

Diagonal walks on plane graphs,  
Gauss words and local duality

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## *General objectives*

Use of **monadic second-order logic (MS logic)** for expressing graph properties and graph constructions.

**Graph decompositions** : tree decomposition, modular decomposition, clique-width and rank decompositions.

### *Motivations and applications :*

1. Context-free Graph Grammars
2. Algorithmic applications (Fixed Parameter Tractability)
3. Decidability of logical theories
4. Graph Theory, structural descriptions

## Graph theoretical constructions

Graph  $\rightarrow$  its modular decomposition

Circle graph  $\rightarrow$  its "split" decomposition from which all its representations by sets of intersecting chords can be defined.

Today's lecture : Self-intersecting curves in the plane represented by their Gauss words  $\rightarrow$  some canonical structure from which all curves with same Gauss word can be defined.

## Relational structures :

Combinatorial objects described by (finite) relational structures over **finite signatures**, their properties and related constructions expressed in MS logic.

# Monadic Second-Order Logic

MSOL = First-order logic over subsets of the domain

Graph  $G = \langle V, \text{edg}(\cdot, \cdot) \rangle = \langle \text{Vertices}, \text{adjacency relation} \rangle$

Logical expression of graph properties :

*Non connectivity* :  $\exists X ( \exists u. u \in X \wedge \exists v. v \notin X$   
 $\wedge [ \forall u, v. \text{edg}(u, v) \Rightarrow ( u \in X \Leftrightarrow v \in X ) ] )$

*Typical Properties* : *k-colorability ( fixed k )*  
*to be a tree*  
*planarity ( by Kuratowski's characterization )*

*but cannot express : equal cardinality of two sets ; bijections.*

# Why Monadic Second-Order Logic is interesting ?

## Context-free Graph Grammars :

- MS logic → Recognizable sets of graphs
- Transductions (no automata)

## Algorithmic applications :

- MS logic → Fixed Parameter Tractable problems  
for tree-width and clique-width

- Query evaluation techniques  
implementation of graphs ;  
linear delay evaluation

## Decidability of logical theories :

Decidability of  $MS_2$  logic for a set of graphs implies bounded *tree-width*,

Decidability of  $C_2MS$  logic for a set of graphs implies bounded *clique-width*,

Decidability of MS logic for a set of matroids implies bounded *branch-width*.

## Combinatorial objects as relational structures

Graphs :  $G = \langle V, \text{edg}(\cdot, \cdot) \rangle = \langle \text{Vertices}, \text{adjacency relation} \rangle$

Plane graphs also called combinatorial maps

$M = \langle D, \alpha, \sigma \rangle$

$D$  = darts = half-edges

$\alpha$  associates the 2 darts of a same edge

$\sigma$  defines the next dart around a same vertex in the considered drawing, according to a fixed orientation of the surface.

Self-intersecting curves = 4-regular planar maps

Double occurrence words :  $W = \langle P, \text{next}(\cdot, \cdot), \text{same-letter}(\cdot, \cdot) \rangle$

$P$  = occurrences, next = next occurrence,

same-letter(x,y)  $\Leftrightarrow$  x and y are occurrences of same letter.

## Monadic 2<sup>nd</sup> order transductions (or "interpretations")

Transformation  $\tau$  of logical structures such that :

$$S \longmapsto T = \tau(S)$$

where  $T$  is defined by **monadic 2<sup>nd</sup> order formulas** inside the structure:  $S \oplus S \oplus \dots \oplus S$

(fixed number of disjoint "marked" copies of  $S$ )

in terms of "parameters" i.e. subsets  $X_1, \dots, X_p$  of the domain of  $S$

*Proposition* : The composition of two monadic 2<sup>nd</sup> order transductions is a **monadic 2<sup>nd</sup> order** transduction.

The inverse image of a monadic 2<sup>nd</sup> order definable set under a monadic 2<sup>nd</sup> order transduction is monadic 2<sup>nd</sup> order definable

## Examples of Monadic 2<sup>nd</sup> order transductions

The mappings from a graph to :

the forest of its biconnected components

the directed acyclic graph of its strongly connected components

all its spanning trees

all its minors

(by using the incidence structure  $\langle V \cup E, \text{inc}(\cdot, \cdot) \rangle$ )

Planar graph drawing = combinatorial map  $\xrightarrow{\hspace{1cm}}$  graph

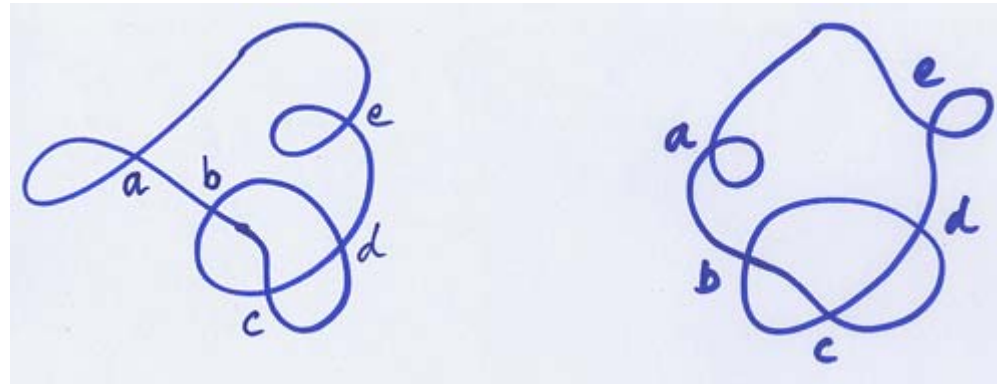
Intersecting chords = double occurrence word  $\xrightarrow{\hspace{1cm}}$  circle graph

Curve on the plane = 4-regular map  $\xrightarrow{\hspace{1cm}}$  Gauss word

MS transductions : easy  $\xrightarrow{\hspace{1cm}}$  harder  $\xleftarrow{\hspace{1cm}}$



Gauss word :  $abcdbcdeea$  representing the curves



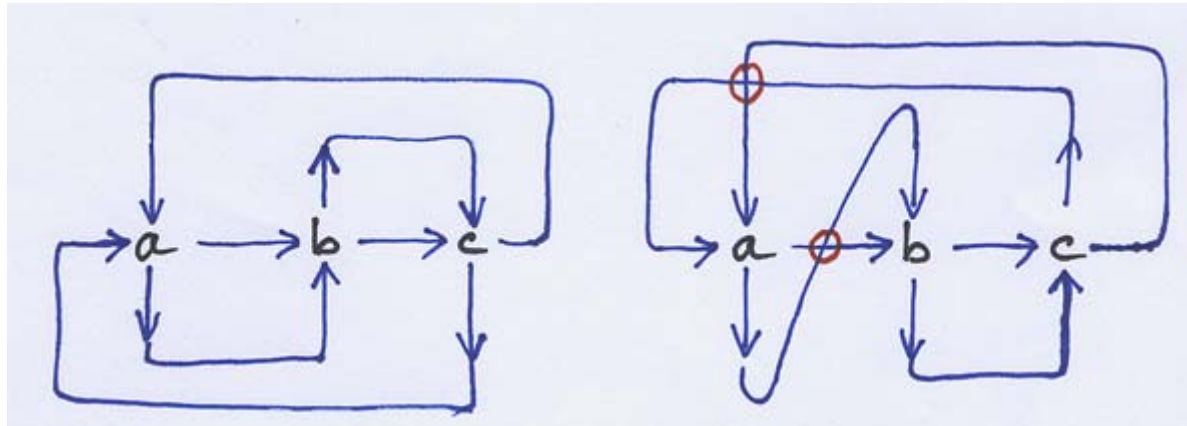
Relational structure for the double occurrence word



Proposition : Gauss words can be characterized by a formula of MS logic.

Proofs : Direct expression of the characterization by Rosensthiel.

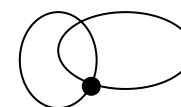
Another proof



Orient a self-intersecting curve. Each occurrence of a letter in the Gauss word gets a sign + or - depending of the direction of the crossing. For the left curve, we get  $a^+b^-c^+a^-b^+c^-$ , a *signed Gauss word*.

From a signed double occurrence word, one gets a unique map (planar or not). For example  $a^+b^+c^-a^-b^-c^+$  gives nonplanar map to the right.

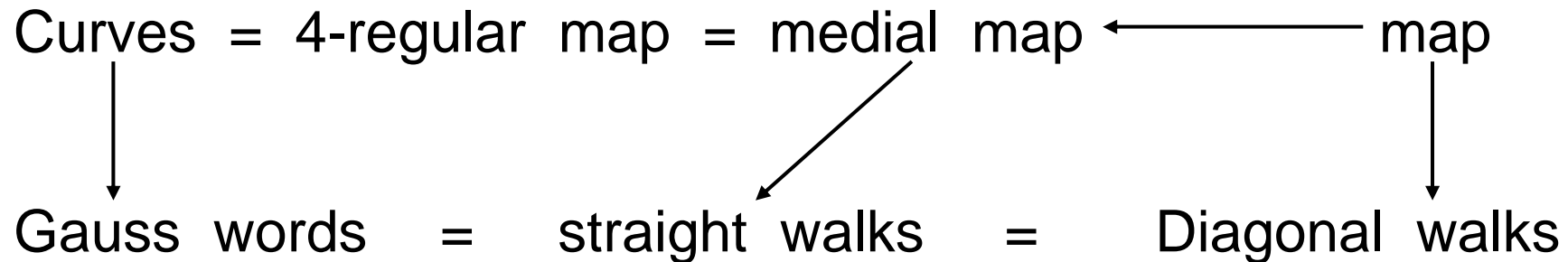
Given a double occurrence word, an MS formula can be written expressing :  
*There is a choice of signs such that the associated map is planar.*



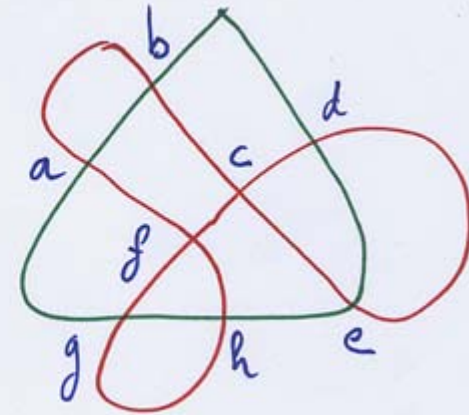
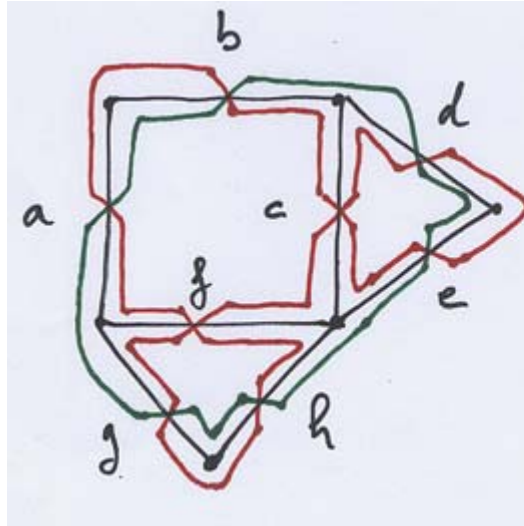
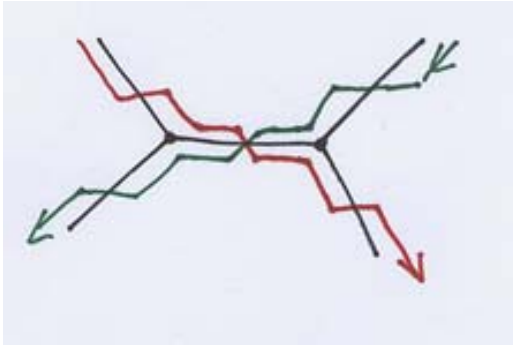
Questions : 1) Can one describe all curves with given Gauss word ?

2) Can one construct them by an MS transduction ?

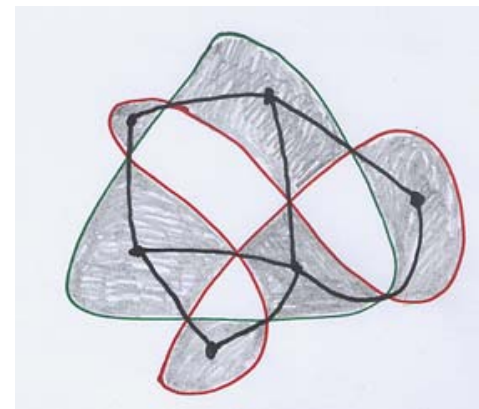
The main picture :



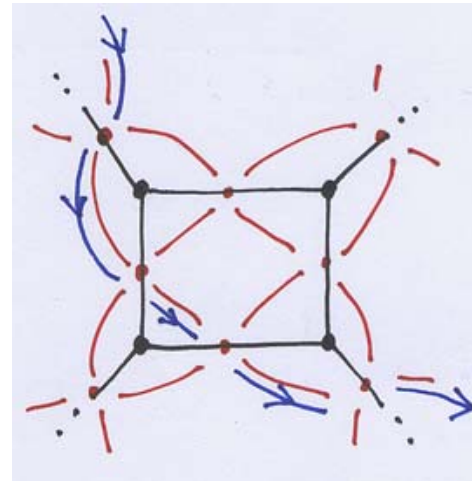
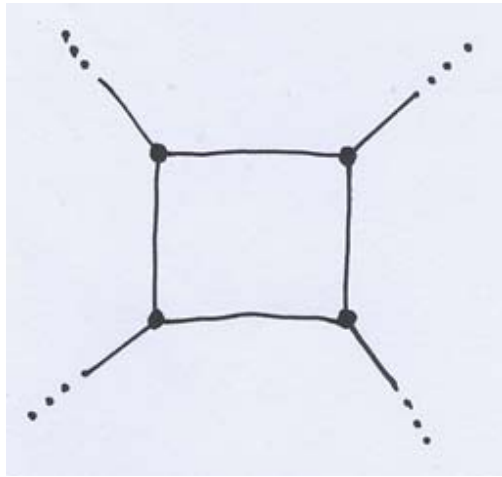
# Diagonal walks



Diagonal walks (they form cycles)  $\rightarrow$  intersecting curves  
and **conversely**

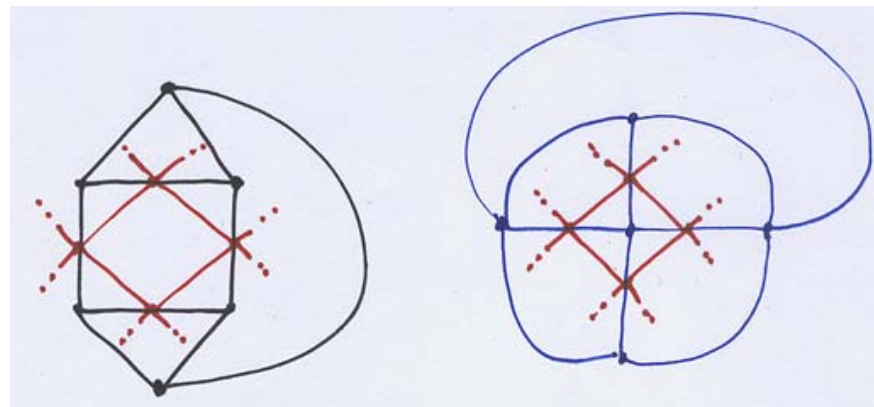


# Medial map

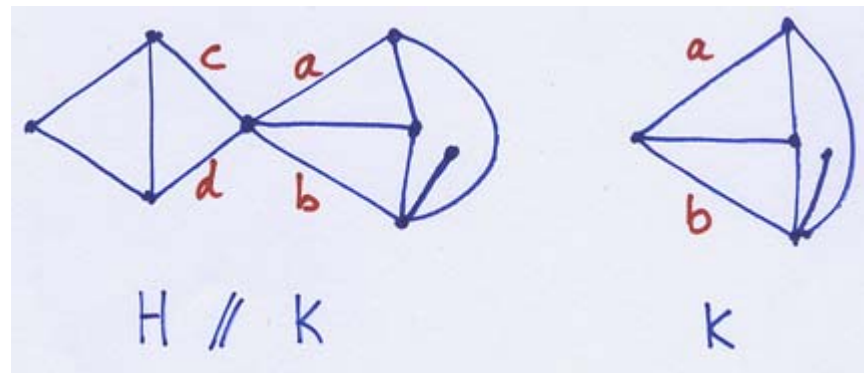


Diagonal walks of a map = **straight walks** of the associated medial map

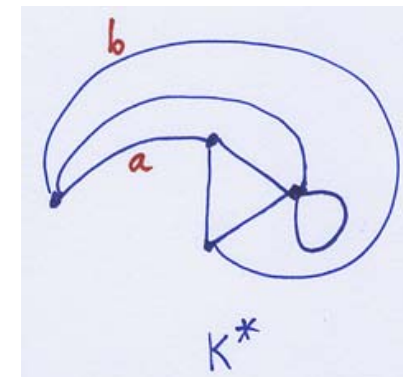
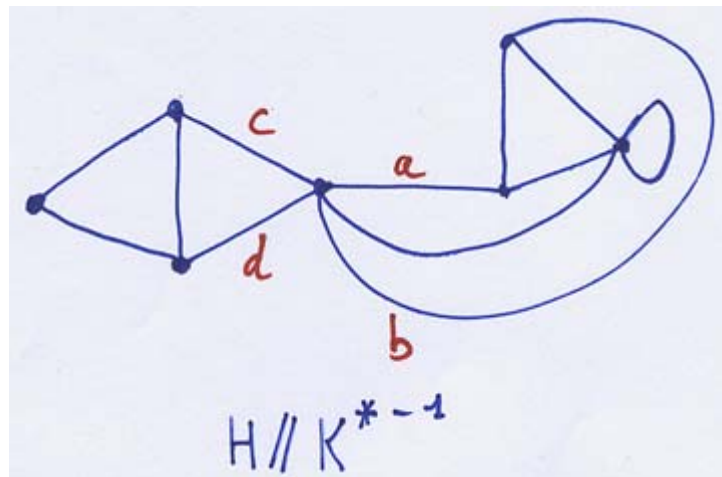
Medial map is preserved under duality :



# Local duality



$H//K^{*-1}$  is a **local dual** of  $H//K$  (\* = dual,  $^{-1}$  = symmetry)



**Proposition** : Diagonal walks are invariant on local duality

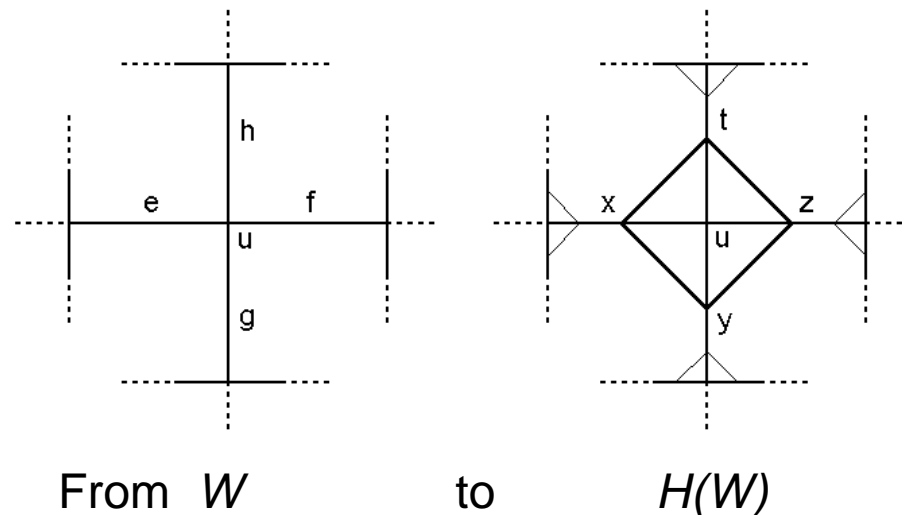
$$Dw(H//K^{*-1}) = \{...cd..., ...ca..., ...db..., ..\}, Dw(H//K^*) = \{...cd..., ...cb..., ...da..., ...\}$$

**Theorem** : Two maps  $M$  and  $N$  with edges in bijection have the same sets of diagonal walks (w.r.t. this bijection) iff one is obtained from the other by finitely many uses of duality, symmetry and local duality, denoted  $M \cong N$ .

**Corollary** : All curves with same Gauss multiword as a given one  $C$  are obtained from the medial maps of the maps  $N \cong M$ , where  $M$  is any of the two dual maps associated with  $C$ .

**Corollary (known)** : A Gauss multiword defines a unique set of closed curves up to homeomorphism iff the corresponding 4-regular graph is 2-connected.

Proof technique : The unicity result : From a double occurrence multiword  $M$ , one gets a 4-regular graph + pairings of opposite edges at each vertex : this is *a weak map*  $W$ , less informative than a map. The pairings of opposite edges can be encoded by additional edges, giving a graph  $H(W)$ . The planar embeddings of  $H(W)$  are the planar embeddings of  $W$ . If the graph  $G$  of  $M$  is 2-connected,  $H(W)$  is 3-connected, hence has at most one planar embedding. So has  $G$ , and  $M$  defines curves in unique way.



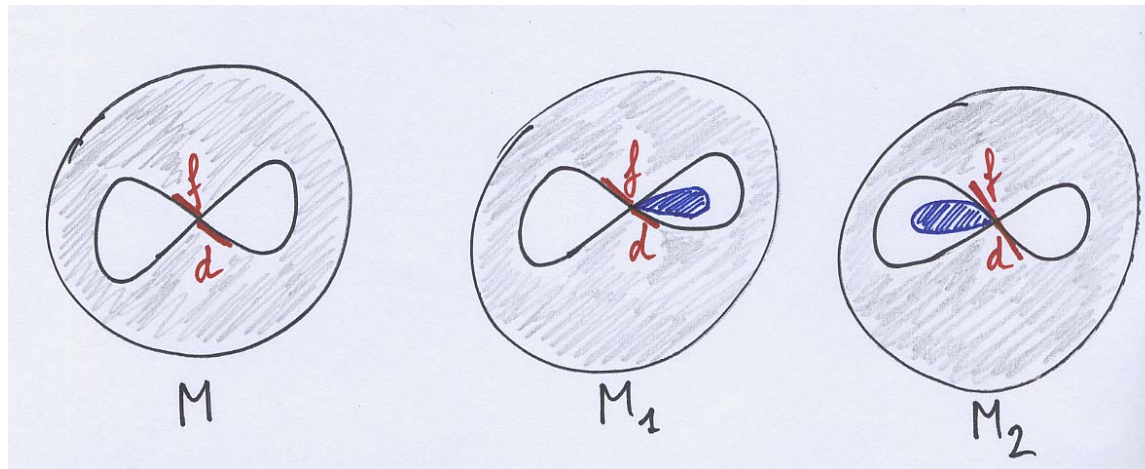


## Proof of main theorem :

If  $M$  and  $N$  are planar maps with  $Dw(M)=Dw(N)$   
then  $M$  or  $M^{-1}$  is locally dual to  $N$ .

By induction on the number of 2-connected components.

We use *gluings of pointed maps*:



In blue : pointed map  $(N,e)$ .  $M_1 = (M,d) // (N,e)$ ,  $M_2 = (M,d) // (N,e)$ .

**Lemma 1** : If  $M = P // Q$  and  $Dw(M) = Dw(N)$ ,

then  $N = R // S$  with  $Dw(R) = Dw(P)$ ,  $Dw(S) = Dw(Q)$ .

Hence by induction :  $R$  or  $R^{-1}$  is locally dual ( $\sim$ ) to  $P$ ,

$S$  or  $S^{-1}$  is locally dual to  $Q$ .

If  $R \sim P$ ,  $S \sim Q$  then  $N = R // S \sim P // Q = M$ .

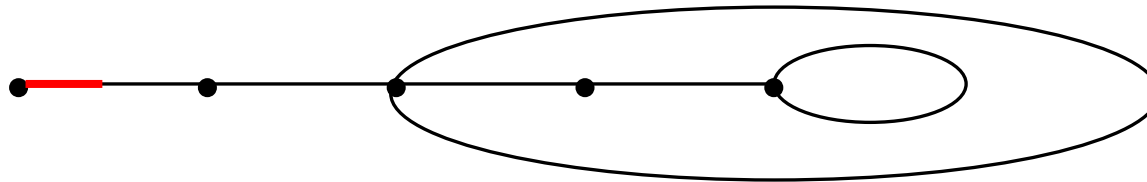
If  $R^{-1} \sim P$ ,  $S^{-1} \sim Q$  then  $N^{-1} = (R // S)^{-1} = R^{-1} // S^{-1} \sim P // Q = M$ .

Last case : If  $R \sim P$ ,  $S^{-1} \sim Q$  then  $Dw(R // S) = Dw(R // S^{-1})$ .

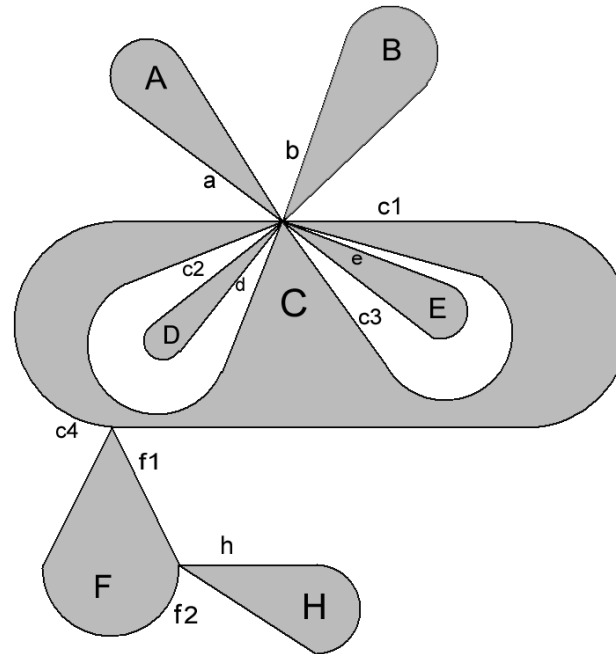
*Lemma 2* : This implies that either  $R$  or  $S$  is *symmetric*.

This gives  $N$  or  $N^{-1}$  locally equivalent to  $M$ , as above.

Symmetric pointed map  $(R,d)$  (dart  $d$  is thick *red*) :  $R$  isom.  $R^{-1}$



# Formalization in MS logic



Insertions of 2-connected maps A,B,C,... at positions defined by darts :

$$M=C[c_1 \uparrow ((B,b) // (A,a)), c_2 \uparrow (D,d), c_3 \uparrow (E,e), c_4 \uparrow (F[f_2 \uparrow (H,h)], f_1)]$$

**Local duality of M:** replacements of components X by  $X^*^{-1}$ . If C is 2-connected:

$$C[c_1 \uparrow M, c_2 \uparrow N, c_3 \uparrow P]^*^{-1} = C^*^{-1}[c_1 \uparrow M^*^{-1}, c_2 \uparrow N^*^{-1}, c_3 \uparrow P^*^{-1}]$$

1) For a map  $M = \langle D, \alpha, \sigma \rangle$ , one can express by MS formula  $\varphi(X)$  :

$X \subseteq D$  is the set of darts of a 2-connected component  $C$  of  $M$

2) The transformation of  $C$  into  $C^*^{-1}$  "inside  $M$ ", giving locally dual  $M' = \langle D, \alpha, \sigma' \rangle$ , can be expressed in logic in terms of  $X$  since :

$$\sigma'(d) = \alpha(\sigma(d)) \quad \text{if } \sigma(d) \in X$$

$$\sigma'(d) = \sigma(d) \quad \text{if } \sigma(d) \notin X$$

Hence, the maps locally dual to  $M$  can be obtained by this transformation applied to sets  $X$  that are the unions of the sets of darts of 2-connected components of  $M$ .

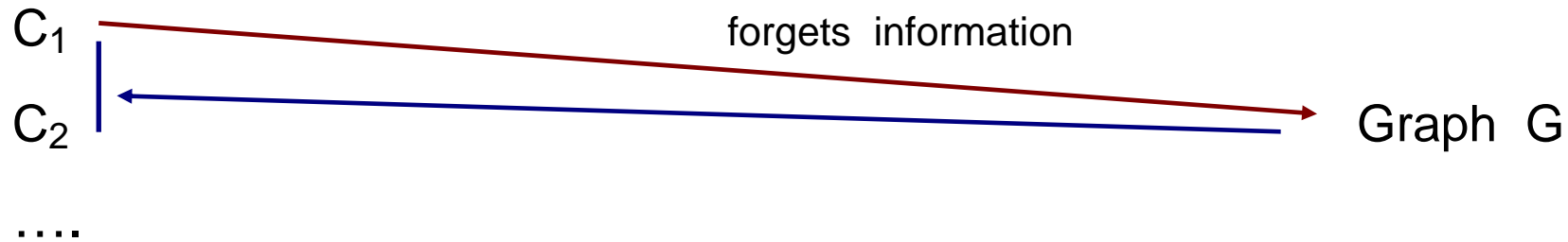
Some open questions :

- 1) Can one describe in a similar way curves and sets of curves with multiple intersections ? Can one describe knot diagrams?
- 2) Gauss multiwords for curves on orientable surfaces can be characterized in MS logic, because the corresponding maps are characterized by finite sets of excluded maps for minor ordering of maps. Can we have a structural description ? When does a double occurrence word characterize a unique curve on a surface ?
- 3) What are the "forbidden" double occ. words for surfaces ?

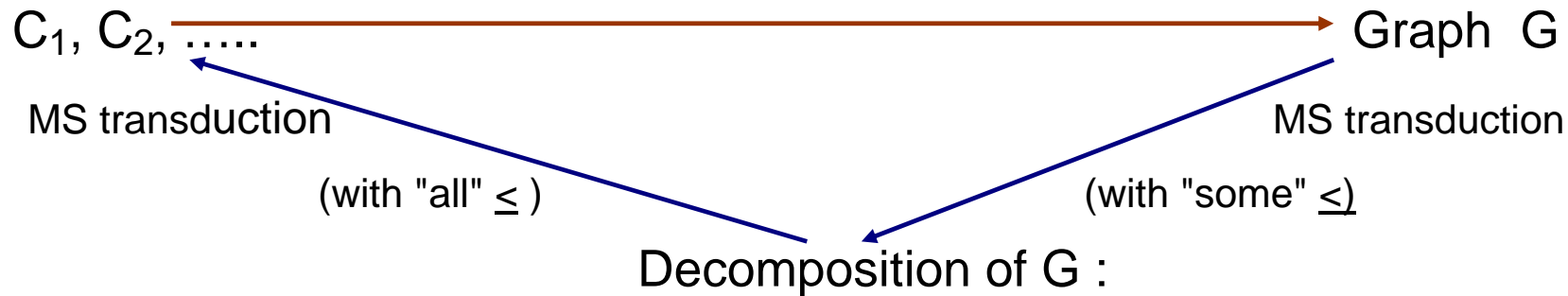
Overview of this method which applies to other cases

Geometrical configurations

(coded by relational structures)



How can one reconstruct  $C_1, C_2, \dots$  from  $G$  ?



tree of basic blocks  $B$  with unique  $C_B$

$B \rightarrow C_B$  : Monadic  $2^{\text{nd}}$  order transduction using the unicity