

# Weighted Trees with Primitive Edge Rotation Groups

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## Abstract

Let  $R, S \in \mathbb{C}[x]$  be two coprime polynomials of the same degree with prescribed multiplicities of their roots. A classical problem of number theory actively studied during the last half-century is, what could be the minimum degree of the difference  $T = R - S$ . The theory of *dessins d'enfants* implies that such a minimum is attained if and only if the rational function  $f = R/T$  is a Belyi function for a bicolored plane map all of whose faces except the outer one are of degree 1. In [16], [17] such maps are called *weighted trees* since they can be conveniently represented by plane trees whose edges are endowed with positive integral weights.

It is well known that the absolute Galois group (the automorphism group of the field  $\overline{\mathbb{Q}}$  of algebraic numbers) acts on dessins. An important invariant of this action is the *edge rotation group*, which is also the monodromy group of a ramified covering corresponding to the Belyi function. In this paper we classify all weighted trees with primitive edge rotation groups. There are, up to the color exchange, 184 such trees, which are subdivided into (at least) 85 Galois orbits and generate 34 primitive groups (the highest degree is 32). This result may also be considered as a contribution to the classification of covering of genus zero with primitive monodromy groups in the framework of the Guralnick–Thompson conjecture (see [8], [7]).

## 1 Preliminaries

The paper consists of two parts. In the first part, we provide a motivation, basic definitions, and a proof of the main result. The second part contains tables with the data describing the dessins and the groups which make up our classification. The tables are followed by figures.

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## 1.1 Motivation

Let  $\alpha, \beta$  be two partitions of an integer  $n$ ,  $\alpha = (\alpha_1, \dots, \alpha_r)$ ,  $\beta = (\beta_1, \dots, \beta_s)$ , and let  $R, S \in \mathbb{C}[x]$  be two coprime polynomials of degree  $n$  having the following factorization pattern:

$$R(x) = \prod_{i=1}^r (x - a_i)^{\alpha_i}, \quad S(x) = \prod_{j=1}^s (x - b_j)^{\beta_j}. \quad (1)$$

In these expressions we consider the multiplicities  $\alpha_i$  and  $\beta_j$ ,  $i = 1, 2, \dots, r$ ,  $j = 1, 2, \dots, s$ , as being given, while the roots  $a_i$  and  $b_j$  are not fixed, though they must all be distinct. Denote  $T = R - S$ . The question is, what could be the minimum possible degree of  $T$ ? This question is a far-reaching generalization of the question raised in 1965 in [4], what is the minimum degree of the difference  $A^3 - B^2$  (when this difference is not identically zero)?

**Assumption 1 (Conditions on  $\alpha$  and  $\beta$ )** We assume that  $\alpha$  and  $\beta$  satisfy the following conditions:

- the greatest common divisor of the numbers  $\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_s$  is 1;
- $r + s \leq n + 1$ .

The case of partitions  $\alpha, \beta$  not satisfying the above conditions can easily be reduced to this case (see [16]). In 1995, Zannier [19] proved the following:

**Theorem 2 (Zannier's bound)** *Under the conditions of Assumption 1, the following statements hold:*

1.  $\deg T \geq (n + 1) - (r + s)$ .
2. *This bound is always attained, whatever are  $\alpha$  and  $\beta$ .*

The *theory of dessins d'enfants* (see, for example, Ch. 2 of [12] or other papers in this volume) implies the following:

**Proposition 3 (Dessins d'enfants)** *The degree  $t = \deg T$  attains the minimum value  $t = (n + 1) - (r + s)$  if and only if the rational function  $f = R/T$  is a Belyi function for a bicolored plane map which has:*

- $n$  edges;
- $r$  black vertices of degrees  $\alpha_1, \dots, \alpha_r$ ;
- $s$  white vertices of degrees  $\beta_1, \dots, \beta_s$ ;
- all its faces except the outer one are of degree 1, where the face degree is defined as half the number of surrounding edges;
- the number of the faces of degree 1 is  $t = (n + 1) - (r + s)$ .

Notice that, according to the Euler formula, the number of faces of a map with  $n$  edges and  $r + s$  vertices is equal to  $t + 1$ , where  $t = (n + 1) - (r + s)$ . Thus, the face degree partition has a *hook form*:

$$\gamma = (\underbrace{1, \dots, 1}_{t \text{ times}}, n - t) = 1^t(n - t)^1.$$

**Definition 4 (Passport)** We call the triple of partitions  $(\alpha, \beta, \gamma)$  the *passport* of the corresponding map. The set of maps with a given passport is called *combinatorial orbit*.

Proposition 3 explains the special interest of maps with the “inner” faces of degree 1. Number theorists are especially interested in the arithmetic nature of the coefficients of the polynomials  $R, S, T$ : for example, when these coefficients are rational? See, in this respect, [3] and the references therein. It is well known that in general the coefficients of Belyi functions are algebraic numbers, and the absolute Galois group  $\Gamma = \text{Aut}(\overline{\mathbb{Q}}|\mathbb{Q})$  (the automorphism group of the field  $\overline{\mathbb{Q}}$  of algebraic numbers) acts on dessins. We return to this question in Sect. 1.4. One of the most important invariants of this action is the *edge rotation group*, see Sect. 1.3. This fact explains our interest in these groups.

## 1.2 Maps with inner faces of degree 1 and weighted trees

A typical bicolored map with all its faces except the outer one being of degree 1 looks like in Fig. 1, left. Such maps have an obvious tree-like form. It is often convenient to represent them as *weighted trees*, that is, bicolored plane trees whose edges are endowed with positive integral weights, as is shown in Fig. 1, right. The degree of a vertex in such a tree is defined as the sum of the weights of the edges incident to this vertex.

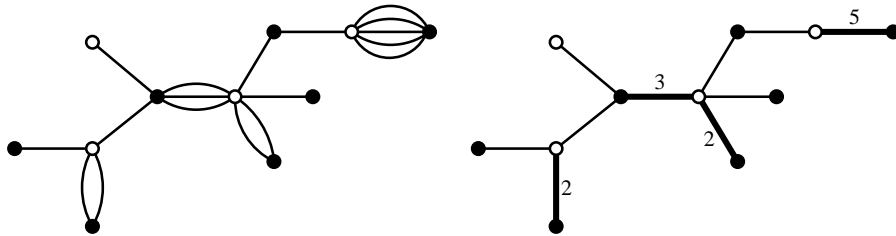


Figure 1: The passage from a map with all its inner faces being of degree 1, to a weighted tree. The weights which are not explicitly indicated are equal to 1; the edges of the weight bigger than 1 are drawn thick.

In this paper we do not use this tree-like structure in any significant way. Nevertheless, we mention this model for the following reasons. First, the notion

of a weighted tree seems to be quite natural and certainly deserves to be studied *per se*. Second, this model is actively used in other publications devoted to the same subject, see, for example, [16], [17]. And, third, when the number of faces of degree 1 is zero we get an *ordinary tree*; we will often meet ordinary trees in this paper.

This is why, in what follows we often call the maps we study here either maps, or weighted trees, or just trees for short.

### 1.3 Edge rotation groups

**Definition 5 (Edge rotation group)** Label the  $n$  edges of a bicolored map by numbers from 1 to  $n$ , and write down two permutations  $a$  and  $b$  as follows. The cycles of  $a$  indicate the cyclic order of edges, in the counter-clockwise direction, around black vertices, and the cycles of  $b$  indicate the cyclic order of edges, in the counter-clockwise direction, around white vertices. Then the *edge rotation group* of the map in question is the permutation group  $G = \langle a, b \rangle$  of degree  $n$ . This group is defined up to a conjugacy inside  $S_n$ .

Note the following properties of the edge rotation group.

- The group is transitive since the map is connected.
- The cycle structures of  $a$  and  $b$  are, respectively,  $\alpha$  and  $\beta$ .
- The faces of the map are determined by the cycles of the permutation  $c = (ab)^{-1}$  (so that  $abc = 1$ ). For a weighted tree, the cycle structure of  $c$  has the form  $\gamma = 1^t(n-t)^1$ .
- The edge rotation group is in fact the *monodromy group* of the ramified covering of the sphere realized by the Belyi function corresponding to the map in question.

**Example 6 (Labeling of a map)** Fig. 2 shows the same map as in Fig. 1, with its edges labeled by numbers from 1 to 18. The corresponding permutations are

$$\begin{aligned} a &= (2, 9, 12, 13, 3)(4, 6)(5, 18, 17, 16, 15)(8, 14)(10, 11), \\ b &= (1, 10, 11, 9)(3, 13, 12, 8, 14, 7, 6)(4, 5, 15, 16, 17, 18), \\ c &= (1, 2, 3, 4, 5, 6, 7, 8, 9, 10). \end{aligned}$$

Notice that the only “long” cycle of the permutation  $c$  can be read out by going around the outer face of the map; in order to obtain this result one must always put the label of an edge on its left side while going from black to white.

Representing bicolored maps as triples of permutations  $(a, b, c)$  such that  $abc = 1$  we may introduce two *braid operations*:

$$\sigma_1 : (a, b, c) \rightarrow (b, b^{-1}ab, c), \quad \sigma_2 : (a, b, c) \rightarrow (a, c, c^{-1}bc).$$

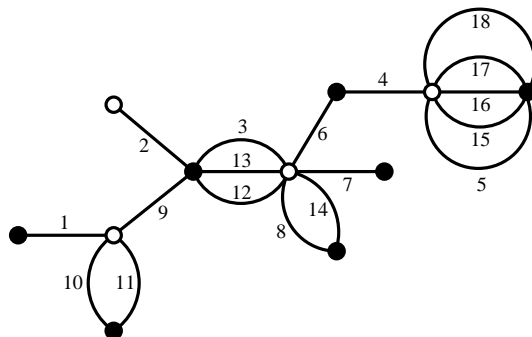


Figure 2: A bicolored map with labeled edges.

Notice that these operations preserve the product of the three permutations in a triple. They also satisfy the braid relation  $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ . Obviously, all the triples obtained by braiding generate the same group.

It is natural to call the operation  $\sigma_1$  *color exchange* since now the permutation  $b$  becomes the first element of the triple and therefore corresponds to the black vertices, the permutation  $b^{-1}ab$  conjugate to  $a$  corresponds to the white vertices, and the permutation  $c$  corresponding to faces remains the same.

The operation  $\sigma_2$  exchanges white vertices with faces, while preserving the black vertices. The operation  $\sigma_1\sigma_2\sigma_1$  gives the triple  $(c, c^{-1}bc, a)$  and thus exchanges black vertices with faces while preserving the white vertices. It is natural to call both operations *dualities*. A map which is isomorphic to one of its duals is called *self-dual*.

**Definition 7 (Primitive group)** A permutation group  $G$  of degree  $n$  acting on a set  $X$ ,  $|X| = n$ , is called *imprimitive* if the set  $X$  can be subdivided into  $m$  disjoint *blocks*  $X_1, \dots, X_m$  of equal size  $|X_i| = n/m$ , where  $1 < m < n$ , such that an image of a block under the action of any element of  $G$  is once again a block. A permutation group which is not imprimitive is called *primitive*. A primitive permutation group not equal to  $S_n$  or  $A_n$  is called *special*.

**Theorem 8 (Ritt's theorem)** A ramified covering is a composition of two or more coverings of smaller degrees if and only if its monodromy group is imprimitive.

**Remark 9 (Primitive groups are few)** There are 301 transitive permutation groups of degree  $n = 12$ , but only six of them are primitive ( $S_n$  and  $A_n$  included); there are (exactly) 25 000 transitive permutation groups of degree  $n = 24$ , but only five of them are primitive (this information is taken from [9]). When  $n$  is prime, all groups of degree  $n$  are primitive, but they are not numerous as well. To compare, there are 1854 transitive groups of degree 28, only

8 groups of degree 29, and 5712 groups of degree 30. What is more, a group generated by a pair of randomly chosen permutations is either  $S_n$  or  $A_n$  with a probability which tends to 1 as  $n \rightarrow \infty$ . This is why special groups have special interest.

In 1990, Guralnick and Thompson [8] conjectured that, apart from the cyclic groups  $C_p$  and dihedral groups  $D_{2p}$  for  $p$  prime, the number of special groups admitting a planar presentation is finite. This conjecture was proved in 2001 by Frohardt and Magaard [7], not only for the genus zero but for an arbitrary genus, by exhibiting an upper bound for the degrees of the groups in question. However, this bound is largely overestimated, so a complete classification of planar special groups is still not achieved. In our paper, we classify these groups for the case of weighted trees.

#### 1.4 Fields of definition and Galois action

It is well known that a Belyi function of a plane dessin is unique up to a linear fractional transformation of its argument. One can choose the transformation to make the coefficients of the Belyi function to be algebraic numbers. To every dessin there corresponds a specific number field which is called its field of definition. There are some subtleties in this notion, but in the specific case studied in our paper, it can be defined in a simple way.

**Definition 10 (Field of definition)** For a plane dessin all of whose faces except the outer one are of degree 1, the *field of definition* is the smallest number field to which belong the coefficients of the Belyi function of the dessin.

The Galois group  $\Gamma = \text{Aut}(\overline{\mathbb{Q}}|\mathbb{Q})$  acts on Belyi functions by replacing all their coefficients by their algebraically conjugate ones. This action descends to an action on dessins.

The passport of the dessin is invariant under this action. Another important invariant of the Galois action is the edge rotation group. Thus, if in the same combinatorial orbit there are dessins with different edge rotation groups then this combinatorial orbit splits into several Galois orbits. Even more so, the character table of the group contains an important information about the field of definition. These considerations give us an additional impetus to study edge rotation groups.

Algebraic conjugation applied to a pair of dual dessins gives us again a pair of dual dessins. The same is true for the color exchange. Therefore the self-duality or the property of being invariant under the color exchange are additional invariants of the Galois action.

## 2 Classification of special weighted trees

### 2.1 Gareth A. Jones's classification

The edge rotation groups of weighted trees, that is, of plane maps with all their faces except of the outer one being of degree 1, contain permutations with the cycle structure  $1^t(n-t)^1$ . Motivated by our study of weighted trees, Gareth Jones classified all special permutation groups containing such a permutation, see [10]. In particular, he had shown that in all such cases the number of fixed points  $t \leq 2$ . This property is based on two results. The first is an old theorem by Jordan (1871) [11] stating that a primitive group containing a permutation with the cycle structure  $1^t(n-t)^1$  is  $(t+1)$ -transitive. The second is the complete classification of multiply transitive groups: it is based on the classification theorem of finite simple groups.

The classification due to Jones looks as follows (we use standard notation for cyclic, affine and projective groups and for the Mathieu groups):

**Theorem 11 (G. Jones)** *Let  $G$  be a primitive permutation group of degree  $n$  not equal to  $S_n$  or  $A_n$ . Suppose that  $G$  contains a permutation with cycle structure  $1^t(n-t)^1$ . Then  $t \leq 2$ , and one of the following holds:*

0.  $t = 0$  and either
  - (a)  $C_p \leq G \leq \text{AGL}_1(p)$  with  $n = p$  prime, or
  - (b)  $\text{PGL}_d(q) \leq G \leq \text{P}\Gamma\text{L}_d(q)$  with  $n = (q^d - 1)/(q - 1)$  and  $d \geq 2$  for some prime power  $q = p^e$ , or
  - (c)  $G = \text{L}_2(11), \text{M}_{11}$  or  $\text{M}_{23}$  with  $n = 11, 11$  or  $23$  respectively.
1.  $t = 1$  and either
  - (a)  $\text{AGL}_d(q) \leq G \leq \text{A}\Gamma\text{L}_d(q)$  with  $n = q^d$  and  $d \geq 1$  for some prime power  $q = p^e$ , or
  - (b)  $G = \text{L}_2(p)$  or  $\text{PGL}_2(p)$  with  $n = p + 1$  for some prime  $p \geq 5$ , or
  - (c)  $G = \text{M}_{11}, \text{M}_{12}$  or  $\text{M}_{24}$  with  $n = 12, 12$  or  $24$  respectively.
2.  $t = 2$  and  $\text{PGL}_2(q) \leq G \leq \text{P}\Gamma\text{L}_2(q)$  with  $n = q + 1$  for some prime power  $q = p^e$ .

The set of groups listed in the above theorem is infinite. When, in addition to the existence of elements with cycle structure  $1^t(n-t)^1$ , we impose a planarity condition, we get a finite set of groups (with two exceptions considered below).

### 2.2 Two infinite series

For  $n = p$  prime, all transitive groups of degree  $p$  are primitive. According to Theorem 11, all the groups  $G$  between the cyclic group  $C_p$  and the affine group  $\text{AGL}_1(p)$  contain a cycle of length  $p$  and therefore, in principle, might be edge rotation groups of ordinary trees. But, according to [2], only two of them are indeed realized by trees, see Fig. 3:

- the cyclic group  $C_p$  of order  $p$ , which is the edge rotation group of the star-tree with  $p$  edges;
- the dihedral group  $D_{2p}$  of order  $2p$ , which is the edge rotation group of the chain-tree with  $p$  edges.

In what follows, we avoid mentioning these two cases.

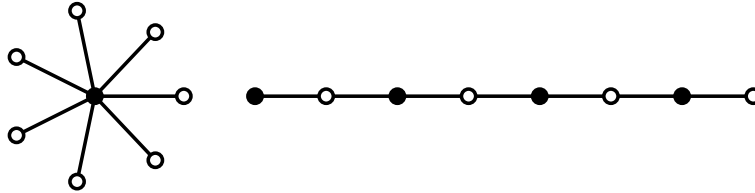


Figure 3: The edge rotation group for the star-tree is the cyclic group  $C_p$ , and for the chain-tree it is the dihedral group  $D_{2p}$ . If  $p$  is prime both groups are primitive.

Thus, among the subgroups of  $AGL_1(p)$  only the group  $AGL_1(p)$  itself remains of interest for us since, according to the case 1(a) of Theorem 11, this group contains permutations with the cycle structure  $1^1(p-1)^1$ , and therefore it could be an edge rotation group of a map with  $p$  edges, one face of degree 1, and  $p$  vertices. However, it is easy to see that this can only happen for  $p = 5$  and  $p = 7$ . Indeed, the cycle structures of the elements of  $AGL_1(p)$  are all of the form  $l^k 1^1$  where  $lk = p - 1$ . This partition has  $k + 1$  parts, and a pair of such partitions has  $k_1 + k_2 + 2$  parts,  $k_1, k_2 \geq 1$  being divisors of  $p - 1$ . Now, if  $k_1 = k_2 = (p - 1)/2$ , both partitions become equal to  $1^{12^k}$ , and we get a chain tree. Otherwise, the biggest possible number of parts in two partitions (which should give us the number of vertices) is  $m = \frac{p-1}{2} + \frac{p-1}{3} + 2$ , and the inequality  $m \geq p$  leads to  $p \leq 7$ .

### 2.3 Main theorem

**Theorem 12 (Classification)** *The complete list of special weighted trees, not taking into account stars and chains, is the list given in the tables of Sect. 3. It contains, up to the color exchange, 184 trees, which are subdivided into (at least) 85 Galois orbits and generate 34 different edge rotation groups.*

**Remark 13 (Color exchange)** One should be careful when counting trees “up to the color exchange” in the case when  $\alpha = \beta$ , that is, when the black partition is equal to the white one. There are several such orbits in our list: 6.7, 8.1, 8.7, 8.9, 9.7, 12.10, 12.12, 24.5 (see the generators and the pictures below). Almost all of them are invariant under the color exchange, except the trees in orbits 8.1 and 9.7 for which the color exchange gives us a different tree in the same orbit (see Fig. 9 and Fig. 16).



**Remark 14 (Duality)** There is a number of dual and self-dual weighted trees in our list:

- dessins 5.1, 6.2, 6.9, 8.5, 8.14 and 10.3 are self-dual;
- dessins 6.3 and 6.4, 6.5 and 6.6, 6.7 and 6.8, 8.12 and 8.13, two dessins in orbit 9.5 are pairwise dual.

**Remark 15 (Galois orbits)** The complete set of combinatorial Galois invariants is unknown (and probably there is none). We observe a couple of combinatorial orbits (9.5 and 9.7) with the same edge rotation groups splitting into two Galois orbits. The orbit 9.7 consists of 4 trees, two of which are invariant under the color exchange, but the other two are not; this gives us a combinatorial explanation of the splitting. At the same time we do not see any combinatorial reason to explain splitting of the “orbit” 9.5.

We have computed all fields of definitions for  $n \leq 12$  except for orbits 12.3, 12.8 and 12.9. Also, Belyi functions for certain ordinary trees with  $n > 12$  edges were computed in [14] (for  $n = 23$ ) and [6] (for  $n = 21, 31$ ). There remain several cases for  $n > 12$  when we cannot guarantee that, what appears as an “orbit” in our list, is indeed a single Galois orbit and not a union of several Galois orbits. That is why we say with caution “*at least* 85 Galois orbits”.

## 2.4 Proof of the main theorem

The case  $t = 0$ , that is, the case of *ordinary trees*, was already settled in [2]; the complete list of the corresponding 48 trees may be found there (we also list them in our tables).

All genus zero generating sets for the *affine groups* (case 1(a) of Theorem 11) are listed in [13]; see also Wang [18]. Taking the generating sets containing the triples of permutations, one of which has a cycle structure  $1^1(n-1)^1$ , permits us to settle this case.

The case 1(c) of Theorem 11, that is, the case of Mathieu groups  $M_{11}$ ,  $M_{12}$  and  $M_{24}$ , can easily be handled using the GAP computer system.

All the remaining groups (the cases 1(b) and 2 of Theorem 11) are subgroups of  $\mathrm{P}\Gamma\mathrm{L}_2(q)$  for some prime power  $q = p^e$ , acting on  $n = q + 1$  points. Notice the subscript 2: the projective geometry in question is always a projective *line*. We will need the following lemma which can be found in Müller [15] or in Adrianov [1].

**Lemma 16 (Number of cycles)** *Let  $q = p^e$  be a prime power,  $e > 1$ , and let  $h > 1$  be the least prime divisor of  $e$ . Then a non-identity permutation  $g \in \mathrm{P}\Gamma\mathrm{L}_2(q)$  cannot have more than  $l = p^{e/h} + 1$  fixed points. A non-identity permutation  $g \in \mathrm{P}\Gamma\mathrm{L}_2(p)$  with  $p$  prime cannot have more than 2 fixed points. If no element of a permutation group of degree  $n$  has more than  $l$  fixed points then an element of order  $k$  can have at most  $\frac{n-l}{k} + l$  cycles.*

A map with  $n$  edges and at most two faces of degree 1 must have at least  $n - 1 = q$  vertices. If both permutations  $a$  and  $b$  are involutions we get a chain-tree. If at least one of them is of order  $k \geq 3$  and thus have, roughly,  $n/k$  cycles, then, once again roughly, we get not more than  $5n/6$  vertices, which is not enough.

More exactly, for  $q = p^e$ ,  $e > 1$ , denote  $u = \sqrt{q}$ , so that  $q = u^2$ . According to the above lemma, a non-identity element of  $\text{PTL}_2(q)$  cannot have more than  $u + 1$  fixed points, and thus an element of order  $k$  cannot have more than  $\frac{(q+1)-(u+1)}{k} + (u + 1)$  cycles. Therefore, the total number of cycles in two permutations is not bigger than

$$\frac{q-u}{2} + (u+1) + \frac{q-u}{3} + (u+1) = \frac{5}{6}(u^2 - u) + 2(u+1).$$

The quadratic inequality

$$\frac{5}{6}(u^2 - u) + 2(u+1) \geq u^2$$

leads to

$$u \leq \frac{7 + \sqrt{97}}{2} < \frac{17}{2} = 8.5$$

so that  $q \leq 72.25$ . The biggest prime power satisfying this inequality is 64.

In the same way, for the group  $\text{PTL}_2(p)$  with  $p$  prime we have

$$\frac{(p+1)-2}{2} + 2 + \frac{(p+1)-2}{3} + 2 = \frac{5(p-1)}{6} + 4 \geq p,$$

which leads to  $p \leq 19$ .

There remains a finite number of groups to study. Some of them can be easily ruled out by hand, as in Example 17 below. The remaining cases were treated by GAP.

Theorem 12 is proved. □

**Example 17 (Groups eliminated “by hand”)** Let us take  $q = 64 = 2^6$ . According to Lemma 16, the upper bound for the number of fixed points is  $2^3 + 1 = 9$ . The number of cycles in an involution is then bounded by  $(64 - 8)/2 + 8 = 36$ , and the number of cycles in a permutation of order 3 is bounded by  $(64 - 7)/3 + 7 = 26$ ; for all the other permutation the number of cycles is even less than that. However,  $36 + 26 = 62$  vertices is not enough in order to create a map with  $n = q + 1 = 65$  edges and with one or two faces of degree 1.

Similar considerations permit us to handle the case  $q = 27 = 3^3$ . The number of fixed points is bounded by  $3 + 1 = 4$ , and the total number of cycles in two permutations, at least one of which is not an involution, is bounded by

$(27 - 3)/2 + 3 + (27 - 3)/3 + 3 = 26$ , which is not enough in order to create a map with 28 edges and with one or two faces of degree 1.

Note, however, that this simple approach fails for  $q = 49$ . The number of fixed points is bounded by  $7 + 1 = 8$ ; the total number of cycles in two permutations is bounded by  $(49 - 7)/2 + 7 + (49 - 7)/3 + 7 = 49$ . This number of vertices is *a priori* sufficient in order to create a map with 50 edges and with two faces of degree 1. Therefore, in order to rule out the group  $\text{PTL}_2(49)$  we need to study it using GAP.

**Remark 18 (Cross-verification)** As in any experimental work, in order to be on the safe side it is useful to attack the problem from various perspectives and to see if the results thus obtained are coherent. In our work, we did the following:

- We looked through *all* primitive groups up to degree 127 (not only those listed in Theorem 11) using the GAP system. Thus, Jones’s classification has also received an independent experimental confirmation.
- Many, though not all of the above groups were also studied using the Maple package `group`.
- For the groups of degree up to 11 we made use of the catalogue [5].
- Whenever possible, we have computed Belyi functions in order to verify if indeed a combinatorial orbit in question splits into several Galois orbits. For example, there exist 16 trees of weight 10 with the passport  $(1^2 8^1, 1^2 2^4, 1^2 8^1)$ . They split into four Galois orbits, and one of them does consist of a single tree corresponding to the group  $\text{PGL}_2(9)$ : see orbit 10.3.
- For a given passport, the number of distinct solutions given by Maple was compared with the total numbers of trees in the corresponding combinatorial orbit. The latter was constructed using GAP.

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### 3 Tables and figures

The GAP system contains a data library of all primitive permutation groups of degree  $< 2500$ . The command `PrimitiveGroup(n,k)` returns the  $k$ th element of the list of primitive groups of degree  $n$ .

There are several trees in our list, which turn out to be the only realizations of their passports. We call such trees *unitrees*; their complete classification was obtained in [16].

When there is a pair of trees mirror symmetric to each other we present only one of them, both in a permutation form and as a figure. Such trees necessarily belong to the same Galois orbit since they can be obtained from each other by a complex conjugation, which is an element of the absolute Galois group  $\Gamma$  (in fact, the only element explicitly known).

Every figure contains a distinguished edge: it is the *root edge*, which means that it is labeled by 1. This permits to easily establish the correspondence between permutations and figures. The choice of the root edge does not always look geometrically natural but it is not our choice: the permutations were generated by GAP.

Fields of definition are computed only for the degrees  $n \leq 12$ , with a very few exceptions for greater degrees.

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**AGL<sub>1</sub>(5) of order 20** PrimitiveGroup(5,3)

**5.1.**  $(1^{12^2}, 1^{14^1}, 1^{14^1})$ . Number of trees: **2**.

$$a = (1, 5)(2, 3) \quad b = (2, 3, 4, 5)$$

Field:  $\mathbb{Q}(\sqrt{-1})$ . There are no other trees with this passport.

---

**L<sub>2</sub>(5) of order 60** PrimitiveGroup(6,1)

**6.1.**  $(1^{22^2}, 3^2, 1^{15^1})$ . Number of trees: **1**.

$$a = (3, 5)(4, 6) \quad b = (1, 2, 3)(4, 5, 6)$$

Field:  $\mathbb{Q}$ . This is a unitree of series  $D$ , see [16].

**6.2.**  $(1^{22^2}, 1^{15^1}, 1^{15^1})$ . Number of trees: **1**.

$$a = (3, 5)(4, 6) \quad b = (1, 5, 3, 2, 4)$$

Field:  $\mathbb{Q}$ . There exist two more trees with this passport, defined over  $\mathbb{Q}(\sqrt{-15})$ .

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**PGL<sub>2</sub>(5) of order 120** PrimitiveGroup(6,2)

**6.3.**  $(1^{22^2}, 1^{24^1}, 6^1)$ . Number of trees: **1**.

$$a = (1, 5)(2, 6) \quad b = (3, 4, 5, 6)$$

Field:  $\mathbb{Q}$ . There exists one more tree with this passport, which is symmetric with the symmetry of order 2 and defined over  $\mathbb{Q}$ .

**6.4.**  $(1^2 2^2, 6^1, 1^2 4^1)$ . Number of trees: **1**.

$$a = (3, 5)(4, 6) \quad b = (1, 2, 3, 5, 4, 6)$$

Field:  $\mathbb{Q}$ . This tree is dual to 6.3, so we have the same situation here.

**6.5.**  $(2^3, 1^2 4^1, 1^1 5^1)$ . Number of trees: **1**.

$$a = (1, 3)(2, 6)(4, 5) \quad b = (3, 4, 5, 6)$$

Field:  $\mathbb{Q}$ . This is a unitree of series  $J$ , see [16].

**6.6.**  $(2^3, 1^1 5^1, 1^2 4^1)$ . Number of trees: **1**.

$$a = (1, 2)(3, 4)(5, 6) \quad b = (2, 3, 4, 6, 5)$$

Field:  $\mathbb{Q}$ . This is a unitree of series  $E_2$ , see [16].

**6.7.**  $(1^2 4^1, 1^2 4^1, 1^1 5^1)$ . Number of trees: **1**.

$$a = (1, 3, 6, 2) \quad b = (3, 4, 5, 6)$$

Field:  $\mathbb{Q}$ . This is a unitree of series  $C$ , see [16].

**6.8.**  $(1^2 4^1, 1^1 5^1, 1^2 4^1)$ . Number of trees: **1**.

$$a = (3, 4, 5, 6) \quad b = (1, 2, 4, 3, 6)$$

Field:  $\mathbb{Q}$ . This is a unitree of series  $C$ , see [16].

**6.9.**  $(3^2, 1^2 4^1, 1^2 4^1)$ . Number of trees: **1**.

$$a = (1, 3, 2)(4, 6, 5) \quad b = (3, 4, 5, 6)$$

Field:  $\mathbb{Q}$ . There exists one more tree with this passport, which is symmetric with the symmetry of order 2 and defined over  $\mathbb{Q}$ .

**AGL<sub>1</sub>(7) of order 42**

PrimitiveGroup(7,4)

**7.1.**  $(1^1 2^3, 1^1 3^2, 1^1 6^1)$ . Number of trees: **2**.

$$a = (1, 4)(3, 5)(6, 7) \quad b = (2, 4, 6)(3, 5, 7)$$

Field:  $\mathbb{Q}(\sqrt{-3})$ . There are no other trees with this passport.

**L<sub>3</sub>(2) of order 168**

PrimitiveGroup(7,5)

**7.2.**  $(1^3 2^2, 1^1 3^2, 7^1)$ . Number of trees: **2**.

$$a = (1, 3)(4, 5) \quad b = (2, 3, 5)(4, 7, 6)$$

Field:  $\mathbb{Q}(\sqrt{-7})$ . There are no other trees with this passport.

**7.3.**  $(1^3 2^2, 1^1 2^1 4^1, 7^1)$ . Number of trees: **2**.

$$a = (1, 4)(6, 7) \quad b = (2, 3, 4, 7)(5, 6)$$

Field:  $\mathbb{Q}(\sqrt{-7})$ . There exist two more trees with this passport, defined over  $\mathbb{Q}(\sqrt{21})$ .

**ATL<sub>1</sub>(8) of order 168**

PrimitiveGroup(8,2)

**8.1.**  $(1^2 3^2, 1^2 3^2, 1^1 7^1)$ . Number of trees: **4**.

$$\begin{aligned} a_1 &= (1, 4, 8)(2, 7, 6) & b_1 &= (3, 5, 8)(4, 6, 7) \\ a_2 &= (3, 5, 8)(4, 6, 7) & b_2 &= (1, 4, 8)(2, 7, 6) \end{aligned}$$

Field:  $\mathbb{Q}(\sqrt{14 - 14\sqrt{-3}})$ . There exists one more tree with this passport, see orbit 8.9.

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**ASL<sub>3</sub>(2) of order 1344** **PrimitiveGroup(8,3)**

**8.2.**  $(2^4, 1^2 2^1 4^1, 1^1 7^1)$ . Number of trees: **2**.

$$a = (1, 3)(2, 8)(4, 6)(5, 7) \quad b = (3, 4)(5, 7, 6, 8)$$

Field:  $\mathbb{Q}(\sqrt{-7})$ . There are no other trees with this passport.

**8.3.**  $(1^4 2^2, 4^2, 1^1 7^1)$ . Number of trees: **2**.

$$a = (2, 6)(4, 8) \quad b = (1, 8, 4, 6)(2, 7, 3, 5)$$

Field:  $\mathbb{Q}(\sqrt{-7})$ . There are no other trees with this passport.

**8.4.**  $(1^4 2^2, 2^1 6^1, 1^1 7^1)$ . Number of trees: **2**.

$$a = (2, 3)(6, 7) \quad b = (1, 2)(3, 6, 7, 4, 5, 8)$$

Field:  $\mathbb{Q}(\sqrt{-7})$ . There exist two more trees with this passport, defined over  $\mathbb{Q}(\sqrt{-14})$ .

**8.5.**  $(1^4 2^2, 1^1 7^1, 1^1 7^1)$ . Number of trees: **2**.

$$a = (1, 3)(6, 8) \quad b = (2, 3, 5, 4, 7, 8, 6)$$

Field:  $\mathbb{Q}(\sqrt{-7})$ . There exist three more trees with this passport, defined over the splitting field of the polynomial  $a^3 + 2058a + 364952$ .

**8.6.**  $(1^2 3^2, 1^2 2^1 4^1, 1^1 7^1)$ . Number of trees: **4**.

$$a_1 = (3, 5, 7)(4, 6, 8) \quad b_1 = (1, 2, 7, 8)(4, 6)$$

$$a_2 = (3, 5, 7)(4, 6, 8) \quad b_2 = (1, 8)(2, 4, 7, 5)$$

Field:  $\mathbb{Q}(\sqrt{-455 + 952\sqrt{-14}})$ . There exist six more trees with this passport, which form a unique Galois orbit of degree 6.

**8.7.**  $(1^2 2^1 4^1, 1^2 2^1 4^1, 1^1 7^1)$ . Number of trees: **2**.

$$a = (1, 7)(2, 4, 8, 6) \quad b = (3, 4)(5, 7, 6, 8)$$

Field:  $\mathbb{Q}(\sqrt{-7})$ . There exist 18 more trees with this passport, which form a unique Galois orbit of degree 18.

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**L<sub>2</sub>(7) of order 168** **PrimitiveGroup(8,4)**

**8.8.**  $(2^4, 1^2 3^2, 1^1 7^1)$ . Number of trees: **1**.

$$a = (1, 3)(2, 8)(4, 5)(6, 7) \quad b = (3, 7, 8)(4, 6, 5)$$

Field:  $\mathbb{Q}$ . This is the sporadic unitree  $K$ , see [16].

**8.9.**  $(1^2 3^2, 1^2 3^2, 1^1 7^1)$ . Number of trees: **1**.

$$a = (1, 5, 2)(4, 8, 6) \quad b = (3, 7, 8)(4, 6, 5)$$

Field:  $\mathbb{Q}$ . There exist four more tree with this passport, see orbit 8.1.

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**PGL<sub>2</sub>(7) of order 336** **PrimitiveGroup(8,5)**

**8.10.**  $(1^2 2^3, 4^2, 1^2 6^1)$ . Number of trees: **1**.

$$a = (2, 6)(3, 8)(4, 7) \quad b = (1, 2, 3, 8)(4, 6, 5, 7)$$

Field:  $\mathbb{Q}$ . There exist two more trees with this passport; they are both symmetric with the symmetry of order 2 and defined over  $\mathbb{Q}(\sqrt{-2})$ .

**8.11.**  $(2^4, 1^2 6^1, 1^2 6^1)$ . Number of trees: **1**.

$$a = (1, 3)(2, 6)(4, 8)(5, 7) \quad b = (3, 5, 7, 4, 8, 6)$$

Field:  $\mathbb{Q}$ . There exists one more tree with this passport, which is symmetric with the symmetry of order 2 and defined over  $\mathbb{Q}$ .

**8.12.**  $(1^2 2^3, 1^2 6^1, 1^1 7^1)$ . Number of trees: **1**.

$$a = (1, 6)(2, 7)(4, 8) \quad b = (3, 5, 7, 4, 8, 6)$$

Field:  $\mathbb{Q}$ . There exist five more trees with this passport, defined over the splitting field of the polynomial  $a^5 + 22a^4 + 209a^3 + 1040a^2 + 2624a + 2560$ .

**8.13.**  $(1^2 2^3, 1^1 7^1, 1^2 6^1)$ . Number of trees: **1**.

$$a = (1, 7)(4, 5)(6, 8) \quad b = (2, 3, 8, 6, 7, 5, 4)$$

Field:  $\mathbb{Q}$ . This tree is dual to 8.12, so we have the same situation here.

**8.14.**  $(1^2 3^2, 1^2 6^1, 1^2 6^1)$ . Number of trees: **1**.

$$a = (1, 2, 4)(3, 6, 5) \quad b = (3, 5, 7, 4, 8, 6)$$

Field:  $\mathbb{Q}$ . There exist four more trees with this passport. One of them is symmetric, with the symmetry of order 2, and therefore it is defined over  $\mathbb{Q}$ . Three remaining trees are defined over the splitting field of the polynomial  $a^3 - 6a + 16$ .

**8.15.**  $(1^2 2^3, 1^2 3^2, 8^1)$ . Number of trees: **2**.

$$a_1 = (1, 6)(2, 7)(4, 8) \quad b_1 = (3, 7, 8)(4, 6, 5)$$

$$a_2 = (1, 4)(2, 6)(5, 8) \quad b_2 = (3, 7, 8)(4, 6, 5)$$

Field:  $\mathbb{Q}(\sqrt{2})$ . There exist two more trees with this passport; they are both symmetric with the symmetry of order 2 and defined over  $\mathbb{Q}(\sqrt{-2})$ .

**ATL<sub>1</sub>(9) of order 144**

PrimitiveGroup(9,5)

**9.1.**  $(1^3 2^3, 1^1 4^2, 1^1 8^1)$ . Number of trees: **2**.

$$a = (1, 3)(5, 6)(7, 8) \quad b = (2, 4, 3, 7)(5, 6, 9, 8)$$

Field:  $\mathbb{Q}(\sqrt{-2})$ . There exist six more trees with this passport; they form a unique Galois orbit of degree 6.

**AGL<sub>2</sub>(3) of order 432**

PrimitiveGroup(9,7)

**9.2.**  $(1^3 2^3, 3^3, 1^1 8^1)$ . Number of trees: **2**.

$$a = (4, 7)(5, 8)(6, 9) \quad b = (1, 2, 4)(3, 9, 7)(5, 6, 8)$$

Field:  $\mathbb{Q}(\sqrt{-2})$ . There are no other trees with this passport.

**9.3.**  $(1^3 2^3, 1^1 2^1 6^1, 1^1 8^1)$ . Number of trees: **2**.

$$a = (1, 7)(3, 6)(5, 8) \quad b = (2, 3)(4, 8, 5, 7, 6, 9)$$

Field:  $\mathbb{Q}(\sqrt{-2})$ . There exist 14 more trees with this passport; they form a unique Galois orbit of degree 14.



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 **$L_2(8)$  of order 504**

PrimitiveGroup(9,8)

**9.4.**  $(1^1 2^4, 3^3, 1^2 7^1)$ . Number of trees: **1**.

$$a = (2, 3)(4, 8)(5, 9)(6, 7) \quad b = (1, 2, 5)(3, 8, 4)(6, 9, 7)$$

Field:  $\mathbb{Q}$ . This is the sporadic unitree  $L$ , see [16].**9.5.**  $(1^1 2^4, 1^2 7^1, 1^2 7^1)$ . Number of trees: **2**.

$$a_1 = (1, 6)(2, 5)(3, 4)(8, 9) \quad b_1 = (3, 4, 9, 8, 6, 7, 5)$$

$$a_2 = (1, 9)(2, 4)(3, 5)(6, 8) \quad b_2 = (3, 4, 9, 8, 6, 7, 5)$$

Field:  $\mathbb{Q}$ . This orbit consisting of two trees splits into 2 Galois orbits, both defined over  $\mathbb{Q}$ . We do not see a combinatorial reason to explain this splitting. There exist four more trees with this passport; they form a unique Galois orbit of degree 4.

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 **$P\Gamma L_2(8)$  of order 1512**

PrimitiveGroup(9,9)

**9.6.**  $(1^1 2^4, 1^3 3^2, 9^1)$ . Number of trees: **2**.

$$a = (1, 5)(2, 8)(3, 7)(6, 9) \quad b = (1, 7, 8)(3, 4, 9)$$

Field:  $\mathbb{Q}(\sqrt{-3})$ . There are no other trees with this passport.**9.7.**  $(1^3 3^2, 1^3 3^2, 9^1)$ . Number of trees: **4**.

$$a_1 = (1, 6, 4)(2, 8, 5) \quad b_1 = (1, 7, 8)(3, 4, 9)$$

$$a_2 = (1, 4, 6)(2, 5, 8) \quad b_2 = (1, 7, 8)(3, 4, 9)$$

Field:  $\mathbb{Q}(\sqrt{-3})$ . This orbit consisting of four trees splits into 2 Galois orbits, both defined over  $\mathbb{Q}(\sqrt{-3})$ . There is a simple reason for this splitting: two trees are invariant under a color exchange, while the other two trees are not. There are no other trees with this passport.

**9.8.**  $(1^3 3^2, 1^1 2^1 6^1, 1^2 7^1)$ . Number of trees: **4**.

$$a_1 = (1, 5, 6)(4, 9, 8) \quad b_1 = (1, 3, 7, 4, 8, 9)(2, 6)$$

$$a_2 = (1, 9, 5)(2, 4, 6) \quad b_2 = (1, 3, 7, 4, 8, 9)(2, 6)$$

Field:  $\mathbb{Q}(\sqrt{10 + 2\sqrt{-3}})$ . There exist nine more trees with this passport; they form a unique Galois orbit of degree 9.

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 **$PGL_2(9)$  of order 720**

PrimitiveGroup(10,4)

**10.1.**  $(2^5, 1^1 3^3, 1^2 8^1)$ . Number of trees: **1**.

$$a = (1, 2)(3, 4)(5, 10)(6, 7)(8, 9) \quad b = (2, 3, 10)(4, 9, 8)(5, 7, 6)$$

Field:  $\mathbb{Q}$ . This is the sporadic unitree  $M$ , see [16].**10.2.**  $(2^5, 1^2 4^2, 1^2 8^1)$ . Number of trees: **1**.

$$a = (1, 3)(2, 7)(4, 8)(5, 6)(9, 10) \quad b = (3, 6, 10, 9)(4, 8, 5, 7)$$

Field:  $\mathbb{Q}$ . There exist two more tree with this passport, which are symmetric with the symmetry of order 2 and defined over  $\mathbb{Q}(\sqrt{-1})$ .

**10.3.**  $(1^{2^4}, 1^{2^8}, 1^{2^8})$ . Number of trees: **1**.

$$a = (1, 8)(2, 6)(4, 9)(7, 10) \quad b = (3, 4, 9, 7, 10, 5, 6, 8)$$

Field:  $\mathbb{Q}$ . This example is rich with various combinatorial invariants. There exist, in total, 16 trees with this passport. One of them is our special tree. Two trees are symmetric, with the symmetry of order 2; they are defined over the field  $\mathbb{Q}(\sqrt{-1})$ . The 13 remaining trees split into two Galois orbits, of size 5 and 8, respectively. The orbit of size 5 contains self-dual trees, and the orbit of size 8 contains not self-dual ones.

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**PTL<sub>2</sub>(9) of order 1440** PrimitiveGroup(10,7)

**10.4.**  $(1^{4^2}, 2^{1^8}, 1^{2^8})$ . Number of trees: **1**.

$$a = (1, 4)(2, 5)(3, 6) \quad b = (1, 2, 3, 6, 9, 8, 7, 4)(5, 10)$$

Field:  $\mathbb{Q}$ . There exist nine more trees with this passport; they form a unique Galois orbit of degree 9.

**10.5.**  $(1^{4^2}, 1^{2^4}, 10^1)$ . Number of trees: **1**.

$$a = (1, 4)(2, 5)(3, 6) \quad b = (1, 5, 6, 9)(2, 10, 7, 8)$$

Field:  $\mathbb{Q}$ . There exist eight more trees with this passport. Three of them are symmetric, with a symmetry of order 2; they are defined over the splitting field of the polynomial  $a^3 - a^2 - 8a + 112$ . The five remaining trees form a Galois orbit of degree 5.

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**L<sub>2</sub>(11) of order 660 acting on 11 points** PrimitiveGroup(11,5)

**11.1.**  $(1^{3^4}, 1^{2^3}, 11^1)$ . Number of trees: **2**.

$$a = (1, 7)(2, 8)(5, 10)(6, 9) \quad b = (3, 5, 11)(4, 9, 7)(6, 10, 8)$$

Field:  $\mathbb{Q}(\sqrt{-11})$ . There exist eight more trees with this passport; they form a unique orbit of degree 8.

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**M<sub>11</sub> of order 7920** PrimitiveGroup(11,6)

**11.2.**  $(1^{3^4}, 1^{3^4}, 11^1)$ . Number of trees: **2**.

$$a = (1, 7)(3, 10)(5, 11)(6, 8) \quad b = (2, 7, 9, 10)(3, 4, 8, 5)$$

Field:  $\mathbb{Q}(\sqrt{-11})$ . There exist eight more trees with this passport; they form a unique orbit of degree 8.

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**M<sub>11</sub> of order 7920 acting on 12 points** PrimitiveGroup(12,1)

**12.1.**  $(1^{4^4}, 2^{2^4}, 1^{11^1})$ . Number of trees: **2**.

$$a = (1, 4)(2, 3)(5, 10)(6, 9) \quad b = (1, 11, 7, 2)(3, 5)(4, 12, 6, 9)(8, 10)$$

Field:  $\mathbb{Q}(\sqrt{-11})$ . There exist 28 more trees with this passport; they form a unique Galois orbit of degree 28.

**12.2.**  $(1^{4^4}, 1^{2^5}, 1^{11^1})$ . Number of trees: **2**.

$$a = (1, 10)(2, 12)(4, 7)(5, 11) \quad b = (1, 6, 11, 9, 8)(2, 12, 3, 5, 7)$$

Field:  $\mathbb{Q}(\sqrt{-11})$ . There exist, in total, 45 trees with this passport. Two of them compose the orbit 12.2 with group  $M_{11}$ , two more compose the orbit 12.7 with group  $M_{12}$ , both orbits are defined over  $\mathbb{Q}(\sqrt{-11})$ . All the 41 remaining trees form a unique Galois orbit of degree 41.

**12.3.**  $(1^4 2^4, 1^1 2^1 3^1 6^1, 1^1 11^1)$ . Number of trees: **6**.

$$\begin{aligned} a_1 &= (1, 10)(2, 11)(3, 6)(5, 12) & b_1 &= (1, 5, 7, 6, 8, 4)(2, 11, 10)(9, 12) \\ a_2 &= (1, 12)(2, 3)(6, 7)(8, 10) & b_2 &= (1, 5, 7, 6, 8, 4)(2, 11, 10)(9, 12) \\ a_3 &= (1, 3)(2, 12)(5, 9)(6, 7) & b_3 &= (1, 5, 7, 6, 8, 4)(2, 11, 10)(9, 12) \end{aligned}$$

There exist, in total, 150 trees with this passport. Six of them compose the orbit 12.3 with group  $M_{11}$ , two more trees compose the orbit 12.8 with group  $M_{12}$ . The calculations for so large orbits are difficult to carry out. We believe that the orbit 12.8 is defined over  $\mathbb{Q}(\sqrt{-11})$  and the orbit 12.3 is defined over a cubic extension of  $\mathbb{Q}(\sqrt{-11})$ . We can say nothing about the 142 remaining trees.

**$M_{12}$  of order 95 040**

PrimitiveGroup(12,2)

**12.4.**  $(1^4 2^4, 3^4, 1^1 11^1)$ . Number of trees: **2**.

$$a = (1, 8)(3, 9)(6, 12)(10, 11) \quad b = (1, 5, 6)(2, 8, 3)(4, 12, 7)(9, 11, 10)$$

Field:  $\mathbb{Q}(\sqrt{-11})$ . There exist three more trees with this passport, defined over the splitting field of the polynomial  $a^3 - a^2 - 216a - 1296$ .

**12.5.**  $(2^6, 1^3 3^3, 1^1 11^1)$ . Number of trees: **2**.

$$a = (1, 11)(2, 7)(3, 6)(4, 5)(8, 10)(9, 12) \quad b = (2, 7, 9)(3, 12, 10)(4, 11, 8)$$

Field:  $\mathbb{Q}(\sqrt{-11})$ . There are no other trees with this passport.

**12.6.**  $(2^6, 1^4 4^2, 1^1 11^1)$ . Number of trees: **2**.

$$a = (1, 6)(2, 5)(3, 12)(4, 8)(7, 10)(9, 11) \quad b = (2, 10, 6, 4)(3, 9, 11, 7)$$

Field:  $\mathbb{Q}(\sqrt{-11})$ . There are no other trees with this passport.

**12.7.**  $(1^4 2^4, 1^2 5^2, 1^1 11^1)$ . Number of trees: **2**.

$$a = (1, 10)(3, 5)(6, 12)(7, 9) \quad b = (1, 8, 9, 12, 2)(3, 5, 6, 4, 11)$$

Field:  $\mathbb{Q}(\sqrt{-11})$ . See comments to the orbit 12.2.

**12.8.**  $(1^4 2^4, 1^1 2^1 3^1 6^1, 1^1 11^1)$ . Number of trees: **2**.

$$a = (1, 12)(3, 8)(6, 9)(7, 10) \quad b = (2, 3, 9, 10, 7, 12)(4, 11, 8)(5, 6)$$

Field:  $\mathbb{Q}(\sqrt{-11})$ . See comments to the orbit 12.3.

**12.9.**  $(1^4 2^4, 1^2 2^1 8^1, 1^1 11^1)$ . Number of trees: **4**.

$$\begin{aligned} a_1 &= (1, 11)(3, 8)(5, 12)(9, 10) & b_1 &= (1, 12)(2, 3, 10, 9, 6, 11, 4, 7) \\ a_2 &= (1, 4)(3, 10)(5, 11)(6, 8) & b_2 &= (1, 12)(2, 3, 10, 9, 6, 11, 4, 7) \end{aligned}$$

There exist, in total, 90 trees with this passport. We believe that this orbit is defined over a quadratic extension of  $\mathbb{Q}(\sqrt{-11})$ .

**12.10.**  $(1^3 3^3, 1^3 3^3, 1^1 11^1)$ . Number of trees: **2**.

$$a = (1, 9, 10)(3, 4, 12)(5, 8, 6) \quad b = (2, 7, 9)(3, 12, 10)(4, 11, 8)$$

Field:  $\mathbb{Q}(\sqrt{-11})$ . There exist 34 more trees with this passport.

**12.11.**  $(1^3 3^3, 1^4 4^2, 1^1 11^1)$ . Number of trees: **2**.

$$a = (1, 6, 11)(3, 5, 9)(8, 10, 12) \quad b = (2, 10, 6, 4)(3, 9, 11, 7)$$

Field:  $\mathbb{Q}(\sqrt{-11})$ . There exist 24 more trees with this passport.

**12.12.**  $(1^4 4^2, 1^4 4^2, 1^1 11^1)$ . Number of trees: **2**.

$$a = (1, 12, 11, 5)(3, 10, 8, 9) \quad b = (2, 10, 6, 4)(3, 9, 11, 7)$$

Field:  $\mathbb{Q}(\sqrt{-11})$ . There exist 14 more trees with this passport.

---

**PGL<sub>2</sub>(11) of order 1320**

PrimitiveGroup(12,4)

**12.13.**  $(1^2 2^5, 3^4, 1^2 10^1)$ . Number of trees: **2**.

$$\begin{aligned} a_1 &= (2, 9)(3, 7)(5, 10)(6, 11)(8, 12) \\ b_1 &= (1, 7, 8)(2, 4, 12)(3, 6, 11)(5, 9, 10) \\ a_2 &= (2, 4)(3, 11)(5, 7)(6, 9)(10, 12) \\ b_2 &= (1, 7, 8)(2, 4, 12)(3, 6, 11)(5, 9, 10) \end{aligned}$$

Field:  $\mathbb{Q}(\sqrt{5})$ . There exist two more trees with this passport; they are both symmetric with the symmetry of order 2 and defined over  $\mathbb{Q}(\sqrt{-1})$ .

**12.14.**  $(1^2 2^5, 1^2 5^2, 1^2 10^1)$ . Number of trees: **2**.

$$\begin{aligned} a_1 &= (1, 7)(3, 10)(4, 8)(6, 11)(9, 12) & b_1 &= (2, 12, 9, 7, 4)(3, 6, 11, 5, 8) \\ a_2 &= (1, 9)(3, 4)(5, 6)(8, 11)(10, 12) & b_2 &= (2, 7, 12, 4, 9)(3, 5, 6, 8, 11) \end{aligned}$$

Field:  $\mathbb{Q}(\sqrt{5})$ . There exist 34 more trees with this passport. Six of them are symmetric with the symmetry of order 2, but they split into two Galois orbits since two trees have the edge rotation group of order 3840, while the other four have the group of order 23040. All the 28 remaining trees form a unique Galois orbit of degree 28.

---

**L<sub>3</sub>(3) of order 5616**

PrimitiveGroup(13,7)

**13.1.**  $(1^5 2^4, 1^1 3^4, 13^1)$ . Number of trees: **4**.

$$\begin{aligned} a_1 &= (1, 3)(2, 12)(5, 7)(6, 10) \\ b_1 &= (1, 10, 12)(2, 9, 7)(3, 8, 13)(4, 5, 11) \\ a_2 &= (1, 3)(2, 8)(4, 7)(6, 11) \\ b_2 &= (1, 10, 12)(2, 9, 7)(3, 8, 13)(4, 5, 11) \end{aligned}$$

**13.2.**  $(1^5 2^4, 1^1 2^2 4^2, 13^1)$ . Number of trees: **4**.

$$\begin{aligned} a_1 &= (2, 4)(3, 9)(5, 8)(7, 12) \\ b_1 &= (1, 11, 3, 8)(2, 10, 6, 12)(4, 13)(5, 7) \\ a_2 &= (2, 4)(5, 6)(9, 13)(11, 12) \\ b_2 &= (1, 11, 3, 8)(2, 10, 6, 12)(4, 13)(5, 7) \end{aligned}$$

**13.3.**  $(1^5 2^4, 1^2 2^1 3^1 6^1, 13^1)$ . Number of trees: **4**.

$$\begin{aligned} a_1 &= (1, 9)(2, 12)(4, 5)(6, 11) \\ b_1 &= (1, 8, 5)(2, 7)(3, 11, 12, 13, 4, 10) \\ a_2 &= (2, 8)(3, 9)(4, 5)(6, 10) \\ b_2 &= (1, 8, 5)(2, 7)(3, 11, 12, 13, 4, 10) \end{aligned}$$

---

**L<sub>2</sub>(13) of order 1092** PrimitiveGroup(14,1)

**14.1.**  $(1^{22^6}, 1^2 3^4, 1^1 13^1)$ . Number of trees: **1**.  
 $a = (1, 10)(2, 12)(3, 14)(4, 6)(5, 11)(7, 8)$   
 $b = (2, 14, 12)(3, 7, 6)(4, 9, 10)(8, 11, 13)$

---

**PGL<sub>2</sub>(13) of order 2184** PrimitiveGroup(14,2)

**14.2.**  $(2^7, 1^2 3^4, 1^2 12^1)$ . Number of trees: **2**.  
 $a_1 = (1, 7)(2, 11)(3, 9)(4, 6)(5, 10)(8, 13)(12, 14)$   
 $b_1 = (1, 7, 14)(2, 11, 6)(3, 13, 4)(9, 12, 10)$   
 $a_2 = (1, 8)(2, 12)(3, 13)(4, 10)(5, 7)(6, 11)(9, 14)$   
 $b_2 = (1, 7, 14)(2, 11, 6)(3, 13, 4)(9, 12, 10)$

**14.3.**  $(1^{22^6}, 1^{24^3}, 1^2 12^1)$ . Number of trees: **2**.  
 $a_1 = (1, 13)(2, 14)(3, 10)(4, 7)(5, 12)(8, 11)$   
 $b_1 = (1, 13, 12, 2)(3, 9, 6, 14)(4, 10, 11, 7)$   
 $a_2 = (1, 2)(3, 9)(4, 8)(5, 7)(10, 14)(11, 13)$   
 $b_2 = (1, 13, 12, 2)(3, 9, 6, 14)(4, 10, 11, 7)$

---

**L<sub>4</sub>(2) of order 20 160** PrimitiveGroup(15,4)

**15.1.**  $(1^{32^6}, 1^{32^2} 4^2, 15^1)$ . Number of trees: **2**.  
 $a = (1, 7)(2, 9)(3, 14)(4, 15)(5, 8)(10, 12)$   
 $b = (1, 2, 14, 13)(3, 12)(4, 7, 11, 8)(6, 9)$

**15.2.**  $(1^{72^4}, 1^{12^1} 4^3, 15^1)$ . Number of trees: **2**.  
 $a = (2, 3)(6, 7)(10, 11)(14, 15)$   
 $b = (1, 13, 8, 11)(2, 7, 4, 14)(3, 10, 12, 5)(6, 9)$

**15.3.**  $(1^{72^4}, 1^{12^1} 3^2 6^1, 15^1)$ . Number of trees: **2**.  
 $a = (1, 2)(4, 7)(9, 10)(12, 15)$   
 $b = (1, 15, 14)(2, 5)(3, 10, 12, 4, 13, 11)(6, 8, 9)$

---

**AGL<sub>4</sub>(2) = 2<sup>4</sup>.L<sub>4</sub>(2) of order 322 560** PrimitiveGroup(16,11)

**16.1.**  $(1^{42^6}, 2^4 4^2, 1^{11} 15^1)$ . Number of trees: **2**.  
 $a = (2, 9)(4, 11)(5, 7)(6, 15)(8, 13)(14, 16)$   
 $b = (1, 7, 16, 10)(2, 5)(3, 9)(4, 11, 13, 6)(8, 14)(12, 15)$

**16.2.**  $(1^{82^4}, 1^{15^3}, 1^{11} 15^1)$ . Number of trees: **2**.  
 $a = (1, 15)(3, 13)(5, 11)(7, 9)$   
 $b = (2, 11, 4, 16, 8)(3, 6, 5, 15, 14)(7, 12, 10, 13, 9)$

**16.3.**  $(1^{82^4}, 1^{13^1} 6^2, 1^{11} 15^1)$ . Number of trees: **2**.  
 $a = (1, 16)(3, 14)(5, 12)(7, 10)$   
 $b = (2, 16, 10, 4, 9, 13)(3, 8, 6)(5, 14, 15, 7, 11, 12)$

**16.4.**  $(1^4 2^6, 1^2 2^1 3^2 6^1, 1^1 15^1)$ . Number of trees: **6**.

$$\begin{aligned} a_1 &= (1, 13)(3, 15)(5, 7)(6, 12)(8, 10)(9, 11) \\ b_1 &= (2, 4, 6, 9, 11, 13)(3, 7, 14)(5, 12, 16)(8, 15) \\ a_2 &= (1, 13)(4, 16)(5, 12)(6, 7)(8, 9)(10, 11) \\ b_2 &= (2, 4, 6, 9, 11, 13)(3, 7, 14)(5, 12, 16)(8, 15) \\ a_3 &= (1, 6)(4, 7)(9, 12)(10, 16)(11, 13)(14, 15) \\ b_3 &= (2, 4, 6, 9, 11, 13)(3, 7, 14)(5, 12, 16)(8, 15) \end{aligned}$$

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**AGL<sub>2</sub>(4) of order 5760** PrimitiveGroup(16,12)

**16.5.**  $(1^4 2^6, 1^2 2^1 4^3, 1^1 15^1)$ . Number of trees: **2**.

$$\begin{aligned} a &= (1, 15)(2, 12)(4, 8)(5, 11)(6, 16)(10, 14) \\ b &= (3, 16, 6, 10)(4, 15, 5, 9)(7, 8)(11, 13, 12, 14) \end{aligned}$$

---

**L<sub>2</sub>(16) of order 4080** PrimitiveGroup(17,6)

**17.1.**  $(1^1 2^8, 1^2 3^5, 1^2 15^1)$ . Number of trees: **1**.

$$\begin{aligned} a &= (1, 11)(2, 10)(3, 9)(4, 8)(5, 7)(12, 17)(13, 16)(14, 15) \\ b &= (1, 14, 11)(2, 7, 15)(3, 8, 12)(4, 6, 5)(9, 16, 13) \end{aligned}$$

---

**L<sub>2</sub>(16) : 2 of order 8160** PrimitiveGroup(17,7)

**17.2.**  $(1^5 2^6, 1^1 4^4, 1^2 15^1)$ . Number of trees: **1**.

$$\begin{aligned} a &= (1, 13)(2, 6)(3, 10)(4, 11)(8, 12)(15, 16) \\ b &= (2, 7, 10, 6)(3, 12, 15, 13)(4, 11, 16, 5)(8, 17, 14, 9) \end{aligned}$$

---

**PGL<sub>2</sub>(19) of order 6840** PrimitiveGroup(20,2)

**20.1.**  $(1^2 2^9, 1^2 3^6, 1^2 18^1)$ . Number of trees: **3**.

$$\begin{aligned} a_1 &= (1, 8)(2, 12)(4, 18)(5, 7)(6, 9)(10, 19)(11, 14)(13, 15)(16, 20) \\ b_1 &= (1, 8, 5)(3, 4, 17)(6, 9, 19)(7, 15, 14)(10, 16, 12)(11, 18, 20) \\ a_2 &= (1, 9)(2, 20)(4, 14)(5, 10)(6, 19)(7, 15)(11, 17)(12, 18)(13, 16) \\ b_2 &= (1, 8, 5)(3, 4, 17)(6, 9, 19)(7, 15, 14)(10, 16, 12)(11, 18, 20) \\ a_3 &= (1, 16)(2, 6)(4, 12)(5, 9)(7, 14)(8, 15)(10, 18)(11, 20)(13, 17) \\ b_3 &= (1, 8, 5)(3, 4, 17)(6, 9, 19)(7, 15, 14)(10, 16, 12)(11, 18, 20) \end{aligned}$$

---

**PGL<sub>3</sub>(4) of order 120 960** PrimitiveGroup(21,7)

**21.1.**  $(1^7 2^7, 1^3 2^1 4^4, 21^1)$ . Number of trees: **2**.

$$\begin{aligned} a &= (1, 16)(4, 17)(5, 20)(6, 18)(10, 11)(12, 13)(14, 21) \\ b &= (2, 13, 16, 11)(3, 10, 9, 7)(4, 15, 20, 12)(5, 19)(6, 14, 8, 17) \end{aligned}$$

Field:  $\mathbb{Q}(\sqrt{-7})$  (see [6]).

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**M<sub>23</sub> of order 10 200 960**

PrimitiveGroup(23,5)

**23.1.** ( $1^7 2^8, 1^3 2^2 4^4, 23^1$ ). Number of trees: **4**.

$$a_1 = (2, 5)(3, 21)(4, 17)(8, 16)(10, 18)(11, 19)(12, 23)(14, 20)$$

$$b_1 = (1, 14, 17, 12)(3, 20, 7, 6)(5, 9, 16, 22)(8, 15, 10, 23)(11, 21)(13, 18)$$

$$a_2 = (2, 6)(3, 13)(4, 17)(8, 9)(10, 12)(11, 18)(15, 20)(19, 23)$$

$$b_2 = (1, 14, 17, 12)(3, 20, 7, 6)(5, 9, 16, 22)(8, 15, 10, 23)(11, 21)(13, 18)$$

Field:  $\mathbb{Q}\left(\sqrt{-23/2 - (5/2)\sqrt{-23}}\right)$  (see [14]).

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**M<sub>24</sub> of order 244 823 040**

PrimitiveGroup(24,1)

**24.1.** ( $2^{12}, 1^6 3^6, 1^1 23^1$ ). Number of trees: **2**.

$$a = (1, 19)(2, 24)(3, 12)(4, 5)(6, 8)(7, 22)$$

$$(9, 10)(11, 14)(13, 20)(15, 23)(16, 18)(17, 21)$$

$$b = (1, 14, 18)(4, 7, 21)(6, 10, 9)(8, 20, 23)(11, 17, 13)(12, 24, 16)$$

**24.2.** ( $1^8 2^8, 3^8, 1^1 23^1$ ). Number of trees: **2**.

$$a = (1, 5)(2, 9)(3, 11)(6, 13)(7, 17)(8, 16)(15, 20)(18, 23)$$

$$b = (1, 22, 23)(2, 24, 4)(3, 16, 8)(5, 10, 13)$$

$$(6, 14, 11)(7, 21, 19)(9, 15, 12)(17, 18, 20)$$

**24.3.** ( $1^8 2^8, 1^4 5^4, 1^1 23^1$ ). Number of trees: **2**.

$$a = (1, 6)(2, 13)(3, 19)(5, 9)(7, 10)(8, 21)(12, 20)(18, 23)$$

$$b = (1, 12, 4, 17, 8)(2, 22, 3, 15, 5)(7, 13, 20, 14, 24)(11, 23, 18, 16, 21)$$

**24.4.** ( $1^8 2^8, 1^2 2^2 3^2 6^2, 1^1 23^1$ ). Number of trees: **10**.

$$a_1 = (1, 7)(2, 6)(3, 19)(5, 24)(10, 13)(11, 23)(15, 21)(16, 20)$$

$$b_1 = (1, 17, 8, 14, 13, 9)(2, 19)(3, 22, 5)(4, 21)(6, 10, 12, 23, 20, 16)(11, 18, 15)$$

$$a_2 = (1, 7)(2, 21)(3, 16)(5, 24)(6, 15)(8, 14)(9, 18)(19, 20)$$

$$b_2 = (1, 17, 8, 14, 13, 9)(2, 19)(3, 22, 5)(4, 21)(6, 10, 12, 23, 20, 16)(11, 18, 15)$$

$$a_3 = (1, 22)(2, 13)(3, 5)(6, 14)(7, 23)(8, 24)(15, 16)(17, 21)$$

$$b_3 = (1, 17, 8, 14, 13, 9)(2, 19)(3, 22, 5)(4, 21)(6, 10, 12, 23, 20, 16)(11, 18, 15)$$

$$a_4 = (1, 7)(2, 16)(3, 21)(5, 18)(8, 14)(9, 24)(10, 22)(12, 13)$$

$$b_4 = (1, 17, 8, 14, 13, 9)(2, 19)(3, 22, 5)(4, 21)(6, 10, 12, 23, 20, 16)(11, 18, 15)$$

$$a_5 = (1, 16)(2, 24)(4, 20)(6, 19)(7, 12)(8, 18)(9, 13)(21, 22)$$

$$b_5 = (1, 17, 8, 14, 13, 9)(2, 19)(3, 22, 5)(4, 21)(6, 10, 12, 23, 20, 16)(11, 18, 15)$$

**24.5.** ( $1^6 3^6, 1^6 3^6, 1^1 23^1$ ). Number of trees: **2**.

$$a = (1, 13, 6)(2, 23, 5)(3, 4, 8)(10, 12, 16)(15, 17, 19)(20, 24, 22)$$

$$b = (1, 14, 18)(4, 7, 21)(6, 10, 9)(8, 20, 23)(11, 17, 13)(12, 24, 16)$$

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**$L_5(2)$  of order 9 999 360**

PrimitiveGroup(31,10)

**31.1.**  $(1^7 2^{12}, 1^3 2^6 4^4, 31^1)$ . Number of trees: **6**.

$$\begin{aligned} a_1 &= (2, 24)(5, 11)(7, 12)(8, 23)(10, 29)(13, 19)(14, 15) \\ &\quad (16, 18)(17, 27)(20, 28)(22, 26)(25, 31) \\ b_1 &= (1, 19)(3, 8)(4, 9, 10, 16)(6, 27)(7, 24, 22, 15)(11, 18) \\ &\quad (12, 23)(13, 17, 14, 30)(20, 29)(21, 26, 25, 28) \\ a_2 &= (2, 13)(5, 15)(7, 20)(8, 26)(10, 17)(11, 14)(12, 28) \\ &\quad (16, 25)(18, 31)(19, 24)(22, 23)(27, 29) \\ b_2 &= (1, 19)(3, 8)(4, 9, 10, 16)(6, 27)(7, 24, 22, 15)(11, 18) \\ &\quad (12, 23)(13, 17, 14, 30)(20, 29)(21, 26, 25, 28) \\ a_3 &= (2, 24)(5, 14)(7, 12)(8, 22)(10, 17)(11, 15)(13, 19) \\ &\quad (16, 25)(18, 31)(20, 28)(23, 26)(27, 29) \\ b_3 &= (1, 19)(3, 8)(4, 9, 10, 16)(6, 27)(7, 24, 22, 15)(11, 18) \\ &\quad (12, 23)(13, 17, 14, 30)(20, 29)(21, 26, 25, 28) \end{aligned}$$

Defined over the splitting field of the polynomial  $a^6 + a^5 + 3a^4 + 11a^3 + 44a^2 + 36a + 32$  (see [6]).

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**$ASL_5(2)$  of order 319 979 520**

PrimitiveGroup(32,3)

**32.1.**  $(1^8 2^{12}, 1^2 3^{10}, 1^1 31^1)$ . Number of trees: **6**.

$$\begin{aligned} a_1 &= (1, 23)(2, 24)(7, 17)(8, 18)(9, 25)(10, 26)(11, 13) \\ &\quad (12, 14)(15, 31)(16, 32)(27, 29)(28, 30) \\ b_1 &= (2, 29, 30)(3, 24, 16)(4, 12, 19)(5, 17, 21)(6, 13, 10) \\ &\quad (7, 8, 28)(9, 26, 18)(11, 15, 31)(14, 22, 25)(20, 32, 23) \\ a_2 &= (1, 19)(4, 18)(6, 24)(7, 21)(9, 15)(10, 30)(11, 31) \\ &\quad (12, 14)(13, 25)(16, 28)(26, 32)(27, 29) \\ b_2 &= (2, 29, 30)(3, 24, 16)(4, 12, 19)(5, 17, 21)(6, 13, 10) \\ &\quad (7, 8, 28)(9, 26, 18)(11, 15, 31)(14, 22, 25)(20, 32, 23) \\ a_3 &= (1, 3)(2, 16)(4, 14)(6, 10)(8, 12)(13, 15)(17, 31) \\ &\quad (18, 20)(19, 29)(21, 25)(23, 27)(30, 32) \\ b_3 &= (2, 29, 30)(3, 24, 16)(4, 12, 19)(5, 17, 21)(6, 13, 10) \\ &\quad (7, 8, 28)(9, 26, 18)(11, 15, 31)(14, 22, 25)(20, 32, 23) \end{aligned}$$



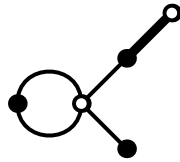
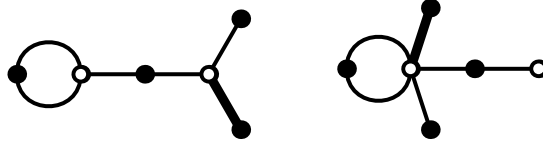


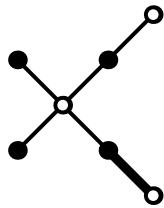
Figure 4: Group  $\mathbf{AGL}_1(5)$ : orbit 5.1 of size 2.



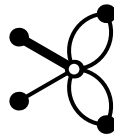
(a) Orbit 6.1.

(b) Orbit 6.2.

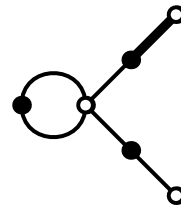
Figure 5: Group  $\mathbf{L}_2(5)$ : two orbits of size 1.



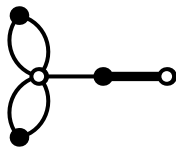
(a) Orbit 6.3.



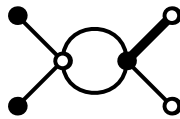
(b) Orbit 6.4.



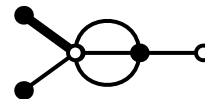
(c) Orbit 6.5.



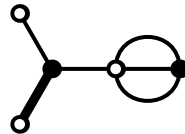
(d) Orbit 6.6.



(e) Orbit 6.7.



(f) Orbit 6.8.



(g) Orbit 6.9.

Figure 6: Group  $\mathbf{PGL}_2(5)$ : seven orbits of size 1.

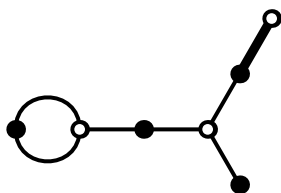
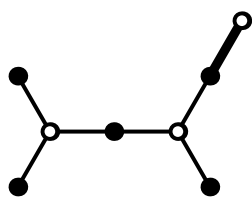
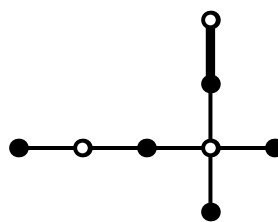


Figure 7: Group  $\mathbf{AGL}_1(7)$ : orbit 7.1 of size 2.



(a) Orbit 7.2.



(b) Orbit 7.3.

Figure 8: Group  $\mathbf{L}_3(2)$ : two orbits of size 2.

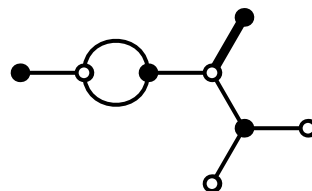
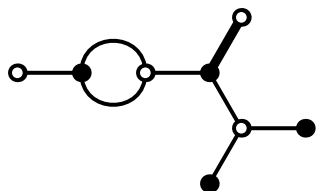
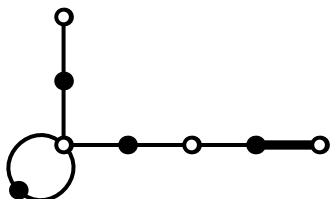
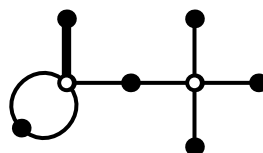


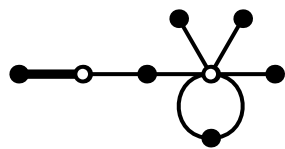
Figure 9: Group  $\mathbf{AFL}_1(8)$ : orbit 8.1 of size 4.



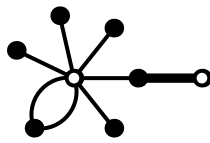
(a) Orbit 8.2.



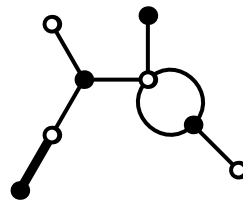
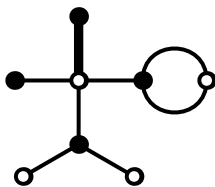
(b) Orbit 8.3.



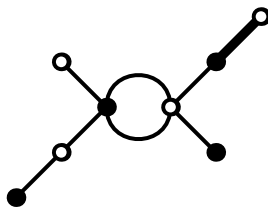
(c) Orbit 8.4.



(d) Orbit 8.5.

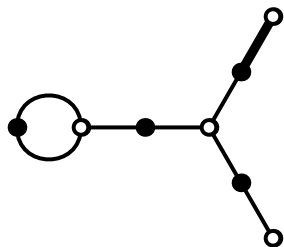


(e) Orbit 8.6.

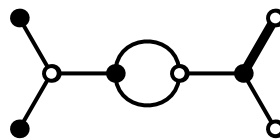


(f) Orbit 8.7.

Figure 10: Group  $ASL_3(2)$ : five orbits of size 2 and one orbit of size 4.

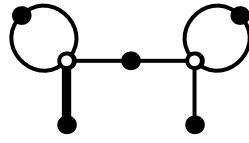


(a) Orbit 8.8.

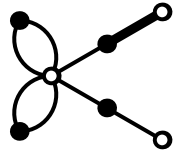


(b) Orbit 8.9.

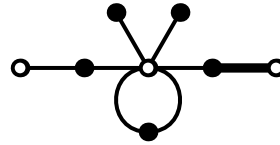
Figure 11: Group  $L_2(7)$ : two orbits of size 1.



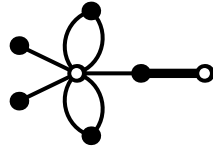
(a) Orbit 8.10.



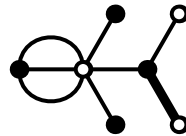
(b) Orbit 8.11.



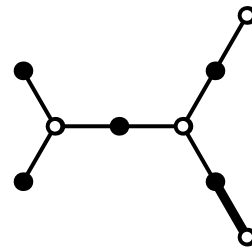
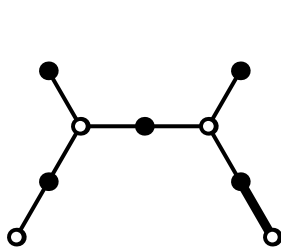
(c) Orbit 8.12.



(d) Orbit 8.13.



(e) Orbit 8.14.



(f) Orbit 8.15.

Figure 12: Group  $\mathbf{PGL}_2(7)$ : five orbits of size 1 and one orbit of size 2.

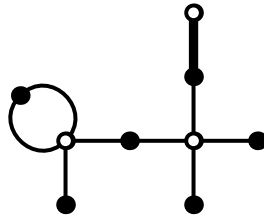
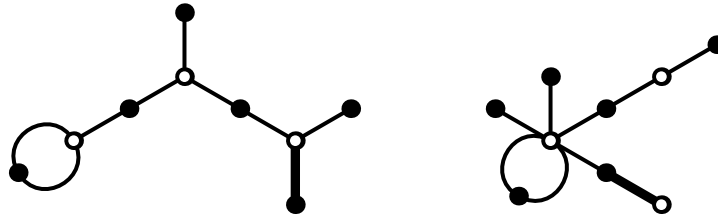


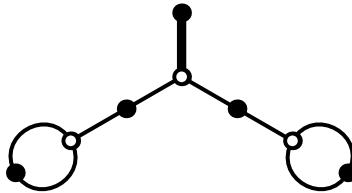
Figure 13: Group  $\mathbf{AGL}_1(9)$ : orbit 9.1 of size 2.



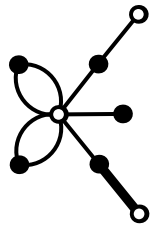
(a) Orbit 9.2.

(b) Orbit 9.3.

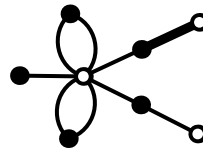
Figure 14: Group  $\mathbf{AGL}_2(3)$ : two orbits of size 2.



(a) Orbit 9.4.

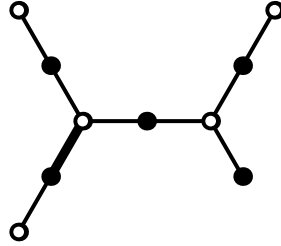


(b) Orbit 9.5(a).

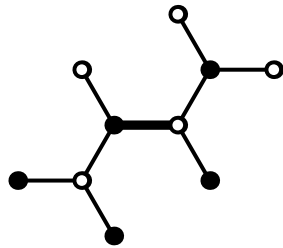


(c) Orbit 9.5(b).

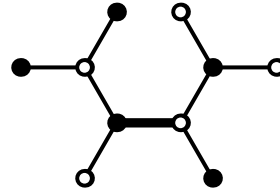
Figure 15: Group  $\mathbf{L}_2(8)$ : three orbits of size 1.



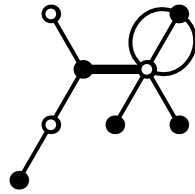
(a) Orbit 9.6.



(b) Orbit 9.7(a).

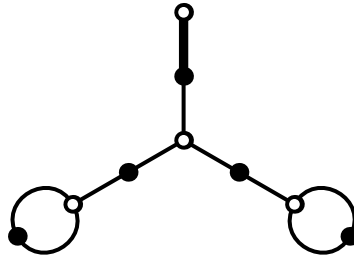


(c) Orbit 9.7(b).

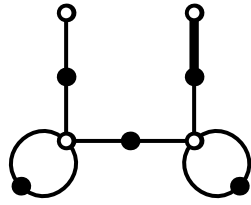


(d) Orbit 9.8.

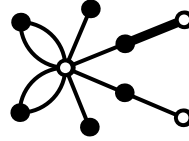
Figure 16: Group  $\mathbf{P}\Gamma\mathbf{L}_2(8)$ : three orbits of size 2 and one orbit of size 4.



(a) Orbit 10.1.

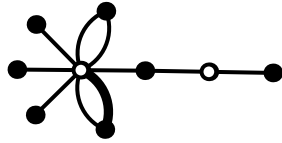


(b) Orbit 10.2.

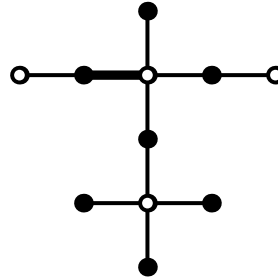


(c) Orbit 10.3.

Figure 17: Group  $\mathbf{PGL}_2(9)$ : three orbits of size 1.



(a) Orbit 10.4.



(b) Orbit 10.5.

Figure 18: Group  $\mathbf{PTL}_2(9)$ : two orbits of size 1.

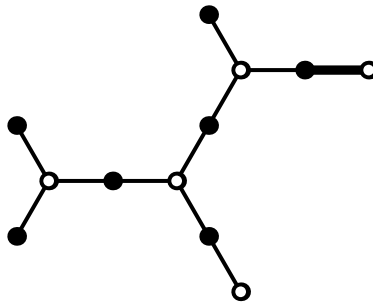


Figure 19: Group  $\mathbf{L}_2(11)$  acting on 11 points: orbit 11.1 of size 2.

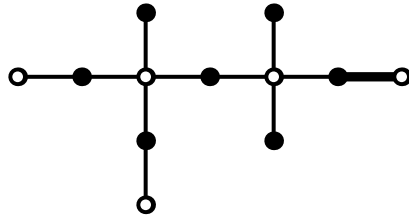
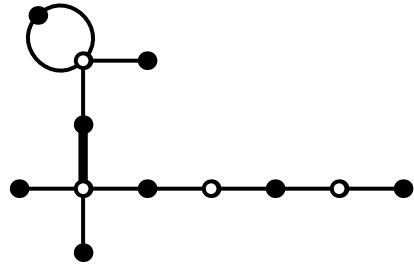
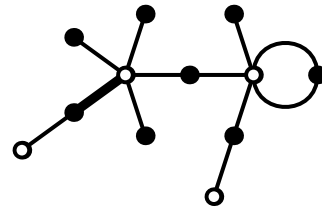


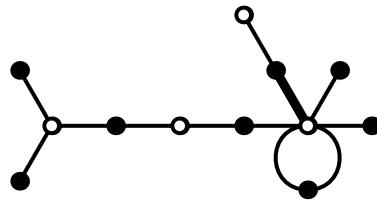
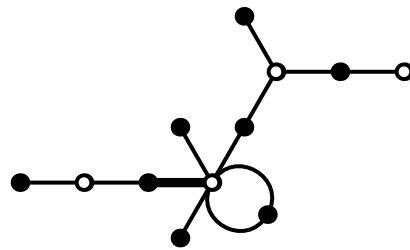
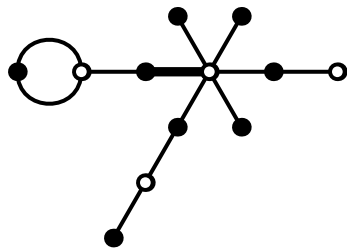
Figure 20: Group  $M_{11}$ : orbit 11.2 of size 2.



(a) Orbit 12.1.



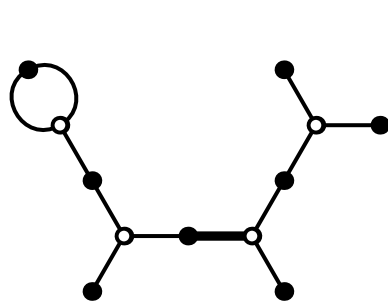
(b) Orbit 12.2.



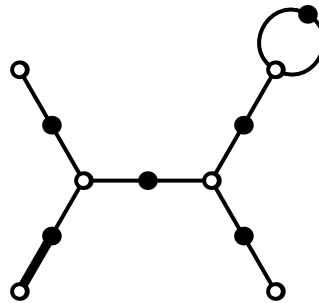
(c) Orbit 12.3.

Figure 21: Group  $M_{11}$  acting on 12 points: two orbits of size 2 and one orbit of size 6.

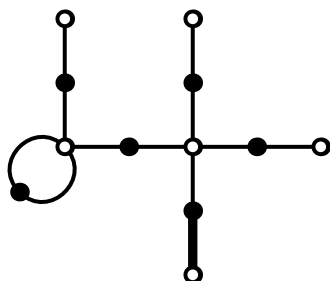




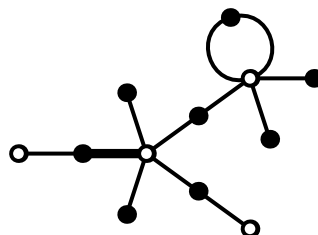
(a) Orbit 12.4.



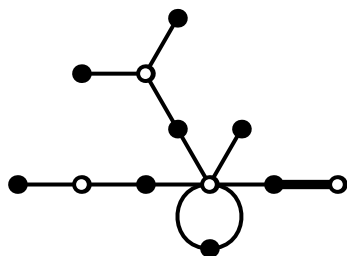
(b) Orbit 12.5.



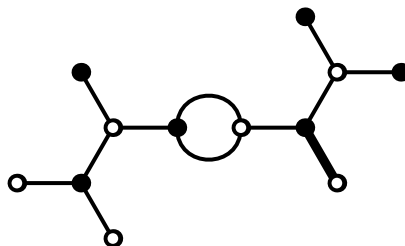
(c) Orbit 12.6.



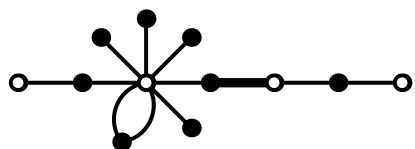
(d) Orbit 12.7.



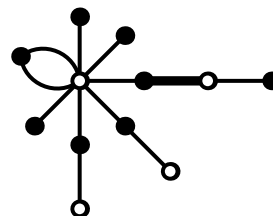
(e) Orbit 12.8.

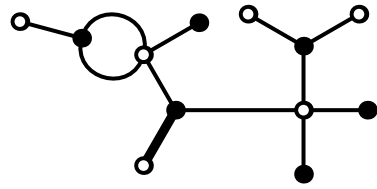


(g) Orbit 12.10.

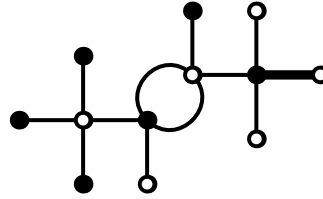


(f) Orbit 12.9.



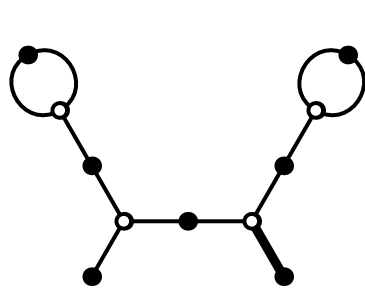


(h) Orbit 12.11.

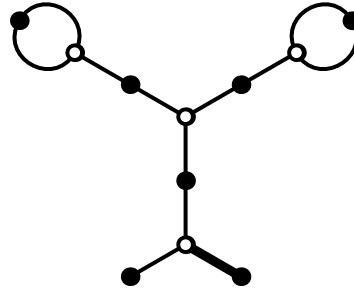


(i) Orbit 12.12.

Figure 22: Group  $M_{12}$ : 8 orbits of size 2 and one orbit of size 4.

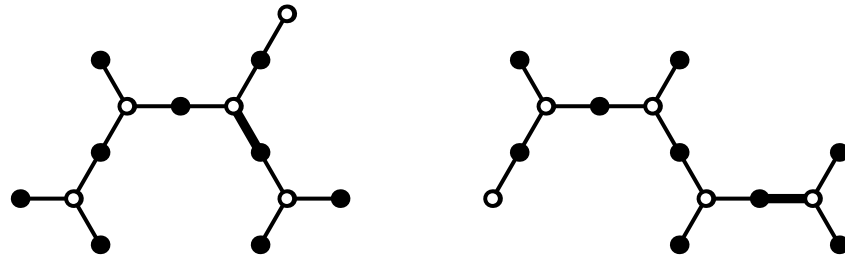


(a) Orbit 12.13.

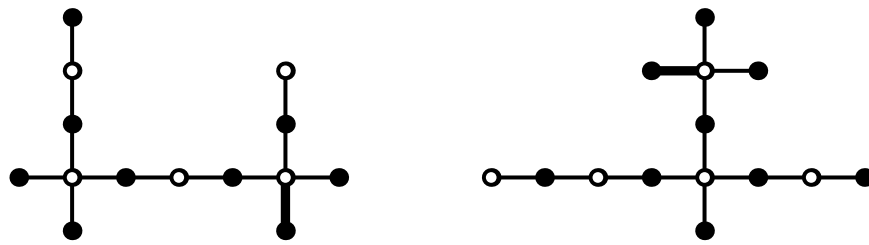


(b) Orbit 12.14.

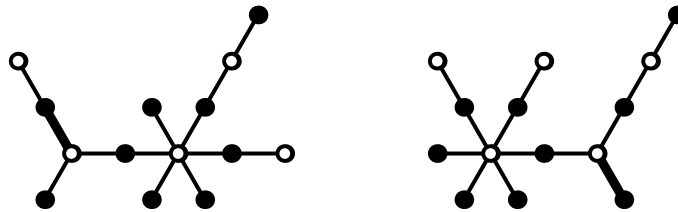
Figure 23: Group  $PGL_2(11)$ : two orbits of size 2.



(a) Orbit 13.1.



(b) Orbit 13.2.



(c) Orbit 13.3.

Figure 24: Group  $L_3(3)$ : three orbits of size 4.

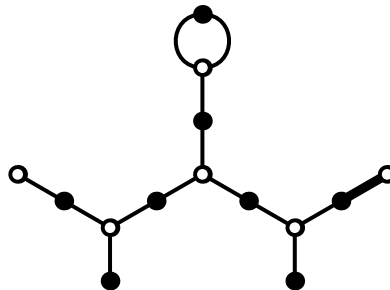
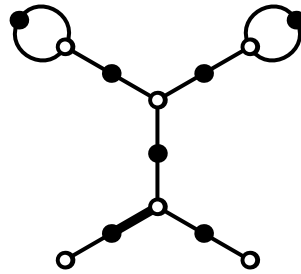
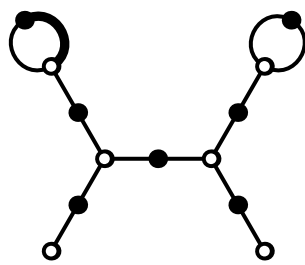
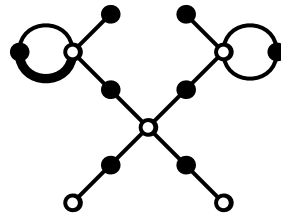
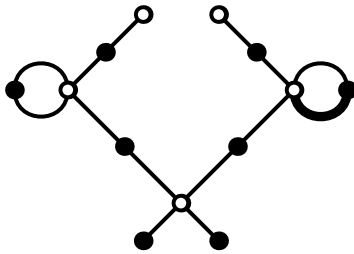


Figure 25: Group  $L_2(13)$ : orbit 14.1 of size 1.

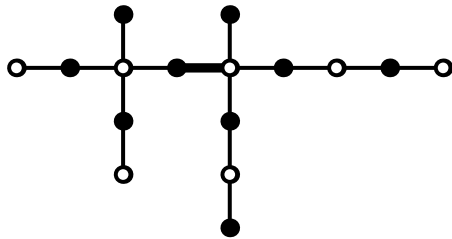


(a) Orbit 14.2.

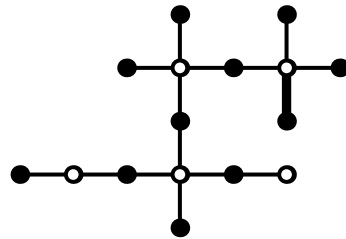


(b) Orbit 14.3.

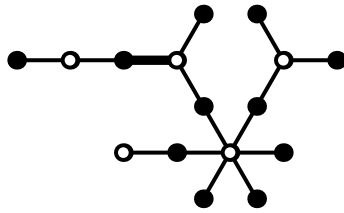
Figure 26: Group  $\mathbf{PGL}_2(13)$ : two orbits of size 2.



(a) Orbit 15.1.

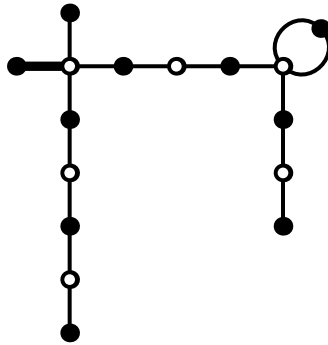


(b) Orbit 15.2.

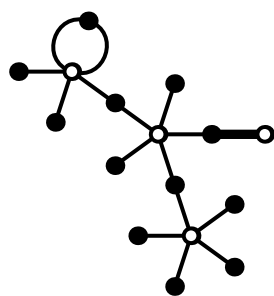


(c) Orbit 15.3.

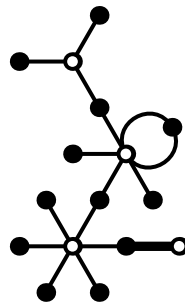
Figure 27: Group  $L_4(2)$ : three orbits of size 2.



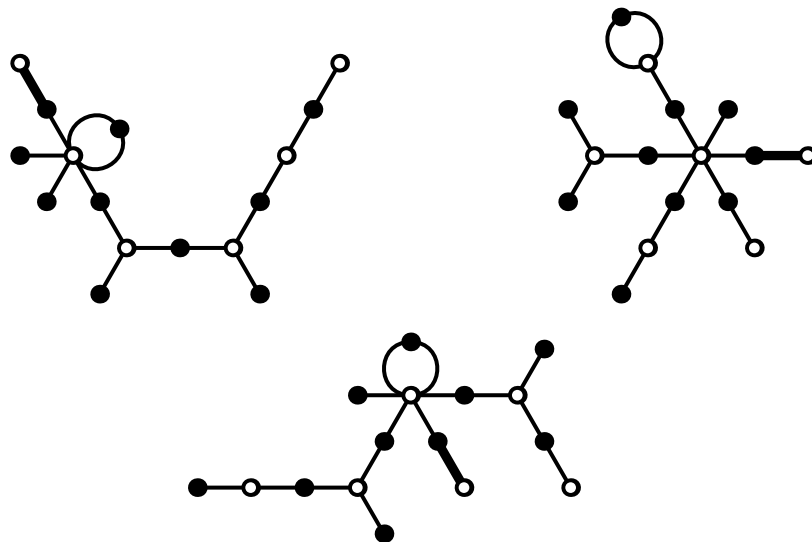
(a) Orbit 16.1.



(b) Orbit 16.2.



(c) Orbit 16.3.



(d) Orbit 16.4.

Figure 28: Group  $\text{AGL}_4(2)$ : three orbits of size 2 and one orbit of size 6.

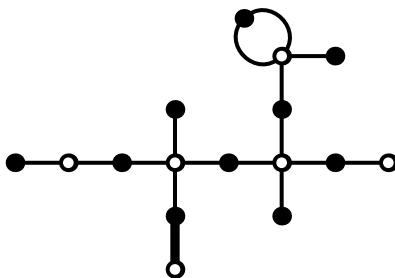


Figure 29: Group  $\text{ATL}_2(4)$ : orbit 16.5 of size 2.

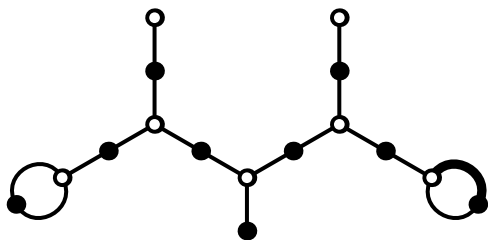


Figure 30: Group  $\text{L}_2(16)$ : orbit 17.1 of size 1.

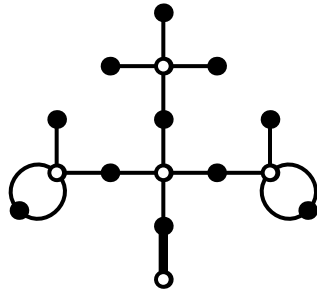


Figure 31: Group  $L_2(16)$  : 2: orbit 17.2 of size 1.

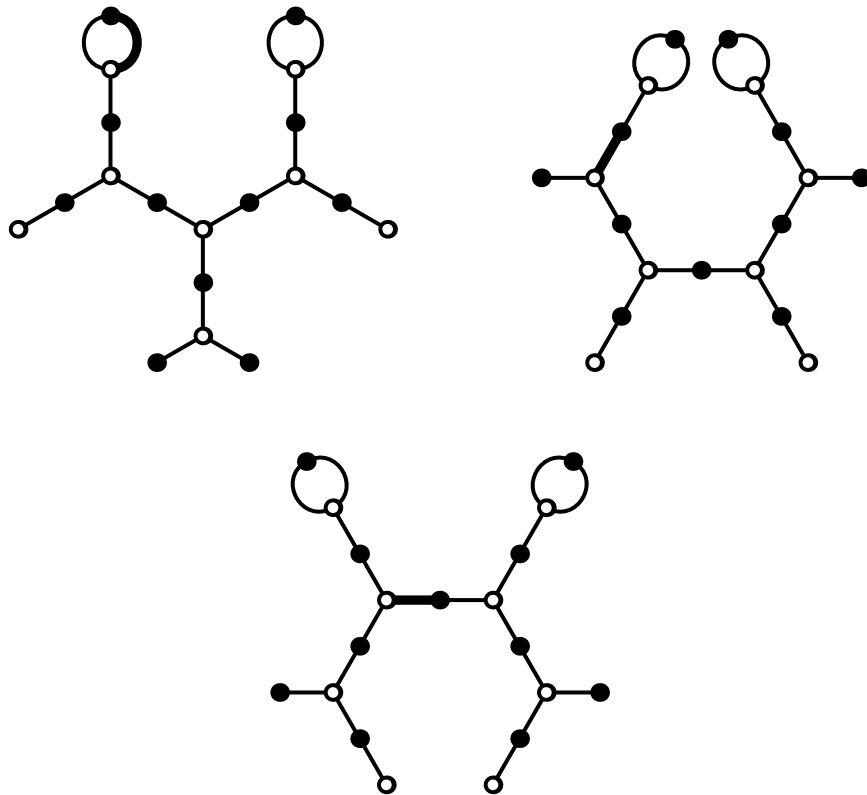


Figure 32: Group  $PGL_2(19)$ : orbit 20.1 of size 3.

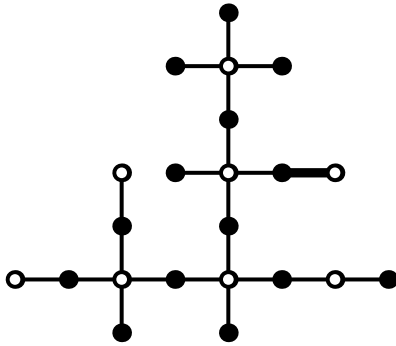


Figure 33: Group  $\mathbf{PTL}_3(4)$ : orbit 21.1 of size 2.

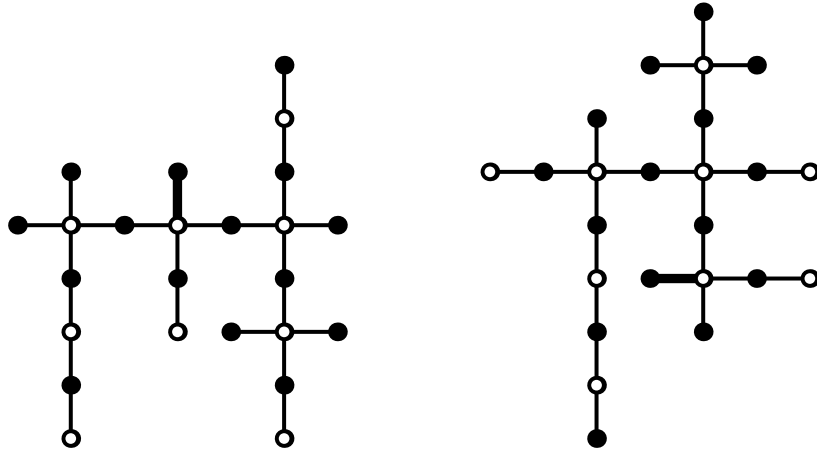


Figure 34: Group  $\mathbf{M}_{23}$ : orbit 23.1 of size 4.



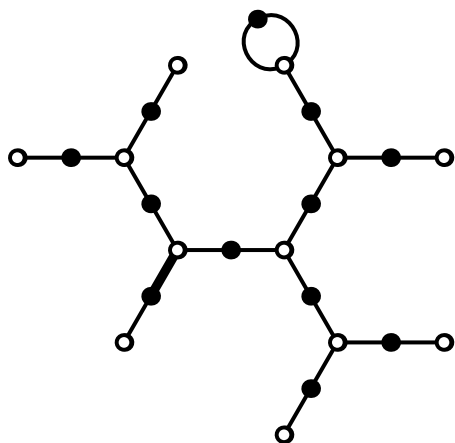


Figure 35: Group  $M_{24}$ : orbit 24.1 of size 2.

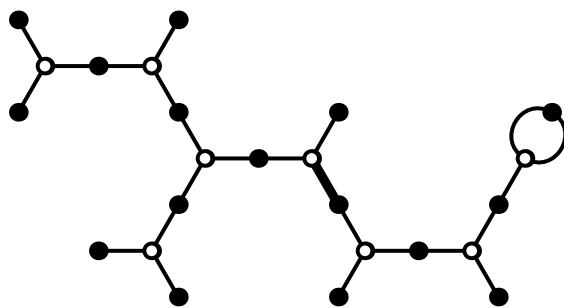


Figure 36: Group  $M_{24}$ : orbit 24.2 of size 2.

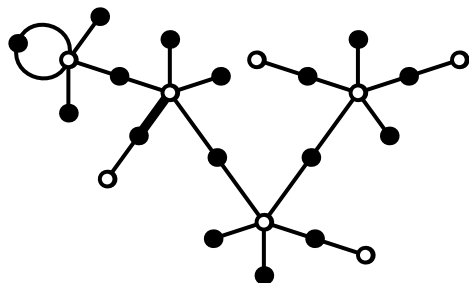


Figure 37: Group  $M_{24}$ : orbit 24.3 of size 2.

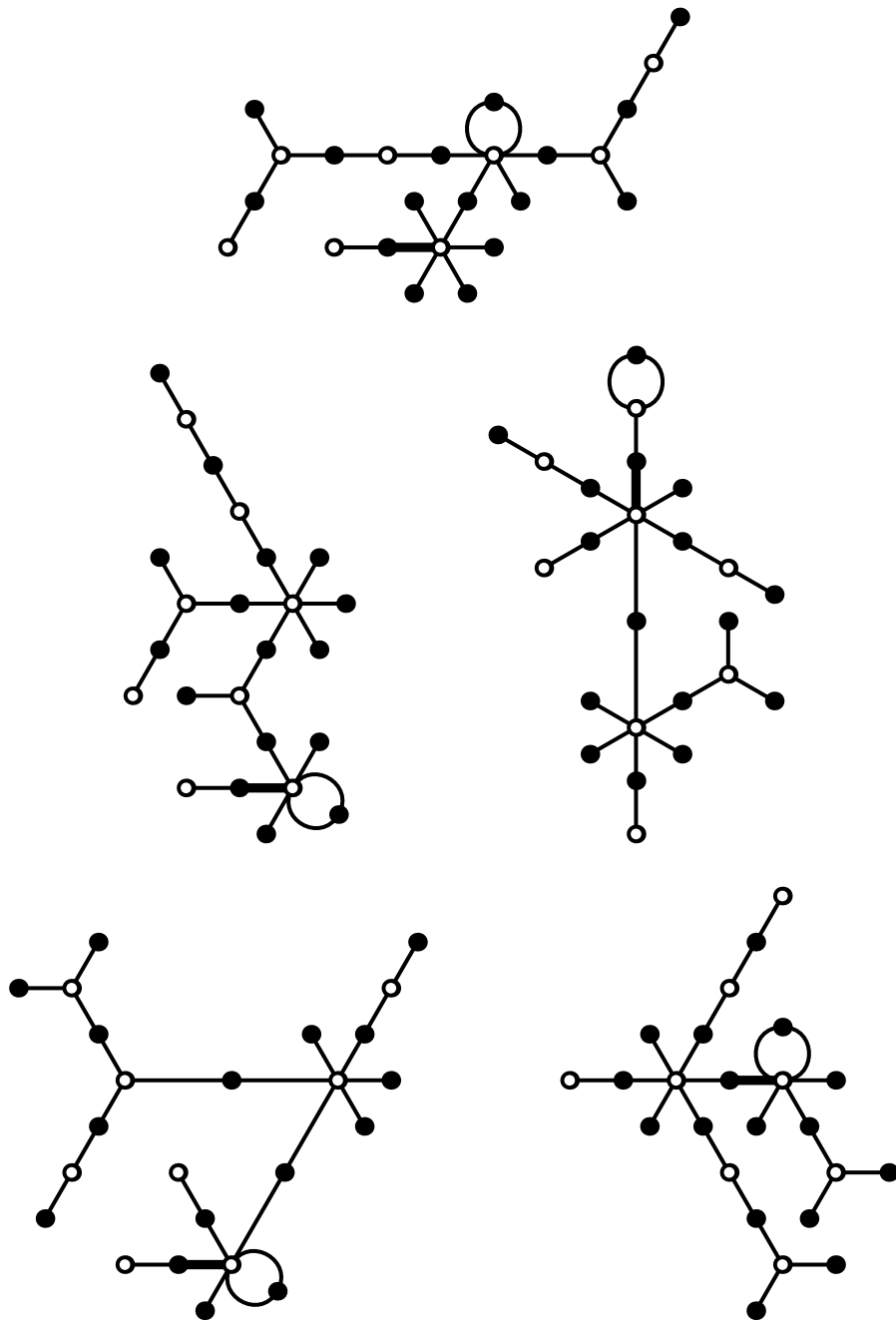


Figure 38: Group  $M_{24}$ : orbit 24.4 of size 10.

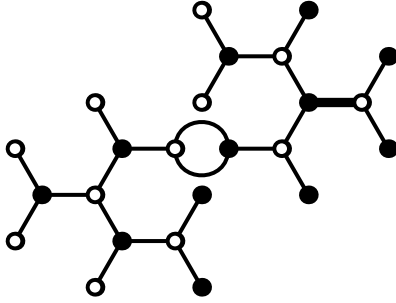


Figure 39: Group  $M_{24}$ : orbit 24.5 of size 2.

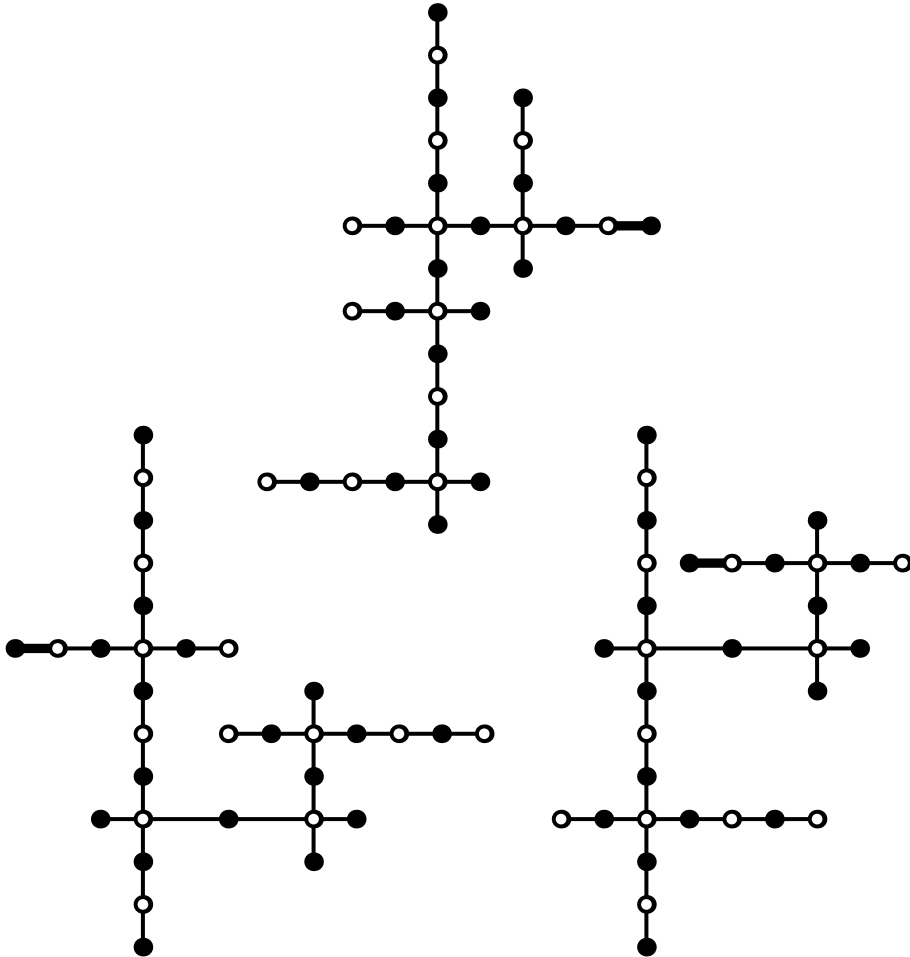


Figure 40: Group  $L_5(2)$ : orbit 31.1 of size 6.

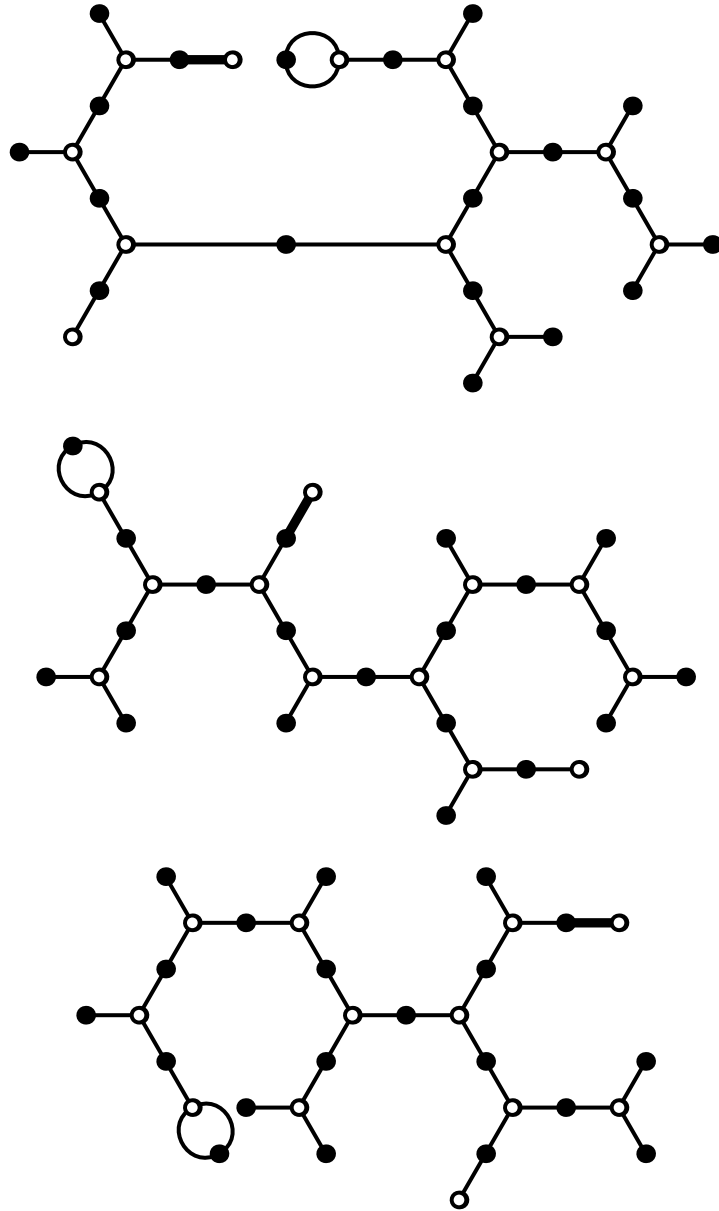


Figure 41: Group  $ASL_5(2)$ : orbit 32.1 of size 6.