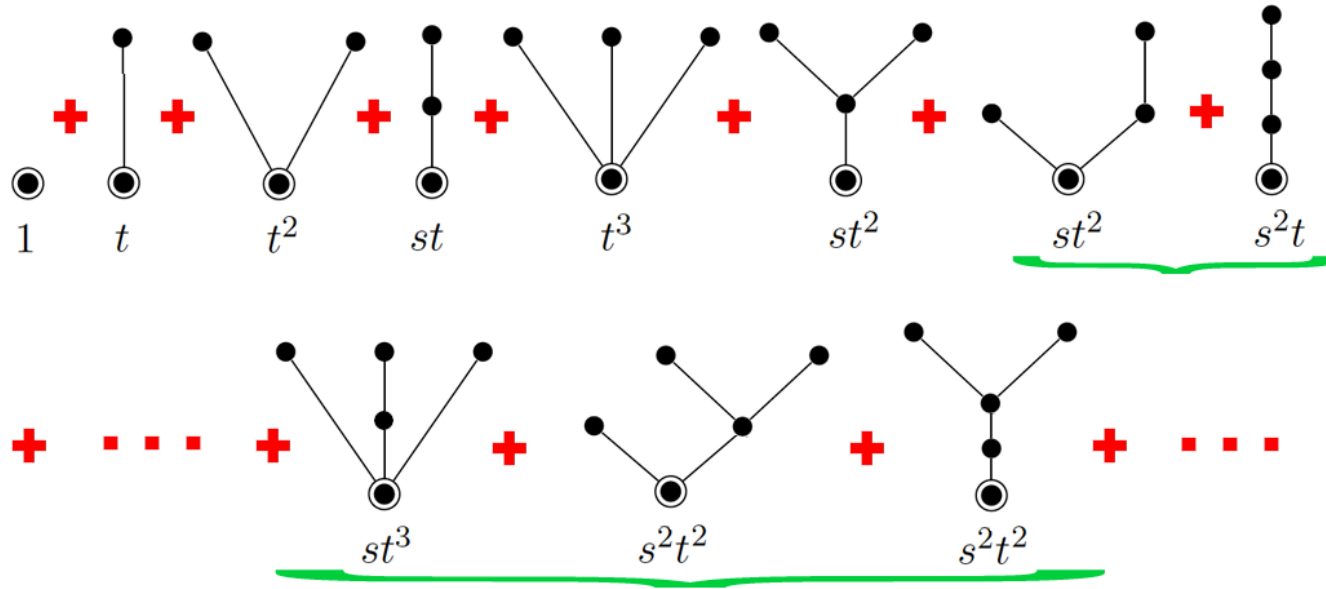


So that the species $C = C(X)$ of cyclic permutations can be written as

$$C(X) = X + \frac{X^2}{\langle (1,2) \rangle} + \frac{X^3}{\langle (1,2,3) \rangle} + \dots + \frac{X^n}{\langle (1,2,3,\dots,n) \rangle} + \dots$$

THIS IS A GENERAL FACT

EXAMPLE : The species of (weighted) rooted trees



MOLECULAR EXPANSION :

$$X + tX^2 + t^2XE_2 + stX^3 + t^3XE_3 + st^2X^2E_2 + (st^2 + s^2t)X^4 + \dots + (st^3 + 2s^2t^2)X^3E_2 + \dots$$

=

$$X + tX^2 + t^2 \frac{X^3}{S_{1,2}} + stX^3 + t^3 \frac{X^4}{S_{1,3}} + st^2 \frac{X^4}{S_{1,1,2}} + (st^2 + s^2t)X^4 + \dots + (st^3 + 2s^2t^2) \frac{X^5}{S_{1,1,1,2}} + \dots$$

MOLECULAR EXPANSION OF ANY SPECIES

Every species $F(X)$ can be written uniquely (up to natural equivalence) as a linear combination of species of the form X^n / H , $H \leq S_n$:

$$\begin{aligned} F(X) &= \sum_{n \geq 0} \sum_{s \in F[n]/S_n} \text{weight}(s) \frac{X^n}{\text{Stab}(s)} \\ &= \sum_{n \geq 0} \sum_{H \in \bullet S_n} f_H \frac{X^n}{H} \end{aligned}$$

where H runs over a system of representatives of the conjugacy classes of subgroups of S_n and f_H are linear combinations of weight monomials.

The species X^n / H are called **molecular species** (irreducible over sum)

MOLECULAR EXPANSION OF ANY 2-SORT SPECIES

Every species $\Omega(X, T)$ can be written uniquely (up to natural equivalence) as a linear combination of species of the form $X^n T^k / H$, $H \leq S_{n, k}$:

$$\begin{aligned} \Omega(X, T) &= \sum_{n, k \geq 0} \sum_{s \in \Omega[n, k] / S_{n, k}} \text{weight}(s) \frac{X^n T^k}{\text{Stab}(s)} \\ &= \sum_{n, k \geq 0} \sum_{H \leq S_{n, k}} \omega_H \frac{X^n T^k}{H} \end{aligned}$$

where H runs over a system of representatives of the conjugacy classes of the subgroups of $S_{n, k}$ and ω_H are linear combinations of weight monomials.

$$S_n \times S_k \cong S_{n, k} \leq S_{n+k}$$

Similar expansions hold for species on an arbitrary number of sorts of singletons

ATOMIC EXPANSION OF ANY SPECIES

(Yeh's Theorem 1986) Every molecular species X^n / H can be written uniquely (up to natural equivalence) as a commutative product

$$\frac{X^n}{H} = \frac{X^{n_1}}{H_1} \cdot \frac{X^{n_2}}{H_2} \cdots \frac{X^{n_k}}{H_k}$$

of atomic species X^{n_i} / H_i (i.e., also irreducible under product).

Consequence : Up to natural equivalence, every species is an element of the « big » half-ring (under + and ·)

$\mathbb{N}[[\text{weight variables, atomic species}]]$

which is also equipped with extra operations (\times , \circ , D , etc).

Similar expansions hold for species on any number of sorts of singletons.

Yeh and Joyal have extended the half-rings

$\mathbb{N}[[\text{weight variables, atomic species}]]$

to more general rings of the form

$\mathbb{Z}[[\text{weight variables, atomic species}]]$ **(weighted virtual species)**

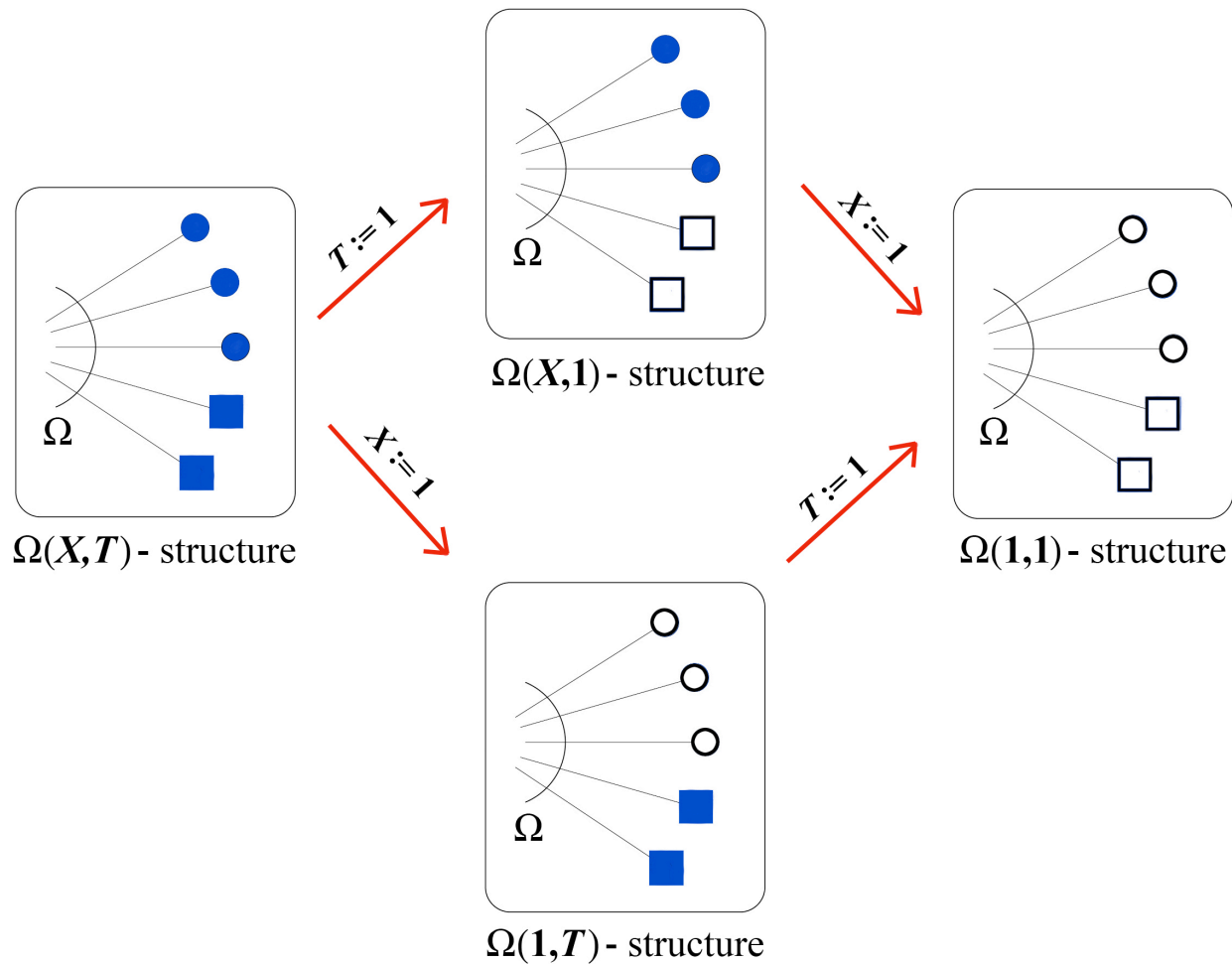
$\mathbb{C}[[\text{weight variables, atomic species}]]$ **(weighted complex species)**

and extended the extra operations (\times , \circ , D , etc).

From now on, the word « species » means **weighted complex species.**

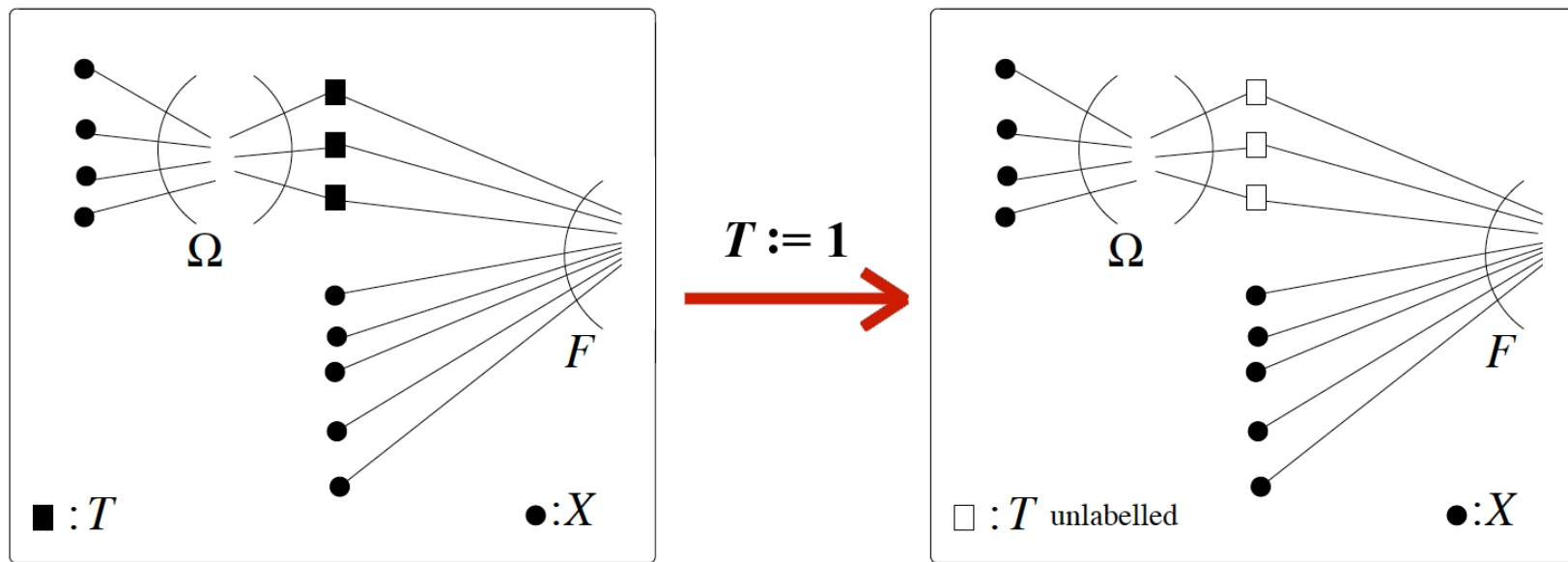
UNLABELLING NODES

Convention : *labelled nodes are colored, unlabelled nodes are not colored (white).*



DEFINITION OF $\Omega(X, D)$

To any 2-sort species, $\Omega(X, T)$, we associate a combinatorial differential operator, $\Omega(X, D)$, acting on species $F(X)$ as follows :

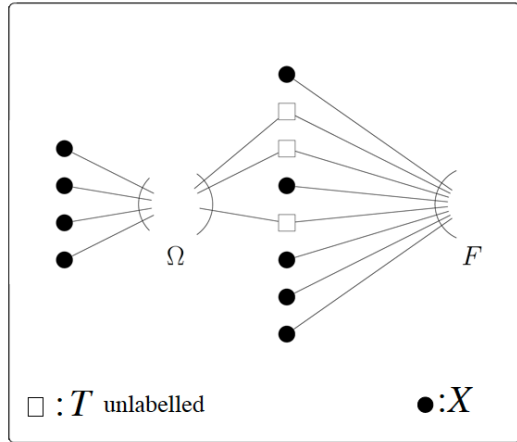


$$\Omega(X, T) \times_T F(X+T)$$

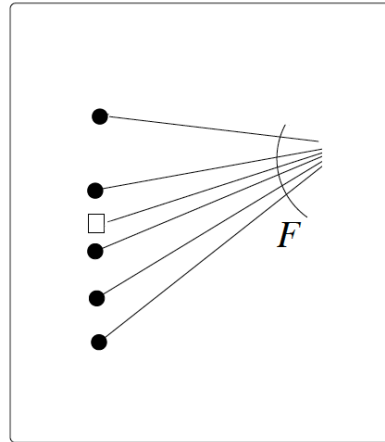
$$\Omega(X, D)F(X)$$

That is : $\Omega(X, D)F(X) \stackrel{\text{def}}{=} \Omega(X, T) \times_T F(X+T) |_{T:=1}$

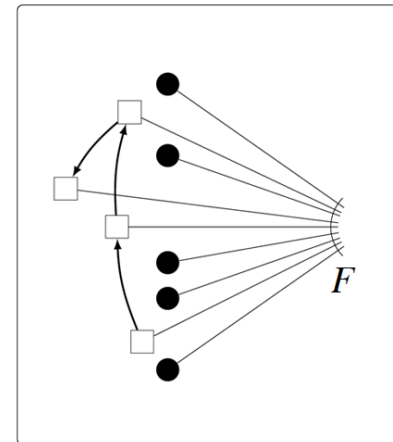
1 - EXAMPLES OF $\Omega(X, D)$



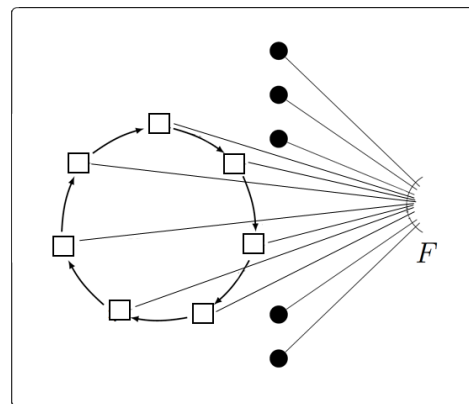
$\Omega(X, D)F(X)$



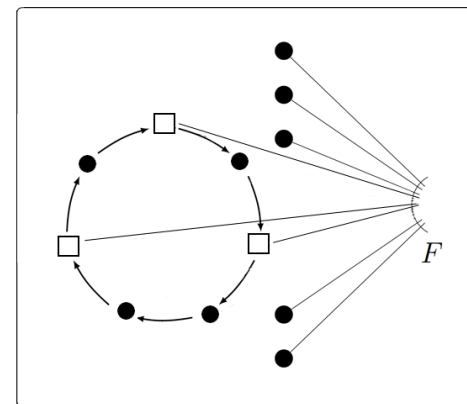
$DF(X)$



$D^4 F(X)$

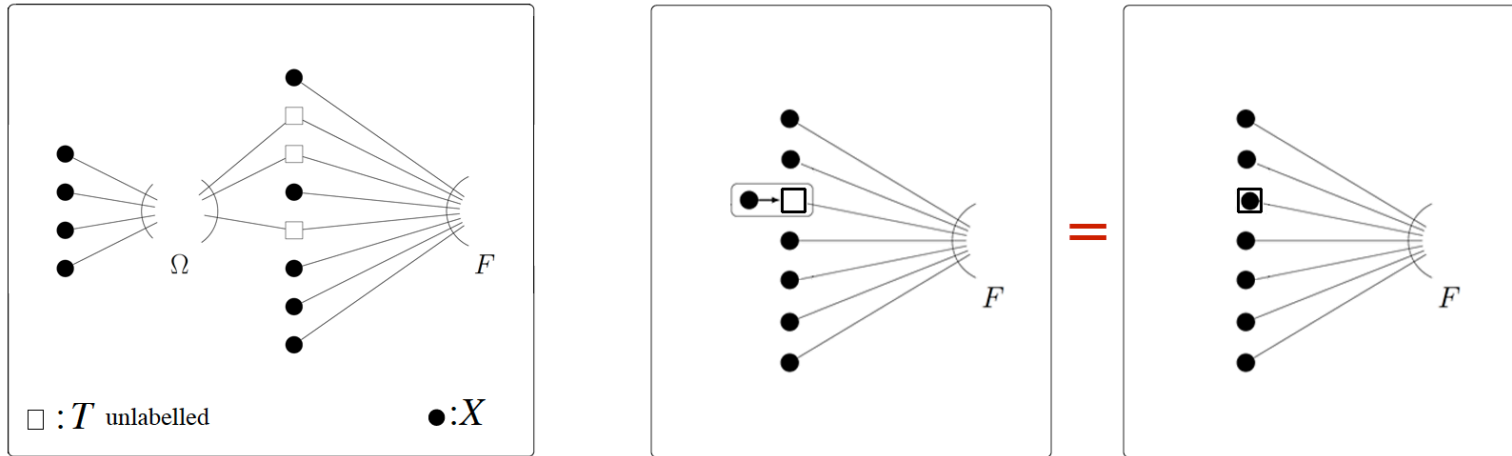


$C(D)F(X)$



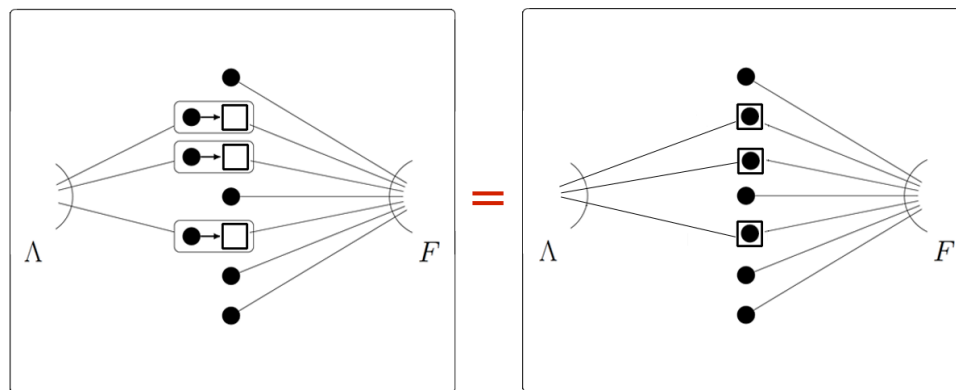
$C(X + D)F(X)$

2 - EXAMPLES OF $\Omega(X, D)$



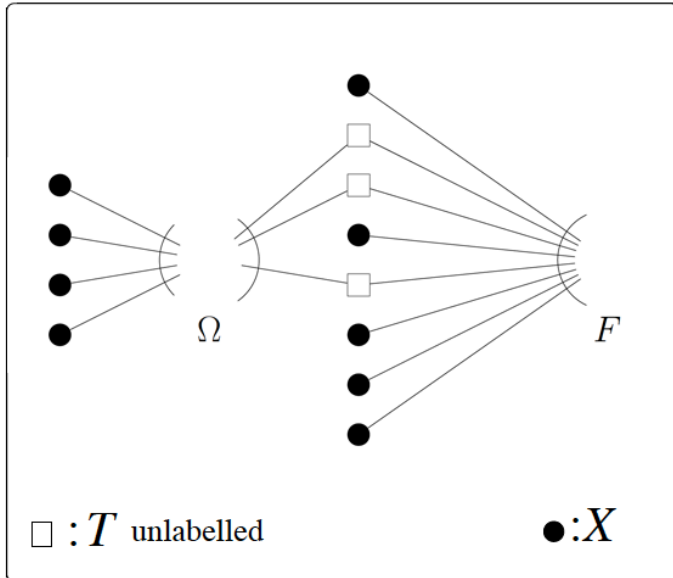
$$\Omega(X, D)F(X)$$

$$XDF(X) = F^\bullet(X) \text{ (pointing)}$$

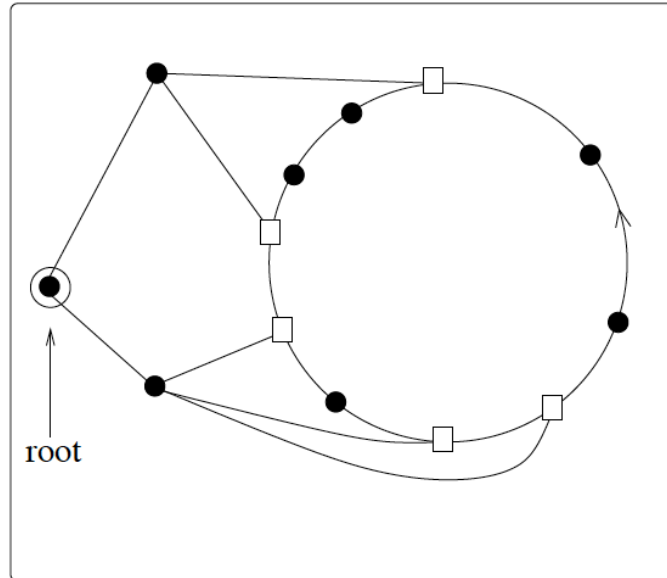


$$\Lambda(XD)F(X) = F^\Lambda(X) \text{ (\Lambda-pointing)}$$

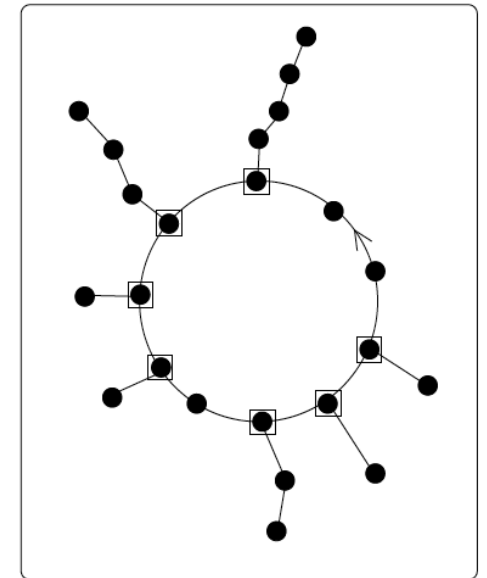
3 - EXAMPLES OF $\Omega(X, D)$



$\Omega(X, D)F(X)$

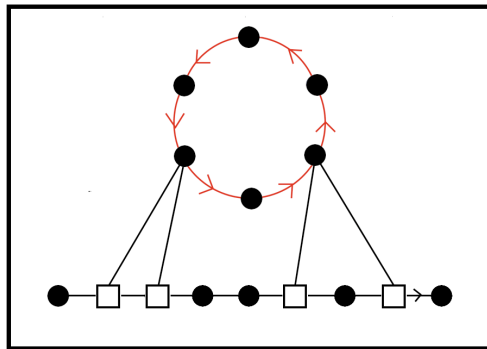


$A(X, D)C(X)$

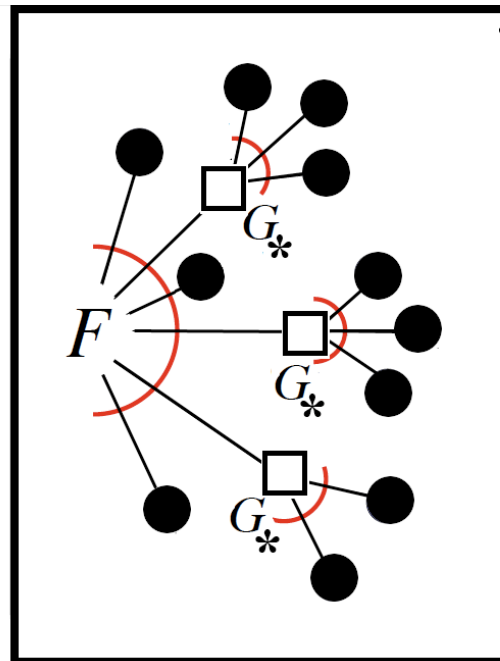


$E(L_{\ge 2}(X)D)C(X)$

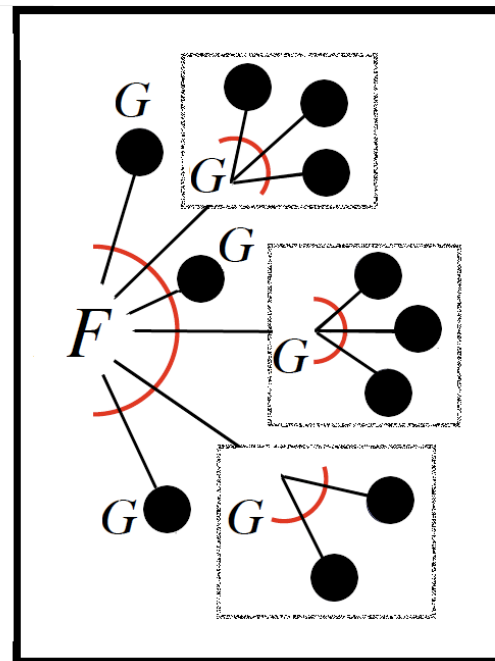
Octopuses



$$C(X + XE_2(D))L(X)$$

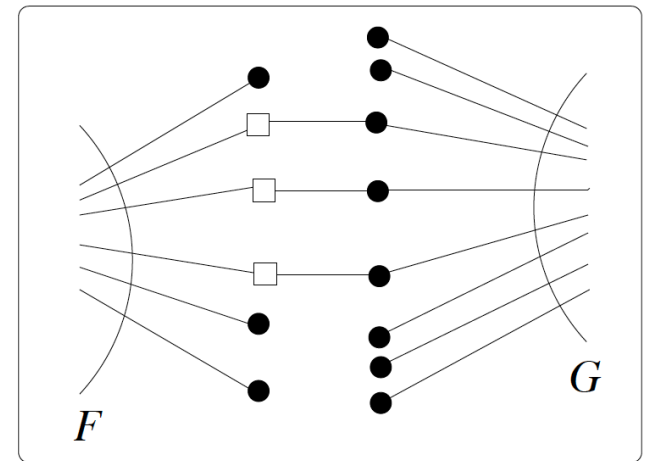
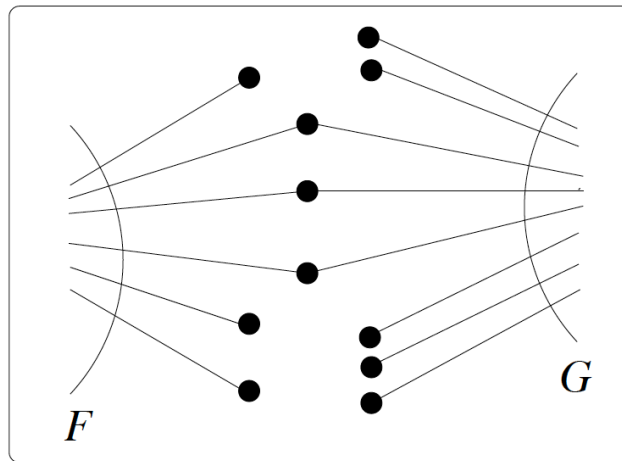
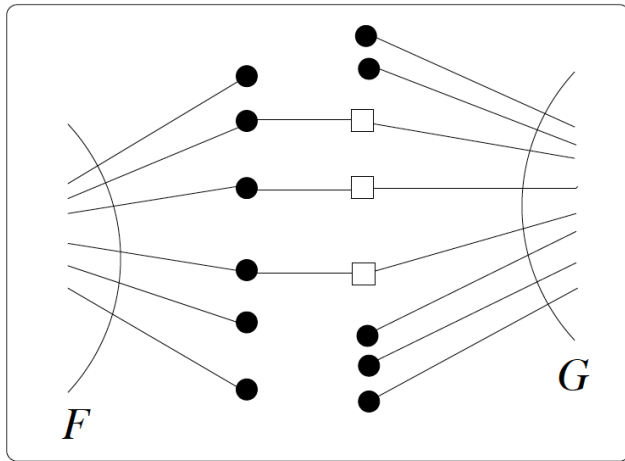


=



$$E(G_*(X)D)F(X) = F(G(X)) \text{ if } G(X) = X + G_*(X)$$

KISS PRODUCT of SPECIES



$$F(X + XD)G(X) = F(X) \dot{\times} G(X) = G(X + XD)F(X)$$

kiss product

The *kiss product* « interpolates » between *Cauchy* and *Hadamard products* :

$$F(X)G(X) \subseteq F(X) \dot{\times} G(X) \qquad F(X) \times G(X) \subseteq F(X) \dot{\times} G(X)$$

GENERALIZED LEIBNIZ RULE

Notation

$$\frac{X^n D^k}{K} F(X) = \frac{X^n F^{(k)}(X)}{K}$$

where $K \leq S_{n,k}$.

Theorem We have

$$\frac{X^n D^k}{K} F(X)G(X) = \sum_{i+j=k} \sum_{L:S_{n,i,j}} \binom{K}{L} \frac{X^n F^{(i)}(X)G^{(j)}(X)}{L}$$

where $L : S_{n,i,j}$ means that L runs through a complete system of representatives of the conjugacy classes of subgroups of $S_{n,i,j}$ and the coefficients $\binom{K}{L}$ are defined by the "addition formula"

$$X^n (T_1 + T_2)^k / K = \sum_{i+j=k} \sum_{L:S_{n,i,j}} \binom{K}{L} X^n T_1^i T_2^j / L$$

MOLECULAR EXPANSION OF ANY COMBINATORIAL DIFFERENTIAL OPERATOR

Since we have a unique molecular expansion

$$\Omega(X, T) = \sum_{n, k \geq 0} \sum_{H \leq \bullet S_{n, k}} \omega_H \frac{X^n T^k}{H},$$

then every combinatorial differential operator $\Omega(X, D)$ has a **canonical** molecular expansion of the form

$$\Omega(X, D) = \sum_{n, k \geq 0} \sum_{H \leq \bullet S_{n, k}} \omega_H \frac{X^n D^k}{H}$$

$X^n D^k / H$ is called **molecular (or atomic)** if $X^n T^k / H$ is **molecular (or atomic)**

TABLES of ATOMIC $\frac{X^n D^k}{H}$

$n + k = 1$: two operators X and D (generating the Weyl algebra under $+$ and \odot)

$n + k = 2$: two operators $X^2 / \langle (12) \rangle$ and $D^2 / \langle (12) \rangle$

$n + k = 3$: 4 operators $X^3 / \langle (123) \rangle$, $X^3 / \langle (132), (12) \rangle$, $D^3 / \langle (123) \rangle$, $D^3 / \langle (132), (12) \rangle$

$n + k = 4$: 13 operators

$$X^4 / \langle (13)(24) \rangle$$

$$X^4 / \langle (14)(23), (13)(24) \rangle$$

$$X^4 / \langle (12)(34), (1324) \rangle$$

$$X^4 / \langle (13)(24), (14)(23), (12) \rangle$$

$$X^4 / \langle (13)(24), (14)(23), (243) \rangle$$

$$X^4 / \langle (13)(24), (14)(23), (243), (12) \rangle$$

$$X^2 D^2 / \langle (12)(34) \rangle$$

(self adjoint)

$$D^4 / \langle (13)(24) \rangle$$

$$D^4 / \langle (14)(23), (13)(24) \rangle$$

$$D^4 / \langle (12)(34), (1324) \rangle$$

$$D^4 / \langle (13)(24), (14)(23), (12) \rangle$$

$$D^4 / \langle (13)(24), (14)(23), (243) \rangle$$

$$D^4 / \langle (13)(24), (14)(23), (243), (12) \rangle$$

TABLES of ATOMIC $\frac{X^n D^k}{H}$

$n + k = 5$: 14 operators $\dots X^3 D^2 / \langle (132), (12)(45) \rangle, X^2 D^3 / \langle (12)(34), (354) \rangle \dots$

$n + k = 6$: 73 operators

$n + k = 7$: 70 operators

$n + k = 8$: 453 operators

Sample from Hugo Tremblay's Tables for $n + k = 8$:

$\dots X^6 D^2 / \langle (3\ 4)(5\ 6), (1\ 2)(3\ 5)(4\ 6)(7\ 8) \rangle$
 $X^6 D^2 / \langle (3\ 4)(5\ 6)(7\ 8), (1\ 2)(3\ 5)(4\ 6) \rangle$
 $X^6 D^2 / \langle (1\ 2\ 3)(4\ 5\ 6), (1\ 4)(2\ 6)(3\ 5)(7\ 8) \rangle$
 $X^6 D^2 / \langle (2\ 3)(5\ 6)(7\ 8), (1\ 2)(4\ 5)(7\ 8) \rangle$
 $X^6 D^2 / \langle (1\ 2)(3\ 4)(5\ 6)(7\ 8), (1\ 3\ 5)(2\ 4\ 6) \rangle$
 $X^6 D^2 / \langle (5\ 6)(7\ 8), (3\ 4)(7\ 8), (1\ 2)(7\ 8) \rangle$
 $X^6 D^2 / \langle (5\ 6)(7\ 8), (3\ 4)(7\ 8), (1\ 2)(3\ 5)(4\ 6)(7\ 8) \rangle$
 $X^6 D^2 / \langle (5\ 6), (3\ 4), (1\ 2)(3\ 5)(4\ 6)(7\ 8) \rangle$
 $X^6 D^2 / \langle (3\ 4)(5\ 6), (3\ 5\ 4\ 6)(7\ 8), (1\ 2)(5\ 6) \rangle$
 $X^6 D^2 / \langle (5\ 6)(7\ 8), (3\ 4)(7\ 8), (1\ 2)(3\ 5)(4\ 6) \rangle$
 $X^6 D^2 / \langle (3\ 4)(5\ 6), (3\ 5\ 4\ 6)(7\ 8), (1\ 2)(5\ 6)(7\ 8) \rangle$
 $X^6 D^2 / \langle (5\ 6)(7\ 8), (1\ 2)(3\ 4), (1\ 3)(2\ 4)(7\ 8) \rangle \dots$

2, 2, 4, 13, 14, 73, 70, 453, 546, 3302, ...

$$\Omega(X, D) = \sum_{n, k \geq 0} \sum_{H \leq \bullet S_{n, k}} \omega_H \frac{X^n D^k}{H}$$

Restricting H to run through trivial subgroups $\{id_{n, k}\} \leq S_{n, k}$, this contains, as a special case :

The Equivalence Principle of Blasiak and Flajolet

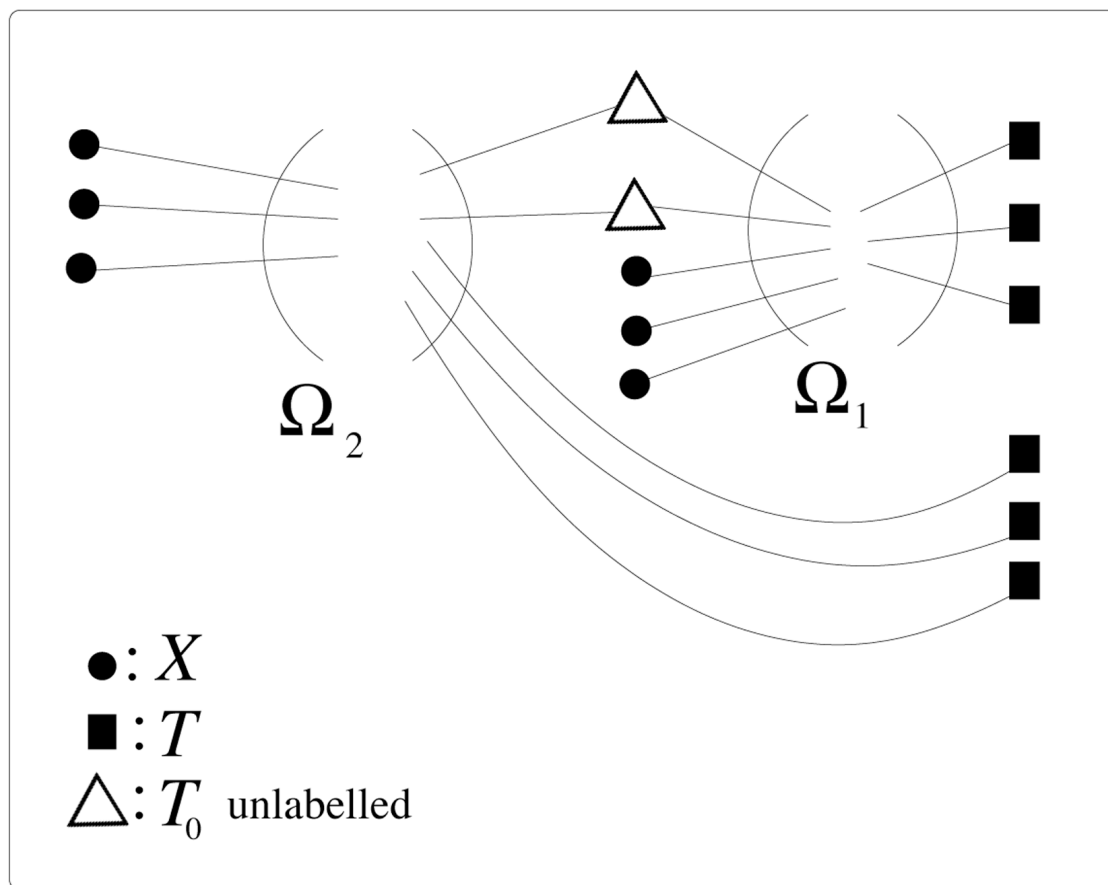
$$\eta = \sum_{r, s} w_{r, s} X^r D^s$$



$$\eta^{\odot n} = \sum_{n, a, b} c_{n, a, b} X^a D^b$$

COMPOSITION OF DIFFERENTIAL OPERATORS

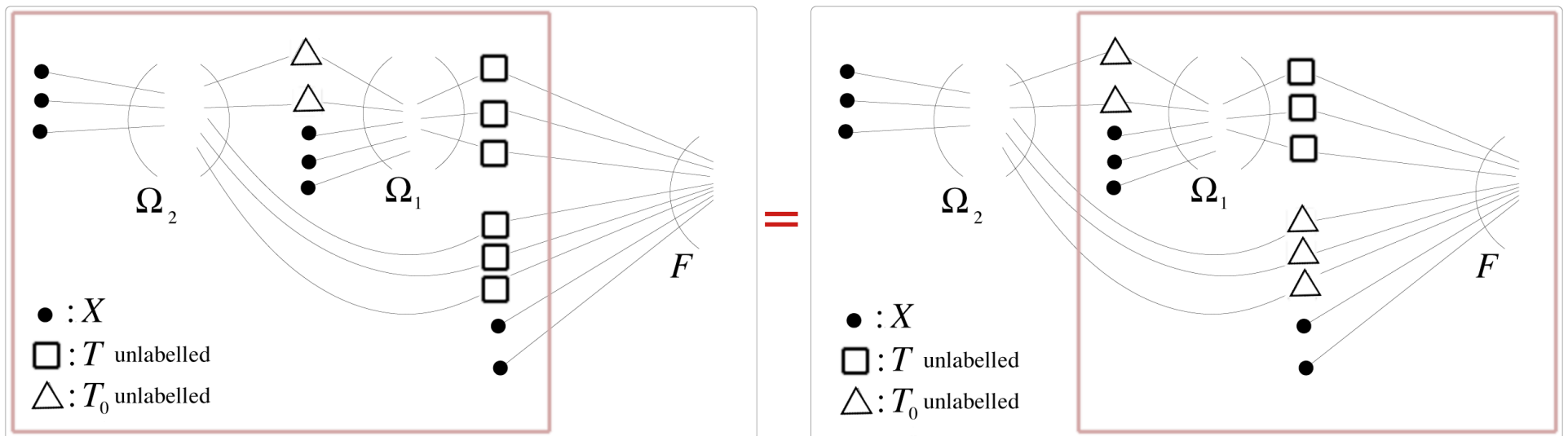
$$\Omega_2(X, D) \odot \Omega_1(X, D) = \Omega_3(X, D)$$



$$\Omega_3(X, T) = \Omega_2(X, T) \odot \Omega_1(X, T) := \Omega_2(X, T + T_0) \times_{T_0} \Omega_1(X + T_0, T)|_{T_0:=1}$$

MONOIDAL ACTION

$$[\Omega_2(X, D) \odot \Omega_1(X, D)] F(X) = \Omega_2(X, D) [\Omega_1(X, D) F(X)]$$



$$(XD)^{\odot n} = \sum_{v \leq n} \left\{ \begin{matrix} n \\ v \end{matrix} \right\} X^v D^v$$

$\left\{ \begin{matrix} n \\ v \end{matrix} \right\} =$ Stirling number of the second kind (**well-known**)

Define $P_n(x) = \sum_{v \leq n} \left\{ \begin{matrix} n \\ v \end{matrix} \right\} x^v.$

Then, $P_0(x) = 1$ and $P_{n+1}(x) = \left(x + x \frac{d}{dx} \right) P_n(x)$ (**also well-known**).

But what is the canonical expression for $E_2(XD)^{\odot n}$?

$$E_2(XD)^{\odot n} = \sum_{v_1+2v_2 \leq 2n} \left\{ \begin{matrix} n \\ v_1, v_2 \end{matrix} \right\} X^{v_1} D^{v_1} E_2(XD)^{v_2}$$

$$\left(\frac{X^2 D^2}{\langle (1,2)(3,4) \rangle} \right)^{\odot n} = \sum_{v_1+2v_2 \leq 2n} \left\{ \begin{matrix} n \\ v_1, v_2 \end{matrix} \right\} X^{v_1} D^{v_1} \cdot \left(\frac{X^{2v_2} D^{2v_2}}{\langle (1,2)(2v_2+1, 2v_2+2), \dots, (2v_2-1, 2v_2)(4v_2-1, 4v_2) \rangle} \right)$$

Define $P_n(x, y) = \sum_{v_1+2v_2 \leq 2n} \left\{ \begin{matrix} n \\ v_1, v_2 \end{matrix} \right\} x^{v_1} y^{v_2}.$

Then, $P_0(x, y) = 1$ **and**

$$P_{n+1}(x, y) = \left(y + x^2 \frac{\partial}{\partial x} + (x^3 + y) \frac{\partial}{\partial y} + x^2 \frac{\partial^2}{\partial x^2} + x^3 \frac{\partial^2}{\partial x \partial y} + x^4 \frac{\partial^2}{\partial y^2} \right) P_n(x, y).$$

CYCLE INDEX SERIES

Proposition Let $G(X) := \Omega(X, D)F(X)$, then

$$Z_G(x_1, x_2, x_3, \dots) = Z_\Omega(x_1, x_2, x_3, \dots; \frac{\partial}{\partial x_1}, 2\frac{\partial}{\partial x_2}, 3\frac{\partial}{\partial x_3}, \dots)Z_F(x_1, x_2, x_3, \dots).$$

Proposition

$$Z_{\Omega_2 \odot \Omega_1} = \sum_{n_1, n_2, \dots} \frac{\left(\left(\frac{\partial}{\partial t_1} \right)^{n_1} \left(2 \frac{\partial}{\partial t_2} \right)^{n_2} \dots Z_{\Omega_2} \right) \left(\left(\frac{\partial}{\partial x_1} \right)^{n_1} \left(2 \frac{\partial}{\partial x_2} \right)^{n_2} \dots Z_{\Omega_1} \right)}{1^{n_1} n_1! 2^{n_2} n_2! \dots}.$$

Convention. Write all the t_j 's to the right of all the x_i 's in $Z_\Omega(x_1, x_2, \dots, t_1, t_2, \dots)$.

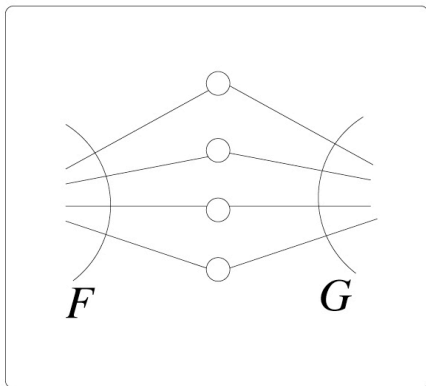
BILINEAR FORM AND ADJOINT OPERATORS

$$\begin{aligned} \langle F(X), G(X) \rangle &\stackrel{\text{def}}{=} F(X) \times_X G(X) |_{X:=1} \\ &= \text{total weight (or number) of all unlabelled } F \times G \text{ - structures} \end{aligned}$$

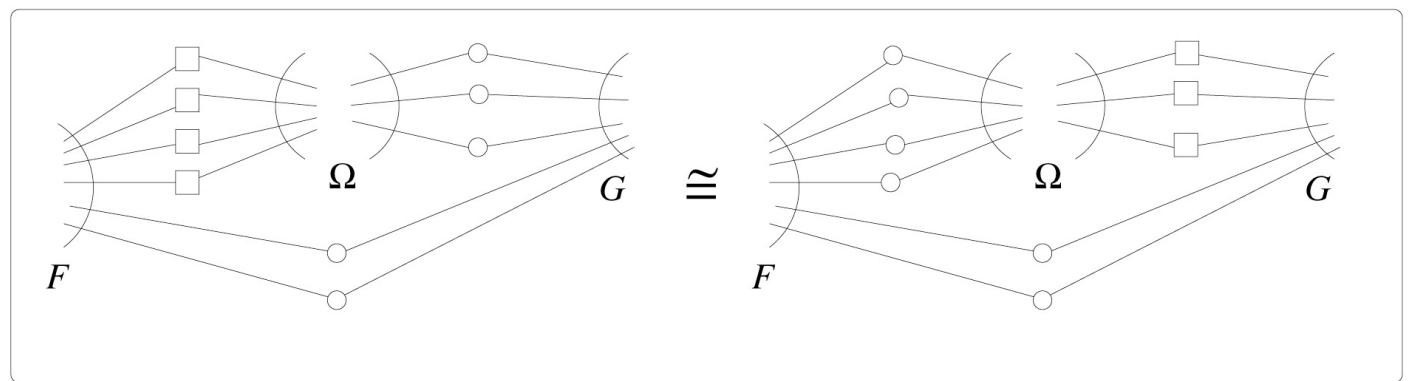
$$\langle \Omega(X, D)F(X), G(X) \rangle = \langle F(X), \Omega(D, X)G(X) \rangle$$

i.e. $(\Omega(X, D))^* := \Omega(D, X)$ is the **adjoint** operator of $\Omega(X, D)$

In particular $D^* = X$ as usual ...



$$\langle F(X), G(X) \rangle$$



$$\langle \Omega(X, D)F(X), G(X) \rangle = \langle F(X), \Omega(D, X)G(X) \rangle$$

COMPUTATION OF $\frac{X^m D^k}{K} \frac{X^n}{H}$

$$\frac{X^m D^k}{K} \frac{X^n}{H} = \frac{X^m T^k}{K} \times_T \frac{(X + T)^n}{H} \Big|_{T:=1}$$

$$\frac{(X + T)^n}{H} = \sum_{k=0}^n \sum_{\omega \in S_{n-k,k} \setminus S_n/H} \frac{X^{n-k} T^k}{\omega H \omega^{-1} \cap S_{n-k,k}}$$

$$\frac{X^a T^k}{A} \times_T \frac{X^b T^k}{B} = \sum_{\tau \in (\pi_2 A) \setminus S_k / (\pi_2 B)} \frac{X^{a+b} T^k}{A \times_{S_k} B^\tau}$$

$$\left[\frac{X^a T^k}{A} \right]_{T:=1} = \frac{X^a}{\pi_1 A}$$

COMMUTATORS, ETC

Make tables and study the commutators

$$[A(X, D), B(X, D)] = A(X, D) \odot B(X, D) - B(X, D) \odot A(X, D)$$

$$[\Omega^*, \Omega] = \Omega^* \odot \Omega - \Omega \odot \Omega^* = ?$$

$$D \odot X - X \odot D = 1 \quad (\text{special case with } \Omega = X)$$

$$\left[\frac{X^a D^b}{H}, \frac{X^c D^d}{K} \right] = ? \quad (\text{molecular expansions})$$

Find new combinatorial models for equations such as

$$[P, Q] = P + Q \quad \text{etc}$$

SOME CURIOUS IDENTITIES BETWEEN DIFFERENTIAL OPERATORS

$$E(aXD) \odot E(bXD) = E((a + b + ab)XD)$$

$$E(-XD) \odot E(-XD) = E(-XD) \quad (\text{ idempotence })$$

$$E(-2XD) \odot E(-2XD) = 1 \quad (\text{ self inverse })$$

$$E(-\frac{1}{2}XD) \odot E(-XD) = E(-XD) \quad (\text{ no effect ! })$$

etc

SPECIAL CASES

$$\Omega_2(X, D) \odot \Omega_1(X, D)$$

noncommutative

1. Combinatorial Hammond differential operator $\Omega(X, D) = \Theta(D)$

$$\Theta_2(D) \odot \Theta_1(D) = \Theta_2 \cdot \Theta_1(D) \quad \text{commutative}$$

2. Multiplication operator $\Omega(X, D) = G(X)$

$$G_2(X) \odot G_1(X) = G_2 \cdot G_1(X) \quad \text{commutative}$$

3. Finite differences operator $\Omega(X, D) = \Phi(X, \Delta), \quad \Delta = E^+(D)$

$$\Phi_2(X, \Delta) \odot \Phi_1(X, \Delta) \quad \text{noncommutative}$$

4. Pointing operator $\Omega(X, D) = \Lambda(XD)$

$$\Lambda_2(XD) \odot \Lambda_1(XD) = \Lambda_2 \times \Lambda_1(XD) \quad \text{commutative}$$

etc

FUTURE EXPLORATIONS

- 1) Study the atomic expansion of $\Omega(X, D)^{\odot n}$ given that of $\Omega(X, D)$
(Xavier Viennot theory of « empilements » may be useful here).
- 2) Find necessary and/or sufficient conditions for the existence, non-existence, multiplicity of solutions of equations of the form $\Omega(X, D)F(X) = G(X)$ and devise methods for solving such equations.
- 3) Analyze connections between the operators $\Omega(X, D)$ and physics
- 4) Find new combinatorial identities and species involving the operations
 $+, \cdot, \times, \circ, \odot, X, D, \Omega(X, D), *, \prod_{X_i:=1}, [A, B],$ etc
- 5) Extend the above to partial differential operators $\Omega(X_1, X_2, \dots, \partial_1, \partial_2, \dots)$
- 6) Implement the above on computer algebra systems (Maple, Sage, etc)
- 7) Extend existing tables
- 8) Etc

BONNE RETRAITE

XAVIER



$$x = \cos(u) \cos(v)^2$$

$$y = \sin(u) \cos(v)$$

$$z = |\cos(u) \cos(v)| + \sin(v)$$

de Gilbert et Hélène