Composition of Plane Trees

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Abstract

The connections recently established between combinatorial bicolored plane trees and Shabat polynomials show that the world of plane trees is incredibly rich with different mathematical structures; see, for example, [4], [5]. In this article we use Shabat polynomials to introduce a new operation, that of a *composition*, for combinatorial bicolored plane trees. The composition may be considered as a generalized symmetry.

1 Introduction. Plane trees and Shabat polynomials

A *tree* is a connected graph without circuits. In order to draw a tree on the plane, we must choose at each vertex a certain cyclic order of branches which corresponds to the "trigonometric" rotation around this vertex. Thus the same tree as an abstract graph may produce several different plane trees.

Definition 1.1 A *plane tree* is a tree with a prescribed cyclic order of edges adjacent to each vertex.

Each tree has natural bipartite structure, i.e., it is bicolorable. Coloring the vertices of a tree in two colors (say, black and white) in one of two possible ways, we obtain a *bicolored tree.*

To any bicolored plane tree we put into correspondence its *set of valencies* $<\alpha,\beta>=<\alpha_1,\alpha_2,\ldots,\alpha_p;\beta_1,\beta_2,\ldots,\beta_q>$. Here α_i (resp. β_j) are the valencies of black (resp. white) vertices. They must satisfy the obvious conditions

$$\sum_{i=1}^{p}\alpha_i=\sum_{j=1}^{q}\beta_j=m,\quad p+q=m+1,$$

where m is the number of edges of the tree.

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Lemma 1.2 *For any $< \alpha, \beta >$ satisfying the above conditions there exists at least one bicolored plane tree for which $< \alpha, \beta >$ is the set of valencies.*

There is a very interesting class of complex polynomials related to bicolored plane trees.

Definition 1.3 A polynomial $P : \mathbb{C} \to \mathbb{C}$ is called a *Shabat polynomial* if it has at most two critical values, i.e., there exist $c_0, c_1 \in \mathbb{C}$ such that

$$P'(z) = 0 \quad \Rightarrow \quad P(z) \in \{c_0, c_1\}.$$

Definition 1.4 Let $P(z)$ and $Q(z)$ be two Shabat polynomials, and let c_0, c_1 and d_0, d_1 be their critical values. We call P and Q *equivalent*, if there exist constants $A, B, a, b \in \mathbb{C}$, $A, a \neq 0$, such that

$$Q(z) = AP(az + b) + B, \quad \text{and} \quad d_0 = Ac_0 + B, \quad d_1 = Ac_1 + B.$$

Theorem 1.5 *There is a bijection between the set of* (combinatorial bicolored plane) *trees and the set of equivalence classes of Shabat polynomials.*

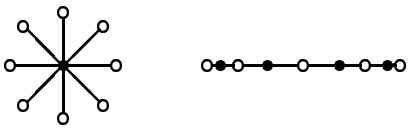
In one direction, having a Shabat polynomial P, we obtain the corresponding tree as $P^{-1}([c_0, c_1])$, where $[c_0, c_1]$ is the segment joining c_0 and c_1 on the complex plane (here $P^{-1}(c_0)$ is the set of black vertices, and $P^{-1}(c_1)$, that of white ones). In the other direction, having a bicolored plane tree, we write the following decompositions:

$$P(z) - c_0 = \lambda \prod_{i=1}^{p} (z - a_i)^{\alpha_i}, \quad P(z) - c_1 = \lambda \prod_{j=1}^{q} (z - b_j)^{\beta_j},$$

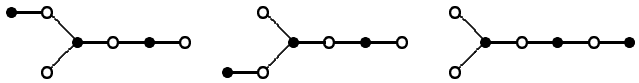
which gives us the system of m algebraic equations with $m + 4$ unknowns. Four additional "degrees of freedom" correspond to the liberty of choosing the numbers A, B, a, b of definition 1.4. Usually we fix the values of c_0 and c_1, and fix in some convenient way the position of the tree itself, by fixing, for example, the positions of any two of its vertices. See details in [5]. The tree corresponding to a polynomial P is denoted A_P; obviously, the number of edges $m = \deg P$. The degree of the system of algebraic equations for the coefficients of P is bounded by the number of bicolored plane trees with the given set of valencies.

Definition 1.6 The *field of definition* of a bicolored plane tree is the smallest number field to which belong the coefficients of the corresponding Shabat polynomial.

Examples 1.7 The polynomial $P(z) = z^m$ has only one critical value $c_0 = 0$. Taking $c_1 = 1$, and taking the pre-image of the segment [0,1], we obtain the "star-tree" (see the figure). The Chebyshev polynomial $T_m(z) = \cos(m \arccos z)$ has two critical values $c_0 = -1$ and $c_1 = 1$. The pre-image of the segment [-1,1] is the "chain-tree" (see the figure). The field of definition in both cases is $\mathbb{Q}$.



For the trees shown in the next figure (note that they all have the same set of valencies $< 3, 2, 1; 2, 2, 1, 1 >$)



the corresponding Shabat polynomial is $P(z) = z^3(z-1)^2(z-a)$, where a is a root of the cubic equation $25a^3 - 12a^2 - 24a - 16 = 0$ (each root corresponds to a specific tree). The field of definition is the field generated by these roots (here it is $\mathbb{Q}(\sqrt[3]{2})$); the critical values are $c_0 = 0$ and $c_1 = -\frac{1}{1500}(a^2 + 52a - 32)$.

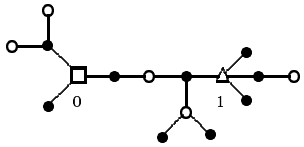
Remark 1.8 To compute a Shabat polynomial for a given tree is not an easy task. For a "random" tree with 24 edges one must solve a system of algebraic equations of a degree greater than million. But, as the reader will see, sometimes one may easily compute Shabat polynomials for some very complicated (though very specific) trees.

2 Composition

For the rest of the paper we fix $c_0 = 0$ and $c_1 = 1$.

Lemma 2.1 *Let $P(u)$ and $Q(z)$ be two Shabat polynomials, and let $P(u)$ satisfy the additional condition $P(0), P(1) \in \{0, 1\}$. Then their composition, i.e., the polynomial $R(z) = P(Q(z))$, is also a Shabat polynomial.*

The proof is a trivial exercise for an application of the chain rule; but the lemma itself gives us a possibility to introduce the operation of composition for the plane trees. The geometrical meaning of the condition $P(0), P(1) \in \{0, 1\}$ is that certain two vertices of the tree A_P are distinguished and positioned at the points $u = 0$ and $u = 1$. It is convenient to mark these points by a square and a triangle, like in the following figure.

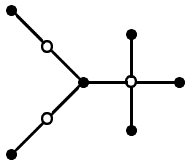


In what follows these vertices will be denoted by **s** and **t** respectively. Then the vertices of A_Q, being pre-images of **s** and **t**, must be marked by squares and triangles instead of black and white. Note that $\deg R = mn$, where $m = \deg P$, $n = \deg Q$.

Examples 2.2 Taking $P(u)$ arbitrary and $Q(z) = z^n$, we obtain the symmetric tree composed of A_P rotated n times around its square vertex. It is well known that the only possible automorphism of a plane tree is a rotational symmetry. Thus the composition of plane trees may be considered as a generalized symmetry (or, borrowing the terminology from physics, as a *hidden symmetry* in the world of plane trees).

Taking $Q(z)$ arbitrary, and $P(u)$ a Chebyshev polynomial normalized in such a way as to have critical values 0 and 1, i.e., $P(u) = \frac{1}{2}(T_m(2u-1)+1)$, we obtain the tree A_Q whose all edges are subdivided into m parts.

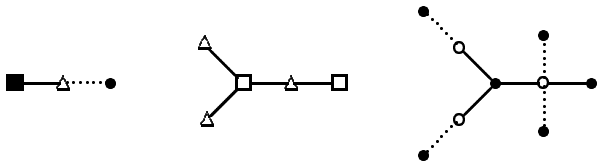
In both cases the structure of a composition is "easily seen". But this is not always the case. For example, the following tree is also a composition of two smaller trees, which is not at all easily seen.



The general procedure of getting the tree A_R from A_P and A_Q looks as follows. Let us decompose the tree A_P into the following parts: the *spine*, i.e., the (unique) path from $\mathbf{s}$ to $\mathbf{t}$; the *body*, i.e., the union of all the branches attached to the spine but not to $\mathbf{s}$ or $\mathbf{t}$; the *head*, i.e., the union of all the branches (except the spine) attached to $\mathbf{t}$; and the *tail*, i.e., the union of all the branches attached to $\mathbf{s}$ (except the spine).

Now take the tree A_Q and make the following operations: (1) instead of each edge of A_Q insert the body of A_P, respecting the direction from the square to the triangle; (2) to each square (resp. triangular) vertex of A_Q attach the tail (resp. the head) of A_P the number of times equal to the valency of the vertex considered.

Example 2.3 The tree shown above is the composition of the following two trees (A_P is on the left, A_Q in the middle, A_R on the right):



Summing up, one may say that the whole tree A_P is inserted instead of each edge of A_Q. (Note that for polynomials it works in the opposite direction: it is Q which is inserted into P.) In other words, in order to get A_R one must take n copies of A_P and glue them together at the vertices $\mathbf{s}$ and $\mathbf{t}$.

3 Cartographic group

A bicolored plane tree A_P may be represented by the pair of permutations π_0 and π_1 on the set of the edges of A_P, where π_0 is the (counterclockwise) rotation of the edges around the black vertices, and π_1, around the white ones.

Definition 3.1 The *effective cartographic group* of a bicolored plane tree A_P is the group $G_P =< \pi_0, \pi_1 >$ generated by the permutations π_0 and π_1.

Remark 3.2 The same permutation group may correspond to different trees.

The group $G_P =< \pi_0, \pi_1 >$ is transitive, because the tree is connected. The set of valencies of the tree is the set of cycle lengths of π_0 and π_1. Let us denote by σ_s and σ_t the similar permutations for the tree A_Q: these are rotations around the square and triangular vertices. The composition procedure described above leads to the following "cartographic description" of the tree A_R.

Theorem 3.3 *Let E and F be the sets of edges of the trees A_P and A_Q respectively. Denote by e_s (resp. by e_t) the edge of the spine of A_P adjacent to* **s** *(resp. to* **t***). Then the tree A_R may be described by the pair of permutations ρ_0 and ρ_1 on the set $E \times F$, where*

$$\rho_0(e,f) = \begin{cases} (\pi_0(e), f) & \text{if } e \neq e_s, e_t, \\ (\pi_0(e), \sigma_s(f)) & \text{if } e = e_s \text{ and } \mathbf{s} \text{ is black,} \\ (\pi_0(e), \sigma_t(f)) & \text{if } e = e_t \text{ and } \mathbf{t} \text{ is black,} \end{cases}$$

and ρ_1 is defined by replacing π_0 by π_1 and "black" by "white".

Remark 3.4 The edges e_s and e_t may eventually coincide; but in this case the vertices **s** and **t** cannot have the same color.

The effective cartographic group $G_R =< \rho_0, \rho_1 >$ of the tree A_R is imprimitive. But the blocks are not the "copies of A_P" inside A_R as one might think. In order to get a block of G_R, one must isolate an edge of A_P and take all its replicas in A_R. For example, the tree shown in the last figure has 2 blocks, each having 4 edges: one block consists of "dotted" edges, the other one, of the "solid" ones. In terms of Theorem 3.3 blocks are the sets $e \times F$, with the edge e fixed.

Thus, the group G_R is a certain subgroup of the *wreath product* of the groups G_P and G_Q, which often but not always coincides with the wreath product itself. This subject deserves a more thourough study.

4 Concluding remarks

Remark 4.1 *Field of definition.* The field of definition of the tree A_R may be obtained from the following information: (1) the field of definition of A_Q; (2) the field to which belong the roots of P corresponding to **s** and **t**.

Remark 4.2 *Inverse problem of enumerative combinatorics.* The usual formulation of a problem of enumerative combinatorics is "compute the number of objects with certain characteristics" (for example, the number of bicolored plane trees with a given set of valencies). The inverse problem is formulated as follows: "Let the number of objects be given in advance; classify the characteristics that lead to exactly this number of objects".

For example, the sets of valencies that produce exactly one tree, as well as the corresponding trees, are completely classified [1] (these trees have the field of definition $\mathbb{Q}$). The same is done for the families of exactly two trees (roughly speaking, they correspond to quadratic extensions of $\mathbb{Q}$). One may observe that both series of examples contain numerous compositions. We therefore suppose that the operation of composition will play an important role in the solution of the inverse enumeration problem.

Remark 4.3 *Hypermaps and Belyi functions.* The analogue of a Shabat polynomial for a planar map is a *Belyi function*, i.e., a rational function with only three critical values 0, 1 and ∞. Belyi functions are also closed under composition, though we must enlarge the set of objects from maps to *hypermaps* (i.e., "maps" whose edges are not necessarily of valency 2, see [3]).

Remark 4.4 *Iterations.* The relations between the iterations of Shabat polynomials (and trees) and the complex dynamics (Julia sets, Fatou sets etc.) are yet to be understood.

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