

Inversions

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joint work with Hayat Cheballah

Summary: We use inversions to investigate the connection between

Alternating sign matrices

and

Totally symmetric selfcomplementary plane partitions

.

Alternating sign matrices

- ▶ Square matrices with coefficients in $\{-1, 0, 1\}$

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

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- ▶ in each line or column, 1 and -1 alternate, starting and ending with 1.

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- ▶ Rewriting problems, ABBA (by Xavier Viennot)



Enumeration of ASMs

The number of ASM of size n is

$$A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$$

$$A_n = 1 \quad 2 \quad 7 \quad 42 \quad 429 \quad 7436 \quad 218348$$

Conjectured by Mills, Robbins, Rumsey.

Proved by D. Zeilberger (1995).

Other proofs by G. Kuperberg, I. Fischer ...

Gog triangles

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$



- ▶ Gog triangles=Gelfand-Tsetlin triangles,

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- ▶ Gog triangles=Gelfand-Tsetlin triangles,
- ▶ strictly increasing along lines.

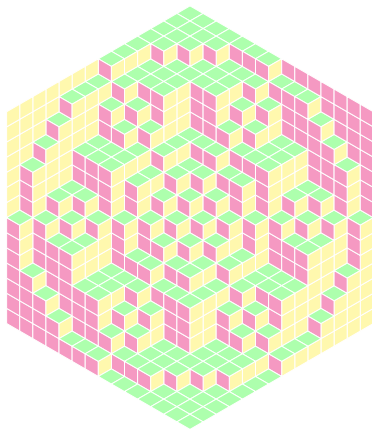
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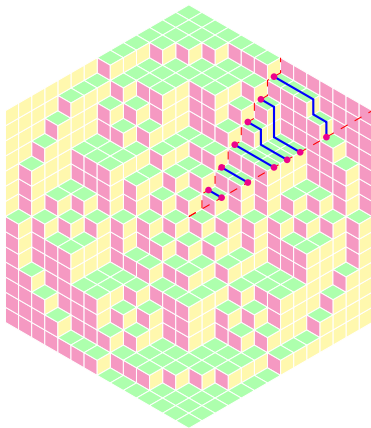
- ▶ Gog triangles=Gelfand-Tsetlin triangles,
- ▶ strictly increasing along lines.
- ▶ Upper line is $1, 2, 3, \dots, n$.

Totally symmetric selfcomplementary plane partitions



Plane partitions having hexagonal symmetry, and equal to their complement in the cube.

TSSCPPs can be encoded by non intersecting lattice paths.



One can use the Lindström Gessel Viennot



formula to compute the number of TSSCPPs:

$$B_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$$

$$B_n = 1 \quad 2 \quad 7 \quad 42 \quad 429 \quad 7436 \quad 218348$$

$$B_n = A_n$$

Problem: find a bijective proof.
VERY DIFFICULT!

Magog triangles

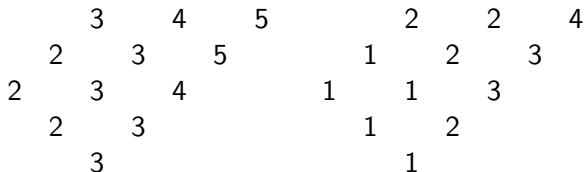
$$\begin{array}{ccccccccc} 1 & & 1 & & 2 & & 2 & & 4 & \leq 5 \\ & & 1 & & 1 & & 2 & & 3 & \leq 4 \\ & \searrow & & & 1 & & 1 & & 3 & \nearrow & \leq 3 \\ & & & & 1 & & 2 & & & & \leq 2 \\ & & & & & & 1 & & & & \leq 1 \end{array}$$

They encode TSSCPPs.

Problem: find a bijection between Gog and Magog triangles.

Gog and Magog trapezoids.

The k right SW-NE diagonals of a Gog (reps. Magog) triangle form a *trapezoid*.



(5, 3) Gog and Magog trapezoids.

By Zeilberger (1995), (n, k) Gog and Magog trapezoids are equienumerated.

Gog

```

1      2      3      4      5
  1      2      3      5
    2      3      4
      2      3
        3
    
```

Magog

```

1      1      2      2      4
  1      1      2      3
    1      1      3
      1      2
        1
    
```

$$1 + 1 + 2 + 2 + 4 - (1 + 1 + 2 + 3) = 3$$

We try to simplify further using a conjecture by C. Krattenthaler.

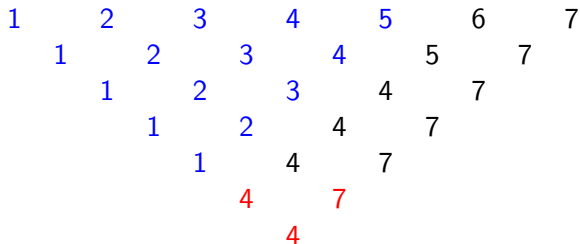
Completion of Gog triangles

A triangle

$$\begin{array}{cc} a & b \\ & c \end{array}$$

$$a \leq c \leq b; \quad a < b$$

has a minimal completion to a Gog trapezoid of size b



Completion of Magog triangles

A triangle

$$\begin{array}{cc} a & b \\ & c \\ a \leq c \leq b \end{array}$$

has a minimal completion to a Magog trapezoid of size $\max(b, c + 1)$

$$\begin{array}{cccccc} 1 & & 1 & & 1 & & 3 & & 4 \\ & 1 & & 1 & & 1 & & 4 & \\ & & 1 & & 1 & & 1 & & \\ & & & 1 & & 1 & & & \\ & & & & 1 & & & & \end{array}$$

It is easy to find a bijection between triangles of same height, preserving the statistic:

if $a < b$

a b is mapped to a b
 c $a + b - c$

and

a b is mapped to a $b - 1$
 a $b - 1$

The bijection splits into two steps.

Step 1:

$$\begin{array}{ccc} a & & b \\ & c & \end{array} \rightarrow \begin{array}{ccc} a & & b \\ & c & \end{array} \quad a < c$$

or

$$\begin{array}{ccc} a & & b \\ & a & \end{array} \rightarrow \begin{array}{ccc} a & & b - 1 \\ & a & \end{array}$$

Step 2:

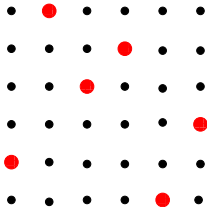
$$\begin{array}{ccc} a & & b \\ & c & \end{array} \rightarrow \begin{array}{ccc} a & & b \\ & a + b - c & \end{array}$$

One can generalize this bijection to more general trapezoids by generalizing the two steps.

Step 1 will lead to investigating *inversions*.

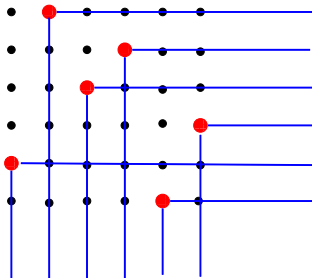
Step 2 will lead to the *Schützenberger involution*.

Inversions of a permutation

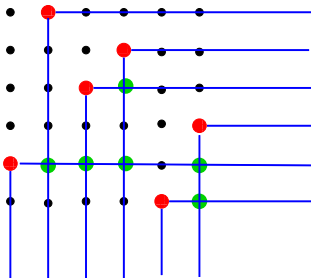


Inversions of a permutation

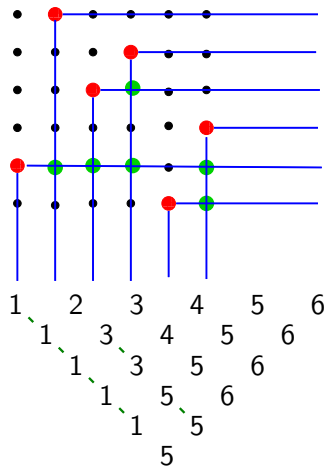
Le soleil de Viennot:



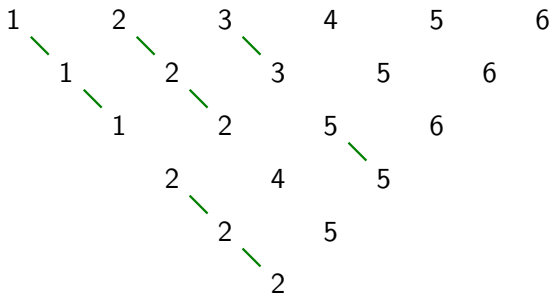
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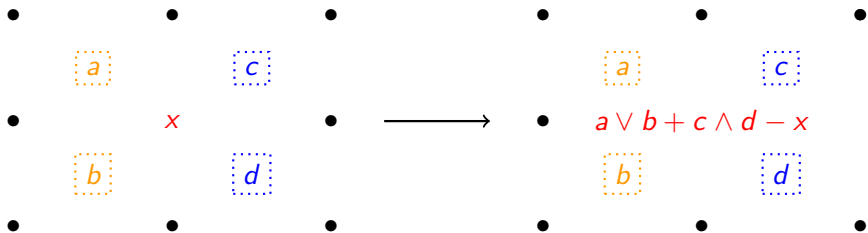


Inversions of a Gog triangle



Already defined by Mills Robbins and Rumsey.

Schützenberger involution



Apply the transform to all entries in row i

$$\begin{array}{cccccc}
 & & 1 & & 1 & & 2 & & 2 & & 4 \\
 & & & 1 & & 1 & & 2 & & 3 & \\
 & & & & 1 & & 1 & & 3 & & \\
 \text{row } 2 & \rightarrow & & & 1 & & 2 & & & & \\
 & & & & & & 1 & & & &
 \end{array}$$

This gives transform s_i

The Schützenberger involution is

$$S = s_1 s_2 s_3 \dots s_{n-1} s_1 s_2 \dots s_{n-2} \dots s_1 s_2 s_1$$

Schützenberger involution on words

Gelfand-Tsetlin \sim SSYT

- ▶ Schützenberger involution lifts to words through RSK.

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- ▶ On a word on letters $1, 2, \dots, n$ replace letter i by $n + 1 - i$, then read the word backwards.

Schützenberger involution on words

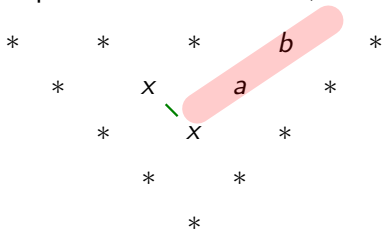
Gelfand-Tsetlin \sim SSYT

- ▶ Schützenberger involution lifts to words through RSK.
- ▶ On a word on letters $1, 2, \dots, n$ replace letter i by $n + 1 - i$, then read the word backwards.
- ▶ Beautiful algorithm for RSK due to Viennot, using the sun:



The general idea is to follow two steps:

- ▶ Step 1: for each inversion, subtract 1 from the NE entries.



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$$\begin{array}{cccccc} * & & * & & * & & b-1 & & * \\ & * & & x & & a-1 & & * \\ & & * & & x & & * \\ & & & * & & * \\ & & & & * & & * \\ & & & & & * & \end{array}$$

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- ▶ Step 1: for each inversion, subtract 1 from the NE entries.

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- ▶ Step 2: Apply the Schützenberger involution.

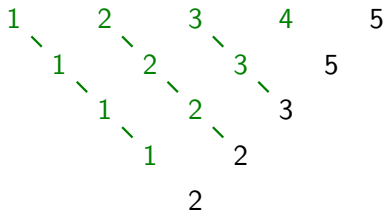
This works for $(n, 1)$ trapezoids.

5
5
3
2
2

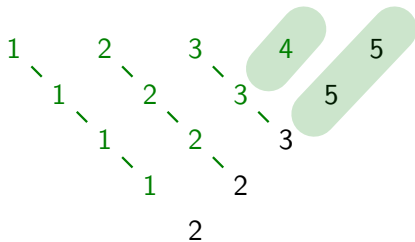
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```
1      2      3      4      5
      1      2      3      5
          1      2      3
              1      2
                  2
```

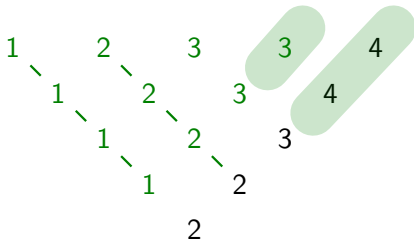
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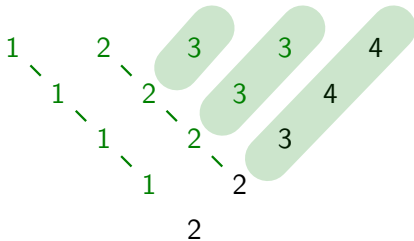
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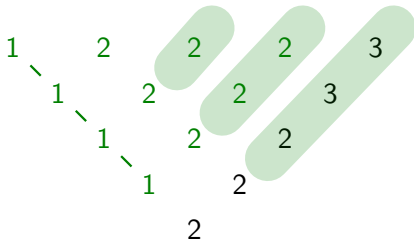
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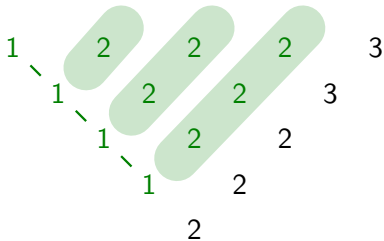
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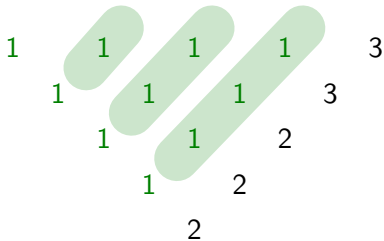
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```
1      1      1      1      3
      1      1      1      3
        1      1      2
          1      2
            2
```

This works for $(n, 1)$ trapezoids.

```
1      1      1      1      3
      1      1      1      2
        1      1      2
          1      2
            1
```

A (complicated) modification of this procedure works for $(n, 2)$ trapezoids.

Hayat Cheballah, P.B. Gog and Magog Triangles, and the Schützenberger Involution Séminaire Lotharingien de Combinatoire, [B66d] (2012)

A *GOGAm triangle* is the image by the Schützenberger involution of a Magog triangle.

Left trapezoids

The k left most NW-SE diagonals of a Gog triangle form a left Gog trapezoid.

```
1      2      3
      1      2      3
        2      3      4
          2      3
            3
```

The k left most NW-SE diagonals of a GOGAm triangle form a left GOGAm trapezoid.

```
1      1      2
      1      1      2
        1      1      3
          1      3
            2
```

Conjecture:

(n, k) Gog and GOGAm trapezoids are equienumerated.

Bijjective proof for $k = 1, 2$.

Remarks:

- ▶ left Magog trapezoids are less interesting.

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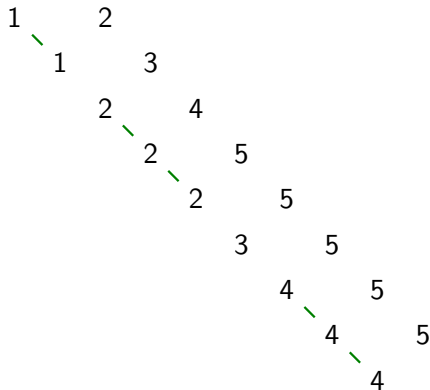
Remarks:

- ▶ left Magog trapezoids are less interesting.
- ▶ One can also define Gog and GOGAm rectangles.

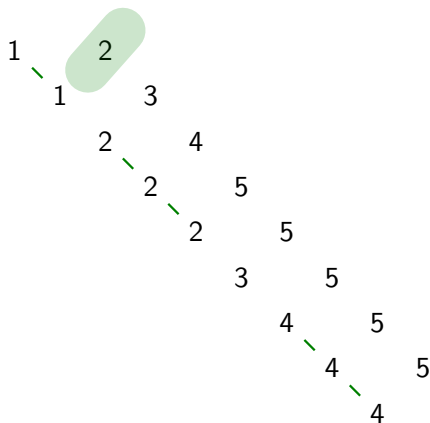
$(n, 2)$ Left Gog \rightarrow GOGAm bijection.

1 2
 1 3
 2 4
 2 5
 2 5
 3 5
 4 5
 4 5
 4

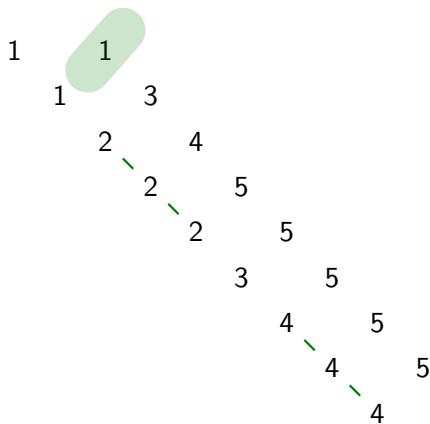
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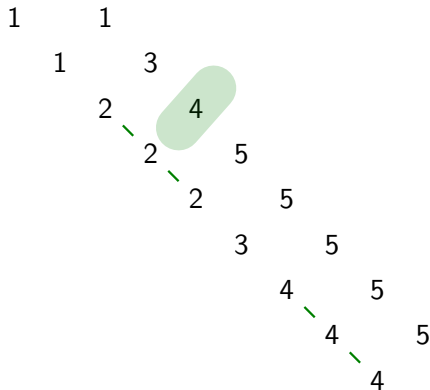
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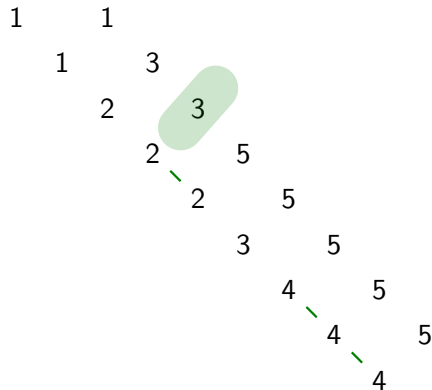
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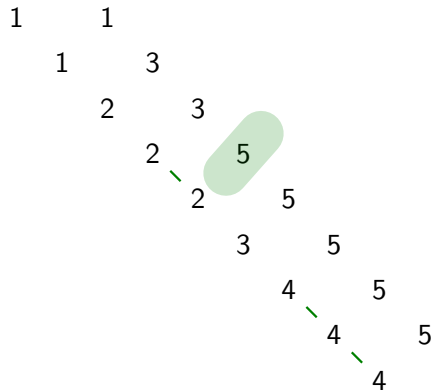
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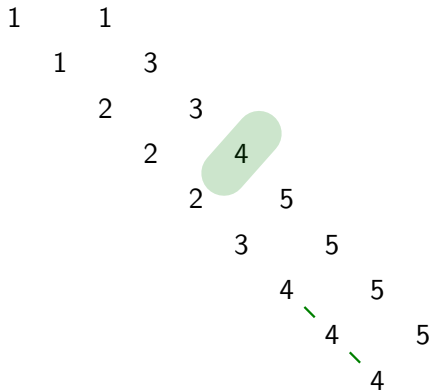
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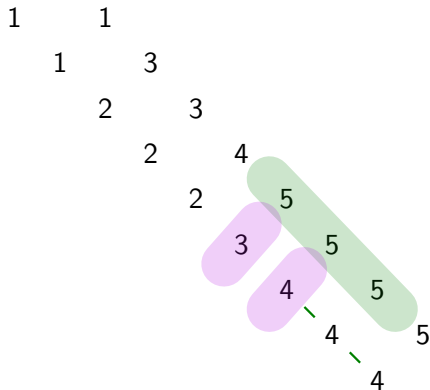
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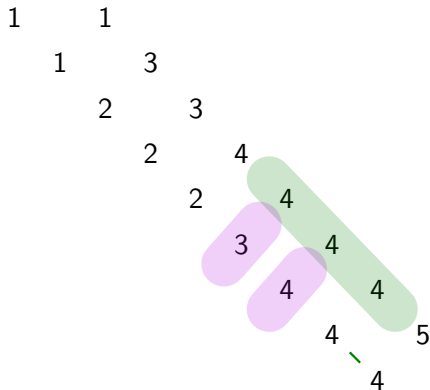
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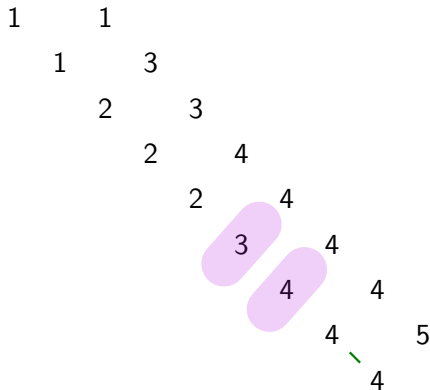
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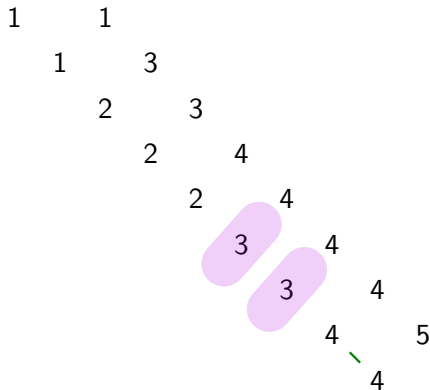
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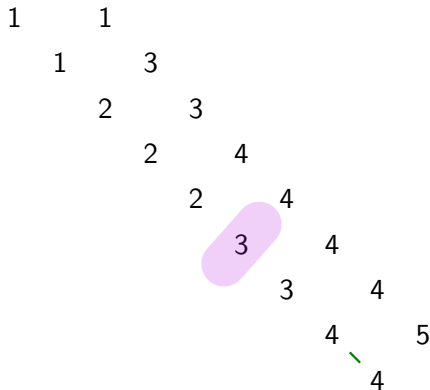
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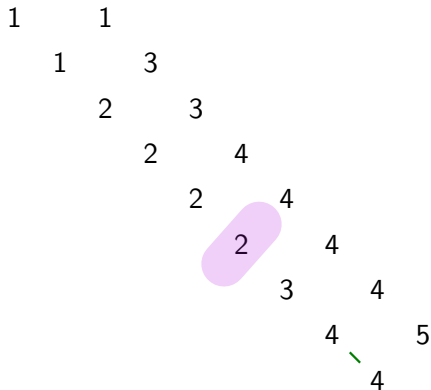
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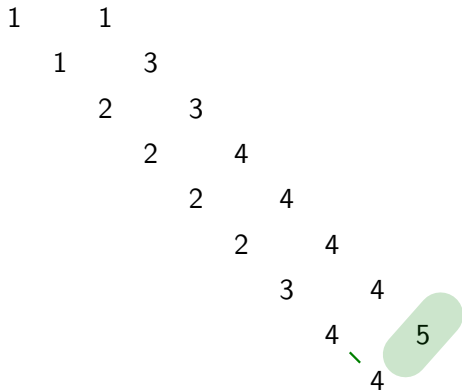
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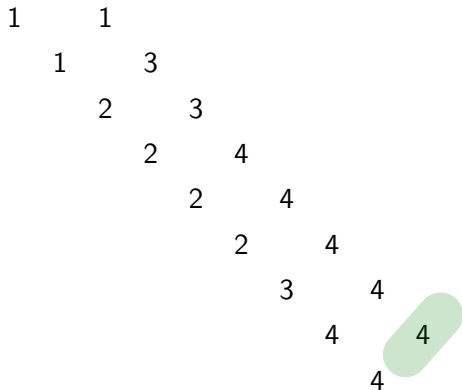
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1 1
 1 3
 2 3
 2 4
 2 4
 2 4
 3 4
 4 4
 4

Thank you!

