

POSITIVITY AND INTEGRABILITY IN CLUSTER ALGEBRAS

[PDF + R. KEDEM 2008-]

Physical Mathematics

Journées Viennot, Labri 2012

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Physical Mathematics

Mathematics is the part of physics
where experiments are cheap.

POSITIVITY AND INTEGRABILITY IN CLUSTER ALGEBRAS

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Physical Combinatorics

- hidden combinatorial structures within physical systems (math solutions)
- hidden integrable structures within combinatorial problems (physical solutions)

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Physical Combinatorics

POSITIVITY

INTEGRABILITY

POSITIVITY AND INTEGRABILITY IN CLUSTER ALGEBRAS

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Physical Combinatorics

POSITIVITY { • unitarity — FT
• reps theory — stat mech
• combinatorics ADE CFT

INTEGRABILITY { • symmetries
• capture generic behavior
(2D, 1+1D, 2+1D)

POSITIVITY AND INTEGRABILITY IN CLUSTER ALGEBRAS

Discrete Integrable Systems

- Reps th'y of integrable quantum spin chains \rightarrow Q, T, Y - systems [Kirillov-Reshetikhin]

= Dynamical Systems with:

- discrete time evolution
- conservation laws
- exact solvability

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Discrete Integrable Systems

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- discrete time evolution
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- exact solvability

Cluster Algebras

- Total positivity in $GL(n)$

- Canonical bases of q -group [Fomin-Zelevinsky]

= Dynamical system with

- multi-directional evolution
- built-in Laurent property
- mysterious positivity

1. DISCRETE INTEGRABLE SYSTEMS: A RUNNING EXAMPLE

"Q system" for A_1

$$Q_{n+1} Q_{n-1} = Q_n^2 - 1$$

- obeyed by sl_2 characters as consequence of tensor product identity $\mathcal{S}_{n+1} \otimes \mathcal{S}_{n-1} + \mathcal{S}_0 = \mathcal{S}_n \otimes \mathcal{S}_n$
- view as discrete time (n) evolution

given initial data (Q_0, Q_1) , find solution?

- INTEGRABILITY: CONSERVED QUANTITY

write the equation as $Q_n^2 - Q_{n+1} Q_{n-1} = 1$

$$W_n = \begin{vmatrix} Q_n & Q_{n-1} \\ Q_{n+1} & Q_n \end{vmatrix} = 1 \quad \text{"discrete Wronskian"}$$

$$\text{then } W_{n+1} - W_n = 0 = \begin{vmatrix} Q_{n+1} + Q_{n-1} & Q_n \\ Q_{n+2} + Q_n & Q_{n+1} \end{vmatrix}$$

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$\Rightarrow \exists c_n$ such that

$$\begin{cases} Q_{n+1} + Q_{n-1} = c_n Q_n \\ Q_{n+2} + Q_n = c_n Q_{n+1} \end{cases}$$

$\Rightarrow c_n = c_{n+1} = c$ conserved quantity (First integral)

$$c = \frac{Q_{n+1}}{Q_n} - \frac{1}{Q_n Q_{n+1}} + \frac{Q_n}{Q_{n+1}} = \frac{Q_1}{Q_0} - \frac{1}{Q_0 Q_1} + \frac{Q_0}{Q_1}$$

• EXACT SOLVABILITY

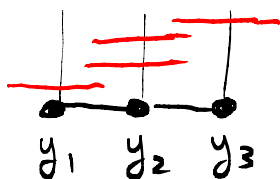
Set $y_1 = \frac{Q_1}{Q_0}$, $y_2 = -\frac{1}{Q_0 Q_1}$, $y_3 = \frac{Q_0}{Q_1}$

$$\begin{cases} Q_{n+1} Q_{n-1} = Q_n^2 - 1 \\ Q_{n+1} - c Q_n + Q_{n-1} = 0 \end{cases}$$

\downarrow
 $c = y_1 + y_2 + y_3$

$$F(t) = \sum_{n=0}^{\infty} t^n Q_n = \frac{1 - t(y_2 + y_3)}{1 - t(y_1 + y_2 + y_3) + t^2 y_1 y_3} \cdot Q_0$$

Viennot's heaps (Pyramids)



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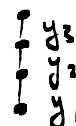
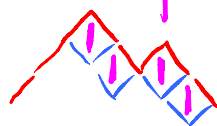
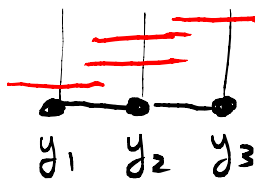
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$$= \frac{Q_0}{1 - t \frac{y_1}{1 - t \frac{y_2}{1 - t y_3}}}$$

"continued fraction"

[Viennot's heap \rightarrow path correspondence]



• EXACT SOLVABILITY

Set $y_1 = \frac{Q_1}{Q_0}$, $y_2 = -\frac{1}{Q_0 Q_1}$, $y_3 = \frac{Q_0}{Q_1}$

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$$= \frac{Q_0}{1-t \frac{y_1}{1-t \frac{y_2}{1-t y_3}}}$$

"continued fraction"

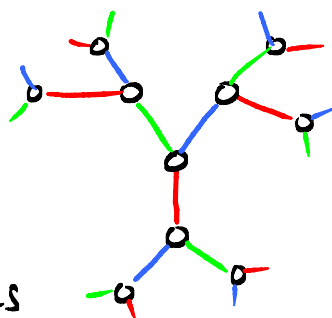
THM $\frac{Q_n}{Q_0} = \left\{ \begin{array}{l} \text{partition fn of paths } 0 \rightarrow 0 \text{ on } [n] \\ \text{with } 2n \text{ steps, weight } y_i \text{ if } i \text{ is } i_1 \end{array} \right\}$



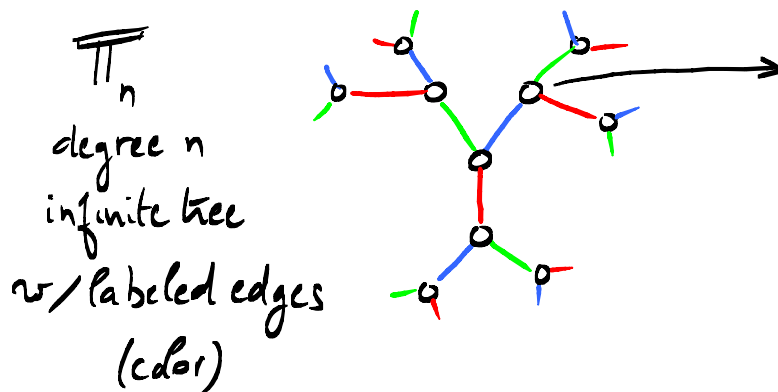
"discrete path integral"

2. CLUSTER ALGEBRAS : DEFINITION

\mathbb{T}_n
degree n
infinite tree
w/ labeled edges
(color)



2. CLUSTER ALGEBRAS : DEFINITION



→ rank n cluster algebra
 = generated by all n -vectors in \mathbb{T}_n

at each vertex, 2 data

1. n -vector $(x_1, \dots, x_n) = \vec{x}$
cluster

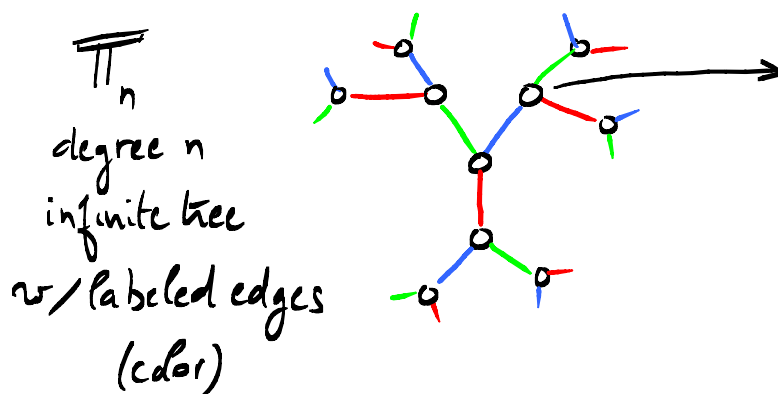
2. $n \times n$ skew sym matrix

$B_{ij} \in \mathbb{Z}$ exchange matrix

+ MUTATION RULES

$(\vec{x}, B) \xrightarrow{\mu} (\vec{x}', B')$

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+ MUTATION RULES

$(\vec{x}, B) \xrightarrow{\mu} (\vec{x}', B')$

The mutation structure guarantees the Laurent
property: \vec{x} at any vertex = Laurent polynomial of
 \vec{x} at any other vertex.

MUTATIONS

$$(\vec{x}, \vec{B}) \xrightarrow{k} (\vec{x}', \vec{B}')$$

1. QUIVER MUTATION (B)

• B-matrix \Leftrightarrow quiver $(B_{ij} > 0 \Leftrightarrow \overset{B_{ij} \text{ arrows}}{i \Rightarrow j})$

• Mutation in the k direction:

(i) reflect arrows incident to k

(ii) for each path $i \rightarrow k \rightarrow j$ via k , create $i \rightarrow j$

short cut \Rightarrow total: $i \xleftarrow{k} j$

(iii) eliminate opposite pairs $i \rightleftharpoons j \rightarrow i \cdot j$

Example

apply μ_2



$$B = \begin{matrix} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \end{matrix}$$

Example
 apply μ_3



$$B = \begin{matrix} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \end{matrix}$$

Example
 apply μ_3



$$B = \begin{matrix} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \end{matrix}$$

etc...

Example



$$B = \begin{matrix} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \end{matrix}$$

apply μ_3

etc...

2. CLUSTER MUTATION ($\vec{x} = (x_1, x_2, \dots, x_k, x_{k+1}, \dots, x_n)$)

$$\mu_k(x_i) = x_i \quad \text{if } k \neq i$$

$$\mu_k(x_k) = \frac{1}{x_k} \left\{ \prod_{\text{arrow } i \rightarrow k} x_i + \prod_{\text{arrow } k \rightarrow i} x_i \right\}$$

Example

Apply μ_2 on \vec{x}

$$\vec{x} = (x_1, x_2, x_3) \quad B = \textcircled{1} \rightarrow \textcircled{2} \rightarrow \textcircled{3}$$

Example

Apply μ_2 on B

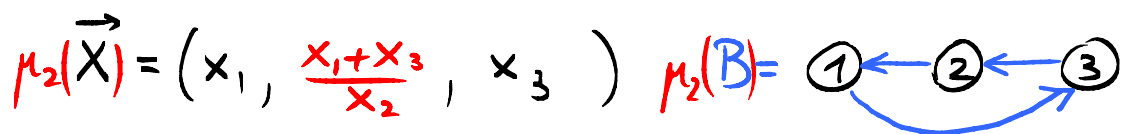
$$\mu_2(\vec{X}) = (x_1, \frac{x_1 + x_3}{x_2}, x_3)$$



Example

Apply μ_2 on B

$$\mu_2(\vec{X}) = (x_1, \frac{x_1 + x_3}{x_2}, x_3)$$



Example

Apply μ_3 on $\mu_2(\vec{x})$

$$\mu_2(\vec{x}) = \left(x_1, \frac{x_1 + x_3}{x_2}, x_3 \right) \quad \mu_2(B) = \textcircled{1} \leftarrow \textcircled{2} \leftarrow \textcircled{3}$$

Example

Apply μ_3 on $\mu_2(\vec{x})$

$$\mu_3 \mu_2(\vec{x}) = \left(x_1, \frac{x_1 + x_3}{x_2}, \frac{x_1 + \frac{x_1 + x_3}{x_2}}{x_3} \right) \quad \mu_2(B) = \textcircled{1} \leftarrow \textcircled{2} \leftarrow \textcircled{3}$$

etc ...

N.B. μ_k are involutions (check)

Noncommutative version = quantum Cluster Alg.

PROPERTIES

THM [Fomin-Zelevinsky]: \forall sequence $i_1, i_2 \dots i_k \in \{1, \dots, n\}$
the mutated cluster $\mu_{i_k} \circ \mu_{i_{k-1}} \dots \circ \mu_{i_1}(\vec{x})$ is a
Laurent polynomial of \vec{x} ($\text{pol}(x_1, x_1^{-1}, x_2, x_2^{-1} \dots x_n, x_n^{-1})$)

CONJ The polynomial has non-negative integer coefficients. (combinatorial?)

CLASSIFICATIONS

finite cluster algebras \Leftrightarrow a mutated quiver is an oriented
Dynkin diagram of a classical Lie algebra (ADE)
B-finite \Leftrightarrow "Triangulations" + 11 exceptional cases

APPLICATIONS

- Triangulations of Teichmüller space (Positive)



B codes the triangulation; \vec{x} = geodesic lengths

"flip"
[Fomin-Thurston
Shapiro]
[Musiker, Schiffl
Williams]
[DF, Kedem]

- Discrete Integrable systems / somas sequences
definition of ASM [Mills-Robbins-Rumsey]
- Conway-Coxeter Frieze patterns [Assem, Bergeron, Reutenauer.]
- Totally positive $GL(n)$ coordinate patches [Fomin, Zelevinsky]
- Canonical Bases of quantum group [Lusztig, Berenstein, Zelevinsky]
- Triangulated categories [Keller]
- DT invariants of top string theory [Kontsevich-Saielbman]
- New quantum dilogarithm identities [Keller]

3. THE A_1 Q-system as a rank 2 CLUSTER ALGEBRA

$$Q_{n+1} Q_{n-1} = Q_n^2 - 1 \Leftrightarrow R_{n+1} R_{n-1} = R_n^2 + 1$$

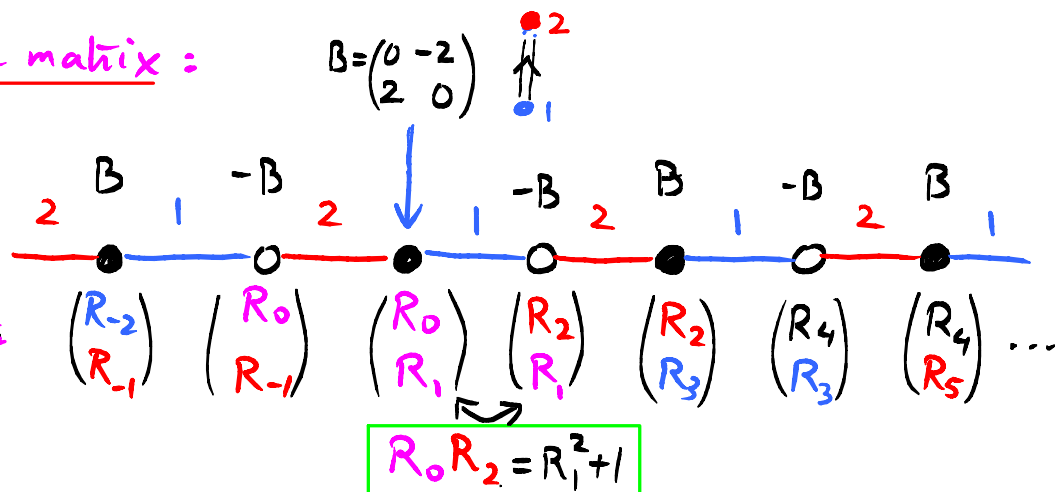
1. exchange matrix:

$$B = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$$

$\Pi_2 =$

2. clusters:

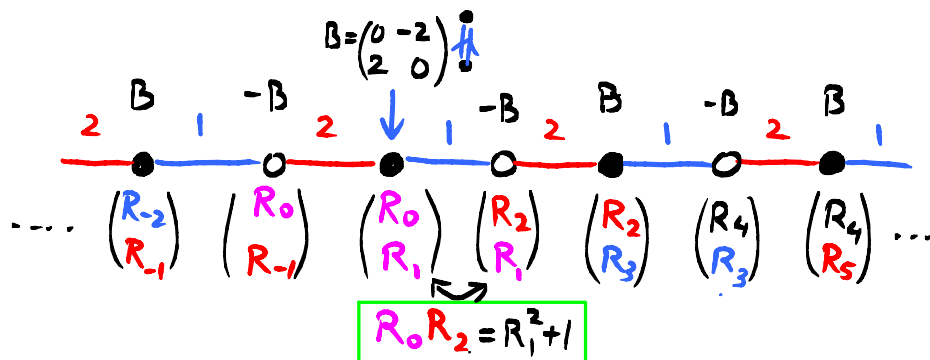
= initial data



3. Mutations:

3. THE A_1 Q-system as a rank 2 CLUSTER ALGEBRA

$$Q_{n+1} Q_{n-1} = Q_n^2 - 1 \Leftrightarrow R_{n+1} R_{n-1} = R_n^2 + 1$$



(Recall: R_n/R_0 = Part fctn of paths w/weights $\begin{cases} y_1 = R_1/R_0 \\ y_2 = 1/R_0 R_1 \\ y_3 = R_0/R_1 \end{cases}$)
 \Rightarrow POSITIVITY!

4. DISCRETE INTEGRABLE SYSTEMS

4.1. Q-system from XXX generalized q-spin chain = 1 per Lie algebra simply-laced case (A, D, E)

$$Q_{\alpha, n+1} Q_{\alpha, n-1} = Q_{\alpha, n}^2 - \prod_{\substack{\beta \sim \alpha \\ C_{\alpha\beta} = -1}} Q_{\beta, n} \quad (\alpha=1, 2, \dots, r) \\ n \in \mathbb{Z}$$

satisfied by characters \leadsto reps with Young diagrams:

$\alpha \left\{ \begin{array}{|c|} \hline \square \\ \hline \end{array} \right\}$ (Kirillov-Reshetikhin modules).

initial data $\{Q_{\alpha, 0}, Q_{\alpha, 1}\}$ solution?

A_r Q-system

$$\textcircled{A_r} \begin{cases} R_{\alpha, n+1} R_{\alpha, n-1} = R_{\alpha, n}^2 + R_{\alpha+1, n} R_{\alpha-1, n} \\ R_{0, n} = R_{r+1, n} = 1 \end{cases} \quad (\alpha=1, 2, \dots, r) \\ n \in \mathbb{Z}$$

• INTEGRABILITY: \exists r independent conserved qties

Lemma:

$$R_{\alpha, n} = \det_{1 \leq i, j \leq \alpha} (R_{1, n+i-j-\alpha})$$

discrete Wronskians $\alpha \times \alpha$.
(with $R_{1, n}$ on diagonal)

$$R_{\alpha, n} = \begin{vmatrix} R_{1, n-\alpha+1} & R_{1, n-\alpha+2} & \dots & R_{1, n} \\ \vdots & \vdots & \ddots & \vdots \\ R_{1, n-1} & \dots & \dots & R_{1, n+1} \\ R_{1, n} & R_{1, n+1} & \dots & R_{1, n+\alpha-1} \end{vmatrix} \quad \begin{matrix} \xrightarrow{+1} \\ \downarrow +1 \end{matrix}$$

Proof:

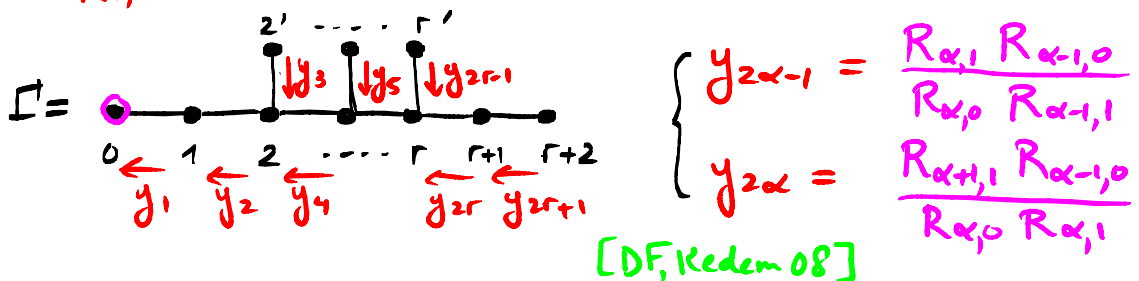
by Desnanot-Jacobi identity on minors of any matrix

$$\begin{array}{c}
 \boxed{\text{size } \alpha+1} \times \boxed{\text{size } \alpha-1} = \boxed{\text{size } \alpha} \times \boxed{\text{size } \alpha} - \boxed{\text{size } \alpha} \times \boxed{\text{size } \alpha} \\
 \text{size: } \begin{array}{ccccc} \alpha+1 & \alpha-1 & \alpha & \alpha & \alpha & \alpha \\ n & n & n-1 & n+1 & n & n \end{array}
 \end{array}$$

- $R_{r+1,n} = 1 = (r+1) \times (r+1)$ Wronskian is constant!
 \Rightarrow linear recursion relation with $r+2$ const. coeffs.
- $R_{r+2,n} = 0$ expand wrt first row e.g.
coeffts = minors = Wronskians with a defect.
- interpret as partition fctrs for hard particles on a graph

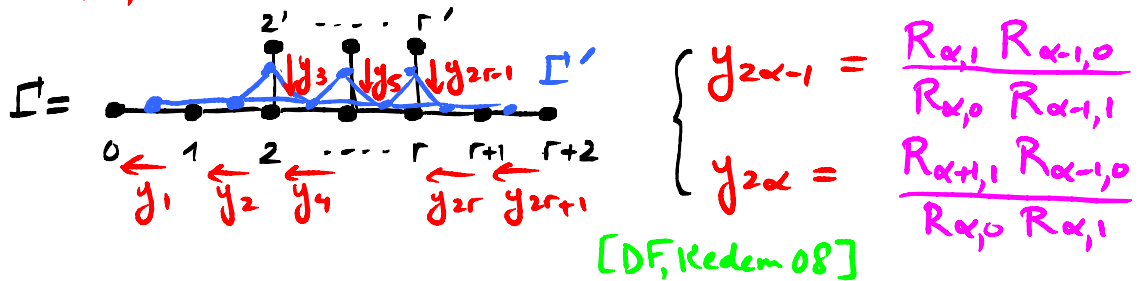
EXACT SOLUTION : PATHS

- $z_n := \frac{R_{1,n}}{R_{1,0}}$ = Partition fctn for paths of length $2n$ on the graph Γ :



EXACT SOLUTION : PATHS

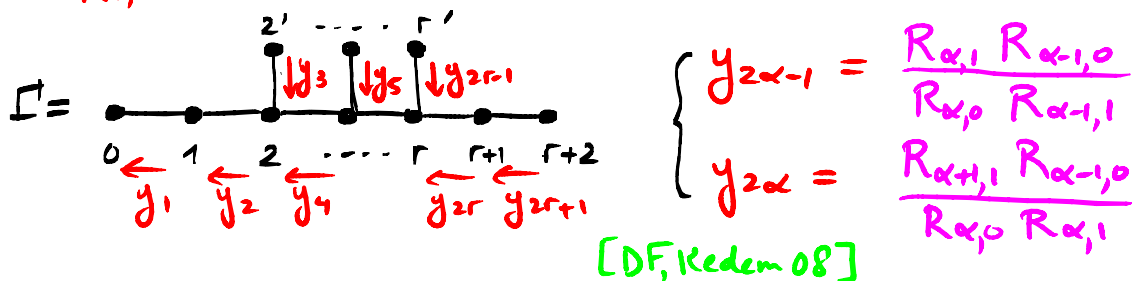
- $z_n := \frac{R_{1,n}}{R_{1,0}}$ = Partition fun for paths of length $2n$ on the graph Γ !



Proof: • conserved quantities = hard particle partition fcn $_{\text{on } \Gamma'}$

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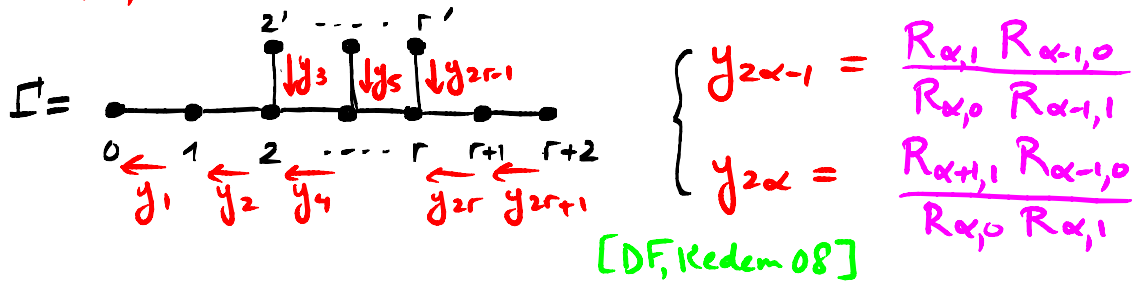


Proof:

- conserved quantities = hard particle partition fctn_{at}
- $\sum z_n t^n =$ heaps (boson-fermion correspondence)
 $\text{bije.} \updownarrow$
 paths on (medial) graph² = I^1

EXACT SOLUTION : PATHS

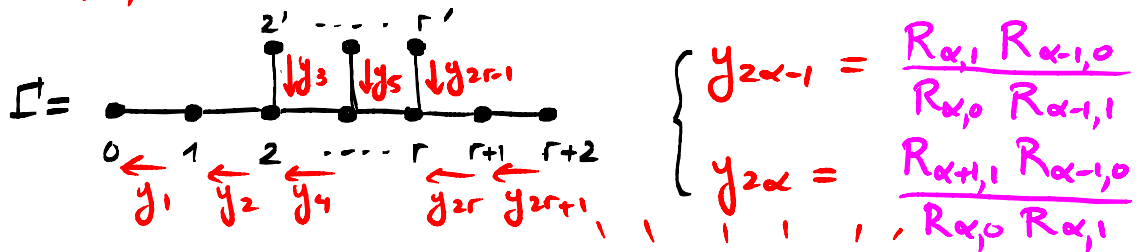
- $z_n := \frac{R_{1,n}}{R_{1,0}}$ = Partition fctn for paths of length $2n$ on the graph Γ :



- $\frac{R_{\alpha,n}}{(R_{1,0})^\alpha} = \begin{vmatrix} z_n & z_{n-1} & \dots & z_{n-\alpha+1} \\ & z_{n+1} & & \\ & & \ddots & \\ & & & z_{n+\alpha-1} & z_n \end{vmatrix}$

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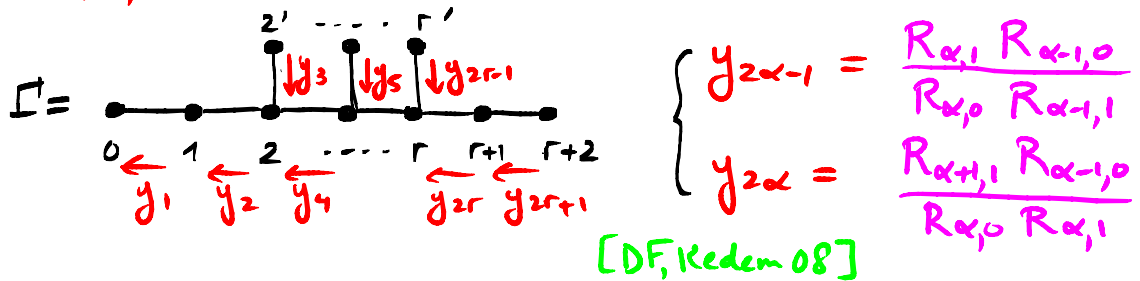


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\Rightarrow POSITIVITY in terms of $\{R_{\alpha,0}, R_{\alpha,1}\}$

Q-SYSTEM AS CLUSTER ALGEBRA

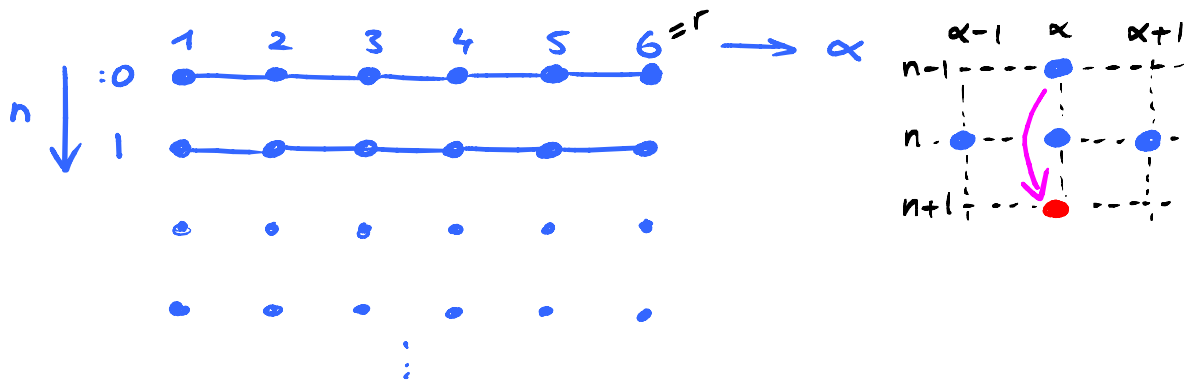
$$R_{\alpha,n+1} R_{\alpha,n-1} = R_{\alpha,n}^2 + R_{\alpha+1,n} R_{\alpha-1,n} ; R_{0,n} = R_{r,n} = 1$$

CLUSTERS = admissible initial data $\{R_{\alpha,0}, R_{\alpha,1}\}_{\alpha=1 \dots r}$
(vectors \vec{x})

MUTATION $\mu_\alpha =$ apply Q-system $R_{\alpha,n-1} \rightarrow R_{\alpha,n+1}$
(\Rightarrow matrix B)

$$R_{\alpha,n+1} R_{\alpha,n-1} = R_{\alpha,n}^2 + R_{\alpha+1,n} R_{\alpha-1,n}$$

ADMISSIBLE INITIAL DATA

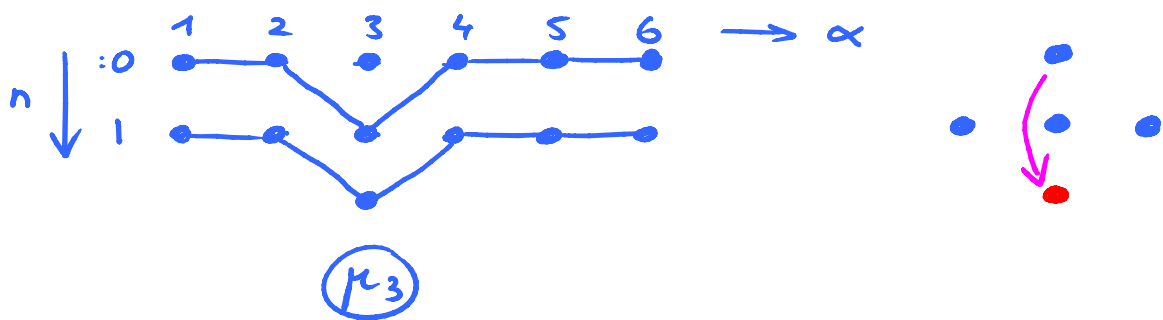


2r initial values

$$\{R_{\alpha,0}, R_{\alpha,1}\}_{\alpha=1,2,\dots,r}$$

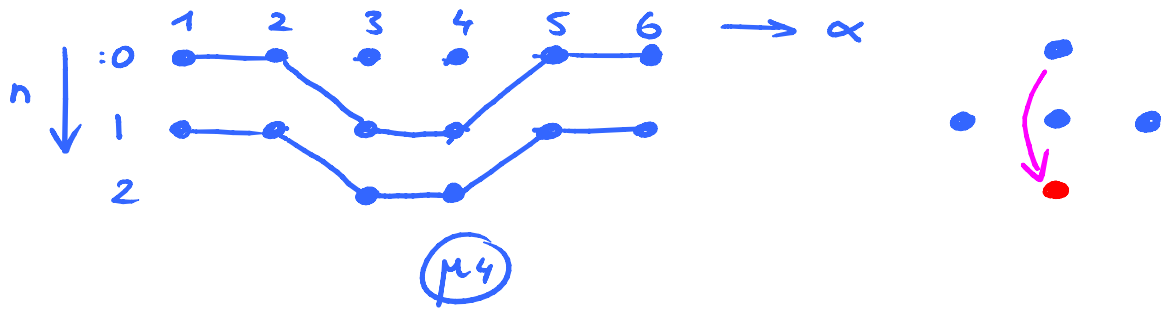
$$R_{\alpha,n+1} R_{\alpha,n-1} = R_{\alpha,n}^2 + R_{\alpha+1,n} R_{\alpha-1,n}$$

ADMISSIBLE INITIAL DATA



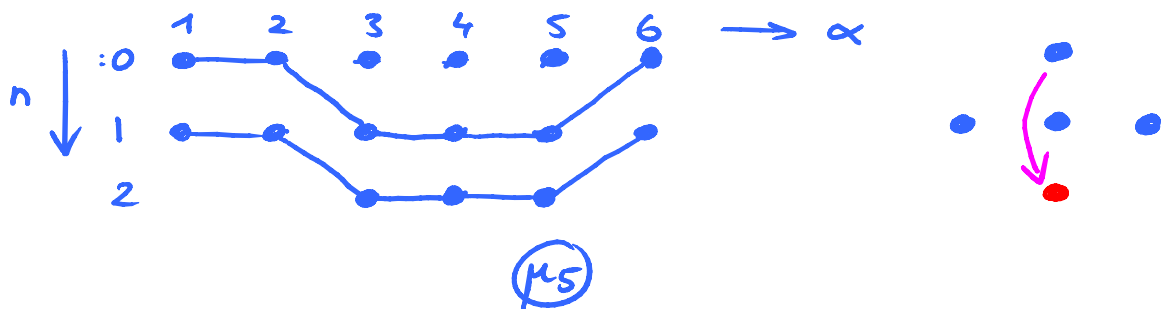
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ADMISSIBLE INITIAL DATA



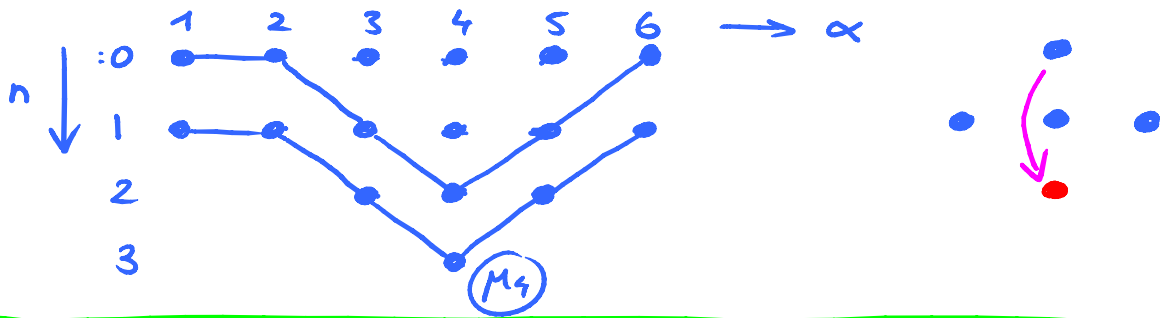
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ADMISSIBLE INITIAL DATA



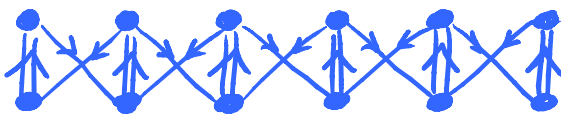
THM The admissible initial data are coded by Motzkin paths $\vec{m} = (m_\alpha)_{\alpha=1}^r$; $m_{i+1} - m_i \in \{0, 1, -1\}$

$$x_{\vec{m}} = (R_{\alpha, m_\alpha}, R_{\alpha, m_\alpha+1})_{\alpha=1}^r$$

= data vectors of the cluster algebra

$$R_{\alpha,n+1} R_{\alpha,n-1} = R_{\alpha,n}^2 + R_{\alpha+1,n} R_{\alpha-1,n}$$

Exchange Matrices

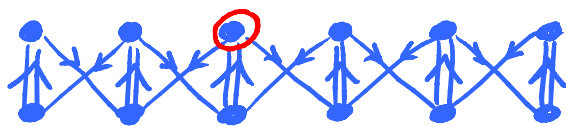


$2r$ vertices

$$B_0 = \begin{pmatrix} \bigcirc & \begin{matrix} -2 & 1 \\ 1 & -2 \\ & \ddots & \ddots \\ & & 1 & -2 \end{matrix} \\ \begin{matrix} 2 & -1 \\ -1 & 2 \\ & \ddots & \ddots \\ & & -1 & 2 \end{matrix} & \bigcirc \end{pmatrix}$$

↑
Cartan matrix of A_r

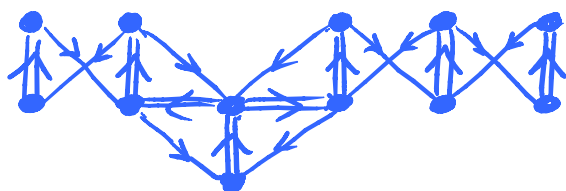
Exchange Matrices



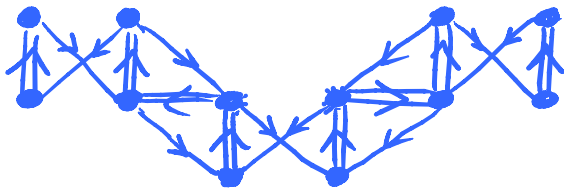
mutate
(μ_3)

$$B_0 = \begin{pmatrix} O & \begin{matrix} -2 & 1 \\ 1 & -2 \\ & \ddots & \ddots \\ & & 1 & -2 \end{matrix} \\ \begin{matrix} 2 & -1 \\ -1 & 2 \\ & \ddots & \ddots \\ & & -1 & 2 \end{matrix} & O \end{pmatrix}$$

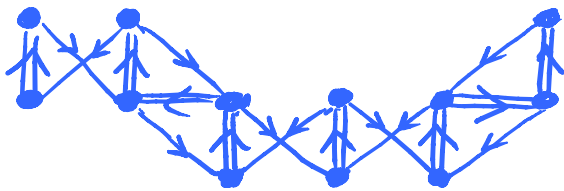
Exchange Matrices



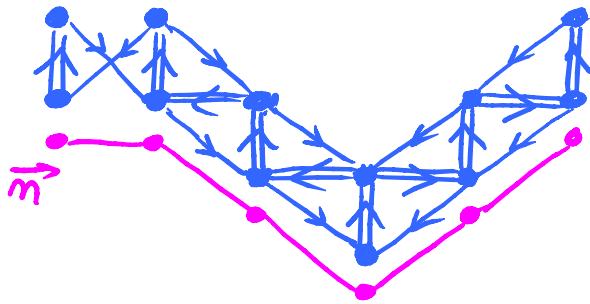
Exchange Matrices



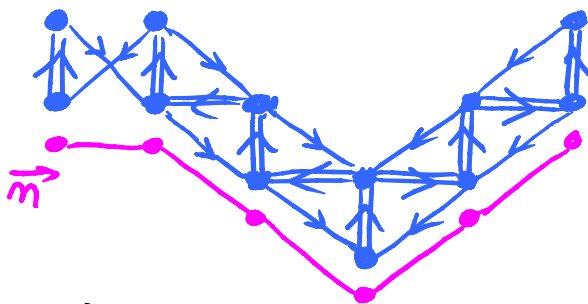
Exchange Matrices



Exchange Matrices



$B = B_{\vec{m}}$ coded by
the Motzkin path



← encodes both initial
data and B matrix for
the Q -system:

$$R_{\alpha, n+1} R_{\alpha, n-1} = R_{\alpha, n}^2 + R_{\alpha+1, n} R_{\alpha-1, n}$$

Solution:

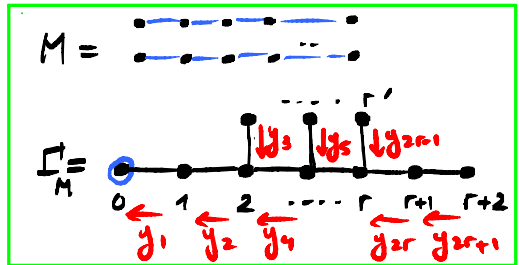
THM (DF Keden) arbitrary initial data $\vec{x}_m = \{R_{\alpha, m}, R_{\alpha, m+1}\}_{\alpha=1,2,\dots,r}$

$\frac{R_{\alpha, n}}{(R_{1,0})^\alpha} = \left\{ \begin{array}{l} \text{Partition fcn for SNIP paths on a mutated graph,} \\ \text{with mutated positive monomial weights of the} \\ \text{mutated initial data, all coded by Motzkin path } \vec{m} \end{array} \right\}$

\Rightarrow POSITIVITY in terms of any init data.

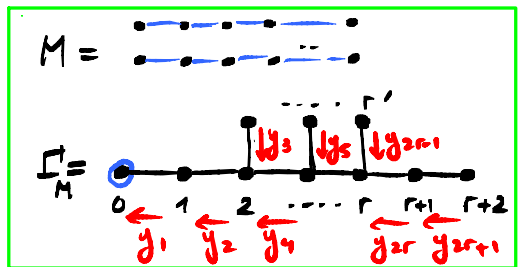
MUTATIONS AS LOCAL REARRANGEMENTS OF CONTINUED FRACTIONS

$$F_0 = \sum_0^{\infty} Q_{1,n} t^n = \frac{Q_{1,0}}{1-t y_1} = C.F.$$



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REARRANGING 1. $1 + \frac{a}{1-a-b} = \frac{1}{1-\frac{a}{1-b}}$

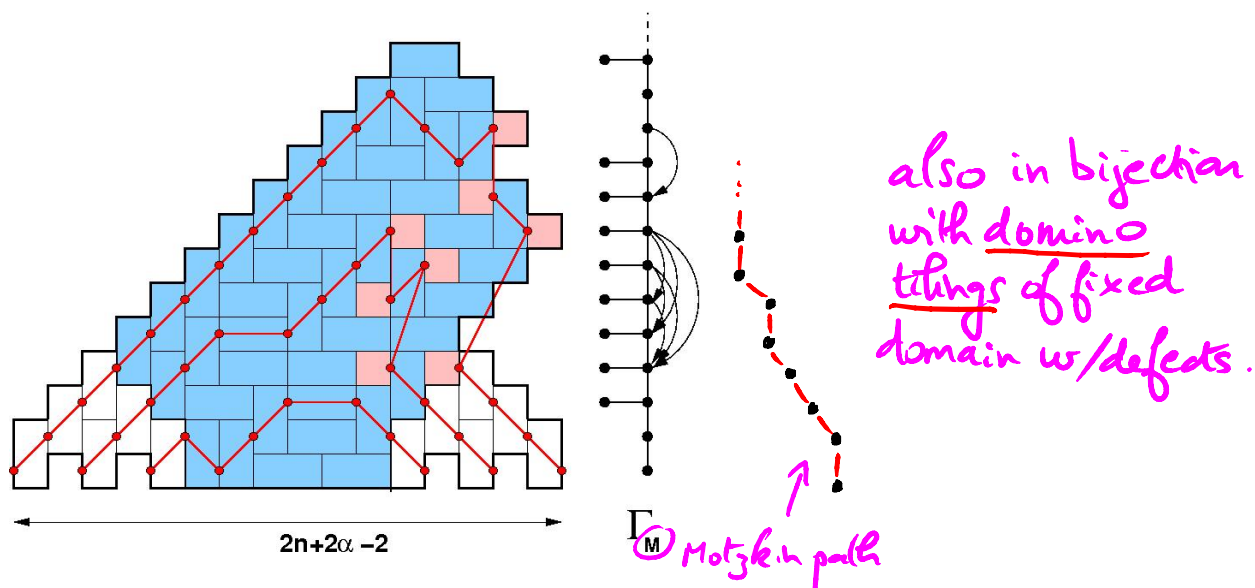
LEMMA 2. $a + \frac{b}{1-c} = \frac{a'}{1-\frac{b'}{1-c'}} \Leftrightarrow \begin{cases} a' = a+b \\ b' = \frac{bc}{a+b} \\ c' = \frac{qc}{a+b} \end{cases}$

MUTATIONS

change
the
structure
of C.F. and
its weights

MUTATIONS are implemented by these transformations, that preserve positivity.

Interpretation: $R_{\alpha,n}$ = partition fctn for α strongly non-intersecting "lattice" paths on a weighted target graph entirely determined by the initial data (Motzkin Path)



EXTENSIONS/CONCLUSION

- NON-COMMUTATIVE CA (rank 2 case)
conjecture by Kontsevich (NC DT invariant theory)
NC A_1 -Q-system

$$R_{n+1} R_n^{-1} R_{n-1} = R_n + R_n^{-1}$$

$R_n \in \mathcal{OT}$
NC algebra

Conj: R_n is a >0 Laurent pol of (R_0, R_1)

Proved [DF Kedem] via path solution:

same paths with NC weights $y_1 = R_1 R_0^{-1}, y_2 = R_1^{-1} R_0^{-1}, y_3 = R_1^{-1} R_0$
(2 conserved qties: $R_{n+1}^{-1} R_n R_{n+1} R_n^{-1} = y_3 y_1; (R_{n+1} + y_3 y_1 R_{n-1}) R_n^{-1} = y_1 + y_3 y_1$)

MORE GENERALLY: NC Discrete Hirota eqn

by reverse engineering our path solutions
(same paths but with NC weights) \Rightarrow
NC Q-systems in the form of a

Non-commutative discrete Hirota eqn [DF+Kedem]

uses quasi-determinants [Gelfand-Retakh]

and quasi-plücker coordinates.

+ NC rearrangement lemmas for NC continued fractions

QUANTUM CASE q -commutations [Berenstein -
 q -Laurent positivity, q -commuting weighted paths. Zelevinski]

OTHER DISCRETE INTEGRABLE SYSTEMS

T-system / Network solutions \Rightarrow POSITIVITY

REARRANGEMENT & YANG-BAXTER

$$\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b' & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a' \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} b' = a + c \\ a' = \frac{ab}{a + c} \\ c' = \frac{bc}{a + c} \end{cases} \quad \text{Same!}$$

REFERENCES

- [DF-Kedem] Q-systems as Cluster Algebras II LMP'09
- [DF] Solution of the A_r T-system for arbitrary boundary EJC'10
- [DF - Kedem] Discrete NC integrability: proof of a conjecture by M. Kontsevich IMRN'10
- [DF - Kedem] NC integrability, paths and quasideterminants Adv Math '11
- [DF - Kedem] Solution of the $q A_1$ T-system for arbitrary boundary CMP'12
- [DF - Kedem] Quantum cluster algebras and fusion products Preprint