

# Tableaux for the enumeration of permutations according to their shape

Matthieu Josuat-Vergès

Université de Marne-la-Vallée

Journées Viennot, 29 juin 2012

## The shape of a permutation

Let  $w = w_1, \dots, w_n$  a permutation.

We take the convention  $w_0 = w_{n+1} = 0$ , i.e.  $w = 0, w_1, \dots, w_n, 0$ .

Let  $x = \sigma(i)$ , then  $x$  is called a :

- ▶ *peak* if  $\sigma(i-1) < \sigma(i) > \sigma(i+1)$ ,
- ▶ *valley* if  $\sigma(i-1) > \sigma(i) < \sigma(i+1)$ ,
- ▶ *double descent* if  $\sigma(i-1) > \sigma(i) > \sigma(i+1)$ ,
- ▶ *double ascent* if  $\sigma(i-1) < \sigma(i) < \sigma(i+1)$ ,

The *shape* of a permutation is the partition of  $\{1, \dots, n-1\}$  in four subsets : { peaks }, { valleys }, { double descents }, { double ascents }.

( $n$  is always a peak.)

If  $\sigma = (0)13254(0)$ , then : 1 is a double ascent, 2 is a valley, 3 is a peak, 4 is a double descent.

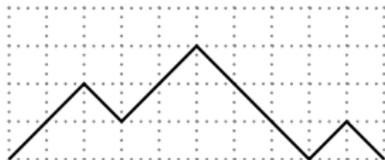
The shape can be encoded in a Dyck path.

Start from  $\nearrow$ , and for  $x$  from 1 to  $n$ ,

- ▶ add  $\nearrow \nearrow$  if  $x$  is a valley,
- ▶ add  $\nearrow \searrow$  if  $x$  is a double ascent,
- ▶ add  $\searrow \nearrow$  if  $x$  is a double descent,
- ▶ add  $\searrow \searrow$  if  $x$  is a peak,

and add a  $\searrow$  at the end.

If  $\sigma = (0)13254(0)$ , this gives :



If  $\sigma = (0)13254(0)$ , then : 1 is a double ascent, 2 is a valley, 3 is a peak, 4 is a double descent.

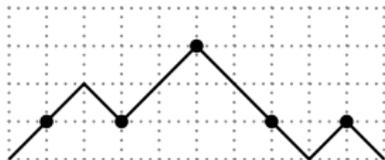
The shape can be encoded in a Dyck path.

Start from  $\nearrow$ , and for  $x$  from 1 to  $n$ ,

- ▶ add  $\nearrow \nearrow$  if  $x$  is a valley,
- ▶ add  $\nearrow \searrow$  if  $x$  is a double ascent,
- ▶ add  $\searrow \nearrow$  if  $x$  is a double descent,
- ▶ add  $\searrow \searrow$  if  $x$  is a peak,

and add a  $\searrow$  at the end.

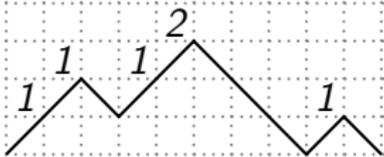
If  $\sigma = (0)13254(0)$ , this gives :



## Théorème (Françon - Viennot)

Let  $D$  a Dyck path, the number of permutations of shape  $D$  is :

$$\prod_{\nearrow \text{ from height } h-1 \text{ to } h} [h/2].$$

For the path  the permutations are 13254 and 15423.

The proof is a bijection between permutations and *subdivided Laguerre histories*.

So

$$n! = \sum_{D \text{ of length } 2n} \prod_{\nearrow \text{ from height } h-1 \text{ to } h} [h/2].$$

In the combinatorial theory of continued fractions and orthogonal polynomials [Flajolet - Viennot], this gives a proof of

$$\sum_{n \geq 0} n! z^n = \frac{1}{1 - \frac{z}{1 - \frac{z}{1 - \frac{2z}{1 - \frac{2z}{\ddots}}}}}$$

## Excursions in Young's lattice

Let  $\triangleleft$  be the cover relation of young diagrams, i.e.  $\mu \triangleleft \lambda$  if  $\mu \subset \lambda$  and  $|\mu| + 1 = |\lambda|$ .

Let

$$U(\lambda) = \sum_{\mu \triangleright \lambda} \mu \quad \text{and} \quad D(\lambda) = \sum_{\mu \triangleleft \lambda} \mu,$$

For example,

$$U\left(\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}\right) = \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} + \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array}, \quad D\left(\begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}\right) = \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} + \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}.$$

Let  $f_\lambda = \#\text{SYT}(\lambda)$ , then :

$$U^n \emptyset = \sum_{|\lambda|=n} f_\lambda \lambda, \quad D^n \lambda = f_\lambda \emptyset, \quad D^n U^n = \sum_{|\lambda|=n} f_\lambda^2 \emptyset = n! \emptyset.$$

$$\left( \square, \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}, \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \iff \begin{array}{|c|c|} \hline \boxed{2} & \\ \hline \boxed{1} & \boxed{3} \\ \hline \end{array} \right)$$

Let  $f_\lambda = \#\text{SYT}(\lambda)$ , then :

$$U^n \emptyset = \sum_{|\lambda|=n} f_\lambda \lambda, \quad D^n \lambda = f_\lambda \emptyset, \quad D^n U^n = \sum_{|\lambda|=n} f_\lambda^2 \emptyset = n! \emptyset.$$

$$\left( \square, \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}, \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \iff \begin{array}{|c|c|} \hline 2 & \\ \hline 1 & 3 \\ \hline \end{array} \right)$$

An *oscillating tableau* is a sequence  $\lambda^{(0)}, \dots, \lambda^{(2n)}$  such that  $\lambda^{(0)} = \lambda^{(2n)} = \emptyset$ , and either  $\lambda^{(i)} \succ \lambda^{(i+1)}$  or  $\lambda^{(i)} \prec \lambda^{(i+1)}$ . The number of such tableaux is

$$\langle \emptyset, (D + U)^{2n} \emptyset \rangle = (2n - 1)!! = 1 \times 3 \times \dots \times (2n - 1).$$

A *vacillating tableau* is a sequence  $\lambda^{(0)}, \dots, \lambda^{(2n)}$  such that  $\lambda^{(0)} = \lambda^{(2n)} = \emptyset$ , and :  $\lambda^{(i)} \prec \lambda^{(i+1)}$  or  $\lambda^{(i)} = \lambda^{(i+1)}$  ( $n$  odd),  $\lambda^{(i)} \succ \lambda^{(i+1)}$  or  $\lambda^{(i)} = \lambda^{(i+1)}$  ( $n$  even). The number of such tableaux is

$$\langle \emptyset, ((U + I)(D + I))^n \emptyset \rangle = B_n. \text{ (Bell numbers)}$$

Let  $\mu$  a real measure,  $P_n$  its orthogonal polynomials, and  $m_n = \int x^n d\mu$  the moments.

- ▶ If  $P_n$  are the Hermite polynomials,  $m_{2n} = (2n - 1)!!$ .
- ▶ If  $P_n$  are the Charlier polynomials,  $m_n = B_n$  (Bell number).
- ▶ If  $P_n$  are the Laguerre polynomials,  $m_n = n!$ .

This is explained by Viennot's combinatorial theory of orthogonal polynomials.

Compare with :

- ▶ Oscillating tableaux :  $\langle \emptyset, (D + U)^{2n} \emptyset \rangle = (2n - 1)!!$
- ▶ Vacillating tableaux :  $\langle \emptyset, ((U + I)(D + I))^n \emptyset \rangle = B_n$
- ▶ Pairs of standard tableaux :  $\langle \emptyset, D^n U^n \emptyset \rangle = n!$

Let  $\mu$  a real measure,  $P_n$  its orthogonal polynomials, and  $m_n = \int x^n d\mu$  the moments.

- ▶ If  $P_n$  are the Hermite polynomials,  $m_{2n} = (2n - 1)!!$ .
- ▶ If  $P_n$  are the Charlier polynomials,  $m_n = B_n$  (Bell number).
- ▶ If  $P_n$  are the Laguerre polynomials,  $m_n = n!$ .

This is explained by Viennot's combinatorial theory of orthogonal polynomials.

Compare with :

- ▶ Oscillating tableaux :  $\langle \emptyset, (D + U)^{2n} \emptyset \rangle = (2n - 1)!!$
- ▶ Vacillating tableaux :  $\langle \emptyset, ((U + I)(D + I))^n \emptyset \rangle = B_n$
- ▶ Pairs of standard tableaux :  $\langle \emptyset, D^n U^n \emptyset \rangle = n!$

But  $D^n U^n$  is not the  $n$ th power of an operator !

Is there an expression  $X$  in  $D$  and  $U$  such that  $\langle \emptyset, X^n \emptyset \rangle = n!$  ?

In the context of the PASEP model, the partition function can be obtained by computing  $(F + E)^N$  where :

$$FE - EF = F + E.$$

Viennot showed that from this relation, the partition function has a combinatorial interpretation in terms of *alternative tableaux* (in bijection with permutations).

From the relation  $DU - UD = I$ , we can check that :

$$\begin{aligned} F &= D(U + I), \\ E &= D(U + I)U, \end{aligned}$$

satisfy  $FE - EF = F + E$ .

It follows that  $\langle \emptyset, (F + E)^n \emptyset \rangle = \langle \emptyset, (D(U + I))^2)^n \emptyset \rangle = (n + 1)!$ .

# Stammering tableaux

## Définition

A stammering tableau of size  $n$  is a sequence  $\lambda^{(0)}, \dots, \lambda^{(3n)}$ , such that  $\lambda^{(0)} = \lambda^{(3n)} = \emptyset$ , and :

- ▶ if  $i \equiv 0, 1 \pmod{3}$  then either  $\lambda^{(i)} \triangleleft \lambda^{(i+1)}$  or  $\lambda^{(i)} = \lambda^{(i+1)}$ ,
- ▶ if  $i \equiv 2 \pmod{3}$  then  $\lambda^{(i)} \triangleright \lambda^{(i+1)}$ .

For example,

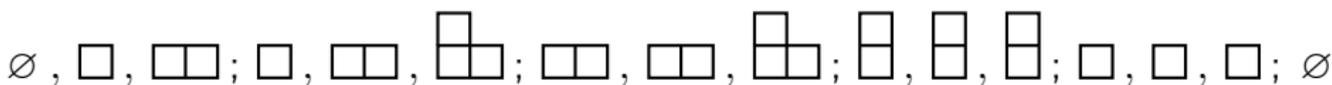
$$\emptyset, \square, \square\square; \square, \square\square, \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}; \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}, \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}, \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}; \square, \square, \square; \emptyset$$

The Dyck path associated with a stammering tableau is as follows.  
 Start from  $\nearrow$ , then read the tableau from  $\lambda^{(0)}$  to  $\lambda^{(3n)}$  and

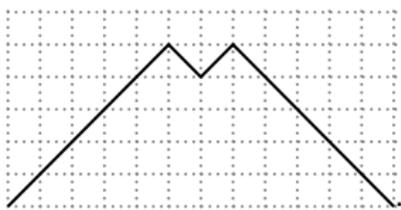
- ▶ add a step  $\nearrow$  if  $\lambda^{(i)} \triangleleft \lambda^{(i+1)}$ ,
- ▶ add a step  $\searrow$  if  $\lambda^{(i)} = \lambda^{(i+1)}$ ,

and add a step  $\searrow$  at the end.

For example,



gives the path



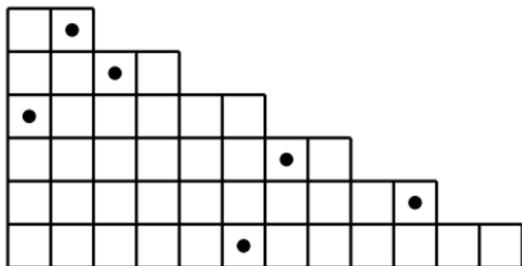
## Théorème

*Let  $D$  a Dyck path of length  $2n$ . Then the number of permutations  $w_1, \dots, w_n$  of shape  $D$  is the number of stammering tableaux  $\lambda^{(0)}, \dots, \lambda^{(3n-3)}$  of shape  $D$ .*

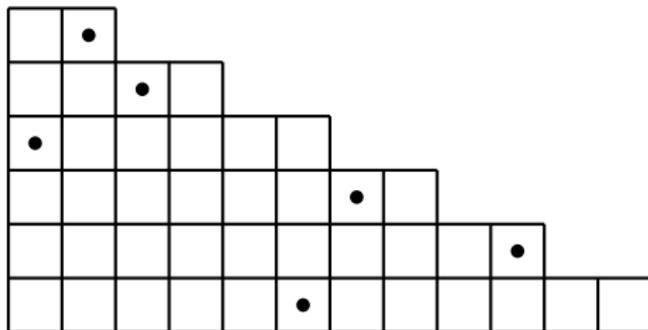
## A first bijection

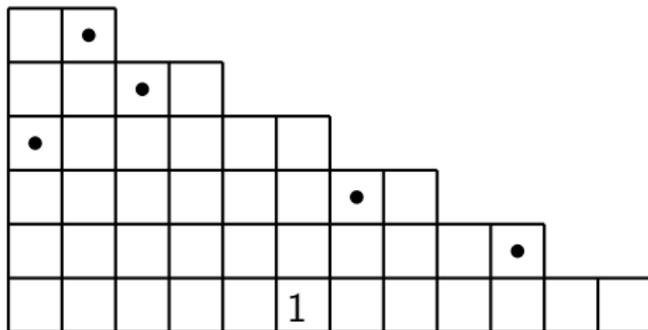
Let  $2\delta_n$  the “double staircase”, i.e. the Young diagram with row lengths  $2n, 2n - 2, \dots, 2$ .

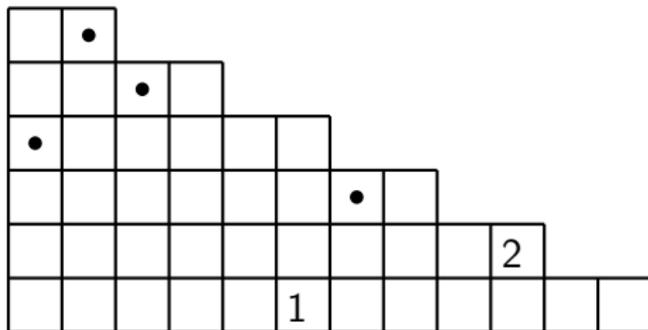
A rook placement is a filling with  $\bullet$ s, with : exactly one  $\bullet$  in each row, at most one  $\bullet$  in each column.

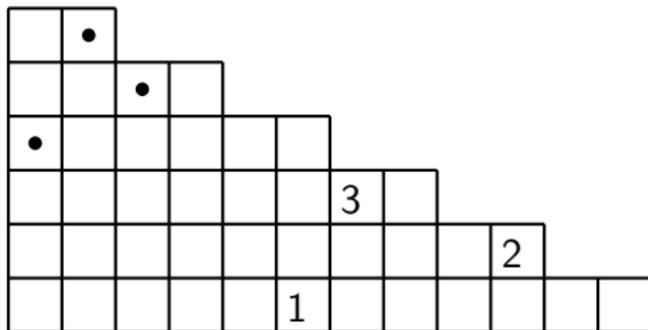


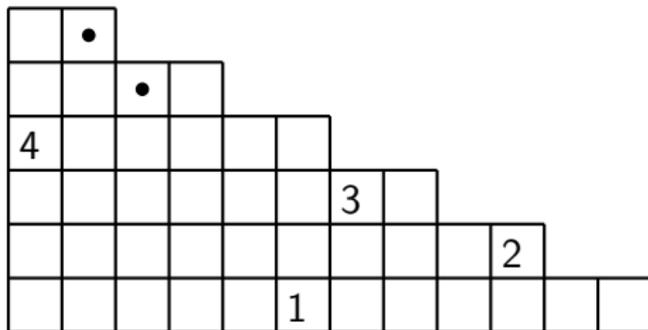
The bijection between stammering tableaux and rook placements is done either with *Fomin's growth diagrams*, *Schensted insertion*, or *Viennot's shadow lines*.

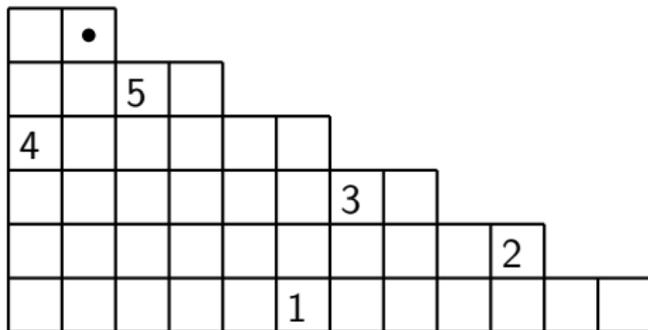


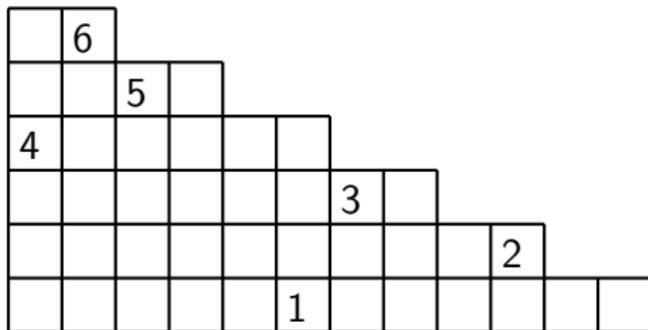


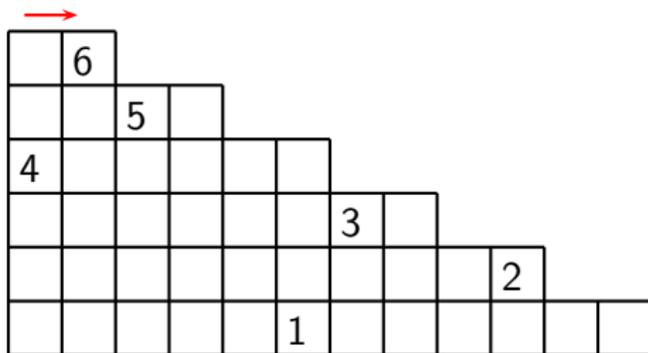


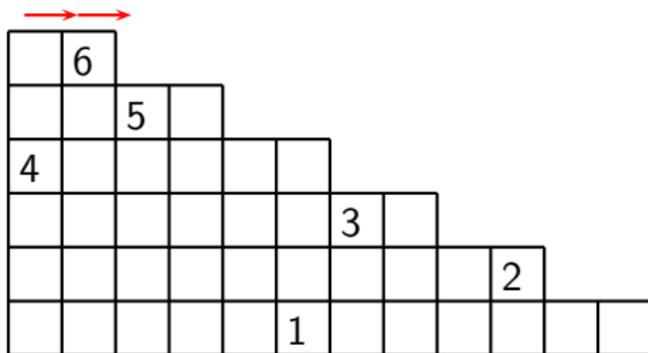




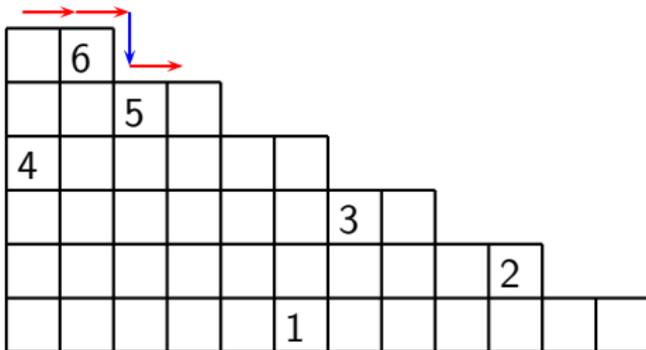




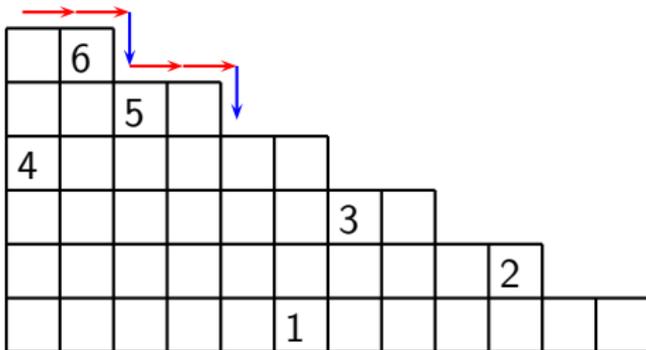


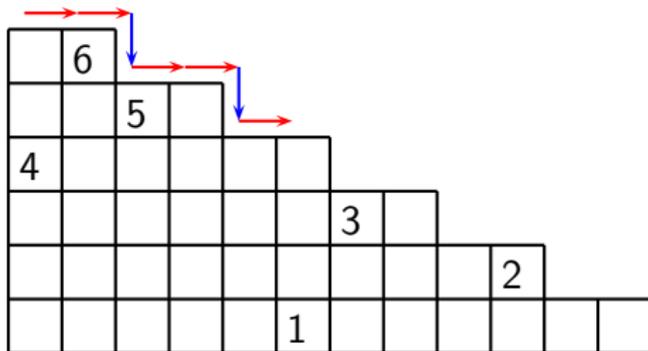


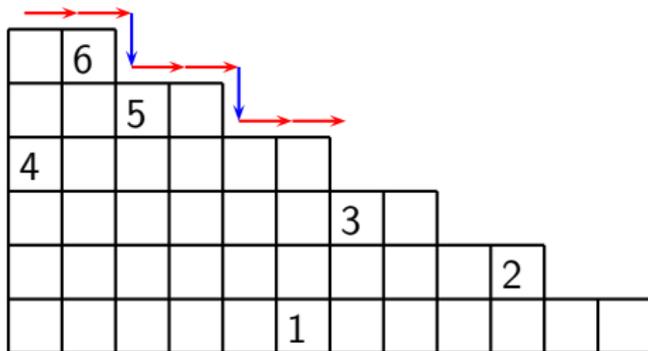


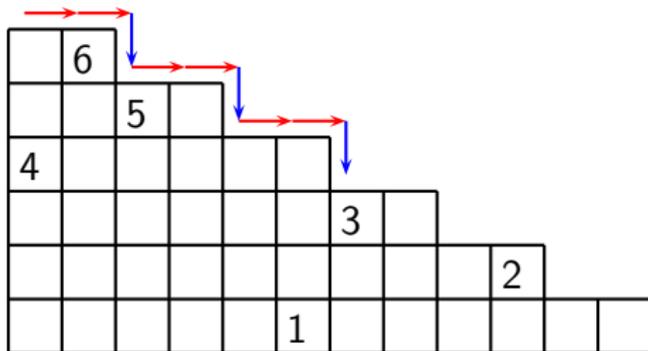


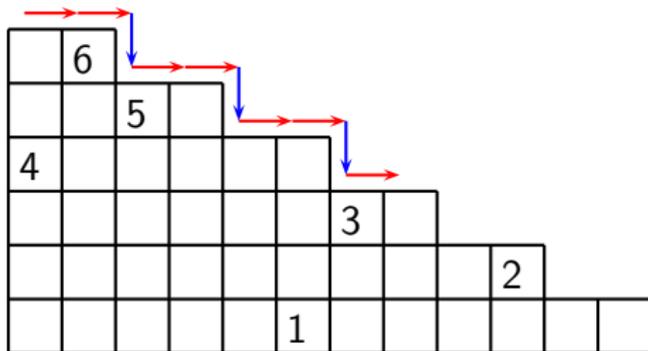


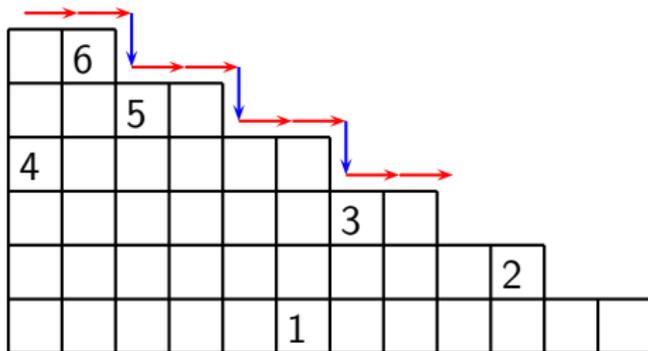


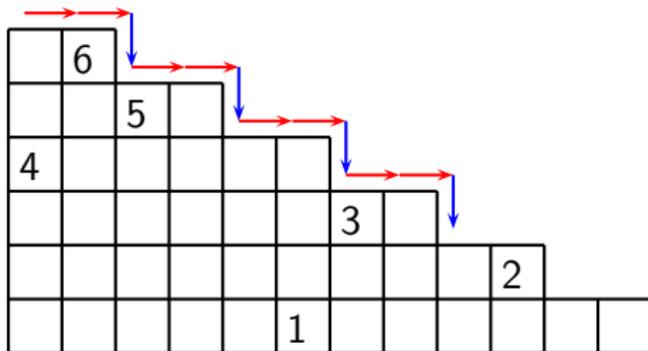


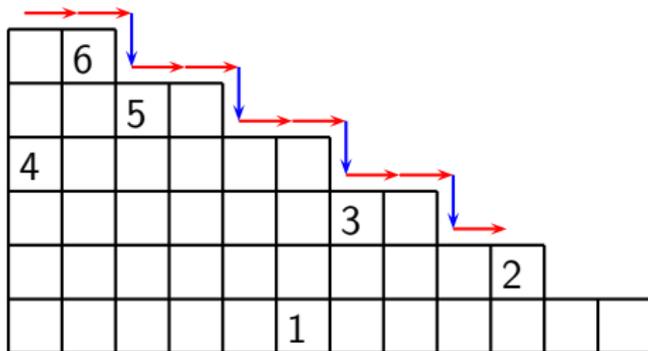


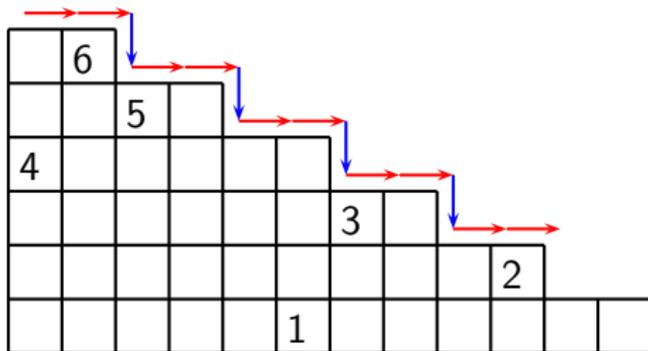


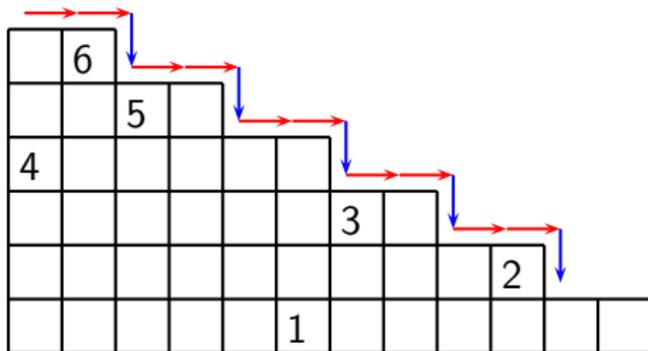


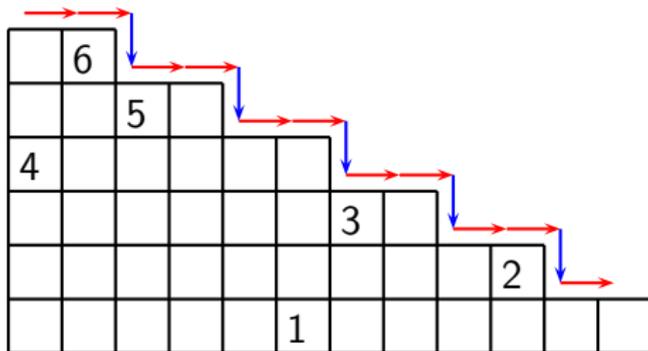


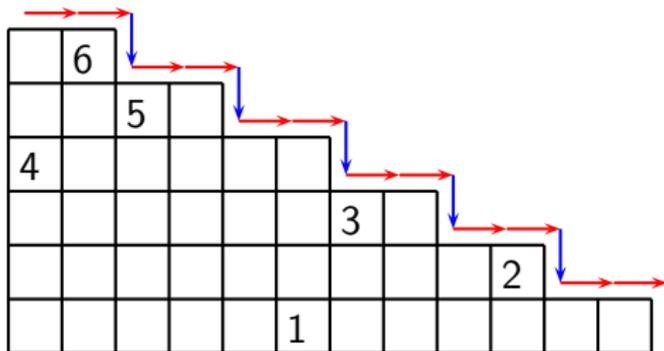


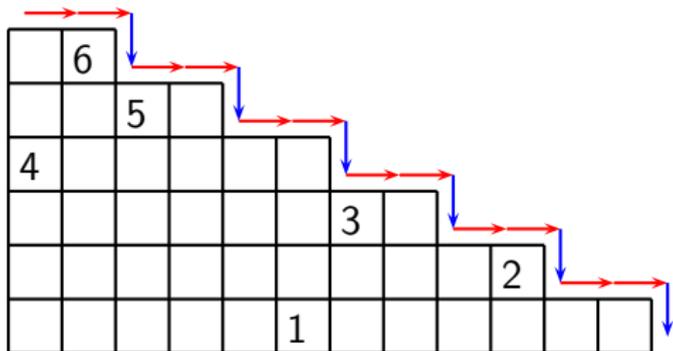




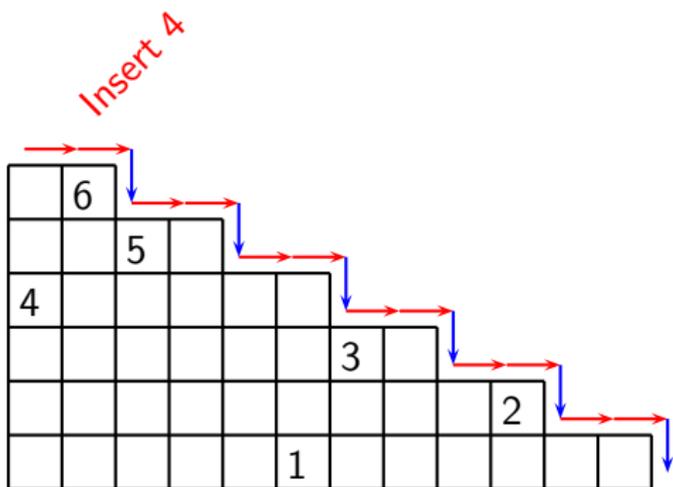




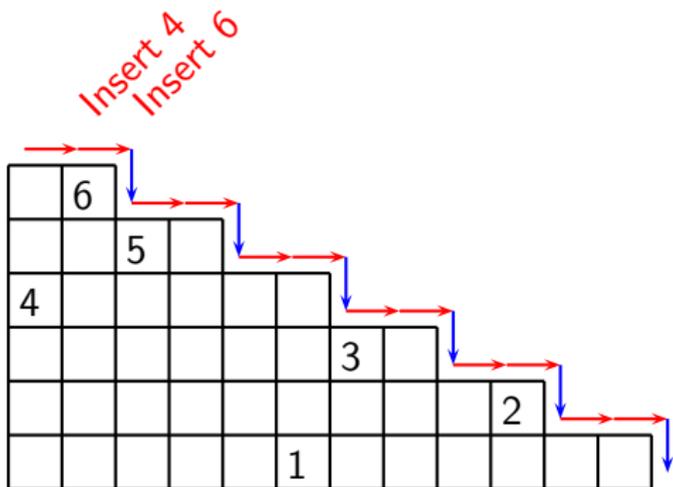




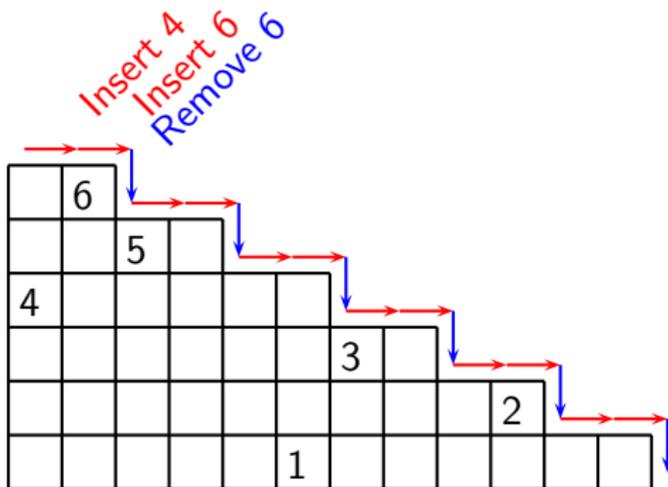
∅



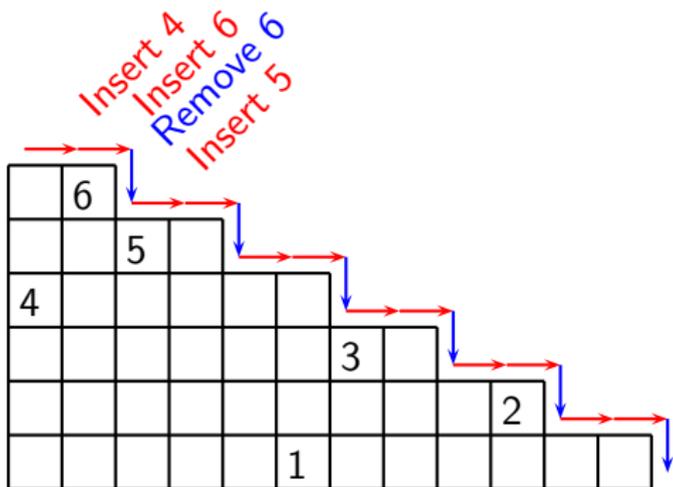
$\emptyset$ , 4



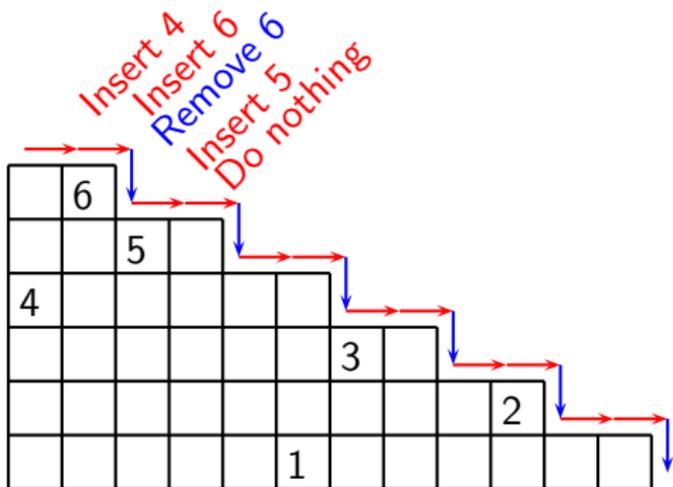
$\emptyset$ ,  $\boxed{4}$ ,  $\boxed{4 \ 6}$



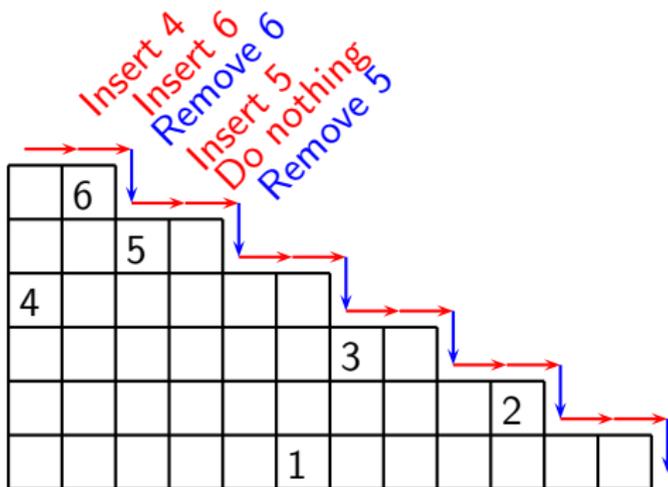
$\emptyset$ , 4, 4 6; 4



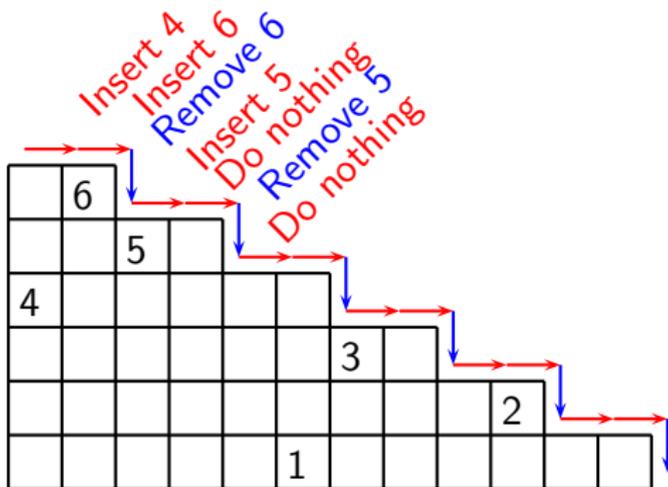
$\emptyset$ , 4, 4 6; 4, 4 5



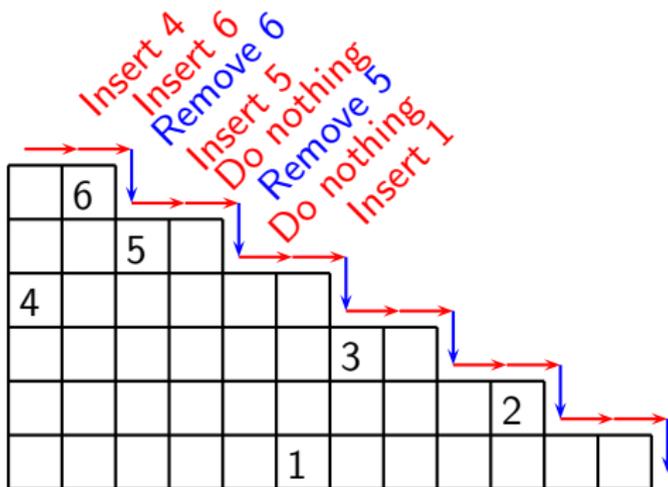
$\emptyset$ , [4], [4 6]; [4], [4 5], [4 5]



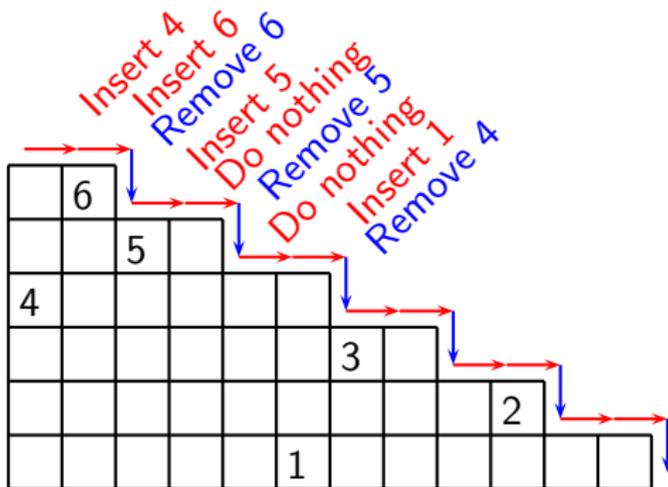
$\emptyset$ , [4], [4 6]; [4], [4 5], [4 5]; [4]



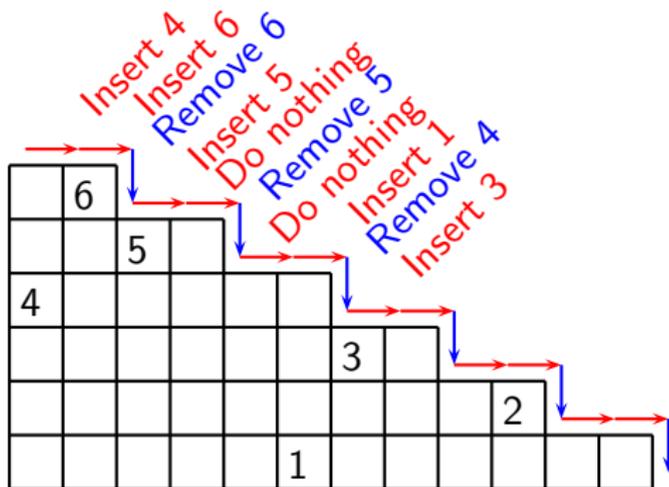
$\emptyset$ , 4, 4 6; 4, 4 5, 4 5; 4, 4



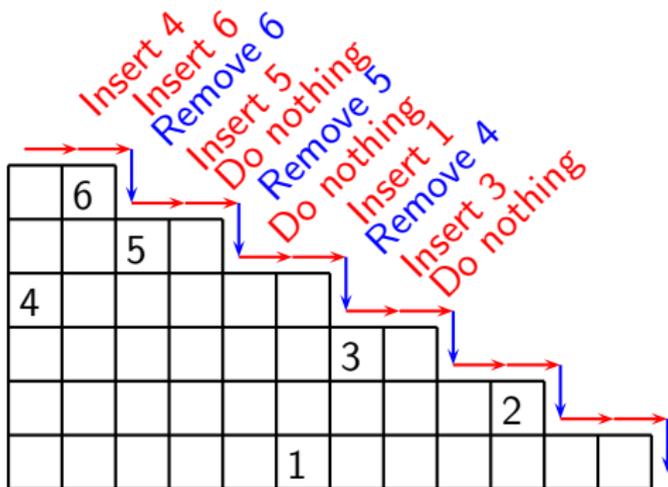
$\emptyset$ ,  $\boxed{4}$ ,  $\boxed{4}\boxed{6}$ ;  $\boxed{4}$ ,  $\boxed{4}\boxed{5}$ ,  $\boxed{4}\boxed{5}$ ;  $\boxed{4}$ ,  $\boxed{4}$ ,  $\boxed{4}$   
 $\boxed{1}$



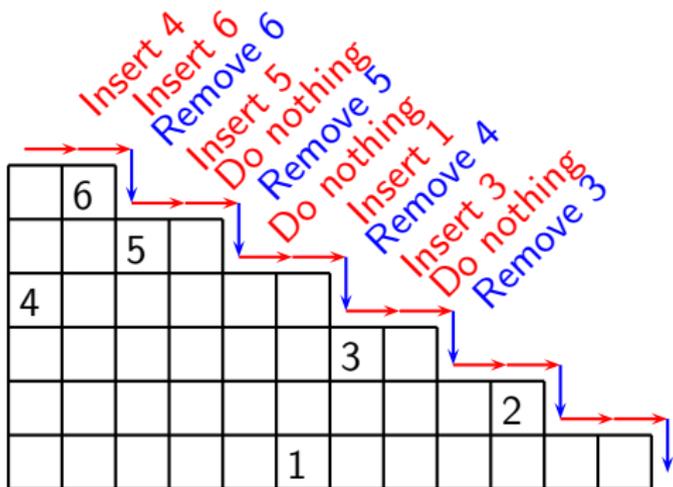
$\emptyset$ , 4, 4 6; 4, 4 5, 4 5; 4, 4, 4 1; 1



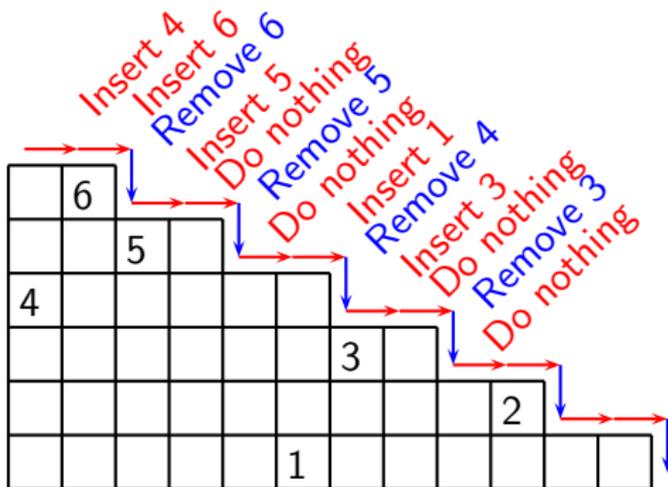
$\emptyset$ , [4], [4 6]; [4], [4 5], [4 5]; [4], [4], [4]; [1], [1 3]



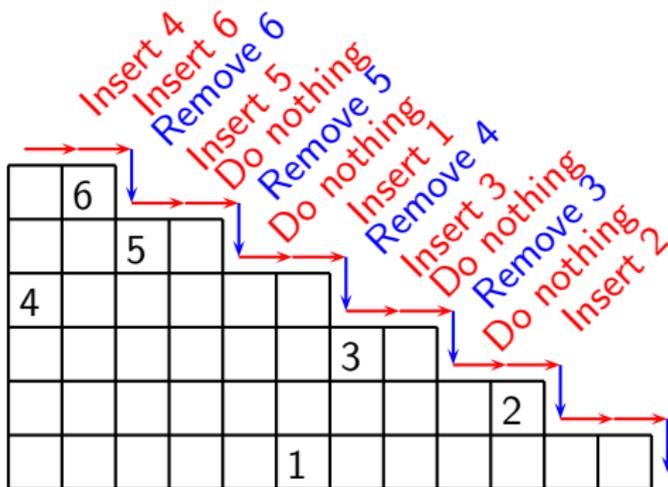
$\emptyset$ ,  $\boxed{4}$ ,  $\boxed{4\ 6}$ ;  $\boxed{4}$ ,  $\boxed{4\ 5}$ ,  $\boxed{4\ 5}$ ;  $\boxed{4}$ ,  $\boxed{4}$ ,  $\boxed{\begin{matrix} 4 \\ 1 \end{matrix}}$ ;  $\boxed{1}$ ,  $\boxed{1\ 3}$ ,  $\boxed{1\ 3}$



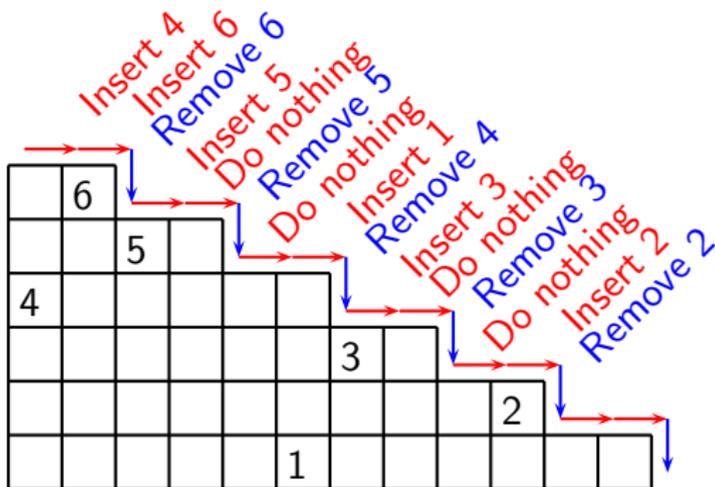
$\emptyset$ ,  $\boxed{4}$ ,  $\boxed{4\ 6}$ ;  $\boxed{4}$ ,  $\boxed{4\ 5}$ ,  $\boxed{4\ 5}$ ;  $\boxed{4}$ ,  $\boxed{4}$ ,  $\boxed{\begin{matrix} 4 \\ 1 \end{matrix}}$ ;  $\boxed{1}$ ,  $\boxed{1\ 3}$ ,  $\boxed{1\ 3}$ ;  $\boxed{1}$



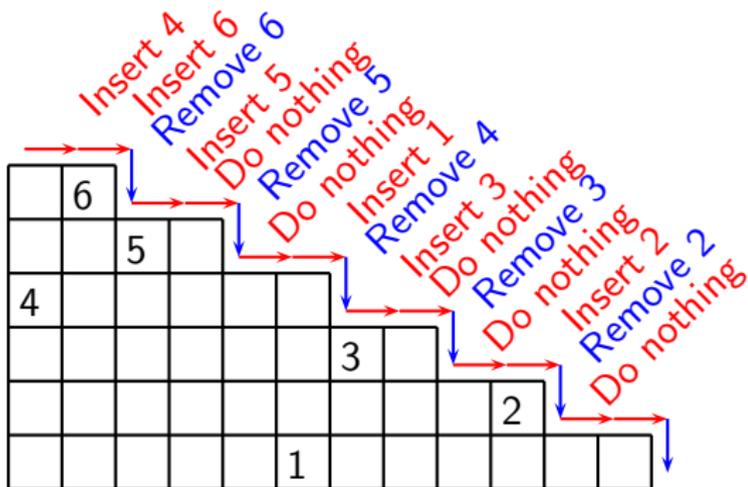
$\emptyset$ ,  $\boxed{4}$ ,  $\boxed{4}\boxed{6}$ ;  $\boxed{4}$ ,  $\boxed{4}\boxed{5}$ ,  $\boxed{4}\boxed{5}$ ;  $\boxed{4}$ ,  $\boxed{4}$ ,  $\boxed{4}$   
 $\boxed{1}$ ;  $\boxed{1}$ ,  $\boxed{1}\boxed{3}$ ,  $\boxed{1}\boxed{3}$ ;  $\boxed{1}$ ,  $\boxed{1}$



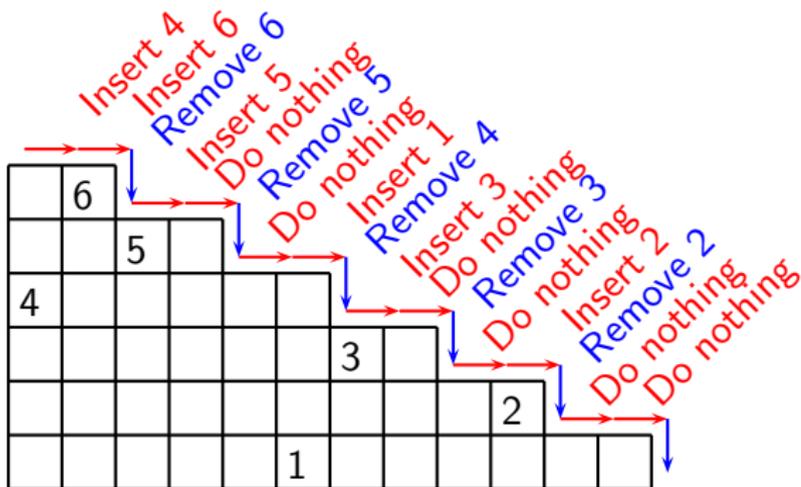
$\emptyset$ ,  $\boxed{4}$ ,  $\boxed{46}$ ;  $\boxed{4}$ ,  $\boxed{45}$ ,  $\boxed{45}$ ;  $\boxed{4}$ ,  $\boxed{4}$ ,  $\boxed{\begin{matrix} 4 \\ 1 \end{matrix}}$ ;  $\boxed{1}$ ,  $\boxed{13}$ ,  $\boxed{13}$ ;  $\boxed{1}$ ,  $\boxed{1}$ ,  $\boxed{12}$



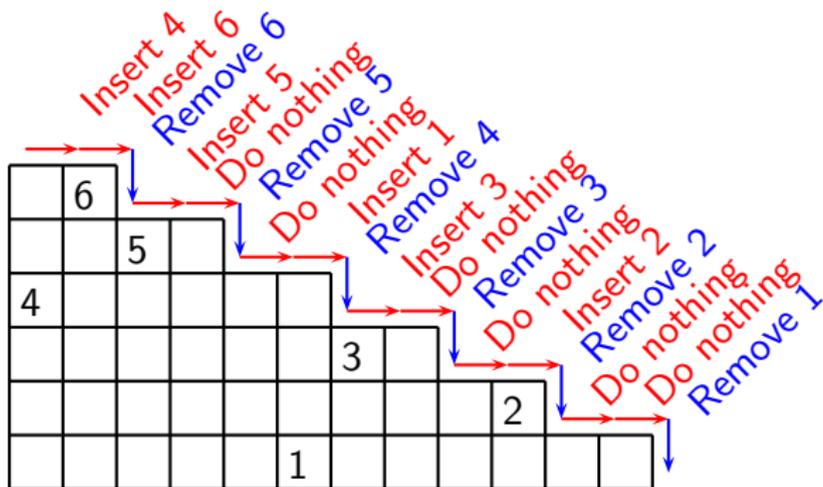
$\emptyset$ ,  $\boxed{4}$ ,  $\boxed{46}$ ;  $\boxed{4}$ ,  $\boxed{45}$ ,  $\boxed{45}$ ;  $\boxed{4}$ ,  $\boxed{4}$ ,  $\boxed{\begin{matrix} 4 \\ 1 \end{matrix}}$ ;  $\boxed{1}$ ,  $\boxed{13}$ ,  $\boxed{13}$ ;  $\boxed{1}$ ,  $\boxed{1}$ ,  $\boxed{12}$ ;   
 $\boxed{1}$



$\emptyset$ ,  $\boxed{4}$ ,  $\boxed{46}$ ;  $\boxed{4}$ ,  $\boxed{45}$ ,  $\boxed{45}$ ;  $\boxed{4}$ ,  $\boxed{4}$ ,  $\begin{array}{c} \boxed{4} \\ \boxed{1} \end{array}$ ;  $\boxed{1}$ ,  $\boxed{13}$ ,  $\boxed{13}$ ;  $\boxed{1}$ ,  $\boxed{1}$ ,  $\boxed{12}$ ;   
 $\boxed{1}$ ,  $\boxed{1}$



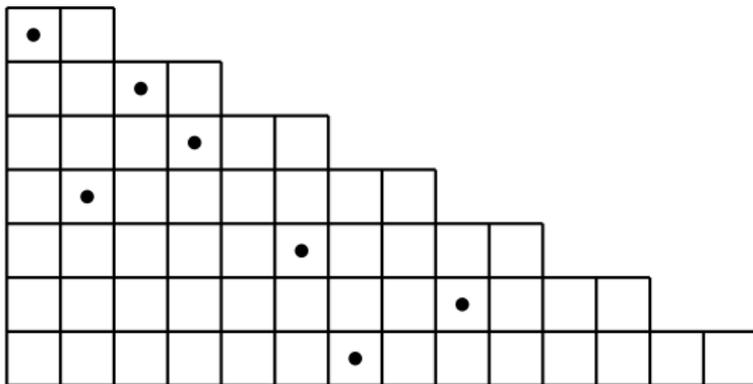
$\emptyset$ , [4], [4 6]; [4], [4 5], [4 5]; [4], [4], [4]; [1], [1 3], [1 3]; [1], [1], [1 2];  
 [1], [1], [1]



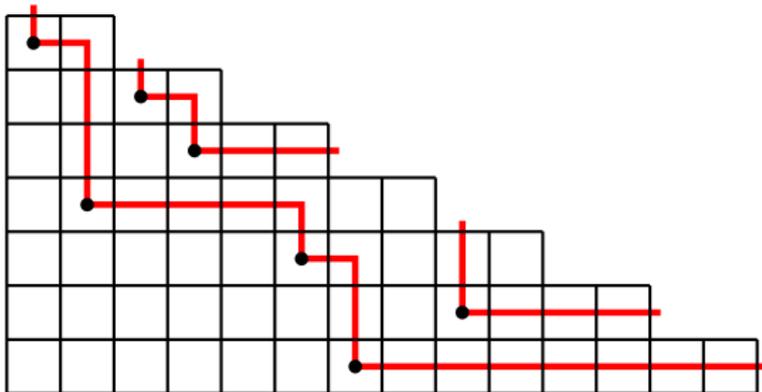
$\emptyset$ ,  $\boxed{4}$ ,  $\boxed{46}$ ;  $\boxed{4}$ ,  $\boxed{45}$ ,  $\boxed{45}$ ;  $\boxed{4}$ ,  $\boxed{4}$ ,  $\begin{array}{c} \boxed{4} \\ \boxed{1} \end{array}$ ;  $\boxed{1}$ ,  $\boxed{13}$ ,  $\boxed{13}$ ;  $\boxed{1}$ ,  $\boxed{1}$ ,  $\boxed{12}$ ;

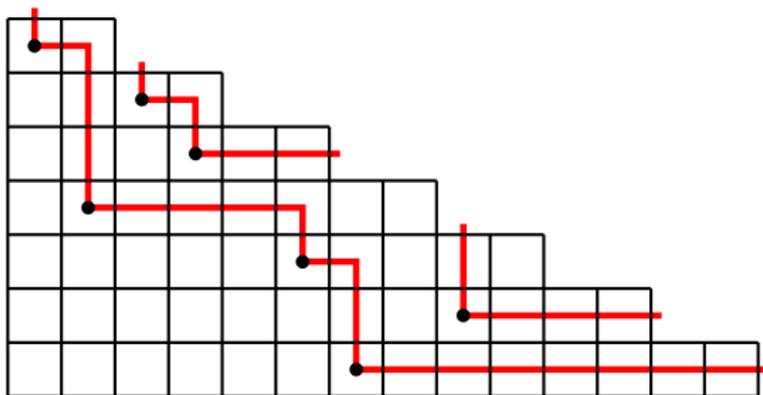
$\boxed{1}$ ,  $\boxed{1}$ ,  $\boxed{1}$ ;  $\emptyset$ .

The same bijection, using shadow lines.



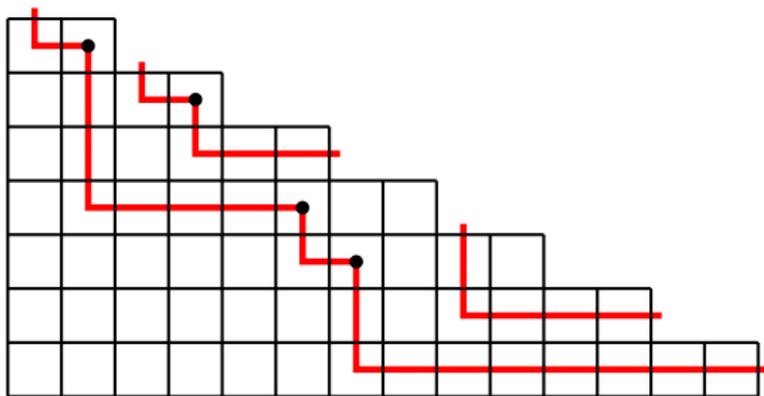






$\emptyset, \blacksquare, \blacksquare; \blacksquare, \blacksquare\blacksquare, \blacksquare\blacksquare; \blacksquare\blacksquare, \blacksquare\blacksquare, \blacksquare\blacksquare; \blacksquare, \blacksquare, \blacksquare; \blacksquare, \blacksquare\blacksquare, \blacksquare\blacksquare; \blacksquare\blacksquare, \blacksquare\blacksquare,$

$\blacksquare\blacksquare; \blacksquare, \blacksquare, \blacksquare; \emptyset$



$\emptyset, \blacksquare, \blacksquare; \blacksquare, \blacksquare\blacksquare, \blacksquare\blacksquare; \blacksquare\blacksquare, \blacksquare\blacksquare, \blacksquare\blacksquare; \blacksquare, \blacksquare, \blacksquare; \blacksquare, \blacksquare\blacksquare, \blacksquare\blacksquare; \blacksquare\blacksquare, \blacksquare\blacksquare,$

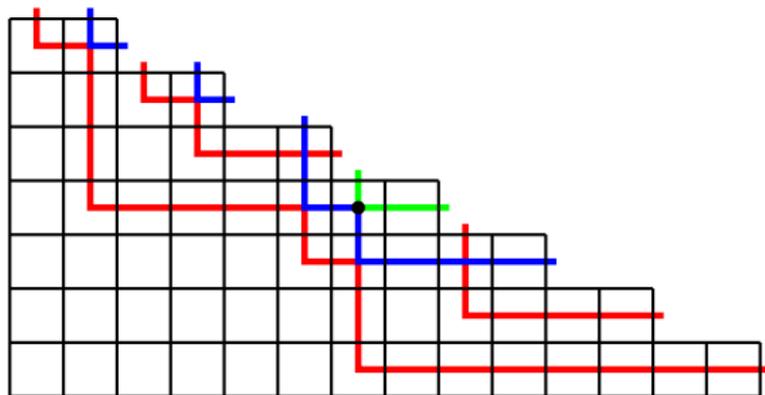
$\blacksquare\blacksquare; \blacksquare, \blacksquare, \blacksquare; \emptyset$







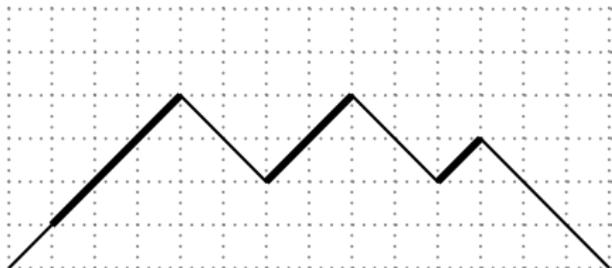
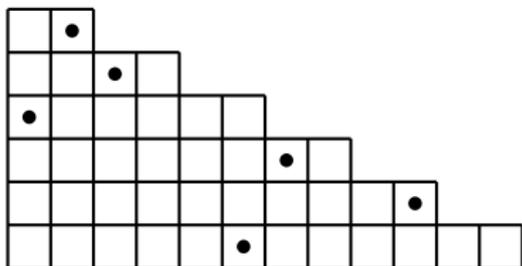




$\emptyset$ ,  $\begin{array}{|c|} \hline \color{red}{\square} \\ \hline \end{array}$ ,  $\begin{array}{|c|} \hline \color{blue}{\square} \\ \hline \color{red}{\square} \\ \hline \end{array}$ ;  $\color{red}{\square}$ ,  $\begin{array}{|c|c|} \hline \color{red}{\square} & \color{red}{\square} \\ \hline \end{array}$ ,  $\begin{array}{|c|} \hline \color{blue}{\square} \\ \hline \color{red}{\square} \\ \hline \end{array}$ ;  $\begin{array}{|c|c|} \hline \color{red}{\square} & \color{red}{\square} \\ \hline \end{array}$ ,  $\begin{array}{|c|c|} \hline \color{red}{\square} & \color{red}{\square} \\ \hline \end{array}$ ,  $\begin{array}{|c|} \hline \color{blue}{\square} \\ \hline \color{red}{\square} \\ \hline \end{array}$ ;  $\begin{array}{|c|} \hline \color{green}{\square} \\ \hline \color{blue}{\square} \\ \hline \color{red}{\square} \\ \hline \end{array}$ ,  $\begin{array}{|c|} \hline \color{green}{\square} \\ \hline \color{blue}{\square} \\ \hline \color{red}{\square} \\ \hline \end{array}$ ;  $\begin{array}{|c|} \hline \color{blue}{\square} \\ \hline \color{red}{\square} \\ \hline \end{array}$ ,  $\begin{array}{|c|c|} \hline \color{blue}{\square} & \color{red}{\square} \\ \hline \end{array}$ ,  $\begin{array}{|c|} \hline \color{blue}{\square} \\ \hline \color{red}{\square} \\ \hline \end{array}$ ;  $\begin{array}{|c|c|} \hline \color{red}{\square} & \color{red}{\square} \\ \hline \end{array}$ ,  $\begin{array}{|c|c|} \hline \color{red}{\square} & \color{red}{\square} \\ \hline \end{array}$ ,

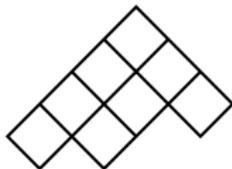
$\begin{array}{|c|c|} \hline \color{red}{\square} & \color{red}{\square} \\ \hline \end{array}$ ;  $\color{red}{\square}$ ,  $\color{red}{\square}$ ,  $\color{red}{\square}$ ;  $\emptyset$

The Dyck path associated with a stammering tableau is apparent in the rook placement :



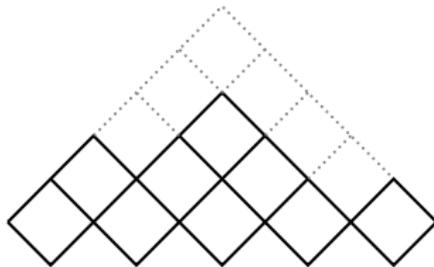
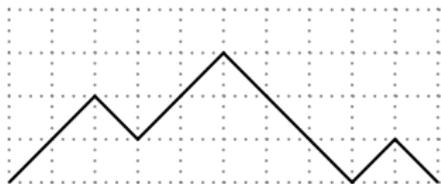
A second bijection :  
From rook placements to chain of Dyck paths.

The Japanese notation for Young diagrams :



The *staircase partition* is  $\delta_n = (n, n - 1, \dots, 1)$ .

A *Dyck path* is a skew shape  $\delta_n/\lambda$  where  $\lambda \subset \delta_{n-1}$ .

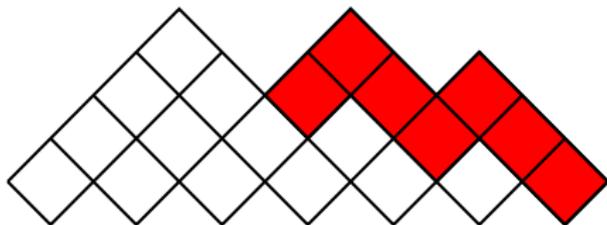
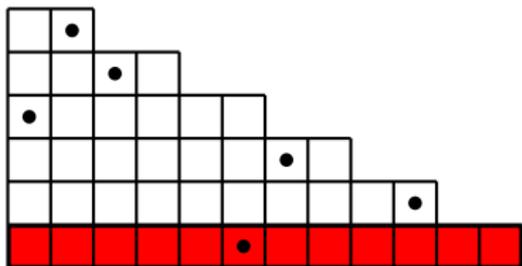


## Définition

*A skew shape is called a ribbon if it is connected and contains no  $2 \times 2$  square. Let  $D$  and  $E$  be two Dyck paths of respective length  $2n$  and  $2n + 2$ . We then denote  $D \sqsubset E$  and say that  $E$  is obtained from  $D$  by addition of a ribbon if  $D \subset E$  and the difference  $E/D$  is a ribbon.*

## Proposition

*Let  $T$  a rook placement in the double staircase with  $n$  rows, and  $T'$  obtained by removing the bottom row. Let  $D, D'$  the Dyck paths associated with  $T$  and  $T'$ . Then  $D$  is obtained from  $D'$  by addition of a ribbon.*



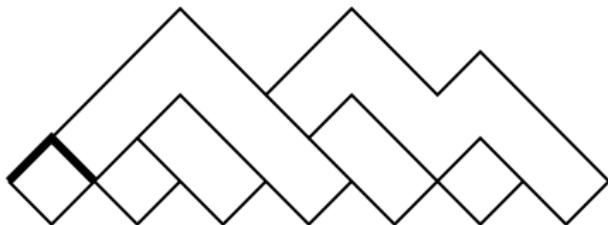
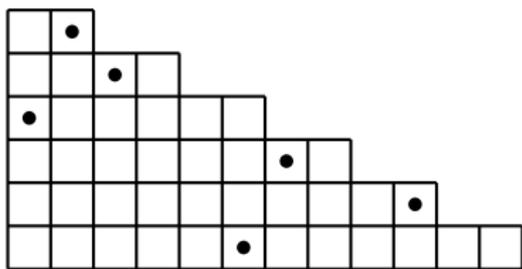
## Définition

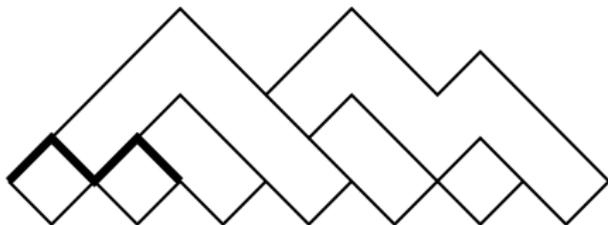
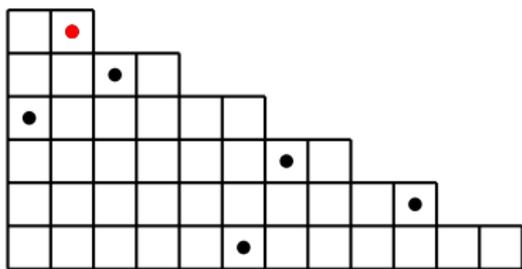
*A  $n$ -chain of Dyck paths is a sequence  $D_1 \sqsubset D_2 \sqsubset \cdots \sqsubset D_n$  where  $D_i$  is a Dyck path of length  $2i$ , obtained by addition of a ribbon to  $D_{i-1}$ . The biggest path  $D_n$  is called the shape of the chain.*

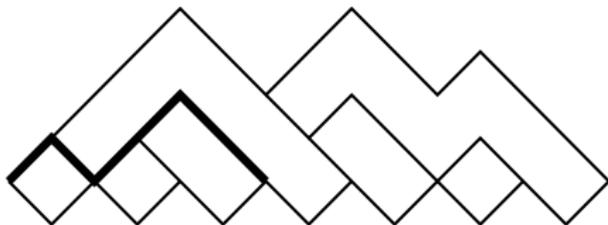
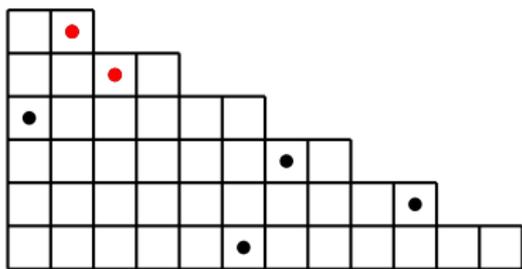
## Proposition

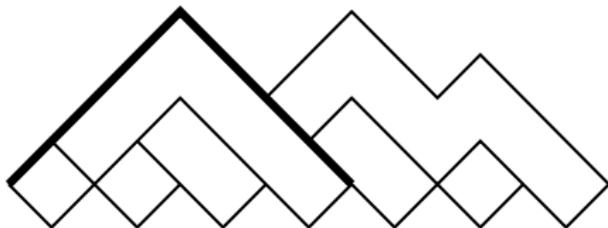
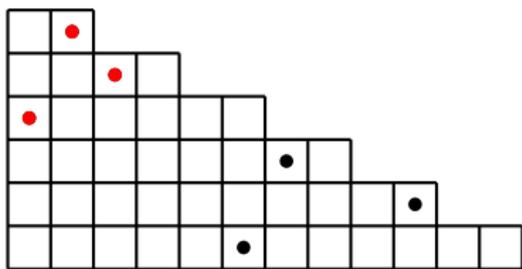
*From a rook placement  $T$  in the double staircase of  $n - 1$  rows, we define a  $n$ -chain  $D_1 \sqsubset D_2 \sqsubset \cdots \sqsubset D_n$  so that  $D_i$  is the Dyck path associated with the rook placement obtained by keeping the  $i - 1$  first rows of  $T$ .*

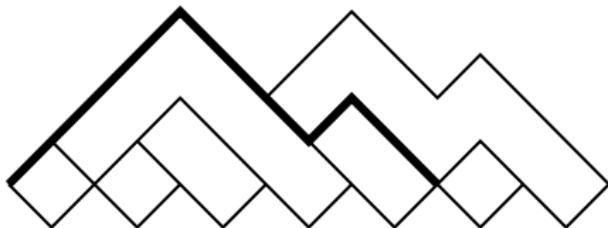
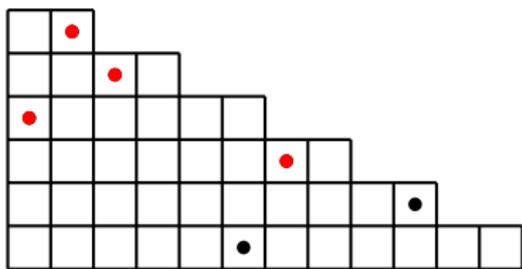
*This defines a bijection between rook placements and  $n$ -chains of Dyck paths.*

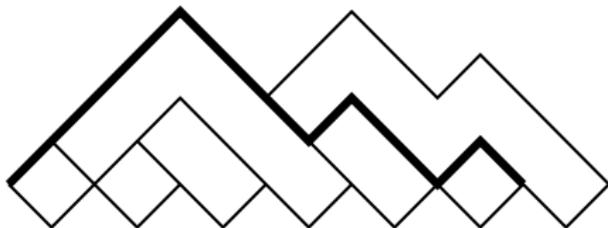
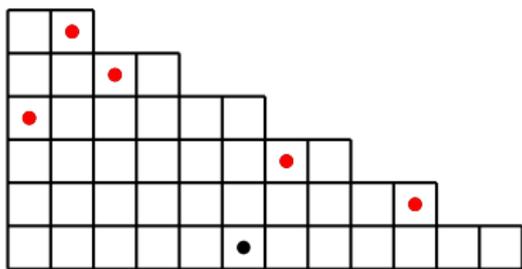


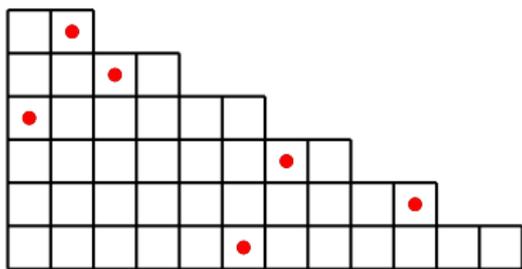








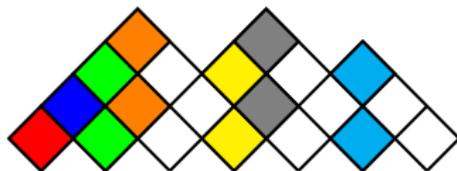




A third bijection :  
From chain of Dyck paths to subdivided Laguerre histories.

## Définition

A subdivided Laguerre history of shape  $D$  is obtained from the Dyck path  $D$  by choosing a cell in each column below a step  $\nearrow$ .

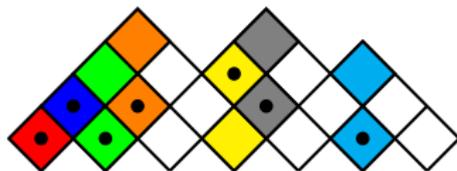


The number of cells in a column is  $\lceil h/2 \rceil$  if the step  $\nearrow$  is from height  $h-1$  to  $h$ , so the number of histories of shape  $D$  is

$$\prod_{\nearrow \text{ from height } h-1 \text{ to } h} \lceil h/2 \rceil.$$

## Définition

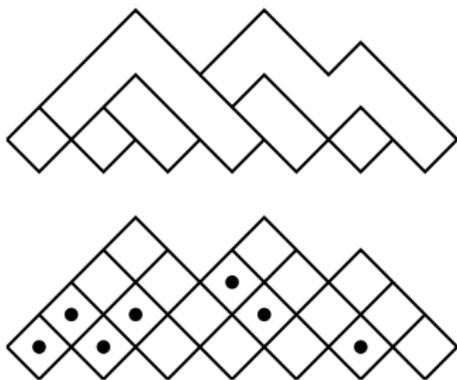
A subdivided Laguerre history of shape  $D$  is obtained from the Dyck path  $D$  by choosing a cell in each column below a step  $\nearrow$ .



The number of cells in a column is  $\lceil h/2 \rceil$  if the step  $\nearrow$  is from height  $h-1$  to  $h$ , so the number of histories of shape  $D$  is

$$\prod_{\nearrow \text{ from height } h-1 \text{ to } h} \lceil h/2 \rceil.$$

The bijection is done by putting a dot in the left extremity of each ribbon.



The chains of Dyck paths present some similarities with Dyck tableaux [Aval, Boussicault, Dasse-Hartaut], which are also equivalent to subdivided Laguerre histories and used to study the 3-parameter PASEP partition function.

Some statistics from the PASEP can be read in the rook placements, for example the deformed relation  $DU - qUD$  leads to count “inversions” and gives the product  $\prod \left[ \left[ \frac{h}{2} \right] \right]_q$ .

But the more general combinatorial results about this partition function still go through the alternative tableaux or permutation tableaux.

Merci Xavier !